# Teacher Resources: Grade 7

## JUMP Math 7

### Contents

**A** Introduction  
Features of the Teacher Resources for Grade 7  
Mental Math  
Contents of Assessment & Practice Books 7.1 and 7.2

### Part 1 Lesson Plans and Blackline Masters

- **B** Unit 1: Number Sense  
- **C** Unit 2: Patterns and Algebra  
- **D** Unit 3: Number Sense  
- **E** Unit 4: Geometry  
- **F** Unit 5: Number Sense  
- **G** Unit 6: Measurement  
- **H** Unit 7: Probability and Data Management  
- **I** Generic Blackline Masters  
- **J** Answer Keys for Assessment & Practice Book 7.1  
- **K** Unit Quizzes and Tests

### Part 2 Lesson Plans and Blackline Masters

- **L** Unit 1: Number Sense  
- **M** Unit 2: Measurement  
- **N** Unit 3: Probability and Data Management  
- **O** Unit 4: Patterns and Algebra  
- **P** Unit 5: Geometry  
- **Q** Unit 6: Number Sense  
- **R** Unit 7: Geometry  
- **S** Unit 8: Probability and Data Management  
- **T** Generic Blackline Masters  
- **U** Answer Keys for Assessment & Practice Book 7.2  
- **V** Unit Quizzes and Tests

- **W** Grade 7 Ontario Curriculum Correlation  
- **X** Grade 7 WNCP Curriculum Correlation
Unit 1  Number Sense

In this unit, students will study repeating decimals, percents, fractions, decimals, and proportions.

BLM Three Types of Percent Problems (p L-34) is a summary BLM for the material in NS7-72, NS7-73, and NS7-74. This BLM summarizes how to solve the different types of percent problems, with examples for each.

Meeting Your Curriculum
Students in Ontario study repeating decimals in Grade 9, so lessons NS7-58 and NS7-60 through NS7-63 are optional for them. For students working with the WNCP curriculum, lessons NS7-80 and NS7-81 are optional.
**NS7-55  Relating Fractions and Decimals (Review)**

**Page 1**

**CURRICULUM EXPECTATIONS**
Ontario: 7m5, 7m7, 7m11, 7m15
WNCP: 7N4, [CN, C]

**VOCABULARY**
decimal
decimal fraction
lowest terms
equivalent fraction
reduced to lowest terms
numerator
denominator

**Goals**
Students will convert between terminating decimals and their equivalent fractions.

**PRIOR KNOWLEDGE REQUIRED**
Can write decimal fractions as decimals
Can write decimals as decimal fractions
Can write a fraction as a decimal fraction when possible

**Review reducing fractions using common factors.** Remind students that you can make an equivalent fraction by multiplying both the numerator (top number) and denominator (bottom number) by the same number. But then dividing the numerator and denominator by the same number gives an equivalent fraction too. So, if the same number goes into both the numerator and the denominator, you can divide both terms by that number to find an equivalent fraction. Have students do this:

a) \( \frac{35}{50} \) (both are divisible by 5, so divide both by 5 to get \( \frac{7}{10} \))
b) \( \frac{30}{40} \) (both are divisible by 10, so divide both by 10 to get \( \frac{3}{4} \))
c) \( \frac{36}{40} \) (both are divisible by 4, so divide both by 4 to get \( \frac{9}{10} \))

**ASK:** When a number divides into another number, what is it called? (a factor of the other number) Remind students that they are dividing by a common factor to reduce the fractions. **ASK:** If you want to get the smallest numbers you can in the numerator and denominator, what special number do you have to divide by? (the greatest common factor, GCF) When a fraction has the smallest possible numbers in the numerator and denominator, how do we describe that fraction? (we say it is in lowest terms) Summarize by saying that to reduce a fraction to lowest terms, you divide its numerator and denominator by their GCF.

**Review decimal fractions and decimals.** Have students

- convert decimal fractions to decimals (EXAMPLE: \( \frac{23}{100} = 0.23 \)).
- convert decimals to decimal fractions (EXAMPLE: \( 5.03 = \frac{5}{100} \)).
- reduce given fractions to lowest terms (EXAMPLE: \( \frac{12}{20} = \frac{6}{10} = \frac{3}{5} \)).
- convert decimals to fractions in lowest terms (EXAMPLE: \( 6.45 = 6\frac{45}{100} = 6\frac{9}{20} \)).
- convert fractions in lowest terms, with terminating decimals, to decimal fractions (EXAMPLE: \( \frac{3}{4} = \frac{75}{100} \)).
• convert fractions with terminating decimals to their decimal equivalent

(EXAMPLE: \( \frac{3}{5} = \frac{6}{10} = 0.6 \)).

Decimal hundredths that are equivalent to a fraction with denominator smaller than 100. Have students do these exercises.

a) Write each decimal as a fraction with denominator 100.
   
i) 0.35 ii) 0.56 iii) 0.13 iv) 0.75 v) 0.47

b) Which fractions from part a) can be reduced to a fraction with denominator smaller than 100? (i, ii, iv) How do you know? (when the numerator has a common factor with 100, then it can be reduced)

Contexts for fractions and decimals. Discuss situations where students have seen fractions or decimals being used. EXAMPLES: recipes, money, time (quarter hour). ASK: Would you write \( \frac{3}{10} \) or \$0.30? Why do you think we use decimal notation for money? (because each dollar is divided into 100 equal parts, so place value makes sense) Would you use \( \frac{1}{4} \) hour or 0.25 hours? Why? (1/4, because an hour is divided into 60 minutes, not 100; also, we always think of an hour as divided into quarters—15-minute intervals) Repeat for \( \frac{1}{4} \) cup or 0.25 cup (1/4 because a cup isn’t 10 or 100 or 1000 of anything) and \( \frac{1}{4} \) m or 0.25 m (0.25 because metres are divided into 100 cm so tenths and hundredths make sense).
Review converting simple fractions to decimals. Have students convert each fraction to a decimal: 1/2, 1/4, 1/5, 1/8, 1/10, 1/20, and 1/25.

**ANSWERS:**

\[
\begin{align*}
\frac{1}{2} &= \frac{5}{10} = 0.5 \\
\frac{1}{4} &= \frac{25}{100} = 0.25 \\
\frac{1}{5} &= \frac{2}{10} = 0.2 \\
\frac{1}{8} &= \frac{125}{1000} = 0.125 \\
\frac{1}{10} &= 0.1 \\
\frac{1}{20} &= \frac{5}{100} = 0.05 \\
\frac{1}{25} &= \frac{4}{100} = 0.04
\end{align*}
\]

Review adding fractions with numerator 1 and the same denominator.

**ASK:** Three people each ate 1/8 of a pizza. How much of the pizza did they eat altogether? (3/8) Write on the board: 1/8 + 1/8 + 1/8 = 3/8. Have students add these fractions:

a) \[\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \]

b) \[\frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \]

c) \[\frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \]

d) \[\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \]

Introduce multiplying a whole number by a fraction. Point out that students have been adding the same number over and over again.

**ASK:** What is a shorter notation for this so that we don’t have to keep writing the same number? (multiplication). Write on the board:

\[4 \times \frac{1}{5} = \frac{4}{5}\]
Now have students go in the other direction: write a given fraction as a whole number multiplied by a fraction with numerator 1. EXAMPLES:

a) \( \frac{3}{5} \)  
b) \( \frac{2}{7} \)  
c) \( \frac{4}{9} \)  
d) \( \frac{3}{10} \)  
e) \( \frac{8}{15} \)

SAMPLE ANSWER: a) \( 3 \times \frac{1}{5} \)

Include improper fractions:

f) \( \frac{6}{5} \)  
g) \( \frac{8}{3} \)  
h) \( \frac{9}{4} \)

Connect multiplying a whole number by a fraction to multiplying the whole number by the corresponding decimal. Show students how they can calculate \( \frac{4}{5} \) by writing it as \( 4 \times \frac{1}{5} \) and then converting the fraction \( \frac{1}{5} \) to a decimal:

\[
\frac{4}{5} = 4 \times \frac{1}{5} \\
= 4 \times 0.2 \\
= 0.8
\]

Have students convert more fractions to decimals using this method:

a) \( \frac{3}{5} \)  
b) \( \frac{1}{2} \)  
c) \( \frac{9}{4} \)  
d) \( \frac{7}{20} \)

ANSWERS:

a) \( 3 \times 0.2 = 0.6 \)  
b) \( 5 \times 0.5 = 2.5 \)  
c) \( 9 \times 0.25 = 2.25 \)  
d) \( 7 \times 0.05 = 0.35 \)

Have students calculate: \( 2 \times \frac{1}{2}, 3 \times \frac{1}{3}, 4 \times \frac{1}{4}, \) and \( 5 \times \frac{1}{5} \). ASK: What do you notice about your answers? (they are always 1) Point out that 2 halves is one whole, 3 thirds is one whole, 4 fourths is one whole, and 5 fifths is one whole.

Now have students use this discovery to check their answers to the exercise at the beginning of the lesson. For example, remind students that they found that \( \frac{1}{8} = 0.125 \). ASK: What should \( 8 \times 0.125 \) be if we’re right that \( \frac{1}{8} = 0.125? \) (it should be 1) Have students check this by long multiplication.

**Adding fractions in two ways: according to the rules for adding fractions and by converting to decimals.** Write \( \frac{1}{4} + \frac{2}{5} \) on the board and work through Workbook page 2 Question 6 a) together: convert both fractions to decimals, add the decimals, and then convert the answer back to fractions. Have students add the fractions to determine if you got the right answer.
Review long division with whole numbers and decimals. Remind students that when doing long division \(a \div b\), they should be careful not to do \(b \div a\). It doesn’t matter which is bigger, \(a\) or \(b\). What matters is which number is being divided by which number.

**ASK:** How would you write the notation for long division to do \(75 \div 3\)? Is it \(3 \overline{)75}\) or \(75 \overline{)}3\)? (it’s \(3 \overline{)75}\)) How would you write the notation for \(2 \div 5\)? Which number is the 2 like—the 3 or the 75? (the 75) How do you know? (because it comes first; it is the number being divided into) **NOTE:** Some students may answer the 3, because it is the smaller number. Emphasize that it doesn’t matter which number is bigger—the notation only tells you which number to divide into which. We are used to dividing a smaller number into a larger number but we can also divide a larger number into a smaller number. Since 2 replaces 75 and 5 replaces 3, we can use our familiarity with the simpler case to get the correct way of writing \(2 \div 5\): \(5 \overline{)2}\).

Remind students that 2 is the same as 2.0 or 2.00 or 2.000. Remind students that to divide a decimal by a whole number, simply line up the decimal point above the division sign and divide as though the decimal is a whole number. Have students use decimal long division to find:

\[
\begin{align*}
\text{a)} & \; 5 \overline{)2.0} \\
\text{b)} & \; 8 \overline{)2.00} \\
\text{c)} & \; 25 \overline{)2.00} \\
\text{d)} & \; 16 \overline{)2.000}
\end{align*}
\]

**Relate fractions to division.** Write on the board the fraction \(12/3\). **ASK:** What number does this represent? (4) How do you know? (because \(12 \div 3 = 4\)) Demonstrate why this works as follows. **ASK:** 12 is the numerator—what does the numerator of a fraction tell us? (the number of parts we are considering) What does the denominator tell us? (the number of parts in one whole) **ASK:** The fraction is \(12/3\). How many parts are in one whole? (3) Then draw on the board:

one whole

\[
\begin{array}{c}
\hline
\hline
\hline
\end{array}
\]
**ASK:** How many parts are we considering altogether? **PROMPT:** What is the numerator of the fraction? (12) Explain that we have to keep adding parts until we have 12 parts altogether. We end up with 4 wholes:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>one whole</td>
<td>two wholes</td>
<td>three wholes</td>
<td>four wholes</td>
</tr>
</tbody>
</table>

By asking how much do we have if we have 12/3, we are asking how many wholes do we have if we have 12 pieces and each whole has 3 pieces. We are dividing a set of 12 pieces into groups of size 3 and asking how many groups we have. This answer is 12 ÷ 3.

Now write on the board the fraction 5/3. **ASK:** How many wholes is that? Is it more than one whole? (yes, one whole is 3 pieces and we have 5) Is it more than two wholes? (no, two wholes is 6 pieces and we have only 5)

Draw on the board:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>one whole</td>
<td></td>
</tr>
</tbody>
</table>

The second picture is only 2/3 of a whole, so we have 1 2/3 wholes. Explain that we are dividing 5 pieces into groups of size 3 (one whole is a group of size 3) and asking how many groups (wholes) we have. The answer is 5 ÷ 3.

Have students write these fractions as division:

- a) \( \frac{1}{4} \)
- b) \( \frac{2}{5} \)
- c) \( \frac{4}{5} \)
- d) \( \frac{3}{10} \)
- e) \( \frac{4}{25} \)
- f) \( \frac{11}{20} \)

Then have students convert the fractions into decimals in two ways:

a) Convert the fraction to a decimal fraction and then to a decimal.
b) Use \( \frac{a}{b} = a \div b \) and do the long division.

Tell students to ensure that they get the same answer both ways; if they don’t, they know to look for a mistake.

**Divide to find decimal equivalents for equivalent fractions.** Have students divide to find:

- \( \frac{3}{4} = 3 \div 4 = \) 
- \( \frac{6}{8} = 6 \div 8 = \) 
- \( \frac{9}{12} = 9 \div 12 = \)

**ASK:** What do you notice about your answers? (they are all the same: 0.75) Why did that happen? (because all the fractions are equivalent)

Ask students to use a calculator to determine whether these fractions are equivalent: 17/200 and 41/500. (no, 17 ÷ 200 = 0.085 and 41 ÷ 500 = 0.082)

Have students redo the problem by finding equivalent fractions for each that have the same denominator. (17/200 = 85/1000 and 41/500 = 82/1000)

Point out that the calculator just did the work of finding the numerator when the denominator is 1000.
Extending patterns in decimals. Have students divide to find 1/40, 2/40, 3/40, 4/40, 5/40. **ANSWERS:** (0.025, 0.05, 0.075, 0.1, 0.125) **ASK:** What kind of pattern do you see? **PROMPT:** Write the decimals to three decimal places. **ASK:** What is the rule for obtaining the next term in the pattern? (add 0.025) Write on the board:

\[
\begin{align*}
6/40 &= \quad \quad \\
7/40 &= \quad \quad \\
8/40 &= \quad \quad \\
9/40 &= \quad \quad \\
10/40 &= \quad \quad \\
\end{align*}
\]

Have students continue the pattern in the decimals. Then have students reduce to lowest terms the fractions that are not already in lowest terms (6/40 = 3/20, 8/40 = 1/5, 10/40 = 1/4) and create for all fractions an equivalent decimal fraction and then an equivalent decimal:

\[
\begin{align*}
6/40 &= \frac{15}{100} = 0.15, \\
7/40 &= \frac{175}{1000} = 0.175, \\
8/40 &= \frac{2}{10} = 0.2, \\
9/40 &= \frac{225}{1000} = 0.225; \\
10/40 &= \frac{25}{100} = 0.25. \\
\end{align*}
\]

Do students get the same answer by turning the fraction into a decimal as they did by extending the pattern? (yes)
**Goal**

Students will use division to represent fractions as repeating or terminating decimals. Students will compare and order decimals, including repeating and terminating decimals.

**PRIOR KNOWLEDGE REQUIRED**

- Can compare and order terminating decimals to thousandths
- Can use long division to write fractions with terminating decimal equivalents as decimals

**MATERIALS**

- 2-cm grid paper

**VOCABULARY**

- quotient
- repeating decimal
- terminating decimal

**PROCESS EXPECTATION**

- Making and investigating conjectures

**Compare the number of digits in the quotient to the number of subtractions in the long division.** Remind students that to write 4/5 as a decimal, they can divide 4 ÷ 5 and use long division. Go through this as a class to get 0.8. **ASK:** How did we know when to stop? (when the “remainder” is 0) Have students do 3 ÷ 16 to write 3/16 as a decimal (0.1875). Provide grid paper to help students align the numbers correctly. **ASK:** How long did it take to get a remainder of 0 this time—how many subtractions in the long division did you have to do? (4 subtractions)

Have students do several long divisions, and then copy and complete this chart:

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Number of decimal places in the quotient</th>
<th>Number of place values in the quotient</th>
<th>Number of subtractions before getting 0 remainder</th>
</tr>
</thead>
<tbody>
<tr>
<td>4/5</td>
<td>0.8</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3/16</td>
<td>0.1875</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>15/8</td>
<td>1.125</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>7/20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3/8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>98/5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7/10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19/25</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**ASK:** What do you notice? Which two columns are always equal? Which two columns are only usually equal? Emphasize that it is the total number of place values in the quotient that tells us the number of subtractions we will need to do before getting a 0 remainder. This is because every time you add a digit, you do another subtraction. When you get 0 as a remainder and there are no more digits from the dividend to bring down, you can stop adding digits to the quotient.

Now have students divide 5 into 15.23. Ask students how many subtractions they did (3 or 4), and how many non-zero digits are in the quotient (3). Tell students that if a digit in the quotient is 0, they can skip that subtraction. Show this on the board:

```
  3.046
5 | 15.230  
  15
  --
  23
- 20
  --
   30
- 30
  --
    0
```

Summarize by emphasizing that the number of non-zero place values in the quotient is equal to the number of subtractions you need to do in the long division, because you have to do a subtraction for every non-zero digit in the quotient.

**PROCESS EXPECTATION**

Looking for a pattern

**NOTE:** WNCP students who have completed JUMP Math Grade 7 Part 1 will be familiar with \( \pi \) from studying the area and circumference of circles. Ontario students will not see \( \pi \) until Grade 8, but can still learn the definition here.

**Introduce repeating decimals.** Write \( \frac{1}{3} \) on the board and **ASK:** What long division can we do to write \( \frac{1}{3} \) as a decimal? (\( 1 \div 3 \)) Have students do this and tell them to stop after they get a remainder of 0. After letting them work for a few minutes, **ASK:** Who is finished? Did anyone get a remainder of 0 yet? Do you think you will ever get a remainder of 0? (no) How can you tell? What pattern do you see in the long division? (we always get \( 3 \times 3 = 9 \) and then \( 10 - 9 = 1 \) as the remainder, so the pattern in the remainders is a very simple repeating pattern: 1 1 1 1…; this never becomes 0) **ASK:** If the remainder never equals 0, how many non-zero digits will the quotient have? (infinitely many digits—the decimal continues forever)

Tell students that some decimals never stop—they keep going for ever and ever. Decimals that do stop are called *terminating* decimals. If your students are familiar with the area and circumference of circles, **ASK:** Where have you seen other decimals that don’t terminate? In any case, tell students that the ratio between the distance around a circle and the distance across a circle, called \( \pi \), is another decimal that doesn’t stop. Write on the board:

\[ \pi = 3.1415926… \]

Tell students that this is just the first few digits—the decimal goes on forever.
ASK: Can you guess what the next digit is? After students guess, tell them it is 5, and write the next digit for them:

\[ \pi = 3.14159265... \]

Repeat having students guess the next digit and then writing what it actually is:

- 3.14159265
- 3.1415926535
- 3.14159265358
- 3.141592653589

If students have a scientific calculator, have them press the “pi” button. What number comes up?

\[ \pi = 3.1415926535897932384626433832795... \]

(NOTE: Different calculators may show a different number of digits, here and below. Just use whatever you see on your calculator display.) Now write on the board:

\[ \frac{1}{3} = 0.33333333333... \]

ASK: Can you guess what the next digit is? (3) Repeat several times. (the next digit is always 3) Tell students that because there is a pattern, the next digit is easy to find. ASK: What type of pattern do you see in the digits? (a repeating pattern) Explain that because the digits form a repeating pattern, the next digit is easy to predict, and the decimal is called a repeating decimal.

Some decimals, like \( \pi = 3.1415926... \), are not terminating and not repeating, but most decimals students will deal with this year will be either terminating or repeating. In fact, mathematicians have shown that any decimal that they can get from a fraction will be either terminating or repeating.

**Finding repeating decimals.** Have students find more decimals by long division and determine whether they are terminating or repeating:

- a) \( \frac{5}{6} \)
- b) \( \frac{7}{12} \)
- c) \( \frac{5}{8} \)
- d) \( \frac{4}{15} \)
- e) \( \frac{5}{11} \)
- f) \( \frac{56}{125} \)

**ANSWERS:**
- a) 0.833333... repeating
- b) 0.5833333... repeating
- c) 0.626 terminating
- d) 0.26666666... repeating
- e) 0.45454545... repeating
- f) 0.448 terminating

ASK: How are these repeating decimals different from \( \frac{1}{3} = 0.333333... \)? Notice that the digit for parts a), b), and d) repeat, but not right away. In part e), there is not a single repeating digit but a repeating pattern (4, 5, then repeat), so this decimal is still called a repeating decimal.

**Calculators can be misleading.** Have students calculate \( \frac{3}{7} \) using long division until they either get 0 as a remainder or they can prove that they won’t because they find a pattern that goes on forever. They should get 0.428571428571... The repeating pattern is 4, 2, 8, 5, 7, 1, then repeat.
Discuss the answers **ASK:** Is 3 ever a digit in 3/7? (no) Tell students that your calculator says it is. Have students calculate $3 \div 7$ on their calculator, then show them what you get on yours: 0.42857142857142857142857142857143. **ASK:** Why does my calculator tell me there is a digit 3 when I know there isn’t one? (because the calculator cannot show an infinite number of digits, so it rounds the answer; the last digit displayed is actually 2, but the next digit is 8, so the calculator rounds the 2 up to get 3)

**Introduce bar notation.** Tell students that mathematicians have invented a way to write repeating decimals exactly, without rounding and without having to write digits forever (because that would take a long time!). Mathematicians have decided to put a bar over the digits that repeat.

Write on the board:

$$0.428571428571428571428571428571428571\ldots \text{ becomes } 0.\overline{428571}.$$  

$$0.83333333 \text{ becomes } 0.\overline{83}$$

Have students write these repeating decimals using bar notation:

a) $0.321321321\ldots$  
b) $0.26666666\ldots$  
c) $0.45454545\ldots$

**ANSWERS:**

a) $0.\overline{321}$  
b) $0.\overline{321}$  
c) $0.\overline{321}$  
d) $0.\overline{4321}$

Tell students that we only put a bar over numbers after the decimal point. So, for example, 0.321032103210321… would be written 0.321, not 0.3210.

Have students write these decimals using bar notation:

a) $351.351351351\ldots$  
b) $42.62626262\ldots$  
c) $538.538383838\ldots$

**ANSWERS:**

a) $351.\overline{351}$  
b) $42.\overline{62}$  
c) $538.\overline{538}$

Write 0.323 on the board, clearly placing the bar over the 2 only. **ASK:** Does this notation make sense? (no, this would say to repeat the 2 forever, but then you can never add a 3 after) Tell students that the bar always goes over the last digits that you write. How many digits it covers depends on which digits repeat.

**Review how to compare terminating decimals.** Remind students how to compare terminating decimals: look for the largest place value where they have different digits. Have students compare:

a) $0.3746$  
b) $0.3728$  
c) $0.4589$  
d) $0.4723$  
e) $0.35$  
f) $0.32141$

**ANSWERS:**

a) $0.3746 > 0.3728$  
b) $0.4589 < 0.4723$  
c) $0.35 > 0.32141$

Use part c) to emphasize that the number of digits after the decimal point tells you nothing about how large a number is; 0.35 has fewer digits after the decimal point than 0.32141, but it is larger. This is in contrast to the number of digits before the decimal point, which tells you a lot about how
large a number is (e.g., 3412.5 is larger than 99.999 because, even though the digits are smaller, there are more digits before the decimal point).

**Introduce how to compare repeating decimals.** Show students how to compare decimals by writing their first few digits after the decimal point at least until they differ (you may need to add zeroes to terminating decimals). See the teaching box on Workbook page 5. Give students grid paper and have them do more problems like Questions 8 and 9 on Workbook page 5.

**EXTRA PRACTICE for Workbook Question 8:**

a) 0.358 0.358  
 b) 0.254 0.25  
 c) 0.754 0.75  
 d) 0.25 0.45  
 e) 0.25 0.23  
 f) 5.123 5.132  
 g) 0.382 0.382  
 h) 0.312 0.312  
 i) 0.318 0.318

**PROCESS ASSESSMENT**

7m6, [CN]
Workbook Question 8

7m1, [R]
Workbook Question 10
NOTE: Many concepts in this lesson assume familiarity with repeating decimals. Ontario students who are not familiar with repeating decimals can still do most of this lesson. Because they will not know the notation for repeating decimals (the bar over digits that repeat), do not use it. To convert fractions to decimals, use a calculator and round the answers to 1, 2, or 3 decimal places, as required. Students can thus compare, order, and subtract terminating decimals to answer all but one question (5 b) on Workbook page 6.

Using decimals to compare fractions. Now that students know how to convert fractions to decimals, and to compare decimals, have students compare fractions by first converting the fractions to equivalent decimals and then comparing the decimals. Allow students to use grid paper to help align the place values. See Workbook page 6 Questions 1, 3, and 7.

**Goal**

Students will use decimal equivalents to compare fractions and to determine fractions that are close to given decimals.

**PRIOR KNOWLEDGE REQUIRED**

- Can compare fractions by finding a common denominator
- Can convert fractions to decimals (repeating or terminating) by using long division
- Can use subtraction to find the difference between two numbers
- Can subtract terminating decimals
- Can compare and order repeating decimals
- Can round decimal numbers

**NOTE:** Many concepts in this lesson assume familiarity with repeating decimals. Ontario students who are not familiar with repeating decimals can still do most of this lesson. Because they will not know the notation for repeating decimals (the bar over digits that repeat), do not use it. To convert fractions to decimals, use a calculator and round the answers to 1, 2, or 3 decimal places, as required. Students can thus compare, order, and subtract terminating decimals to answer all but one question (5 b) on Workbook page 6.

**Using decimals to compare fractions.** Now that students know how to convert fractions to decimals, and to compare decimals, have students compare fractions by first converting the fractions to equivalent decimals and then comparing the decimals. Allow students to use grid paper to help align the place values. See Workbook page 6 Questions 1, 3, and 7.

**Compare fractions and decimals two ways.** Have students compare 4/9 and 0.435, first by writing both as fractions with a common denominator (4/9 = 4000/9000 and 435/1000 = 3895/9000, so 4/9 is larger) and then by writing both as decimals (4/9 = 0.4 = 0.4444… > 0.435 because the largest place value where they differ is hundredths and 4 hundredths is more than 3 hundredths). **ASK:** Did you get the same answer both ways? (yes) Which way did you like better? Have students do more examples both ways, as in Workbook Question 4, but keep in mind that students have not yet learned to convert repeating decimals to fractions, so include only terminating decimals in decimal format for the comparison.

**Which one is closer?** Have students convert these fractions to decimals: 3/11 and 3/13. **ANSWERS:** 0.27 and 0.230769. **ASK:** Which one is closer to 1/4 = 0.25? To guide students, write on the board:

0.27272727...
0.25000000...
0.230769230769…
**ASK:** To know how far one number is from another, what do we have to do? (subtract them) **PROMPT:** What operation will we use? Then tell students that we don’t yet know how to subtract repeating decimals. How can we do this problem without subtracting repeating decimals? **PROMPT:** What do we know how to subtract? (we know how to subtract terminating decimals)

**ASK:** How can we change a repeating decimal into a terminating decimal and still keep the answer the same? (if we round the repeating decimals to a few decimal places after they start being different, and then subtract, we should still get the right answer as to which difference is smaller) Point out that when we compared repeating decimals, we didn’t have to compare all the place values, only the early place values. It’s the same here: we only need to compare the differences in the early place values, and rounding the repeating decimals will allow us to do that.

Have students round each decimal to 5 decimal places and then find the differences:

<table>
<thead>
<tr>
<th>0.27273</th>
<th>0.25000</th>
</tr>
</thead>
<tbody>
<tr>
<td>− 0.25000</td>
<td>− 0.23077</td>
</tr>
<tr>
<td>0.02273</td>
<td>0.01923</td>
</tr>
</tbody>
</table>

So the difference between 1/4 and 3/13 is smaller than the difference between 3/11 and 1/4. This means that 3/13 is closer to 1/4 than 3/11 is. Point out that 1/4 = 3/12, and write down this sequence of fractions:

\[
\frac{3}{1}, \frac{3}{2}, \frac{3}{3}, \frac{3}{4}, \frac{3}{5}, \frac{3}{6}, \frac{3}{7}, \frac{3}{8}, \frac{3}{9}, \frac{3}{10}, \frac{3}{11}, \frac{3}{12}, \frac{3}{13}
\]

It is easy to see the decimal equivalents for the first few terms in this sequence—they are all terminating decimals:

\[
3, 1.5, 1, 0.75, 0.6, 0.5
\]

**ASK:** Do these fractions appear to be getting closer together or further apart? (closer together) Explain that the further we go in the sequence, the closer together the numbers are. **ASK:** Does this agree with the answer we got? (yes, we said that 3/18 is closer to 3/12 than 3/11 is) Discuss the differences between the two methods. Which method do students like better, and why?

Have students do Workbook page 6 Questions 2 and 5.
Introduce the lesson topic. Tell students that they will now investigate how to determine, by looking at a fraction, whether its decimal equivalent will terminate or repeat. (You will work through the Investigation on Workbook page 7 over the course of the lesson.)

Review writing terminating decimals as decimal fractions. Have students write each of these terminating decimals as a decimal fraction:

- a) 0.3
- b) 0.34
- c) 0.342
- d) 0.3425
- e) 0.32457
- f) 0.324579

**ANSWERS:**

- a) \( \frac{3}{10} \)
- b) \( \frac{34}{100} \)
- c) \( \frac{324}{1000} \)
- d) \( \frac{3425}{10000} \)
- e) \( \frac{324457}{1000000} \)
- f) \( \frac{342579}{1000000} \)

Have students do Parts A and B of the Investigation on Workbook page 7.

Tell students that a terminating decimal can be very long, for example: 0.123 456 789 876 543 421

Have students write that terminating decimal as a decimal fraction.

**ANSWER:** 123 456 789 876 543 421/1 000 000 000 000 000 000

**ASK:** How did you know what to write in the denominator? (the number of zeroes is the number of decimal places in the decimal) Tell students that any terminating decimal can be written as a decimal fraction. For example, consider a decimal with two decimal places, such as 3.12. Since there are two decimal places, we can read the number of hundredths: there are 312 hundredths, so 3.12 = 312/100. Similarly, 0.045 has 3 decimal places, so we can read the number of thousandths: there are 45 thousandths, so 0.045 = 45/1000.

Tell students that terminating decimals can also look very much like the start of a repeating decimal. For example, 0.3232323232 is a terminating decimal. Have students write that decimal as a decimal fraction.

**ANSWER:** 323,323,323,323

\[ \frac{1,000,000,000,000}{1} \]
Review writing decimal fractions as terminating decimals. **ASK:** Can any decimal fraction be written as a terminating decimal? (yes) Ask students to articulate the reason, then explain that the smallest place value in the decimal will be determined by the denominator of the decimal fraction. For example, if the decimal fraction has denominator 10,000, then the smallest place value in the decimal will be ten thousandths, which means the decimal has only 4 place values.

Have students write these decimal fractions as terminating decimals:

- **a)** \(\frac{36}{100}\)
- **b)** \(\frac{5}{1000}\)
- **c)** \(\frac{2341}{1000}\)
- **d)** \(\frac{23456}{100000}\)

**ANSWERS:**
- a) 0.36
- b) 0.005
- c) 2.341
- d) 0.23456

**PROCESS EXPECTATION**

**Connecting**

**Representing, Technology**

**Can you always use a calculator to decide whether a fraction has a terminating decimal equivalent?** Have students calculate these quotients using a calculator:

\[
\frac{1}{25} = 1 \div 25 = \text{?} \\
\frac{1}{27} = 1 \div 27 = \text{?} \\
\frac{1}{29} = 1 \div 29 = \text{?}
\]

**ASK:** Which of the fractions looks like it has a decimal equivalent that terminates? (1/25 = 0.04 does) Can you tell for sure from the calculator display for all three fractions?

Tell students that your calculator says that 1/27 = 0.0370370370370370370370370370370370. This could either be a very long terminating decimal or it could be the repeating decimal 0.037. Also, tell students that your calculator says that 1/29 = 0.034482758620689655172413793103448. This also could be a terminating decimal or it could be the repeating decimal 0.0344827586206896551724137931. **SAY:** Notice how close to the end of my calculator display we see the digits start to repeat. Some calculators may not even show enough digits for you to see that they repeat at all! In fact, 1/29 is a repeating decimal with a very long (28 digits!) block that repeats.

**Technology**

**Decide whether a fraction has a terminating or repeating decimal equivalent.** Have students do Workbook page 7 Investigation Part C. Tell students that if, after dividing a numerator by a denominator, the digits do not go to the end of the calculator display—the most digits that the calculator can show—then students can be sure that the equivalent decimal terminates, since the calculator would have shown more digits if there were any more to show. If the digits do go to the end of the calculator display, students won’t be able to tell for sure whether the equivalent decimal repeats or not. Tell students to guess by looking at the digits whether there is a repeating pattern or not.

After students complete the exercise, remind them that a fraction will have a terminating decimal equivalent if it has an equivalent fraction that is a decimal fraction. Then have students look at the fractions they said were repeating. Have them try to find an equivalent decimal fraction and to think about why they can’t. Explain that to prove that it’s not possible to do something, mathematicians often start by trying to do it! Sometimes only when they start trying to do something can they see why it won’t work. After allowing students to work for several minutes, have students work in pairs...
to articulate a reason why 7/15 has no equivalent decimal fraction. Then have students get into groups of four (two pairs) and agree on a reason. Allow each group of four to articulate their reason then summarize the results as follows: The fraction 7/15 is in lowest terms, so to make any equivalent fraction we have to multiply the numerator and denominator by the same number. Since we want to find an equivalent decimal fraction, we’re looking for a fraction that has denominator a power of 10. Suggest trying each possibility in turn. Write on the board:

\[
\begin{array}{c}
7 \times \frac{10}{15} = \frac{70}{150} \\
7 \times \frac{100}{15} = \frac{700}{1500} \\
7 \times \frac{1000}{15} = \frac{7000}{15000}
\end{array}
\]

**ASK:** Does 15 divide evenly into any power of 10? Students can try this by long division and see that there will always be a remainder of 10.

Then have students do Workbook page 7 Investigation Part D. After students finish, tell them that any power of 10 can always be written as a product of 2s and 5s. Now tell students that 15 = 3 × 5. **ASK:** Can 15 × something be equal to a product of 2s and 5s? (no) Why not? (because of the 3) Tell students that 20 = 4 × 5, but 20 does divide evenly into 100, a power of 10. What makes the 3 different from the 4? (4 can be written as 2 × 2, so 20 is still a product of 2s and 5s, but 3 cannot be written as a product of 2s and 5s) Tell students that 3 is prime, and if a number divides evenly into a power of 10, the only prime numbers that divide into it are 2 and 5. **ASK:** Can 15 be written as a product of 2s and 5s? (no, because the prime number 3 divides into it)

Now look again at the fractions from Investigation Part C. They are all in lowest terms. **ASK:** Which fractions have denominators that are products of only 2s and/or 5s? (5/8 because 8 = 2 × 2 × 2 and 13/2000 because 2000 = 2 × 2 × 2 × 5 × 5 × 5—students can find the answers by continually dividing by 2 and then by 5, or they can use prime factorizations if they are familiar with them). Are these the same as the fractions you said were terminating? (yes) Is 3/17 terminating? (no) How do you know? (because 17 is not a product of 2s and 5s)

Now have students complete the Investigation on Workbook page 7.

**EXTRA PRACTICE:**

1. Write out all the fifteenths from 1/15 to 14/15, and then write them in lowest terms. Decide from the denominators which fractions will terminate as decimals, then check on a calculator.

2. Write these fractions in lowest terms and then decide if their decimal equivalents will terminate or not.
   
   \[
   \begin{align*}
   &a) \frac{7}{25} \\
   &b) \frac{13}{20} \\
   &c) \frac{9}{150} \\
   &d) \frac{5}{7} \\
   &e) \frac{6}{21} \\
   &f) \frac{4}{110}
   \end{align*}
   \]

   Check your answers on a calculator.
Does a calculator display the exact value for a fraction? Tell students that calculators never show a decimal as repeating—you will never see the bar above repeating digits on a calculator, so calculators always make decimals look terminating. This means that when you calculate a fraction that has a repeating decimal equivalent on a calculator, it will not give you the exact answer, only an approximation. Have students do the following exercise:

A calculator shows \( \frac{50}{97} = 0.5154639175257732 \). Is this an exact value or an approximation? How do you know? (it is an approximation because 97 is not a product of 2s and 5s, so its decimal equivalent is repeating, not terminating)

Create fractions with a repeating or terminating decimal equivalent. Challenge students to create a fraction with a terminating decimal equivalent that

a) has a 1-digit denominator,
b) has a 2-digit denominator,
c) has a 3-digit denominator,
d) has a 4-digit denominator.

**SAMPLE ANSWERS:**

a) \( \frac{1}{4} \)  
b) \( \frac{11}{100} \)  
c) \( \frac{111}{1000} \)  
d) \( \frac{1111}{1000} \)

Repeat for fractions with a repeating decimal equivalent.  

**SAMPLE ANSWERS:** \( \frac{1}{3} \), \( \frac{1}{33} \), \( \frac{1}{333} \), \( \frac{1}{3333} \). Have students explain how they chose the denominators.

Extension

Why is the decimal representation of every fraction either repeating or terminating? Why can’t it be like pi, which doesn’t terminate or repeat? Think of what actually happens in the long division algorithm. Take for example \( \frac{2}{7} = 2 \div 7 \).

\[
\begin{array}{c|cccccc}
\text{7} & 2.0000000 \\
14 & \\
60 & \\
56 & \\
40 & \\
35 & \\
50 & \\
49 & \\
10 & \\
7 & 30 \\
28 & 20 \\
\end{array}
\]
At each step of the algorithm, we are dividing 7 into ten times whatever remainder we get. We start with remainder 2 because $2 \div 7 = 0$ Remainder 2, and divide 7 into 20 to get 2 Remainder 6. Our next step is to divide 7 into 60 and determine the remainder. As soon as we get a remainder that we’ve already had, then the division algorithm becomes exactly the same as it was from the first time we saw that remainder. If we get 0 as a remainder, the algorithm stops and the decimal terminates, but if we never get 0, then we have to continue forever. But there are only 6 possible remainders when we divide by 7 (not including 0), so if we do the algorithm forever, and each time the remainder is either 1, 2, 3, 4, 5, or 6, we are eventually going to repeat a remainder. Once we do that, the decimal starts repeating. This reasoning works for any number. For example, to find $1/29 = 1 \div 29$, we are always dividing by 29, so our remainder will always be less than 29. It might take a while to find a repeat remainder, but if we go on forever, we eventually will.

No matter what we divide by, if we keep dividing forever (because we never find a 0 remainder), we will eventually get the same remainder twice. The decimal will repeat from the point at which that remainder occurred the first time, because we are now doing the exact same divisions that we did to get all the digits in the decimal from the first time that remainder showed up!

**NOTE:** Some numbers, like π, do not come from fractions, and hence cannot be calculated by using long division of one whole number by another. The decimal for π neither terminates nor repeats.
Review adding and subtracting terminating decimals to thousandths. Remind students that it is important to line up the place values, to ensure they are always adding the same place values together. An easy way to do this is to always line up the decimal point. The decimal point is always between the ones and the tenths, so if the decimal points are lined up, every place value will be lined up. Give students grid paper to help them line up the decimal points and have students solve:

a) 3.451 + 2.764  b) 8.45 + 3.582  c) 13.45 + 1.345  d) 43.218 + 7.469

**Bonus**
8.9452 + 3.479  
9.415 + 23.74 + 12.678

Introduce adding repeating decimals with no regrouping. Write on the board: 0.32 + 0.\(\bar{4}\). **ASK**: How is this different from the other problems we have done so far? (this one asks us to add repeating decimals) Have students solve these problems:

a) 0.32  b) 0.32  c) 0.32  d) 0.32  e) 0.32  
+ 0.4  + 0.44  + 0.444  + 0.4444  + 0.44444

**ANSWERS**: 0.72, 0.76, 0.764, 0.7644, 0.76444

Then have students continue the pattern in the questions and the answers. What are the next two sums?

Process Expectation

Looking for a pattern

**ANSWER**: 0.764. Explain that the sum will have 4 in every place value from thousandths on because in those place values we are always adding 0 (from 0.3200000...) and 4 (from 0.7644444...).
PROCESS EXPECTATION  Connecting

Have students practise adding a repeating decimal with a terminating decimal by lining up the first few decimal places. Point out that this is how we compared repeating decimals too—we pretended that the decimal didn’t go on forever and that was enough to know which was greater. Similarly, adding the first few decimal places might be enough to see the repeating pattern in the sum. Suggest that students write at least eight decimal places so that they can see the repeating pattern in the sum. **EXAMPLES:**

a) $0.42 + 0.\overline{5}$  
b) $0.4\overline{5} + 0.3$  
c) $0.3\overline{2} + 0.47\overline{5}$

**Adding repeating decimals with repeating parts of the same length.**

Then demonstrate how to add two repeating decimals with repeating parts both the same length, by lining up the decimal places. Point out that students should start adding from the left since they cannot start adding from the right—there is no right-most digit. Emphasize that they can do this as long as no regrouping is required. **EXAMPLE:**

\[
\begin{array}{c}
0.52323232\ldots \\
+ 0.16161616\ldots \\
\hline
0.68484848\ldots 
\end{array}
\]

So $0.52\overline{3} + 0.1\overline{6} = 0.6\overline{8}4$

Have students practise.

**EXAMPLES:**

a) $0.27 + 0.\overline{31}$  
b) $4.\overline{13} + 0.\overline{31}$  
c) $0.4\overline{13} + 0.2\overline{31}$

d) $4.\overline{13} + 0.2\overline{31}$  
e) $0.3\overline{45} + 0.2\overline{3}$  
f) $0.3\overline{45} + 0.1\overline{32}$

g) $0.3\overline{45} + 0.4\overline{32}$  
h) $0.2\overline{41} + 0.3\overline{155}$  
i) $0.2\overline{14} + 0.4\overline{52}$

**Adding repeating decimals with repeating parts of different lengths.**

Now write on the board: $0.2\overline{6} + 0.3\overline{12}$. **ASK:** How is this different from what we have done so far? **PROMPT:** Look at the length of the repeating parts. Have we ever looked at a problem like this before? Do you think it will be very different? Tell students that they solve it in the same way as before, but that sometimes the repeating part of the sum might be quite a bit longer than the repeating part of the addends, so they might have to keep track of more decimal places. Have students practise, and suggest that they write out the core in the answer twice, so that they are sure it repeats.

**EXAMPLES:**

a) $0.3\overline{4} + 0.1\overline{32}$  
b) $0.2\overline{14} + 0.1\overline{32}$  
c) $0.3\overline{155} + 0.2\overline{1}$

d) $0.2\overline{41} + 0.3\overline{155}$  
e) $0.2\overline{3} + 0.3\overline{1525}$  
f) $0.2\overline{31} + 0.3\overline{1525}$

**PROCESS EXPECTATION**  Reflecting on what made a problem easy or hard

When students finish, **ASK:** How was adding two decimals with different lengths of repeating blocks different from adding two decimals with the same length of repeating blocks? (the repeating part of the sum was a lot longer than the repeating parts of the addends) Did this surprise you?

**Subtracting repeating decimals with no regrouping.** Tell students that subtracting repeating decimals is just like adding them: line the first few decimal places up and subtract from left to right until you see a pattern in the difference. **EXAMPLE:** $0.8\overline{74} - 0.3\overline{4}$
0.874874874874…
− 0.343434343434…
0.531440531440…

Emphasize that they can subtract from left to right because there is no regrouping required. Have students practise. As above, students should write at least two cores in the answer, to be sure it repeats.

**EXAMPLES:**

a) 0.874 − 0.234  
b) 0.874 − 0.870  
c) 0.874 − 0.34  
d) 0.874 − 0.034

The pattern in ninths. Show students the beginning of the pattern:

\[
\begin{align*}
\frac{1}{9} &= 0.\overline{1} \\
\frac{2}{9} &= 0.\overline{2} \\
\frac{3}{9} &= 0.3 \\
\end{align*}
\]

Have students extend the pattern to express the rest of the ninths in decimal form. Then have students do Workbook page 8 Questions 2 and 3. When students finish, discuss the results: When we don’t need to regroup, adding repeating decimals is easy, but when we need to regroup, we can’t start at the left! So we have to add the decimals to more decimal places each time and hope we can see how the pattern continues. Sometimes it is easier to add the fractions.

Adding and subtracting with regrouping when the decimal being subtracted is terminating. Show students how to subtract 0.4 – 0.27.

First, subtract up to hundredths, because that is where 0.27 ends:

\[
\begin{align*}
0.4 \underline{\phantom{0}} & \quad 0.27 \\
3.14 & \quad 0.27 \\
\underline{3.14} & \quad \underline{0.27} \\
0.174 & \quad \underline{0.27}
\end{align*}
\]

Now we can subtract the rest:

\[
0.44444444…
− 0.27
\]

\[
0.17444444… \quad \text{so } 0.4 - 0.27 = 0.174
\]

Have students practise:

a) 0.7 – 0.18  
b) 0.572 – 0.39  
c) 0.63 – 0.485

**ANSWERS:**

a) 0.597  
b) 0.1827  
c) 0.15136

Encourage students to check their answers on a calculator.
Bonus  Remind students that, to subtract terminating decimals, if the larger decimal has fewer decimal places, they can keep adding 0s after its smallest place value. Then have students do these questions:

a) $0.35 - 0.2$  
b) $0.35 - 0.22$  
c) $0.35 - 0.222$  
d) $0.35 - 0.2222$  
e) $0.35 - 0.22222$  
f) $0.35 - 0.222222$  
g) $0.35 - 0.2222222$  
h) $0.35 - 0.2$

**ANSWERS:**  
a) $0.15$  
b) $0.13$  
c) $0.128$  
d) $0.1278$  
e) $0.12778$  
f) $0.127778$  
g) $0.1277778$  
h) $0.127$

Before students do Workbook page 8 Question 4, display a list of the common fraction-decimal conversions that they will need to do the question:

\[
\begin{align*}
\frac{1}{4} &= 0.25 \\
\frac{1}{2} &= 0.5 \\
\frac{1}{3} &= 0.3 \\
\frac{4}{9} &= 0.4
\end{align*}
\]

**Extensions**

1. Repeating decimals that both have repeating blocks of length 3 can add to a decimal with repeating block of length 1. For example, $0.345 + 0.43\overline{2} = 0.77\overline{7} = 0.\overline{7}$. Challenge students to find an example of two such decimals that add to $0.\overline{8}$. (There are many, e.g., $0.345 + 0.543$.)

Challenge students to find:

- two decimals with repeating blocks of length 2 that add to a decimal with a repeating block of length 1 (SAMPLE ANSWER: $0.41 + 0.36 = 0.7$)
- three decimals with repeating blocks of length 2 that add to a decimal with a repeating block of length 1 (SAMPLE ANSWER: $0.41 + 0.13 + 0.23 = 0.7$)
- two decimals with repeating blocks of length 4 that add to a decimal with a repeating block of length 2 (SAMPLE ANSWER: $0.4123 + 0.3755 = 0.78$)

Then ASK: Can you find two decimals with repeating blocks of length 3 that add to a decimal with a repeating block of length 2? (The answer, perhaps surprisingly, is no. It is worth allowing students to struggle with this first before telling them that mathematicians have shown that this is not possible).

2. **Adding two repeating decimals always results in a decimal that either repeats or terminates.** We can think of a terminating decimal as a repeating decimal that repeats 0s. For example, $0.25 = 0.2500000\ldots$

Investigate: If you know the length of the repeating blocks of each addend, what can you say about the length of the repeating block of the sum? Is there a maximum number it can be?
Have students fill in a chart like the one below. Students can add their own decimals to the chart. (Point out that the repeating pattern in some of the sums will have a very long core, so students will have to write out a lot of digits to be sure that the decimal repeats.)

<table>
<thead>
<tr>
<th>First addend</th>
<th>Length of repeating block</th>
<th>Second addend</th>
<th>Length of repeating block</th>
<th>Sum</th>
<th>Length of repeating block</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>2</td>
<td>0.354</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.34 = 0.3400…</td>
<td>1</td>
<td>0.52</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.342</td>
<td>3</td>
<td>0.251</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.45</td>
<td>2</td>
<td>0.54</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.45</td>
<td>2</td>
<td>0.13212</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The answer is that the length of the repeating block of the sum will be at most the lowest common multiple (LCM) of that for the two addends. The digits after the decimal point form a repeating pattern. If one pattern has core length 2 and another has core length 3, what happens when we add them? Consider the decimals. 0.712 and 0.354. Start at the place value where both are repeating:

\[
0.\overline{712} + 0.\overline{354} = 0.71212121212121212...
\]

Starting at the hundredths, the repeating patterns are 1, 2, repeat and 5, 4, 3, repeat. The first pattern will start over 2 places later, 4 places later, 6 places later, and so on (all the multiples of 2). The second pattern will start over 3 places later, 6 places later, and so on (all the multiples of 3). Simply find the smallest multiple of both to see when we can be sure they will both start over together. Then the sum will repeat from here on too (although it might start repeating earlier as well).
**Goal**
Students will write repeating decimals that begin repeating immediately as fractions.

**CURRICULUM EXPECTATIONS**
Ontario: 7m1, 7m3, 7m5, 7m6, optional
WNCP: 7N4, [R, CN, T]

**VOCABULARY**
terminating decimal
repeating decimal

| Write these fractions on the board: |
|---|---|---|---|
| 1/11 | 2/11 | 3/11 | 4/11 |

Have students use long division to write these fractions as decimals, as in Workbook page 9 Question 1 a). **ASK:** What do you notice about your answers? Is there a pattern? Can you predict what 5/11 will be as a decimal? **PROMPT:** Look at the first two digits after the decimal point: 09, 18, 27, 36. How should we continue this pattern? (add 9) So the next term is 45, which means we expect that 5/11 = 0.45. **ASK:** Does this make sense? Let’s compare both to 1/2 to check. Is 5/11 more or less than 1/2? (less)

How much less—a lot or a little? (just a little) How about 0.45—is it more or less than 1/2? **PROMPT:** What is 1/2 as a decimal? (0.5) Is 0.45 more or less than 0.5? (less) How much less—a lot or a little? (just a little) They are both a little less than one half, so our prediction makes sense. Then have students extend the pattern to find 6/11, 7/11, 8/11, 9/11, 10/11, and 11/11, as in Workbook page 9 Question 1 b).

**Using products to write all the elevenths as decimals.** Write down:

\[
\frac{1}{11} = 0.09 = 0.090909090909\ldots
\]

Remind students that 5/11 = 1/11 + 1/11 + 1/11 + 1/11 + 1/11 = 5 × 1/11. So 5/11 = 5 × 1/11. But 1/11 = 0.09, so 5/11 = 5 × 0.09. Have students calculate each product:

\[
0.09 \times 5 \quad 0.0909 \times 5 \quad 0.090909 \times 5 \quad 0.09090909 \times 5
\]

Then have students predict 0.09 × 5. (ANSWER: 0.45) Express admiration: tell students that you knew they could multiply whole numbers by decimals with lots of decimal places, but you didn’t know they could multiply whole numbers by decimals with infinitely many decimal places!

Then verify that this result is the same as their prediction for 5/11. It is, because we predicted that the digits that repeat would be 9 × 5 = 45, and this is what we found.

Have students finish writing all the elevenths, up to 11/11, as repeating decimals.
Discovering that $0.\overline{9} = 1$. Use the pattern in ninths from Workbook page 8 Question 2 to write 9/9 as a repeating decimal. Then compare the decimal representations of 9/9 and 11/11. **ASK:** Why are these the same? (9/9 and 11/11 are both 1) **ASK:** Does this mean that $1 = 0.9999...$? (yes—if you write 9s forever after the decimal point, this number is equal to 1) **SAY:** Most students don’t learn that $1 = 0.\overline{9}$ until they are in high school but now you’ve proven it twice! Once by finding the pattern for elevenths to find 11/11 and once by finding the pattern for ninths to find 9/9.

Writing fractions with denominator 99 as repeating decimals. Have students do Workbook page 9 Question 2. Then explain that just as $17/99 = 17 \times 1/99$, any fraction with denominator 99 can be written as the product of the numerator and 1/99. Have students predict the decimal representation of 13/99 by using multiplication: $13/99 = 13 \times 0.0\overline{1}$, so successively multiply $0.01 \times 13$, $0.0101 \times 13$, $0.010101 \times 13$, and look for a pattern in the answers. **ANSWER:** The products are 0.13, 0.1313, and 0.131313, so predict $13/99 = 0.\overline{13}$. Then have students check by long division: $99 \div 13$.

Repeat for 8/99 and 58/99. When students finish, emphasize that it is very easy to write fractions with denominator 9 or 99 as decimals: write the numerator with a repeating bar over it, but ensure that the repeating part has the same number of digits as the denominator (so 8/99 = 0.0\overline{8}, not 0.8).

Then ask students to write many fractions with denominator 99 as repeating decimals and to check their answer with a calculator instead of by long division. **EXAMPLES:** 5/99, 52/99, 47/99, 7/99.

Have students calculate the decimal for 1/11 by first writing it as an equivalent fraction with denominator 99. Does this method produce the same results as dividing directly, $1 \div 11$? ($1/11 = 9/99 = 0.0\overline{9}$. It fits!)

Making connections between numbers that are close together. Have students compare how to find the equivalent decimal for fractions with denominator 99 and fractions with denominator 100. For example, compare the decimal representations for 35/99 and 35/100. They are $0.\overline{35} = 0.35353535...$ and 0.35. **ASK:** Are these decimals close to each other? (yes) Why does this make sense? (because the fractions are close to each other)

Writing fractions as decimals using equivalent fractions with denominator 9, 10, 99, or 100. Have students find an equivalent fraction with denominator 99, and then write the fraction as a decimal, as in Workbook page 9 Question 4. Students should check their answers with a calculator. **EXAMPLES:**

\[
\begin{array}{cccc}
2 & 7 & 50 & 28 \\
33 & 11 & 66 & 44 \\
\end{array}
\]

Have students find an equivalent fraction with denominator 9, 10, 99, or 100, and write the equivalent decimals, again checking with a calculator. **EXAMPLES:**

\[
\begin{array}{cccc}
12 & 9 & 32 & 7 \\
18 & 11 & 25 & 33 \\
4 & 5 & 50 & 66 \\
30 & 22 & 33 & 33 \\
\end{array}
\]

**PROCESS EXPECTATION**
Connecting

**PROCESS EXPECTATION**
Connecting

**PROCESS EXPECTATION**
Representing

---

Number Sense 7-62
Writing repeating decimals that begin repeating right away as fractions.

Now start with the repeating decimal and write it as a fraction. **Ask:** What is $0.0\overline{1}$ as a fraction? Tell students that the answer is right in their workbook, on page 9. (1/99) **Ask:** What is $0.0\overline{2}$ as a fraction? (2/99)

Write on the board:

$$0.0\overline{1} + 0.0\overline{1} = 0.0101010101\ldots + 0.0101010101\ldots$$

$$= 0.0202020202\ldots$$

So $0.0\overline{2} = 1/99 + 1/99 = 2/99$. **Ask:** What is $0.0\overline{3}$ as a fraction? (1/99 + 1/99 + 1/99 = 3/99) What is $0.1\overline{2}$ as a fraction? (12 × 1/99 = 12/99)

Write on the board: $0.1\overline{1} = 1/9$ and $0.0\overline{1} = 1/99$. Have students use these to convert decimals to fractions, as on Workbook page 9 Question 5.

After students finish Workbook Question 6, point out that any repeating decimal that begins repeating right away can be written as a fraction with a denominator of the form 9, 99, 999, etc.

Tell students that you divided two numbers, $a \div b$, on your calculator and it displayed the answer as:

$$0.63636363636363636363636363636364$$

Have students write two different fractions $a/b$ which you could have been calculating, one of which has a terminating decimal equivalent and the other a repeating decimal equivalent. **Hint:** Use your answer to Workbook page 9 Question 1.

**Answers:**

<table>
<thead>
<tr>
<th>Terminating decimal</th>
<th>Repeating decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>63 636 363 636 363</td>
<td>7 363 636 363 636</td>
</tr>
<tr>
<td>10 000 000 000 000</td>
<td>11 000 000 000 000</td>
</tr>
</tbody>
</table>
Multiplying and dividing repeating decimals by 10, 100, and 1000.
Review multiplying and dividing terminating decimals by 10, 100, and 1000. Then have students calculate the products in Questions a)–e) and then look for a pattern to find the product in f):

a) $3.54 \times 100$

b) $3.544 \times 100$

c) $3.5444 \times 100$

d) $3.54444 \times 100$

e) $3.544444 \times 100$

f) $3.5\bar{4} \times 100$

Then point out that you can use the same method to multiply and divide repeating decimals by 10, 100, and 1000 as you use to multiply and divide terminating decimals by 10, 100, and 1000—simply shift the decimal point 1, 2, or 3 places to the right or left. Demonstrate with two EXAMPLES:

$$3.5\bar{4} \times 100 = 3.54444444\ldots \times 100$$

$$= 354.44444\ldots$$

$$= 354.\bar{4}$$

$$3.5\bar{4} \div 100 = 3.54444444\ldots \div 100$$

$$= 0.035444444$$

$$= 0.035\bar{4}$$

Have students do Workbook page 10 Question 2.

Multiplying fractions by 10. Show students how to multiply a fraction by 10 by writing the multiplication as repeated addition. Write on the board:

$$10 \times \frac{3}{71} = \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71}$$

$$= \frac{3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3}{71}$$

$$= \frac{10 \times 3}{71}$$
So to multiply a fraction by 10, multiply its numerator by 10. Have students multiply these fractions by 10:

a) \( \frac{3}{99} \)  

b) \( \frac{4}{9} \)  

c) \( \frac{7}{999} \)  

d) \( \frac{82}{999} \)  

**ANSWERS:**

a) \( \frac{30}{99} \)  

b) \( \frac{40}{9} \)  

c) \( \frac{70}{999} \)  

d) \( \frac{820}{999} \)  

**Connecting**

**Making connections between equivalent numbers.** Have students find the equivalent decimals for the fractions above, and the equivalent decimals for the fractions found when multiplying by 10. For **EXAMPLE:** \( \frac{30}{99} = 0.30 \) and \( \frac{30}{99} = 0.3\overline{0} \). Have students multiply the decimal for \( \frac{30}{99} \) by 10: Do they get the decimal for \( \frac{30}{99} \)? (yes)

Then have students add the decimal representations for \( \frac{2}{99} \) and \( \frac{30}{99} \). 
\( (0.02 + 0.30 = 0.32) \) **ASK:** What fraction should this be equivalent to? (32/99) Is it? (yes)

**Dividing fractions by 10.** Tell students that we can divide decimals by 10 and now we want to be able to divide fractions by 10. Write on the board:

\[
\frac{1}{3} \div 10 = \left( \frac{1}{3} \right) \div 10 \\
= \frac{1}{3} \div 10 \\
= \frac{1}{3} \div (3 \times 10) \\
= \frac{1}{3} \div 30 \\
= \frac{1}{30}
\]

Point out that we just multiplied the denominator by 10 to divide the fraction by 10. Have students divide each of these fractions by 10, by multiplying the denominator by 10.

a) \( \frac{4}{9} \)  

b) \( \frac{56}{99} \)  

c) \( \frac{17}{999} \)  

d) \( \frac{8}{99} \)  

**ANSWERS:** a) \( \frac{4}{90} \)  

b) \( \frac{56}{990} \)  

c) \( \frac{17}{9990} \)  

d) \( \frac{8}{990} \)  

**Dividing fractions by 100 and 1000.** **ASK:** If to divide a fraction by 10 you multiply its denominator by 10, how can you divide a fraction by 100? (multiply its denominator by 100) How would you divide a fraction by 1000? (multiply its denominator by 1000) Have students practise.

a) \( \frac{13}{99} \div 10 \)  

b) \( \frac{4}{9} \div 100 \)  

c) \( \frac{8}{3} \div 1000 \)  

**ANSWERS:** a) \( \frac{13}{990} \)  

b) \( \frac{4}{900} \)  

c) \( \frac{8}{3000} \)  

**Finding decimal equivalents for fractions with denominator 9, 90, 900, or 9000.** Remind students that \( \frac{4}{9} = 0.\overline{4} \). **ASK:** What decimal is equivalent to \( \frac{4}{90} \)? **PROMPT:** We divided the fraction by 10, so let’s divide its equivalent decimal by 10.
\[ 0.\overline{4} \div 10 = 0.4444\ldots \div 10 = 0.04444\ldots \]

So \( \frac{4}{90} = 0.04 \). Ask students to write these fractions as decimals:

a) \( \frac{4}{900} \)

b) \( \frac{4}{9000} \)

c) \( \frac{7}{90} \)

d) \( \frac{7}{9000} \)

e) \( \frac{8}{900000} \)

**Finding decimal equivalents for fractions with denominator a power of 10 times 99 or 999.** Tell students that now that they know how to find decimal equivalents for fractions with denominator 9, 99, or 999, and can divide fractions by 10, 100, and 1000, they can find decimal equivalents for lots of different fractions. Have students try writing these fractions as decimals (do a) and b) together as a class):

a) \( \frac{18}{990} \)

b) \( \frac{29}{3330} \)

c) \( \frac{95}{1110} \)

d) \( \frac{35}{9900} \)

e) \( \frac{21}{3300} \)

Students should check their answers on a calculator.

Now write on the board: \( \frac{83}{90} \). Tell students that you want to find \( \frac{83}{90} \) as a decimal. **ASK:** What is different about this problem? (you have to divide an improper fraction by 10 to get it) Remind students that an improper fraction can be easily changed to a decimal by first changing it to a mixed number, and then changing the fractional part to a decimal. Write on the board:

\[
\frac{83}{90} = \frac{83}{9} \div 10 = 9.\overline{2} \div 10 \quad \text{(since } \frac{2}{9} = 0.\overline{2})
\]

\[= 0.92\]

Now have students use this method to write these fractions as decimals:

a) \( \frac{718}{990} \)

b) \( \frac{895}{1110} \)

c) \( \frac{837}{3330} \)

Students should check their answers with a calculator.

**ANSWERS:**

a) \( \frac{718}{99} \div 10 = 7.\overline{25} \div 10 = 7.25 \div 10 = 0.725 \)

b) \( \frac{895}{111} \div 10 = 8.\overline{7} \div 10 = 8.73 \div 10 = 0.873 \)

c) \( \frac{837}{3330} \div 10 = 2.\overline{71} \div 10 = 2.713 \div 10 = 0.2713 \)

**Writing any repeating decimal as a fraction.** Review writing repeating decimals that begin repeating right away as fractions. Have students find the equivalent fractions for these repeating decimals:

a) \( 0.47 \)

b) \( 0.3\overline{02} \)

c) \( 0.\overline{7} \)

d) \( 0.0\overline{82} \)

**ANSWERS:**

a) \( \frac{47}{99} \)

b) \( \frac{3\overline{02}}{999} \)

c) \( \frac{7}{9} \)

d) \( \frac{8\overline{2}}{999} \)
As always, students should check the answers on a calculator by dividing the numerator by the denominator.

Now have students find equivalent fractions for:

a) 0.047  
   b) 0.0302  
   c) 0.07  
   d) 0.0082

**ANSWERS:**

a) \( \frac{47}{990} \)  
   b) \( \frac{302}{9990} \)  
   c) \( \frac{7}{90} \)  
   d) \( \frac{82}{9990} \)

Show students how to find the equivalent fraction for 0.247. First, write on the board: 0.247 = 0.2 + 0.047. **SAY:** We know how to find the fractions for both of these decimals and we know how to add fractions, so we know how to do this problem:

\[
0.247 = 0.2 + 0.047 \\
= \frac{1 + 47}{5 + 990} \\
= \frac{99 \times 2 + 47}{99 \times 2 \times 5} \\
= \frac{198 + 47}{990} \\
= \frac{245}{990}
\]

Have students find 245 ÷ 990 on their calculators to check this answer. Then have students use this method to find an equivalent fraction (not necessarily reduced) for each of these repeating decimals:

a) 0.347  
   b) 0.4302  
   c) 0.27  
   d) 0.5082  
   e) 0.2547  
   f) 0.16302  
   g) 0.757  
   h) 0.13082

**ANSWERS:**

a) \( \frac{344}{990} \)  
   b) \( \frac{4298}{9990} \)  
   c) \( \frac{25}{90} \)  
   d) \( \frac{5077}{9990} \)  
   e) \( \frac{2522}{9900} \)  
   f) \( \frac{16286}{99900} \)  
   g) \( \frac{682}{900} \)  
   h) \( \frac{13069}{99900} \)

Have students check their answers using a calculator.

**Extension**

Write always, sometimes, or never in the blanks. Justify your answer.

a) Adding two terminating decimals will ______________ result in a terminating decimal.

b) Adding two repeating decimals will ______________ result in a repeating decimal.

c) Adding a terminating decimal and a repeating decimal will ______________ result in a repeating decimal.
d) Adding two repeating decimals will __________________ result in a
decimal that goes on forever but does not repeat.

ANSWERS:

a) always—a decimal that does not terminate does not have a smallest
place value—it goes on forever!—and so cannot be the sum of two
terminating decimals, which each have smallest place values. For
example, adding 2 hundredths + 35 ten thousandths will result in a
smallest place value of ten thousandths (0.02 + 0.0035 = 0.0235).

b) sometimes—for example, 2/7 + 1/7 = 3/7 results in a repeating
decimal, but 2/3 + 1/3 = 1 results in a terminating decimal.

c) always—after the point where the terminating decimal stops, the sum
will be the same as the repeating decimal, which means that it also
goes on forever and becomes a repeating decimal.

d) never—adding two fractions always results in a fraction! (See
Extension 2 from NS7-61 for a more complete explanation of why
adding two repeating decimals always results in a repeating or
terminating decimal.)
NS7-64 Percents
Workbook page 11

CURRICULUM EXPECTATIONS
Ontario: 6m27; review, 7m3, 7m6, 7m7
WNCP: 7N3, [CN, R, C]

VOCABULARY
percent
ratio
fraction

Goals
Students will write whole number percents as ratios that compare a number or amount to 100 and as fractions with denominator 100.

PRIOR KNOWLEDGE REQUIRED
Can find equivalent fractions
Can reduce fractions to lowest terms
Can name ratios from pictures
Can relate ratios to fractions

Introduce the word percent. Ask students what the word per means in these sentences:

Rita can type 60 words per minute.
Anna scores 3 goals per game.
John makes $10 per hour.
The car travels at a speed of up to 140 kilometres per hour.

Then write percent on the board. Point out that percent is made up of two words: per and cent. Ask: Has anyone seen the word cent before? What does it mean? Does anyone know a French word that is spelled the same way? What does that word mean? (cent is French for 100) Explain that percent means “for every 100,” “out of 100,” or “out of every 100.” For example, a score of 84% on a test would mean that you got 84 out of every 100 marks or points. Another example: if a survey reports that 72% of people read the newspaper every day, that means 72 out of every 100 people surveyed read the newspaper daily.

Have students rephrase the percents in the following statements using the phrases “for every 100” or “out of every 100.”

a) 52% of students in the school are girls (For every 100 students, 52 are girls OR 52 out of every 100 students in the school are girls.)

b) 40% of tickets sold were on sale (For every 100 tickets sold, 40 were on sale OR 40 out of every 100 tickets were on sale.)

c) Alejandra scored 95% on the test (For every 100 possible points, Alejandra scored 95 points on the test OR Alejandra got 95 out of every 100 points on the test.)

d) About 60% of your body weight is water (For every 100 kg of body weight, about 60 kg is water OR 60 kg out of every 100 kg of body weight is water.)

A percent is a ratio that compares a number to 100. Explain to students that a percent is a part-to-whole ratio that compares a number to 100. For example, 45% = 45 : 100. Remind students that a ratio refers to “for every.” If there are 74 girls for every 100 people at a party, 74% of people at the
party are girls. However, if there were 74 girls for every 100 boys, we could
not say that 74% of the people are girls. We could say that the number
of girls is 74% of the number of boys. But since girls are not part of the
group of boys, we cannot say that 74% of the boys are girls. Have students
complete Workbook p. 11.

PROCESS ASSESSMENT

Have students rewrite each sentence using percents.

7m6, [CN]

a) Sally got 83 marks for every 100 possible marks on a test.
b) The ratio of boys to students in a class is 65 to 100.
c) 36 out of every 100 people surveyed said they would vote for Annette.

ANSWERS:

a) Sally got 83% on a test.
b) 65% of the students in a class are boys.
c) 36% of people surveyed said they would vote for Annette.

Have students rewrite each sentence in terms of a ratio comparing a
number to 100.

a) 30% of people in a city are visible minorities.
b) John got 80% on his math test.
c) Jacob paid the friendly waiter a 25% tip.

ANSWERS:

a) For every 100 people in a city, 30 are visible minorities.
b) For every 100 possible marks on a test, John got 80 marks.
c) For every $100 Jacob paid for the food, he gave $25 to the friendly
waiter as a tip.

PROCESS EXPECTATION

Reflecting on what made a problem easy or hard

ASK: Is it easier to use percents or ratios when expressing one quantity
as part of another quantity? (percents) Why? (it is less awkward to say
“percent of something” than to say “for every 100 somethings, this many
were like this”)

PROCESS EXPECTATION

Whole number percents are fractions with denominator 100. Explain
to students that a percent is just a short way of writing a fraction with
denominator 100. Have students write each fraction as a percent:

a) $\frac{28}{100}$  b) $\frac{9}{100}$  c) $\frac{34}{100}$  d) $\frac{67}{100}$  e) $\frac{81}{100}$  f) $\frac{3}{100}$

Then have students write each percent as a fraction:

a) 6%  b) 19%  c) 8%  d) 54%  e) 79%  f) 97%

PROCESS ASSESSMENT

Workbook Question 7
Adding and Subtracting Percents

First, review writing percents as fractions with denominator 100. Then review adding and subtracting fractions with denominator 100. Then combine the two steps to add and subtract percents.

The total percent is always 100%. Tell students that one student wrote a test and got 85% of the questions right. **ASK:** What percent did the student get wrong? (15%) How do you know? (because the total percent is 100%, so I subtracted 100% − 85%) Have students do Workbook p. 12 Question 4. Emphasize that students should total the percents given and then subtract that total from 100% to find the missing percent.

Money and percents. Remind students that percent means “out of every 100.” **ASK:** How many pennies are in a dollar? (100) **ASK:** What percent of a dollar is a penny? (1%) What percent of a dollar is a dime? (10%, because a dime is worth 10 pennies)

Writing percents as decimal hundredths. Review writing percents as fractions with denominator 100. Then review writing fractions with denominator 100 as decimals. Combine the two steps to write percents as decimals.

Writing decimal hundredths as percents. Review writing decimal hundredths as fractions with denominator 100. Then review writing fractions with denominator 100 as percents. Combine the two steps to write decimal hundredths as percents.

Writing decimal tenths as percents. Have students write various decimal tenths as percents by first changing the decimal to a fraction with denominator 100.

**EXAMPLES:**

a) \( \frac{2}{10} = \frac{20}{100} = 20\% \)  

b) 0.3  

c) 0.9  

d) 0.7  

e) 0.5
Writing decimals as percents by moving the decimal point two places to the right. Have students look at their answers to writing decimal hundredths and tenths as percents. **ASK:** How can you get the percent from the decimal? (multiply by 100, or move the decimal point two places to the right) Have students write decimal tenths or hundredths as percents directly, by moving the decimal point two places to the right.

**EXAMPLES:**

a) 0.43  
   b) 0.9  
   c) 0.07  
   d) 0.74  
   e) 0.2

What percent is shaded? Explain to students that they can find a percent of a figure just as they can find a fraction of a figure. Ask students to decide first what fraction and then what percent of each figure is shaded:

- a)  
  - What percent is shaded?
  - ANSWERS: a) \(\frac{4}{10} = \frac{40}{100} = 40\%\) shaded

- b)  
  - What percent is shaded?
  - ANSWERS: b) \(\frac{1}{4} = \frac{25}{100} = 25\%\) shaded

- c)  
  - What percent is shaded?
  - ANSWERS: c) \(\frac{7}{20} = \frac{35}{100} = 35\%\) shaded

Rounding decimals to the nearest whole number percent. Review rounding decimals to the nearest hundredth. Then write the hundredth as a whole number percent.

**Word Problems Practice:**

a) Jane got 16 out of 25 marks on a test. What was her percentage grade?

b) Ron sold 17 of his 20 books at a yard sale. What percentage of his books did he sell?

c) Nomi collects sports cards. 20% of her collection is baseball cards and 35% of her collection is hockey cards. What percentage of her collection is neither hockey cards nor baseball cards?

**ANSWERS:**

a) 64%  
   b) 85%  
   c) 45%
NS7-67 Fractions and Percents
Workbook page 14

**Goals**

Students will change fractions to percents by reducing fractions when necessary. Students will compare fractions to percents that serve as good benchmarks, and will convert between fractions, percents, and decimals.

**PRIOR KNOWLEDGE REQUIRED**

- Can convert a fraction $a/b$ to a decimal by dividing $a ÷ b$.
- Can find equivalent fractions by multiplying the numerator and denominator by the same number.
- Can convert a fraction with denominator 100 to a percent.

**MATERIALS**

1 cm grid paper

**Changing reduced fractions to percents.** Write the fraction $3/5$ on the board and have a volunteer find an equivalent fraction with denominator 100. **ASK:** If 3 out of every 5 students at a school are girls, how many out of every 100 students are girls? (60) What percent of the students are girls? (60%)

Write on the board: \[
\frac{3}{5} = \frac{60}{100} = 60%.
\]

Then have volunteers find the equivalent fraction with denominator 100 and the equivalent percent for more fractions with denominator 5.

**EXAMPLES:** \[
\frac{4}{5}, \quad \frac{1}{5}
\]

Repeat for more simple fractions, that is, fractions with denominators that divide evenly into 100. **EXAMPLES:**

<table>
<thead>
<tr>
<th>a)</th>
<th>b)</th>
<th>c)</th>
<th>d)</th>
<th>e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2}{5}$</td>
<td>$\frac{7}{10}$</td>
<td>$\frac{9}{20}$</td>
<td>$\frac{37}{50}$</td>
<td>$\frac{18}{25}$</td>
</tr>
</tbody>
</table>

**ANSWERS:** a) 40% b) 70% c) 45% d) 74% e) 72%

Have students use the equivalent percents to put the above fractions in order from least to greatest.

**ANSWERS:** $\frac{2}{5} < \frac{9}{20} < \frac{7}{10} < \frac{18}{25} < \frac{37}{50}$

**Fractions that need reducing before the denominator divides evenly into 100.** Write the fraction $9/15$ on the board. Tell students that you want to find an equivalent fraction with denominator 100. **ASK:** How is this fraction different from previous fractions you have changed to percents? (The denominator does not divide evenly into 100.) Is there any way to find an equivalent fraction whose denominator does divide evenly into 100?
(Reduce the fraction by dividing both the numerator and the denominator by 3. Now the denominator is a factor of 100.) Write on the board:

\[
\frac{9}{15} = \frac{3}{5} = \frac{60}{100} = 60\%
\]

Summarize the steps for finding the equivalent percent of a fraction.
1. Reduce the fraction so that the denominator is a factor of 100.
2. Find an equivalent fraction with denominator 100.
3. Write the fraction with denominator 100 as a percent.

Have students write various fractions as percents:

a) \(\frac{3}{12}\)  
   b) \(\frac{6}{30}\)  
   c) \(\frac{24}{30}\)  
   d) \(\frac{3}{75}\)  
   e) \(\frac{6}{15}\)  
   f) \(\frac{36}{48}\)  
   g) \(\frac{60}{75}\)

ANSWERS: a) 25%  
           b) 20%  
           c) 80%  
           d) 4%   
           e) 40%  
           f) 75%  
           g) 80%

Changing fractions to percents using division. Have students change the fractions above to percents, but this time by using division to obtain the decimal. EXAMPLE: \(\frac{3}{12} = 3 \div 12 = 0.25 = 25\%\).

Changing percents to fractions in lowest terms. First write the percent as a fraction with denominator 100, and then reduce to lowest terms.

Which percent is a fraction closest to? Show students a double number line with fractions above and percents below:

Have students copy the number line onto 1 cm grid paper (using a ruler) with 2 mm representing 1 percent (so that 2 cm represent 10 percent) and mark on it the fractions from Workbook p. 14 Questions 6. Students should first convert each fraction to a percent. The easiest way to do this is to convert each fraction to a fraction with denominator 100. You could also encourage your students to check their answers using long division. If any students struggle with motor skills, have the double number line pre-drawn for them. Students can then use the number line to complete Question 6.

Not all fractions can be written as a whole number percent. Tell students that you want to change \(\frac{3}{7}\) to a percent. But 7 does not divide evenly into 100 and \(\frac{3}{7}\) is already reduced! You can't reduce the fraction to make the denominator divide evenly into 100. So you have to use division to change the fraction to a decimal: 

\[\frac{3}{7} = 3 \div 7 = 0.428571 \approx 0.43 = 43\%\]

Have students write these fractions to the nearest whole number percent by following these steps:
Step 1) division \(a/b = a \div b\);

Step 2) round the resulting decimal to the nearest hundredths;

Step 3) Write the decimal hundredth as a whole number percent.

**Estimating fractions as percents.** Write the fraction \(9/40\) on the board. Tell students that you want to estimate what percent this is close to. **SAY:** Let’s first think about other fractions that are easy to turn into percents. Which fractions can you think of that are easy to turn into percents? (1/2, 1/4, 1/5, 1/10, 1/20, 1/25, 1/50, and any fraction with these denominators) Is \(9/40\) close to one of these fractions? **PROMPT:** What fraction with denominator 40 is equivalent to 1/4? Is 9/40 close to that fraction? (10/40 = 1/4, and this is close to 9/40 because 10 is close to 9) What percent is 10/40? (25%) Should 9/40 be a little more or a little less than 25%? (a little less) How do you know? (because 9 is a little less than 10) Have students find \(9 \div 40\) on a calculator or using long division. (9/40 = 0.225 ≈ 0.23 = 23%) **ASK:** Is this a little less than 25%? (yes) Explain that this tells us that our estimate was good. Have students practise estimating more fractions as percents:

a) \(\frac{23}{30}\)  
b) \(\frac{43}{70}\)  
c) \(\frac{22}{35}\)  
d) \(\frac{43}{80}\)

**ANSWERS:**

a) a little less than \(24/30 = 8/10 = 80\%\)  
b) a little more than \(42/70 = 6/10 = 60\%\)  
c) a little more than \(21/35 = 3/5 = 60\%\)  
d) a little less than \(44/80 = 11/20 = 55\%\)

Have students divide to change each fraction to a decimal (round to two decimal places) and then to a percent. Was their estimate close to the answer?

**ANSWERS:**

a) \(23 \div 30 \approx 0.77 = 77\%\)  
b) \(43 \div 70 \approx 0.61 = 61\%\)  
c) \(22 \div 35 \approx 0.63 = 63\%\)  
d) \(43 \div 80 \approx 0.54 = 54\%\)

Yes, the estimate was close in all cases.

Then tell students that so far, they have only had to change the numerator to find an equivalent fraction with denominator 2, 4, 5, 10, 20, 25, 50, or 100. Sometimes it is convenient to start by changing the denominator. For example, write on the board the fraction \(45/99\). **ASK:** What denominator that is easy to work with is close to 99? (100) What makes it easy to work with denominator 100? (a fraction with denominator 100 can be easily changed to a percent because percents are out of 100) Is the fraction \(45/99\) close to \(45/100\)? (yes) How do you know? (they are both 45 parts, and the parts are almost the same size) Which fraction is bigger? (45/99 because ninety-ninths are slightly bigger than hundredths)

**PROCESS EXPECTATION**

Mental math and estimation, Using logical reasoning

**PROCESS EXPECTATION**

Reflecting on what made the problem easy or hard, Changing into a known problem
Have students calculate $45/99$ on a calculator, round the answer to the nearest hundredth, and then convert it to a whole-number percent. $(45/99 = 0.4545454545\ldots \approx 0.45 = 45\%)$ Notice that although $45/99$ is slightly more than 45%, it is 45% to the nearest whole number percent—it is closer to 45% than to 46%. Now have students practise estimating with these examples:

\begin{align*}
\text{a)} \quad & \frac{20}{49} \approx 0.41 \quad \text{b)} \quad \frac{17}{24} \approx 0.71 \\
\text{c)} \quad & \frac{17}{52} \approx 0.33 \quad \text{d)} \quad \frac{12}{19} \approx 0.63
\end{align*}

\text{ANSWERS:}

\begin{align*}
\text{a)} & \quad \text{a little more than } 20/50 = 40\% \\
\text{b)} & \quad \text{a little more than } 17/25 = 68\% \\
\text{c)} & \quad \text{a little less than } 17/50 = 34\% \\
\text{d)} & \quad \text{a little more than } 12/20 = 60\%
\end{align*}

Have students use a calculator to change each fraction to a decimal (rounded to two decimal places) and then to a percent. Was their estimate close to the answer?

\text{ANSWERS:}

\begin{align*}
\text{a)} \quad & 20 \div 49 \approx 0.41 = 41\% \\
\text{b)} \quad & 17 \div 24 \approx 0.71 = 71\% \\
\text{c)} \quad & 17 \div 52 \approx 0.33 = 33\% \\
\text{d)} \quad & 12 \div 19 \approx 0.63 = 63\%
\end{align*}

Yes, the estimate was close in all cases.

\text{Extensions}

\text{1. ASK:} How many degrees are in a circle? (360) If I rotate an object $90^\circ$ counter-clockwise, what percent of a complete turn has the object made? $(90/360 = 1/4 = 25/100 = 25\%)$

\text{2. Investigate:} Which fractions can be written as whole number percents and which need to be rounded to whole number percents.

Try these examples plus two more of your own.

\begin{align*}
\text{a)} \quad & \frac{7}{25} \\
\text{b)} \quad & \frac{18}{30} \\
\text{c)} \quad & \frac{7}{9} \\
\text{d)} \quad & \frac{21}{28} \\
\text{e)} \quad & \frac{21}{35} \\
\text{f)} \quad & \frac{8}{14}
\end{align*}

Articulate how you can tell without writing the fraction as a decimal whether it is equivalent to a whole number percent or not. \text{ANSWER:} If the fraction can be reduced so that the denominator divides evenly into 100, then the fraction is equivalent to a whole number percent.
**Visual Representations of Percents**

**Comparing Fractions, Decimals, and Percents**

Workbook pages 15–16

**Goals**

Students will develop visual representations of percents and will compare fractions, percents, and decimals.

**Prior Knowledge Required**

Can compare and order fractions with like and unlike denominators
Can compare and order percents
Can compare and order decimals
Can convert between fractions, percents, and decimals

**Materials**

a metre stick

**Visual Representations of Percents.** Draw a hundreds block on the board and have students write what part of the block is shaded in three different ways: a fraction, a decimal, and a percent.

**Example:**

\[
\frac{39}{100}, 0.39, 39\%
\]

Now draw the shapes below on the board and have students write which fraction is shaded. To change the fraction to a percent, students should find an equivalent fraction with denominator 100 to change to a decimal and then a percent.

**Answers:**

a) \(\frac{7}{10} = \frac{70}{100} = 0.70 = 70\%\)

b) \(\frac{1}{5} = \frac{20}{100} = 0.20 = 20\%\)

c) \(\frac{9}{25} = \frac{36}{100} = 0.36 = 36\%\)

d) \(\frac{14}{20} = \frac{70}{100} = 0.70 = 70\%\)

e) \(\frac{11}{20} = \frac{55}{100} = 0.55 = 55\%\)
**PROCESS EXPECTATION**

Visualizing

**Estimating percents of line segments.** Draw a line on the board and ask a volunteer to mark 50% on the line:

```
50%
```

Use a metre stick to check the volunteer’s estimate. Then invite students to draw 5 lines of different lengths independently on a blank sheet of paper and to mark a different percent on each one: for example, 20% on the first line, 75% on the second line, and so on. To check their estimates, students can use regular rulers or “percent” rulers made from elastics. (A percent ruler is a wide elastic band on which you make 11 evenly spaced markings labelled with percents: 0, 10, 20, … 100. To use this ruler, stretch the elastic so that the 0 and 100 markings line up with the ends of a line segment. You could make 3 or 4 such rulers and have students share them.)

**Bonus**

Draw two lines such that 20% of the first line is longer than 50% of the second line.

Now draw a line segment and identify what percent of a line the segment represents. Invite a volunteer to extend the line segment to its full length, that is, to show 100%.

**EXAMPLES:**

<table>
<thead>
<tr>
<th>Percent</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>50%</td>
<td></td>
</tr>
<tr>
<td>40%</td>
<td></td>
</tr>
</tbody>
</table>

Notice that 50% = 1/2, so the given line is 1 of 2 equal parts—simply draw another equal part. Also, 40% = 4/10 = 2/5, so the given line is 2 of 5 equal parts—divide the given line into two equal parts and draw three more identical parts.

Draw on the board a line one metre long, and have students estimate the percent of various marks on the number line (to the nearest 10%). Then, using the metre stick, draw another line of the same length divided into 10 equal parts above or below the first line, so that students can check their estimates.

**Comparing simple fractions, decimal hundredths, and whole number percents.** Review comparing fractions with the same denominator. Then review changing simple fractions, decimal hundredths, and whole number percents to fractions with denominator 100. Then have students do Workbook p. 16 Question 2.

Then have students redo three or four parts of Question 2 (they can choose the parts) by changing all the fractions and percents to decimals. **ASK:** Did you get the same answer both ways?
After students do Workbook p. 16 Question 3, have students redo the question by changing all the numbers to decimal fractions with the same denominator. **ASK:** Did you get the same answer both ways? Ensure that students understand that if they didn’t get the same answer both ways, they should check their answer with another student or with you.

Tell students that Sally wrote five math tests this year. Each test had a different number of questions, all worth 1 mark each. Her marks were:

<table>
<thead>
<tr>
<th>Test 1</th>
<th>Test 2</th>
<th>Test 3</th>
<th>Test 4</th>
<th>Test 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>6</td>
<td>18</td>
<td>24</td>
<td>17</td>
</tr>
<tr>
<td>25</td>
<td>10</td>
<td>24</td>
<td>30</td>
<td>20</td>
</tr>
</tbody>
</table>

Have students answer these questions:

a) Change each of Sally’s grades to percent and decimal form. Copy and complete this chart:

<table>
<thead>
<tr>
<th>Test Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fraction</td>
<td>17/25</td>
<td>6/10</td>
<td>18/24</td>
<td>24/30</td>
<td>17/20</td>
</tr>
<tr>
<td>Percent</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Decimal</td>
<td>0.68</td>
<td>0.6</td>
<td>0.75</td>
<td>0.8</td>
<td>0.85</td>
</tr>
</tbody>
</table>

b) Decide which form of Sally’s grades (fraction, percent, or decimal) you would use to answer Sally’s questions below. Justify your choice.

i) Sally wants to know if her grades are improving.

ii) To study for a cumulative test, Sally will study only the questions she got wrong. How many questions does she have to study?

iii) Sally wants to know her average (mean) test score.

c) Answer each question in part b.

**ANSWERS:**

a)  
<table>
<thead>
<tr>
<th></th>
<th>Percent</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Decimal</td>
<td>68%</td>
<td>60%</td>
<td>75%</td>
<td>80%</td>
<td>85%</td>
</tr>
<tr>
<td></td>
<td>0.68</td>
<td>0.6</td>
<td>0.75</td>
<td>0.8</td>
<td>0.85</td>
</tr>
</tbody>
</table>

b)  

i) Percents and decimals are definitely easier to compare than fractions with unlike denominators, and percents are easiest because they are whole numbers.

ii) The fractions tell you how many questions Sally got wrong: subtract the numerator (number of questions she got right) from the denominator (total number of questions).
iii) Percents and decimals are definitely easier to add than fractions with unlike denominators, and percents are probably easiest because they are whole numbers.

c) i) Her grades tend to be improving—the only exception is from the first to second test, but all the rest are improving and are better than the first test.

   ii) \[8 + 4 + 6 + 6 + 3 = 27\]

   iii) mean score = \[(68 + 60 + 75 + 80 + 85) ÷ 5 = 368 ÷ 5 = 0.736 \approx 74\%\]

Word Problems Practice:

a) If 13 out of 20 students in a class like skiing, what percent of the students like skiing?

b) Audrey got 16/20 on a math test, 19/25 on a science test, and 78% on a history test. On which test did she do best?

c) In Bilal’s city, 54% of the population are visible minorities. In Bilal’s class, 18 of 30 students are visible minorities. Does Bilal’s city or Bilal’s class have a greater percentage of visible minorities?

ANSWERS:

a) 65%

b) Audrey got 80% in math, 76% in science, and 78% in history, so she did best on her math test.

c) In Bilal’s class, \[18/30 = 6/10 = 60\%\] of students are visible minorities. This is greater than the city-wide average of 54%.
Finding one tenth of a number using base ten materials. Tell students that you will use one thousands block (the big cube) to represent one whole. Remind students that a thousands block is called a thousands block because it is made up of 1 000 little cubes, but it doesn’t always have to represent 1 000.

ASK: If the thousands block represents one whole, what does a hundreds block represent? (one tenth) A hundreds block has one tenth the number of cubes of a thousands block, so if the thousands block is a whole, the hundreds block is one tenth of a whole, or 1/10. ASK: If the thousands block represents a whole, what does a tens block represent? (one hundredth) What does a ones block represent? (one thousandth) These relationships are summarized in the teaching box on Workbook p. 17.

Ask students to identify the fraction and the decimal each model represents:

a) \[ \frac{1}{10} \]  b) \[ \frac{1}{100} \]  c) \[ \frac{3}{100} \]

d) 0.005, \[ \frac{5}{1000} \]  e) 0.033, \[ \frac{33}{1000} \]  f) 0.136, \[ \frac{136}{1000} \]

ANSWERS:

Gr7_TG_Final.indd 46
29/08/11 5:37 PM
Then tell students that you want to make a model of the number 1.6, again using the thousands block as one whole. **ASK:** What do I need to make the model? (1 thousands block, 6 hundreds blocks) How can I show 1/10 of 1.6? (one tenth of a thousands block is a hundreds block, and one tenth of a hundreds block is a tens block, so I need 1 hundreds block and 6 tens blocks to make 1/10 of 1.6) What number is 1/10 of 1.6? (0.16, since this is what the base ten materials show) Find 1/10 of more numbers together, then have students do so independently using base ten materials.

**EXAMPLES:**
- a) 0.1  
- b) .01  
- c) 0.1  
- d) 7  
- e) 2.3  
- f) 0.41  
- g) 5.01

**ANSWERS:**
- a) 0.1  
- b) 0.001  
- c) 0.01  
- d) 0.7  
- e) 0.23  
- f) 0.041  
- g) 0.501

**Finding one tenth of a number by moving the decimal point one place left.** Remind students that when they find one tenth of a number, each place value becomes worth one tenth as much: if there were 2 ones, there are now 2 tenths; if there were 3 tenths, there are now 3 hundredths. **ASK:** How can you move the decimal point to make each place value worth one tenth as much? (move the decimal point one place left) Demonstrate this with 1/10 of 2.3 = 0.23. **ASK:** Does this remind you of a rule for dividing by something? (yes, to divide by 10, move the decimal point one place left) Emphasize that to find 1/10 of anything, you divide it into 10 equal parts; to find 1/10 of a number, you divide the number by 10.

**Finding 10% of a number by moving the decimal point one place left.** Have students convert 1/10 to a decimal (1/10 = 0.1) and to a percent (1/10 = 10/100 = 10%). Emphasize that by finding 1 tenth of a number, they are finding 10% of it. Ask your students to find 10% of each number by just moving the decimal point.

- a) 40  
- b) 4  
- c) 7.3  
- d) 500  
- e) 408  
- f) 3.07  
- g) 432.5609

**Finding multiples of 10% of a number.** Show this number line:

![Number line](image)

Have a volunteer fill in the missing numbers on the number line. (because 10% of 30 is 3, skip count by 3s to find the remaining percents) Then ask students to look at the completed number line and identify 10% of 30, 40% of 30, 90% of 30, and 70% of 30.

Repeat the exercise for a number line from 0 to 21. (Since 10% of 21 is 2.1, skip count by 2.1. It might be easier to skip count by 21 and divide by 10 as you go: 21 (the first mark is 2.1), 42 (the next mark is 4.2), 63 (the next mark is 6.3) and so on.) **ASK:** If you know 10% of a number, how can you find 30% of that number? (multiply 10% of the number by 3) Tell students that you would like to find 70% of 12. **ASK:** What is 10% of 12? (1.2) If I
know that 10% of 12 is 1.2, how can I find 70% of 12? (multiply 1.2 × 7)
Have students use this method to find:

a) 60% of 15    b) 40% of 40    c) 60% of 4    d) 20% of 1.5

e) 90% of 8.2    f) 70% of 4.3    g) 80% of 5.5

Visualizing percents. Have students calculate:

a) 40% of 33    b) 30% of 42    c) 60% of 85    d) 90% of 21

ANSWERS:       a) 13.2    b) 12.6    c) 51    d) 18.9

Have students check their answers using BLM Percent Strips. Students should cut out the percent strip from the bottom of the sheet and line it up with the other strips. Here is the answer for part a) above:

<table>
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<tr>
<th>0%</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
<th>50%</th>
<th>60%</th>
<th>70%</th>
<th>80%</th>
<th>90%</th>
<th>100%</th>
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<td>0</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td>15</td>
<td>18</td>
<td>21</td>
<td>24</td>
<td>27</td>
<td>30</td>
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<td>24</td>
<td>27</td>
<td>30</td>
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<td>36</td>
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</tbody>
</table>

So 40% of 33 is a little more than 13, as calculated.

Finding 5% and 15% of a number using 10% of the number. ASK: If I know 10% of 42 is 4.2, how can I find 5% of 42? (5% is half of 10%, so if 10% is 4.2, then 5% is 2.1) Have students find 5% of the following numbers by first finding 10% then dividing by 2. (Students should use long division on a separate piece of paper.)

a) 80    b) 16    c) 72    d) 50    e) 3.2    f) 2.34

ANSWERS:       a) 4    b) 0.8    c) 3.6    d) 2.5    e) 0.16    f) 0.117

SAY: I know that 10% of 42 is 4.2, and 5% of 42 is 2.1. What is 15% of 42? (15% of a number is 10% of the number plus 5% of the number, so 15% of 42 is 4.2 + 2.1 = 6.3. Have students verify this answer using BLM Percent Strips by lining up the percent strip with the “42” strip. ASK: Does 10% look like it lines up with 4.2? (yes) Does 15% look like it lines up with 6.3? (yes)

Then have students calculate 15% of each number below by finding 10% and 5% and then adding:

a) 60    b) 240    c) 12    d) 7.2    e) 3.80    f) 6.10

Tell students that when going out to eat at a restaurant, people are expected to leave a tip. Depending on how good the service is, the tip could be anywhere from 10% to 20% of the bill, before taxes. Assuming you want to leave a 15% tip, have students calculate, in their heads, what tip they should leave for various total costs of a meal. Remind students that 15% = 10% + 5%.

a) $30    b) $22    c) $18    d) $35    e) $47
ANSWERS:
a) $4.50  
b) $3.30  
c) $2.70  
d) $5.25  
e) $7.05

As a check on their calculations, have students order the meal costs from smallest to greatest and the tips they calculated from smallest to greatest—these should be in the same order! (c, b, a, d, e)

Relate the different percents to each other. Remind students that 5% is half of 10%, and that 15% is the sum of 10% and 5%. **ASK:** If I know 4% of a number is 32, how can I find 2%? (find half of 32, so 2% is 16) What is 1%? (1% is half of 2% = 16, so 1% is 8) What is 100%? (800) What is the number? (800, since 100% of a number is just the number) **ASK:** Could I have gotten that directly from knowing that 4% of the number is 32? **PROMPT:** What can I multiply 4% by to get 100%? (25) So calculate 32 × 25 = 800 to get 100% of the number.

**ASK:** If 30% of a number is 27, what is the number? To guide students, suggest that they first find 10% of the number, then find the number. (10% is 9, so the number is 90)

**Easy percents to find.** Tell students that since percents can be changed to fractions, and some fractions of numbers are easy to find, some percents are easy to find too. **ASK:** What is 20% as a fraction? (1/5) How can you find 20% of a number easily, without using 10% of the number? (divide the number by 5) Have students calculate 20% of the numbers from BLM Percent Strips (33, 42, 85, and 21) by dividing the number by 5, and then have them check their answer using the BLM. Repeat for 25% of the numbers (25% = 1/4, so divide the number by 4) Have students find 75% of the same numbers using 25% of each number (which they already found). Students can check their answer on BLM Percent Strips.

**Finding 1% of a number.** Have students convert 1% to a decimal and to a fraction. (1% = 0.01 = 1/100) **ASK:** Finding 1% of a number is the same as dividing the number by what? (100) How can you move the decimal point to find 1% of a number? (move it two places left) Have students find 1% of various numbers. **EXAMPLES:**

<table>
<thead>
<tr>
<th>a) 27</th>
<th>b) 3.2</th>
<th>c) 773</th>
<th>d) 12.3</th>
<th>e) 68</th>
</tr>
</thead>
</table>

**ANSWERS:**

<table>
<thead>
<tr>
<th>a) 0.27</th>
<th>b) 0.032</th>
<th>c) 7.73</th>
<th>d) 0.123</th>
<th>e) 0.68</th>
</tr>
</thead>
</table>

**Finding any whole number percent of a number.** **SAY:** I know that 1% of 400 is 4. **ASK:** What is 2% of 400? (2 × 4 = 8) What is 3% of 400? (3 × 4 = 12) What is 17% of 400? (17 × 4 = 68) Explain that to find any percent of a number, you can find 1% of the number and then multiply. Have students use this method to calculate:

<table>
<thead>
<tr>
<th>a) 64% of 33</th>
<th>b) 37% of 42</th>
<th>c) 94% of 85</th>
<th>d) 83% of 21</th>
</tr>
</thead>
</table>

Students can check their answers visually using BLM Percent Strips.
Tell students that the Harmonized Sales Tax (HST) used in Ontario is 13%.
Have students determine the amount of HST on something that costs:

a) $33  b) $42  c) $85  d) $21

Alternatively, use the tax rate in your region instead of the Ontario HST.

ASK: How can you use the percent strips to check your answer?

NOTE: Have students keep their percent strips to refer to next class.
NS7-71 Further Percents
Workbook page 19

CURRICULUM EXPECTATIONS
Ontario: 7m1, 7m2, 7m3, 7m4, 7m5, 7m7, 7m28
WNCP: 7N3, [CN, R, ME, C]

VOCABULARY
none

Goals
Students will use estimation to determine if the percent they calculated is reasonable. Students will discover and explain the commutativity of percents (e.g., 30% of 10 is 10% of 30).

PRIOR KNOWLEDGE REQUIRED
Can find any whole number percent of any given number
Can add percents
Can convert between fractions, percents, and decimals
Knows that multiplication commutes

MATERIALS
BLM Percent Strips (p L-103)
BLM Price List (p L-104)

Review determining percents of numbers. In the last lesson, students learned to find the percents of numbers by using division to find 1% and then multiplying. For example, to find 64% of 33, they found 1% of 33 by calculating 33 ÷ 100. Then they multiplied 1% of 33 by 64 to find 64% of 33 since 64% of a number is 64 times greater than 1% of the number. So they calculated: 64 × 33 ÷ 100.

Connect this method to finding a fraction of a whole number. We know that 64% = 64/100, so 64% of 33 is 64/100 of 33. But this is just 64 × 33 ÷ 100, as above. Have students use this method to calculate 53% of 12 (53 × 12 ÷ 100 = 6.36).

Using easy percents to check if an answer is reasonable. Tell students that you want to know if the answer is reasonable. Have students take out their percent strips from last class. ASK: Can I check using the percent strips? (no) Why not? (because we don’t have a percent strip for 12) ASK: What is another way to check if our answer is reasonable? PROMPT: Is there a percent of 12 that is close to 53% and easy to calculate? (yes, 50%) Will this estimate be lower or higher than the actual answer? (50% of 12 is 6, which is lower than the actual answer because 50% is less than 53%) Is the answer we found just a little higher than 6? (yes) Conclude that the answer is reasonable. Have students calculate the following percents and then use an easy percent to determine if their answers are reasonable.

a) 76% of 24 (should be slightly more than 3/4 of 24, which is 6 × 3 = 18)
b) 19% of 25 (should be slightly less than 1/5 of 25, which is 25 ÷ 5 = 5)
c) 48% of 76 (should be slightly less than 1/2 of 76, which is 38)
d) 11% of 32 (should be slightly more than 1/10 of 32, which is 3.2)
Commutativity of percents. Have students predict which is greater: 20% of 60 or 60% of 20. Tell students that the first is a smaller percentage of a larger number, so you’re not sure which is bigger. Then have students calculate both. (both are 12) SAY: These are both the same. In hindsight, is there a reason we should have been able to predict this? PROMPT: How does 20% of 60 compare to 20% of 20? (It is 3 times greater) How does 60% of 20 compare to 20% of 20? (It is also 3 times greater) So they are both equal.

Have students calculate and compare:

a) 30% of 50 and 50% of 30  
b) 40% of 20 and 20% of 40  
c) 70% of 90 and 90% of 70  
d) 80% of 60 and 60% of 80  
e) 50% of 40 and 40% of 50  
f) 36% of 24 and 24% of 36  
g) 17% of 35 and 35% of 17  
h) 29% of 78 and 78% of 29  
i) 48% of 52 and 52% of 48

What pattern do students see? Challenge them to figure out why this pattern holds. PROMPT: What calculations do you need to do to get 24% of 36? (24 × 36 ÷ 100) How about 36% of 24? (36 × 24 ÷ 100) What rule can you use to explain why these calculations will get the same answer? (multiplication is commutative—e.g. 36 × 24 = 24 × 36)

NOTE: In the following discussion and questions, we presume the sales tax is 13%. If the sales tax in your region is different, please adjust the questions accordingly. For example, we calculate 10% and 30% in preparation for calculating 10% and 3%. If your sales tax is 12%, you could change this to 10% and 20% in preparation for calculating 10% and 2%.

Estimating percents of whole numbers and decimals. Tell students that you want to estimate 30% of 48. Discuss different strategies.

1. 10% of 48 is about 5, so 30% is about 15.
2. 30% of 48 = 48% of 30, which is about 50% of 30, or 15.
3. 30% of 48 is about half of 30% of 100, which is 30, so 30% of 48 is about 15.

Have students estimate:

a) 1% of 732  
b) 3% of 732  
c) 3% of 7.32

ANSWERS: a) 7  
b) 21  
c) 0.21

ASK: What is a good estimate for 10% of 732? (73 or 70—73 is a more accurate estimate but 70 is easier to work with) What about 13% of 732? (73 + 21 = 94 or 70 + 21 = 91) And 13% of 7.32? (0.91 or 0.9)

Remind students that 1% of a dollar is a cent. ASK: What is 1% of $12? (12 cents) Have students estimate:
a) 13% of $12.49  

ANSWERS:

a) 1% is about 12¢, so 13% is about
13 \times 12¢ = 10 \times 12 + 3 \times 12 = 120 + 36 \approx 160,
so 13% of $12.49 is about $1.60.

b) 13% of 851¢ is about 85¢ + 25¢ = 110¢. So 13% of $8.51 is about $1.10.

c) 13% of 900¢ is 90¢ + 27¢, or about 120¢, so 13% of $9.00 is about $1.20.

d) 13% of 2599¢ is about 13% of 2600¢, or about 260¢ + 75¢ = 335¢, so 13% of $25.99 is about $3.35.

Point out that rounding $25.99 to $26 will not affect the answer, since it will only be off by 13% of 1 cent.

ASK: Which two answers are close together? (13% of 851 is close to 13% of 900) Why does this make sense? (because 851 is close to 900) Which answer is about twice as much as another answer? (13% of $25.99 is about twice as much as 13% of $12.49) Why does this make sense? ($25.99 is about $26 and $12.49 is close to $13, and 26 is twice as much as 13, so 13% of the first quantity will be about twice as much as 13% of the second)

Have students calculate the exact sales tax on the amounts above. ASK: My calculator tells me that 13% of $12.49 is 1.6237—what is the sales tax on a book that costs $12.49? (the sales tax is $1.62) Why do we need to round to two decimal places? (because the lowest coin we have is a penny, worth one hundredth of a dollar, we can't pay .37 of a cent.)

Have students use a calculator to add a sales tax of 13% to the following prices. They should round the total price to two decimal places.

a) original price: $27.85
b) original price: $26.44
c) original price: $119.99
d) original price: $74.00

ANSWERS: a) $3.62   b) $3.42   c) $15.60   d) $9.62

ASK: Which question did you not need to round for? Why not? (d, because 13% of 74 has only 2 decimal places)

Solve the following problem together as a class: A book costs $18.49. The salesperson tells you that the total price is $22.37. If books are taxed at 13% of the cost of the book, is the total price reasonable?

ASK: What whole number price that is easy to work with is close to $18.49? (19 or 20) Tell students that you find 20 easier to work with, even though it is not as close to 18.49 as 19 is. Have students calculate 13% of $20. (10% is
2, 1% is 0.2, 3% is 0.6, and 13% is 10% + 3% = $2.60) **ASK:** The tax on $20 would be $2.60. Will the tax on $18.49 be more or less than $2.60? (less) Why? (because 13% of 18.49 is less than 13% of 20) To estimate the total cost of the book, students can add the tax on a $20-dollar book to $18.49 and know that the cost will be a little less than the answer: $18.49 + $2.60 is about $18.50 + $2.50 = $21.00, so the total price of the book should be a little less than $21. But the salesperson told you that the price is more than $22! This is highly unreasonable. Have students use a calculator to determine the actual exact cost of the book. (13% of $18.49 is $2.40 so the total cost is $20.89)

Either the salesperson made an honest mistake, or she is pocketing the extra money. **ASK:** If the salesperson is pocketing the extra money, how much did she pocket? ($22.37 − $20.89 = $1.48)

Have students determine, without using a calculator, which total prices (after taxes) are reasonable if the tax is 13%:

<table>
<thead>
<tr>
<th>Original Price</th>
<th>Price After Taxes</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $15.00</td>
<td>$16.95</td>
</tr>
<tr>
<td>b) $23.00</td>
<td>$26.99</td>
</tr>
<tr>
<td>c) $21.49</td>
<td>$24.28</td>
</tr>
<tr>
<td>d) $3.24</td>
<td>$3.66</td>
</tr>
<tr>
<td>e) $5.83</td>
<td>$7.29</td>
</tr>
<tr>
<td>f) $832.99</td>
<td>$961.48</td>
</tr>
</tbody>
</table>

**ANSWERS:**

a) 10% is $1.50 and 3% is $0.45, so 13% is $1.95. Yes, $16.95 is reasonable.

b) 10% is $2.30 and 3% is $0.69, so 13% is about $3.00, but the price quoted is nearly $4 more than the original price. No, this is not reasonable.

c) 10% is about $2.10 and 3% is about $0.60, so the total price should be about $21.50 + $2.70 or about $24.20, so yes, this is reasonable.

d) 10% is about $0.32, and 3% is about $0.09, so the total price should be about $0.40 cents more than original, as it is, so the price given is reasonable.

e) 10% is about $0.60 and 3% is about $0.18 so the total price should be less than $0.80 more than the original price, but it is over a dollar more, so the price given is not reasonable.

f) 10% is about $83 and 3% is about $25, so the total price should be about $110 more than the original price, but it is about $130 more, so the price given is not reasonable.

Have students use a calculator to check their estimates. Were they correct about which prices after taxes are reasonable?
ACTIVITY

The Honest Cashier

In this whole-class activity, five students will act as cashiers and the rest will be shoppers in a bookstore. One cashier is honest and calculates the total price of the books correctly all the time. The remaining cashiers calculate the total price incorrectly at different rates. This activity will give students lots of practice calculating prices after taxes (in other words, calculating percents). The five cashiers will not get this practice, so the game should be repeated with different cashiers.

Invite five volunteers to be the cashiers. The cashiers behave as follows:

Cashier 1 calculates the total price correctly all the time.

Cashier 2 adds $1.00 to the total price every time.

Cashier 3 calculates the total price correctly half the time, and adds $1.00 to the price the other half of the time.

Cashier 4 calculates the total price correctly one quarter of the time, and adds $1.00 to the price the other three quarters of the time.

Cashier 5 calculates the total price correctly three quarters of the time, and adds $1.00 to the price the other one quarter of the time.

Each cashier receives a copy of BLM Price Chart. The BLM lists 50 prices before and after taxes (the tax rate is 13%). The total prices have been calculated according to the roles listed above. On each cashier’s copy, the teacher circles the column of prices that the cashier should use. The cashiers can determine which cashier they are (1, 2, 3, 4, or 5) by looking at the list. Take the cashiers aside to discuss what makes this easy to determine.

Then have the cashiers stand around the room and invite the shoppers to make purchases. You can attach a different price to 50 different books, or just write the different prices on slips of paper. Shoppers should follow these rules:

1. If the cashier cheats the student, the student lines up at a different cashier.
2. If the cashier is honest, the student lines up again at the same cashier.
3. Once you enter a line, you cannot leave it even if you see other people ahead of you were cheated.

As students make purchases, they check the total price the cashier charged them to see if it is correct, and line up again (with the same or a different book), the line up for the honest cashier will get longer and the line ups for dishonest cashiers will get shorter. The goal is to have all the shoppers lined up at the honest cashier.

Variation: Students try to figure out which cashier is playing which role. They confer with each other about which cashiers cheated them to help them decide. All students must agree before they are allowed to make their guess.

PROCESS EXPECTATION

Mental math and estimation, Using logical reasoning

PROCESS ASSESSMENT

7m7, [C]
Workbook Question 4
Extensions

1. Sara says that to find 10% of a number, she divides the number by 10, so to find 5% of a number, she divides the number by 5. Is she right? Explain. (No—5% of a number is 5/100 or 1/20 of the number, so to find 5%, or 1/20, of the number, she should divide it by 20.)

2. DISCUSS: Does it make sense to talk about 140% of a number? What would it mean? Lead the discussion by referring to fractions greater than 1. Discuss what 100% and 40% of a number mean separately. Could 50% of a number be obtained by adding 20% and 30% of that number? Could 140% be obtained by adding 100% and 40% of the number?

3. Revisit this problem: A book costs $18.49, and the salesperson tells you that the total price is $22.37. If books are taxed at 13% of the cost, is the total price reasonable?

Tell students that sometimes it is easy to do such problems by using fractions as benchmarks. Since taxes are 13%, it is useful to use benchmarks that are near 13%. For example:

1/5 is 20%
1/6 is a little less than 17%
1/7 is a little more than 14%
1/8 is halfway between 12% and 13%.

To identify which benchmark is best to use in a particular case, think about what the number you are working with is easy to divide by. In this case, the cost of the book is about $18, and 18 is easily divided by 6. So using a benchmark of 1/6 would work well in this case. We know that 1/6 of $18 is $3, so if the tax rate is 17%, we expect to be taxed about $3. But we were taxed at 13% and the tax was almost $4, so right away we can see that the total price is unreasonable.

Have students redo problems a) – f) above (determining which prices are reasonable), using this new strategy.

ANSWERS:

a) $15.00 divided by 8 is almost $2, so yes, this is reasonable.

b) $23 divided by 8 is almost $3, but the price quoted is $4 more than the original price, so no, this is not reasonable.

c) $21 ÷ 7 is $3, so if the tax rate was 14%, we would expect about $3 in taxes. In fact, we were charged about $2.70 in taxes, which is reasonable since the tax rate is 13%.

d) $3.20 ÷ 8 = $0.40, so the total price should be about $0.40 more than the original, which it is, so the given price is reasonable.
e) 1/6 of $6 is $1, so if the original price was $6 and we were taxed at 17% we should expect $1 in taxes. In fact, we were charged more for a lower original price and a lower tax rate, so the given price is not reasonable.

f) The taxes we pay should be less than $832 \div 7$, which in turn is less than $840 \div 7 = $120, so $120 should be more than the taxes we pay, but in fact we are paying more than that, so no the given price is not reasonable.
Review equivalent ratios using pictures. The following picture shows that 6 : 9 = 2 : 3.

6 of the 9 circles are shaded.
2/3 of the 9 circles are shaded.
So 6 is 2/3 of 9.

Have students do Workbook Questions 1, 2, and 3. Have students write the ratio as part : whole.

Writing ratios with missing parts. Have students write each number in the correct place in the proportion, but replace the missing number with a question mark.

a) 3 is 1/2 of what number? (1 : 2 = 3 : ?)
b) 4 is 1/3 of what number? (1 : 3 = 4 : ?)
c) 6 is 2/5 of what number? (2 : 5 = 6 : ?)
d) What number is 3/4 of 20? (3 : 4 = ? : 20)
e) What number is 4/5 of 20? (4 : 5 = ? : 20)
f) What number is 2/7 of 21? (2 : 7 = ? : 21)

Then have students write their answers in fraction form as well.

For example, the answer for part a) is \( \frac{1}{2} = \frac{3}{?} \).

Write on the board: 12 is how many fifths of 30? Underline “how many fifths” and point out that this is the same as “?/5.” The denominator tells you that the size of the parts is a fifth, and the numerator, the unknown, tells you the number of fifths. So “12 is how many fifths of 30” is another way of saying “12 is ?/5 of 30.” This is now easy to change to an equivalent ratio using the
method of Workbook p. 20 Question 3. \( : 5 = 12 : 30 \) Have students write equivalent ratios for these questions:

a) 8 is how many thirds of 12?

b) 21 is how many quarters of 28?

c) 18 is how many tenths of 30?

Again, have students write the fraction form as well.

**Writing percent statements in terms of ratios.** Remind students that asking how many hundredths is like asking for \( ?/100 \). **ASK:** What is another name for a fraction with denominator 100? **PROMPT:** What do we use to compare numbers to 100? (a percent) Since students can write fraction statements as equivalent ratios, and a percent is just a fraction with denominator 100, students can now write percent statements as equivalent ratios. Have students write proportions for these questions:

a) 19 is how many hundredths of 20?

b) 13 is how many hundredths of 50?

d) 36 is how many hundredths of 60?

Have students use their answers above to write proportions for these questions (without solving them):

a) 19 is what percent of 20?

b) 13 is what percent of 50?

c) 36 is what percent of 60?

Have students write the proportion in terms of fractions as well. For example, \( 19/20 = ?/100 \).

Have students write these questions as a proportion, in both ratio and fraction form:

a) What is 15% of 40? \( : 40 = 15 : 100 \), or \( ?/40 = 15/100 \)

b) What is 32% of 50? \( : 50 = 32 : 100 \), or \( ?/50 = 32/100 \)

c) What is 75% of 48? \( : 48 = 75 : 100 \), or \( ?/48 = 75/100 \)

Repeat for these problems:

a) 24 is 80% of what number?

b) 62 is 25% of what number?

c) 12 is 30% of what number?

And then combine all three types of questions above. (These are summarized on **BLM Three Types of Percent Problems**). Have students do Workbook p. 21 Questions 5 and 6.
Word Problems Practice:

Have students use their answer to each problem below to obtain the answer to the next problem. Discuss the similarities and differences between each problem and the next.

a) 12 is how many fifths of 30?
b) How many fifths of 30 is 12?
c) 12 is what percent of 30?
d) What percent of 30 is 12?
e) A shirt costs $30, and $12 was taken off. What percent was taken off?

ANSWERS: a) 2  b) 2  c) 40  d) 40  e) 40

Have students solve these word problems.

1. A shirt costs $25. After taxes, it cost $30. What percent of the original price are the taxes?

2. A shirt costs $40. After taxes, it cost $46. At what rate was the shirt taxed?

3. A shirt costs $40. It was on sale for $28. What percent was taken off?

**Bonus**

A shirt costs $20. It was on sale at 15% off. A 15% tax was then added. What was the final price?

**ANSWER:** The sale price is $17. After a 15% tax, the price becomes $19.55.

**Extensions**

1. Include non-whole numbers as the value of a percent. **EXAMPLES:**

   a) What percent of 30 is 16.5?
   b) What percent of 18 is 2.7?
   c) What percent of 14 is 2.8?

2. Give word problems involving non-whole numbers as the value of a percent.

   a) A book that costs $18 came to $20.70 after taxes.
      i) How much were the taxes?
      ii) What percent is the tax?
   b) The regular price of a book is $18. The sale price is $12.60.
      i) How much was taken off the regular price?
      ii) What percent was taken off the regular price?
NS7-73 Using Proportions to Solve Percent Problems
NS7-74 Percent Problems
Workbook pages 22–24

CURRICULUM EXPECTATIONS
Ontario: 7m3, 7m5, 7m7, 7m28
WNCP: 7N3, [CN, C, R]

VOCABULARY
proportion
equivalent ratio
percent

Goals
Students will use proportions to solve percent problems

PRIOR KNOWLEDGE REQUIRED
Can write equivalent statements for proportions
Can write percents as ratios to 100


Using proportions to solve percent problems. Have students rewrite each question as a proportion in fraction form, and then solve the proportion:

a) What percent of 60 is 12?
b) What is 24% of 20?
c) 6 is what percent of 20?
d) 6 is 30% of what number?

Continue with proportions that require reducing one of the ratios before it can be solved.

a) What percent of 80 is 16?
b) What is 60% of 15?
c) 9 is what percent of 30?
d) 6 is 40% of what number?

When students do Workbook p. 23 Question 8, note that they should be comparing the proportions in Question 7 to parts a–e of Question 4. (The proportions in Question 4 parts a–e can be solved directly without reducing either ratio; the proportions in Question 7 cannot be.)

Adding percents. Explain that because percents are just fractions with denominator 100, we can add and subtract percents the same way we add and subtract fractions with the same denominator. For example, $23\% + 45\% = \frac{23}{100} + \frac{45}{100} = \frac{68}{100} = 68\%$. Have students solve these problems:

a) $30\% + 40\% = \ ____$
b) $32\% + 27\% = \ ____$
c) $46\% + 38\% = \ ____$
d) $20\% + \ ____ = 60\%$
e) $24\% + \ ____ = 37\%$
f) $38\% + \ ____ = 72\%$
g) $50\% - 10\% = \ ____$
h) $78\% - 24\% = \ ____$
i) $62\% - 45\% = \ ____$
j) $43\% - \ ____ = 18\%$
k) $34\% - 12\% + 56\% = \ ____$
Extension

Teach students to cross-multiply to solve proportions. Write on the board:

\[
\frac{6}{10} = \frac{?}{15}
\]

Have students solve the proportion. \(\frac{6}{10} = \frac{9}{15}\)

Then tell students that you find whole numbers easier to work with than fractions, and if you multiply both fractions by the same number, they will still be equal. To make both fractions whole numbers, the easiest way is to multiply by the product of the denominators: \(10 \times 15\). So:

\[
\frac{6}{10} \times 10 \times 15 = \frac{9}{15} \times 10 \times 15
\]

So \(6 \times 15 = 9 \times 10\)

Ask students to find both products to verify that they are equal. (yes, they are both 90)

Have students write the equation they get by multiplying both sides by the product of the denominators:

a) \(\frac{9}{12} = \frac{15}{20}\)  
b) \(\frac{4}{10} = \frac{6}{15}\)  
c) \(\frac{5}{6} = \frac{10}{12}\)

ANSWERS: a) \(9 \times 20 = 15 \times 12\)  
b) \(4 \times 15 = 10 \times 6\)  
c) \(5 \times 12 = 6 \times 10\)

Have students check that each equation is correct by calculating both products.

ANSWERS: a) 180  
b) 60  
c) 60

Have students write the equation they get by multiplying both sides by the product of the denominators when the fractions are given in terms of variables: \(\frac{a}{b} = \frac{c}{d}\) so \(a \times d = b \times c\). Tell students that this is called cross-multiplying because we can draw a cross that tells us what to multiply:

Now, have students decide which of these are true by cross-multiplying:

a) \(\frac{3}{4} = \frac{42}{60}\)  
b) \(\frac{4}{7} = \frac{24}{42}\)  
c) \(\frac{3}{5} = \frac{513}{860}\)

ANSWERS:

a) Does \(3 \times 60 = 4 \times 42\)? No, 180 ≠ 168, so not true

b) Does \(4 \times 42 = 7 \times 24\)? Yes, 168 = 168, so true

c) Does \(3 \times 860 = 5 \times 513\)? No, 2580 ≠ 2565, so not true

Now, show students how to use cross-multiplying to replace the fractions in a proportion with products: \(\frac{3}{5} = \?/35\) becomes \(3 \times 35 = 5 \times \?\), so \(\? = 3 \times 35 \div 5 = 21\).

Have students solve proportions in the Workbook using this method.
Recognizing the part and the whole. To solve questions involving fractions, ratios, and percents, students need to be able to recognize the part and the whole (and to express the ratio of the part to the whole as a fraction). Use simple word problems to illustrate and work through different cases.

EXAMPLES:

a) There are 5 boys among 9 children. (In this case, you are given the whole and one of the parts.) **ASK:** What fraction of the children are boys? (5/9) What fraction of the children are girls? (first, subtract the number of boys from the number of children to find the number of girls: 9 − 5 = 4. Therefore, 4/9 of the children are girls.)

b) There are 5 boys and 6 girls. (In this case, you are given two parts). **ASK:** What is the fraction of boys and girls? (first find out how many children there are (this is the whole): 5 + 6 = 11, so 5/11 of the children are boys and 6/11 are girls)

c) The ratio of boys to girls is 3 : 4. **ASK:** What is the fraction of boys and girls? (For every 3 boys there are 4 girls, so 7 children in total (i.e., add the parts of the ratio: 3 + 4 = 7). So 3/7 of the children are boys and 4/7 are girls)

Finding indirect information. Ask students how many boys (b) are in their class today, how many girls (g), and how many children (c) altogether:

\[
b = \quad g = \quad c =
\]

**ASK:** Did you count everyone one by one or was there an easier way once you found the number of boys and girls? Tell students that sometimes they are not given all the information that they need, but can find the information indirectly. Ask students to fill in the numbers of boys, girls, and children given various pieces of information:

a) 7 girls and 8 boys  \( b = \quad g = \quad c = \)
b) 6 girls in a class of 20  \( b = \quad g = \quad c = \)
c) 12 boys in a class of 30  \( b = \quad g = \quad c = \)
d) 17 girls in a class of 28  \( b = \quad g = \quad c = \)
Then have students determine the numbers of boys, girls, and children, the fraction of girls, and the fraction of boys in these classes:

a) There are 6 boys and 5 girls.
b) There are 14 boys in a class of 23.
c) There are 15 girls in a class of 26.

Now have students write the fraction and percent of girls and boys in these classes.
a) There are 3 boys and 7 girls in a class.
b) There are 7 boys and 20 children in a class.
c) There are 8 girls and 25 children in a class.
d) The ratio of boys to girls is 1:3.
e) The ratio of girls to boys is 2:3.
f) The ratio of boys to girls is 12:13.
g) The ratio of boys to girls is 13:12.
h) The ratio of girls to boys is 13:12.

ANSWERS:
a) \(\frac{7}{10} = 70\% \) girls; \(\frac{3}{10} = 30\% \) boys
b) \(\frac{13}{20} = 65\% \) girls; \(\frac{7}{20} = 35\% \) boys
c) \(\frac{8}{25} = 32\% \) girls; \(\frac{17}{25} = 68\% \) boys
d) \(\frac{3}{4} = 75\% \) girls; \(\frac{1}{4} = 25\% \) boys
e) \(\frac{2}{5} = 40\% \) girls; \(\frac{3}{5} = 60\% \) boys
f) \(\frac{13}{25} = 52\% \) girls; \(\frac{12}{25} = 48\% \) boys
g) \(\frac{12}{25} = 48\% \) girls; \(\frac{13}{25} = 52\% \) boys
h) \(\frac{13}{25} = 52\% \) girls; \(\frac{12}{25} = 48\% \) boys

ASK: Which two of the last three questions have the same answer? (parts f) and h) have the same answer) Can you think of a question that has the same answer as part g)? (The ratio of girls to boys is 12:13.)

Determining the number of girls and boys given the total and a fraction or ratio. Write this problem on the board: There are 28 children in a class and 3/7 are girls. How many girls and how many boys are in the class?

Have students write the fraction as a part-to-whole ratio: There are 3 girls for every 7 \(\text{children}\). (children) Show students how to model the situation. Represent girls with shaded circles and boys with unshaded circles:

\[\circ \circ \circ \circ \circ \circ \circ \] 3 of the 7 students are girls.
We are told that there are 28 students, so we keep drawing 7 students (3 girls and 4 boys) until we have drawn 28 students:

![Diagram of students drawn]

**ASK:** How many girls are in the class? (12) How many boys are in the class? (16) Have students determine the number of girls and boys in more classes:

a) There are 30 children in a class and 3/5 are girls.
b) There are 36 children in a class and 4/9 are girls.
c) There are 21 children in a class and 4/7 are boys.

Then go back to the original problem (3/7 are girls in a class of 28) and show students how they can solve it without drawing a model. To find the number of girls, they must solve this proportion:

\[
\frac{3}{7} = \frac{?}{28}
\]

To solve, notice that \(7 \times 4 = 28\), so multiply the numerator by 4 as well:

\[
\frac{3 \times 4}{7 \times 4} = \frac{12}{28}, \text{ so } ? = 12
\]

**ASK:** How does the model show that there are \(3 \times 4\) girls? (the shaded circles are in 4 rows of 3) How does the model show that there are \(7 \times 4\) students? (the circles are in 4 rows of 7) Have students redo the other three questions (a – c) by solving proportions instead of drawing the model.

Now write this problem on the board: There are 36 children in a class. The ratio of boys to girls is 4 : 5. How many boys and how many girls are in the class?

Demonstrate using a model to solve this one. There are 4 boys for every 5 girls, so draw 4 unshaded circles for every 5 shaded circles. Continue until there are 36 circles in total:

![Diagram of students drawn]

The model shows that there are 16 girls and 20 boys. Now demonstrate using proportions to solve the problem, and make the connection between the diagram and solving the proportion, as above. Then have students do the problems below, both by using a model and by solving a proportion.

a) There are 18 children in a class. The ratio of boys to girls is 7:2.
b) There are 18 children in a class. The ratio of girls to boys is 2:7.
c) There are 18 children in a class. The ratio of boys to girls is 2:7.

Stop to discuss which 2 of the 3 are the same (parts a and b) and how to formulate another question the same as part c.

d) There are 30 children in the class and 60% are girls.
e) There are 45 children in the class and 40% are girls.

**Extension**

You can estimate what percent one number is of another by changing one or both of the numbers slightly. For example, to estimate what percent 5 is of 11, change the 11 to 10, because 11 is close to 10 and 10 is easy to find percents of. Since 5 is 50% of 10, 5 is close to 50% of 11.

The chart below shows the lengths (in feet) of different types of whales at birth (when they are called calves) and as adults. Approximately what percent of the adult length is the calf’s length? Did you need to know how long a foot is to answer this question?

<table>
<thead>
<tr>
<th>Type of Whale</th>
<th>Killer</th>
<th>Humpback</th>
<th>Narwhal</th>
<th>Fin Backed</th>
<th>Sei</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calf Length (feet)</td>
<td>7</td>
<td>15</td>
<td>5</td>
<td>20</td>
<td>14</td>
</tr>
<tr>
<td>Adult Length (feet)</td>
<td>32</td>
<td>48</td>
<td>15</td>
<td>78</td>
<td>50</td>
</tr>
</tbody>
</table>

SAMPLE SOLUTIONS: 7 is 25% of 28 and 20% of 35; 32 is closer to 35 than to 28, so estimate that 7 is about 22% of 32 OR 8 is 25% of 32, so 7 is slightly less, say 23% of 32.

Students should check their estimates with a calculator.

**EXAMPLE:** \(7 \div 32 = 0.21875 \approx 0.22 = 22\%\).
Drawing linear models. Note that if we are given that there are 2/3 as many boys as girls, as in the teaching box on Workbook p. 27, then we can draw two bars, one of which is 2/3 the length of the other. The easiest way to do that is to draw 2 equal length bars end to end for boys and 3 of the same bars, end to end, for girls. For solving word problems, it is convenient to have both bars divided into unit bars of the same length. This is what is done in the teaching box. Have students use the following questions to practice drawing linear models before they do Workbook p. 27 Question 2:

- Draw two bars so that:
  - a) Bar A is 1/4 of Bar B
  - b) Bar A is 3/5 of Bar B

Then have students draw bars to represent girls and boys if:

- c) The bar for boys is 3/4 of the bar for girls
- d) The bar for girls is 4/5 of the bar for boys
- e) The bar for boys if 2/5 as long as the bar for girls
- f) The bar for girls is 5/9 as long as the bar for boys

Solving problems when given the total number of boys and girls.
Teach this case as in the teaching box on Workbook p. 27. That is, find the number of students in one unit of the bars by dividing: number of students ÷ number of units in the bars.

EXTRA PRACTICE: Find the number of boys and girls by drawing a model.

- a) There are 30 students on a bus. There are 1/2 as many girls as boys.
- b) There are 32 students on a bus. There are 3/5 as many boys as girls.
- c) There are 35 students on a bus. There are 2/3 as many boys as girls.
ANSWERS:

a) G
   B

There are 3 units and 30 students, so $30 \div 3 = 10$ students in each unit. So there are 10 girls and 20 boys.

b) B
   G

There are 8 units and 32 students, so $32 \div 8 = 4$ students in each unit. There are 12 boys and 20 girls.

c) B
   G

There are 5 units and 35 students, so $35 \div 5 = 7$ students in each unit. There are 14 boys and 21 girls.

Solving problems when given the difference between the number of boys and girls. In this case, we have a different way to determine the number that each unit in the bars represent. For example, suppose we are given that there are 4/7 as many boys as girls:

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Tell students that we don’t know how many students there are in total, but we know that there are 6 more girls than boys. How can we tell how many students each unit on the bars represent? PROMPT: There are 6 extra girls. How many extra units does the bar for girls have? (3) So 3 units represent 6 students. How many students does 1 unit represent? ($6 \div 3 = 2$) Since we know how many students each unit represents, and how many units represent boys and girls, we can calculate how many boys and girls there are. There are $4 \times 2 = 8$ boys and $7 \times 2 = 14$ girls.

Have students model the following situations to determine the number of boys and girls:

a) There are 4/5 as many boys as girls. There are 3 more girls than boys.

b) There are 3/5 as many boys as girls. There are 10 more girls than boys.

c) There are 3/7 as many girls as boys. There are 12 more boys than girls.

ANSWERS:

a) 12 boys and 15 girls  
   b) 15 boys and 25 girls  
   c) 9 girls and 21 boys

Students should verify that their answers satisfy the given information.

EXAMPLE: 12 is 4/5 of 15 and there are indeed 3 more girls than boys.
Solve the same problems by using a chart. Show students how they could solve the same type of problem using a chart instead. **EXAMPLES:**

a) There are 4/7 as many boys as girls. There are 6 more girls than boys.

Continue adding 4 boys and 7 girls until there are 6 more girls than boys.

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
<th>How many more girls than boys</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>6</td>
</tr>
</tbody>
</table>

There are 8 boys and 14 girls.

b) There are 3/5 as many girls as boys. There are 40 students altogether.

Continue adding 5 boys and 3 girls until there are 40 students altogether.

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
<th>How many boys and girls in total</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>16</td>
</tr>
<tr>
<td>15</td>
<td>9</td>
<td>24</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>32</td>
</tr>
<tr>
<td>25</td>
<td>15</td>
<td>40</td>
</tr>
</tbody>
</table>

There are 25 boys and 15 girls.

Discuss which method (draw a linear model or use a chart) students like better.

**Finding the whole from the part.** Write on the board: 2/3 of a number is 100. What is the number? **ASK:** Is 100 the part or the whole? (the part) What is the whole? (the number that we don’t know) Tell students that this is a part-to-whole ratio, and write on the board:

\[
\frac{2}{3} = \frac{100}{?} = \frac{\text{part}}{\text{whole}}
\]

Have students solve the proportion. (? = 150) Then have students do Workbook p. 28 Question 1.

Now write on the board: 2/3 of the beads in a box are red. 100 beads are red. How many beads are in the box? (This is the same problem as before written a little differently. Use the following sentences and prompts to show this.) Then write:

2/3 of the number of beads in the box is the number of red beads in the box.

**ASK:** What is the number of red beads in the box? (100) Now, underneath the previous sentence, write:

2/3 of the number of beads in the box is 100.

Now underline part of the sentence: 2/3 of the number of beads in the box is 100. Tell students this underlined part is what we want to know, so now underneath the previous sentence, write:

2/3 of what number is 100?

This is exactly the problem we solved earlier.
PROCESS EXPECTATION

Changing into a known problem

Have students do Workbook p. 28 Question 2 by recognizing that the problems are just like those in Question 1 and rewriting them accordingly.

Problems with three groups instead of two. Write the following problem on the board:

2/5 of the people (boys, girls, adults) at the park are boys.

There are 3 more girls than boys.

There are 7 adults.

How many people are at the park?

Do this problem together both ways—using a model and using a chart. Start with the chart. It should have a column for the number of each type of person (boys, girls, adults), as well as the total number of people. SAY: We know that there are 2 boys for every 5 people, so we can add 2 boys for every 5 people in our chart. Begin filling the chart in:

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
<th>Adults</th>
<th>People</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td>15</td>
</tr>
</tbody>
</table>

ASK: If there are 2 boys, how many girls are there? (5) How do you know? (there are 3 more girls than boys) How many adults are there? (7) How do you know? (it says so right in the question) Fill in the first row completely. ASK: Is this possible? Can there be 2 boys, 5 girls, 7 adults, and 5 people altogether? (no, there would be 14 people altogether) Have students continue in this way until they find a combination that works. Students should add rows to the chart as needed.

When students finish, challenge them to find a model to help solve the problem. One possible model could be to draw a bar divided into fifths and then add the given information to it:

Have students use this model, or their own, to solve the problem. When students finish, ask them again which way they like better—using a chart or using a model. A model in this case makes the problem quite easy. From the picture, it is clear that 1/5 of the total number of people is 10, so there are 50 people altogether.

PROCESS EXPECTATION

Selecting tools and strategies

Have students decide which method to use—modelling or a chart—to do these problems. Then do the problems both ways. Was their prediction about which way would be easier correct?
a) There are 5 adults at a park.
   There are 7 more girls than boys at a park.
   3/7 of the people at the park are boys.
   How many people are at the park?

b) 80% of the people at a park are children.
   There are 24 more children than adults at the park.
   How many people are at the park?

c) There are 18 boys at a park.
   3/5 of the people (boys, girls, and adults) are girls.
   There are 6 fewer adults than boys.
   How many people are at the park?

ANSWERS: a) 84   b) 40   c) 75

Note that all of these problems are quite easy using modeling, and more difficult using a chart.

Other contexts. Draw a model and then solve these problems:


b) In a parking lot, 3/4 of the vehicles are cars, 1/5 are trucks, and the rest are buses. There are 4 buses. How many vehicles are in the parking lot? (80)

c) Katie has a rock collection. She found 2/5 of her rocks in British Columbia, 1/3 in Alberta, and the rest in Ontario. She found 8 more rocks in Alberta than she did in Ontario.
   i) What fraction of her rocks did she find in Ontario? (4/15)
   ii) What fraction of the total number of rocks do the 8 rocks represent? Hint: 8 is the difference between the number found in Alberta and the number found in Ontario. (1/15)
   iii) How many rocks does Katie have altogether? (15 × 8 = 120)

d) At a school, 3/7 of the people are boys, 2/5 are girls, and the rest are adults. There are 6 more students of one gender than the other.
   i) Are there more boys or girls? (3/7 = 15/35 and 2/5 = 14/35, so there are more boys)
   ii) What fraction of the total number of people do the 6 extra students of one gender represent? (3/7 − 2/5 = 15/35 − 14/35 = 1/35)
   iii) How many people are there in the school altogether? (6 × 35 = 210)
In Jacob’s coin collection:
1/5 are from China,
1/4 are from Europe,
some are from Canada,
and the rest are from Mexico.

There are 12 more coins from Europe than from Mexico.
There are 48 more coins from Canada than from China.
How many coins does Jacob have altogether?

**ANSWER:** 360

**Extension**

a) 3/5 of 4/5 of a number is 60. What is the number? (Solve in two steps:
If 3/5 of something is 60, then the something is 100, so 4/5 of a number is 100. This means the number is 125.)

b) 2/3 of 3/4 of a number is 36. What is the number? (72)

c) 2/3 of 3/4 of 4/5 of 5/6 of 6/7 of a number is 18. What is the number? (Solve in five steps: If 2/3 of (3/4 of 4/5 of 5/6 of 6/7 of a number) is 18, then the part in brackets is 27. So 3/4 of 4/5 of 5/6 of 6/7 of a number is 27. Continue in this way.)
NS7-78 Multiplying Fractions by Whole Numbers

Workbook pages 29–30

CURRICULUM EXPECTATIONS
Ontario: 7m1, 7m2, 7m3, 7m5, 7m19, 7m25
WNCP: essential for 7N2, [ME, CN]

VOCABULARY
variable

Goals
Students will multiply fractions with whole numbers and vice versa.
Students will use mental math to solve problems involving the multiplication of fractions.

PRIOR KNOWLEDGE REQUIRED
Can find a fraction and percent of a whole number
Can evaluate expressions involving whole numbers using order of operations

MATERIALS
BLM Multiplying With Counters (p L-106)
several counters for each student

Multiplication as a short form for addition. See Questions 1–3 on Workbook p. 29: present and solve similar problems in the same sequence. Notice that the addition involves all identical fractions, and hence like denominators, so the addition itself is quite simple.

Write on the board: Jade is having a pie party. (If your students are familiar with π, you could tell them that Jade is celebrating pi day, officially celebrated March 14th because $\pi \approx 3.14$.) She has seven different pies, each cut into 8 pieces. At the end of the party, each pie has 3 pieces left. Jade decides to put them into fewer pie plates by filling up as many as she can. How much pie does she have left?

Show students how to model Jade’s actions. Draw 7 pies, each with 8 pieces, on the board, or use an overhead transparency if available. Place 3 paper counters (if using the board) or real counters (if using an overhead) in each pie, to demonstrate how much Jade has left.

ASK: How does this model show $\frac{3}{8} + \frac{3}{8} + \frac{3}{8} + \frac{3}{8} + \frac{3}{8} + \frac{3}{8} + \frac{3}{8}$? (we are adding all the amounts left in each pie to find the total amount of pie left) How does it show $7 \times \frac{3}{8}$? (there is $\frac{3}{8}$, 7 times) Then demonstrate moving all the counters to fill as many pie plates as you can:
There are \( \frac{5}{8} = \frac{21}{8} \) pies. So \( 7 \times \frac{3}{8} = \frac{21}{8} = \frac{5}{8} \).

**PROCESS ASSESSMENT**

7m1, [R]

**PROCESS EXPECTATION**

Generalizing from examples

Give students counters and have them do **BLM Multiplying With Counters**.

After students finish the BLM, have students predict the rule for multiplying a whole number with a fraction. (to find the numerator of the product, multiply the whole number with the numerator of the fraction; the denominator of the product is the denominator of the fraction)

**Developing the formula for** \( a \times \frac{b}{c} \). **Use an EXAMPLE:**

\[
5 \times \frac{3}{15} = \frac{3}{15} + \frac{3}{15} + \frac{3}{15} + \frac{3}{15} + \frac{3}{15} = \frac{3 + 3 + 3 + 3 + 3}{15} = \frac{5 \times 3}{15}
\]

Since all the numerators are the same, we can write the repeated addition as multiplication:

\[
\frac{3 + 3 + 3 + 3}{15} = \frac{5 \times 3}{15}
\]

**ASK:** How would you write \( \frac{7}{15} \) as a single fraction, using the numbers 7, 4, and 15? \( \frac{7 \times 4}{15} \)

How would you write \( a \times \frac{4}{5} \) as a single fraction? \( \frac{a \times 4}{5} = \frac{4a}{5} \)

How would you write \( a \times \frac{b}{c} \) as a single fraction? \( \frac{a \times b}{c} \)

**Extra practice for Question 5:**

a) \( 2 \times \frac{4}{9} \)  
  b) \( 3 \times \frac{2}{5} \)  
  c) \( 7 \times \frac{3}{5} \)  
  d) \( 2 \times \frac{8}{11} \)

**ANSWERS:**  
  a) \( \frac{8}{9} \)  
  b) \( \frac{6}{5} \)  
  c) \( \frac{21}{5} \)  
  d) \( \frac{16}{11} \)

Have students practise multiplying a whole number by an improper fraction using this method.

**EXAMPLE:**

\[
4 \times \frac{7}{2} = \frac{7}{2} + \frac{7}{2} + \frac{7}{2}
\]

\[
= \frac{7 + 7 + 7}{2}
\]

\[
= \frac{4 \times 7}{2}
\]

\[
= \frac{28}{2} = 14
\]
Extra practice:

a) \( \frac{9}{2} \times 4 \)  

b) \( \frac{8}{3} \times 6 \)  

c) \( \frac{11}{4} \times 8 \)  

d) \( \frac{11}{3} \times 15 \)  

**ANSWERS:**

a) \( 9 \times 4 \div 2 = 18 \)  

b) \( 8 \times 6 \div 3 = 16 \)  

c) \( 11 \times 8 \div 4 = 22 \)  

d) \( 11 \times 15 \div 3 = 55 \)  

Discuss which is easier to do mentally: \( (11 \times 15) \div 3 \) or \( 11 \times (15 \div 3) \). Why? (dividing first is easier because it allows you to work with smaller numbers)  

Tell students that you want to multiply 10 by \( \frac{3}{5} \). Have students discuss two different strategies:  

Change it into a known problem by converting the mixed number to an improper fraction.  

**EXAMPLE:** 
\[
10 \times \frac{3}{5} = 10 \times \frac{18}{5} \\
= 10 \times 18 \div 5 \\
= 18 \times 2 \\
= 36
\]  

Multiply directly. Explain that since \( \frac{3}{5} = 3 \div 5 \), we can use the distributive property.  

**EXAMPLE:** 
\[
10 \times \frac{3}{5} = 10 \times \left(3 + \frac{3}{5}\right) = 10 \times 3 + 10 \times \frac{3}{5} \\
= 30 + 10 \times 3 \div 5 \\
= 30 + 30 \div 5 \\
= 30 + 6 \\
= 36
\]  

What is \( \frac{a \times b}{a} \)? Challenge students to predict the general answer before doing Question 6 and then use Question 6 to check their prediction.  

“Of” can mean multiply. Discuss situations where the word “of” means multiply. For example, with whole numbers, 2 groups of 3 means \( 2 \times 3 \) objects. “Of” can mean multiply with fractions too: \( \frac{1}{2} \) of 6 means \( \frac{1}{2} \) of a group of 6 objects, or \( \frac{1}{2} \times 6 \).  

Review finding a fraction of a whole number (see NS7-22) and then have students use this to multiply a fraction with a whole number. Have students develop the general formula:  
\[
\frac{a}{b} \times c = \frac{a}{b} \text{ of } c = \left(a \div b\right) \times c = a \times c \div b
\]
A special property of multiplying whole numbers and fractions. Write on the board: Is 5/8 of three pizzas more than, less than, or the same amount as 3/8 of five pizzas? Have students answer this question using a model and using the formula above.

**ANSWER:** Using a model: To find 5/8 of three pizzas, draw three pizzas with 5/8 of each shaded. This shows 15 pieces of size 1/8. To find 3/8 of five pizzas, draw five pizzas with 3/8 of each shaded. This also shows 15 pieces of size 1/8, so we have the same amount of pizza in both cases. Using the formula: Since \(3 \times 5 = 5 \times 3\) (because multiplication commutes), \(3 \times 5 \div 8 = 5 \times 3 \div 8\), so 3/8 of 5 = 5/8 of 3.

Connect this to the following property of percents that students saw earlier: 20% of 60 = 60% of 20. This can be rewritten as 20/100 of 60 = 60/100 of 20, or, in terms of multiplication, as 20/100 \(\times\) 60 = 60/100 \(\times\) 20.

Use fractions and percents of numbers to compare and order fractions and percents. Have students find the following fractions and percents of 132, and then use the answers to write the fractions and percents in order.

- a) 25%  
- b) 3/11  
- c) 1/3  
- d) 40%  
- e) 36%

**ANSWERS:**
- a) 33  
- b) 36  
- c) 44  
- d) 52.8  
- e) 47.52

So, 25% < 3/11 < 1/3 < 36% < 40%

**Using distance to multiply a mixed number with a whole number.** Tell students that because 1/2 is halfway between 0 and 1, 1/2 \(\times\) 6 is halfway between 0 \(\times\) 6 = 0 and 1 \(\times\) 6 = 6. So 1/2 \(\times\) 6 is 3—this is exactly 1/2 of 6, as we expect.

Challenge students to find \(3\frac{1}{2} \times 6\). **ASK:** \(3\frac{1}{2}\) is halfway between which two whole numbers? (3 and 4) Explain that \(3\frac{1}{2}\) \(\times\) 6 should be halfway between 3 \(\times\) 6 and 4 \(\times\) 6 because \(3\frac{1}{2}\) is halfway between 3 and 4. Since 21 is halfway between 18 and 24, \(3\frac{1}{2} \times 6 = 21\).

\[
\begin{align*}
18 & \quad 19 & \quad 20 & \quad 21 & \quad 22 & \quad 23 & \quad 24 \\
3 \times 6 & & & & & & \\
4 \times 6 & & & & & & \\
\end{align*}
\]

Now tell students that you want to find \(3\frac{1}{3} \times 6\). **ASK:** What two whole numbers is \(3\frac{1}{3}\) between? (3 and 4) Explain that \(3\frac{1}{3}\) \(\times\) 6 should be 1/3 of the way from 3 \(\times\) 6 to 4 \(\times\) 6. Have a volunteer show where \(3\frac{1}{3} \times 6\) should be on the number line above. (at 20)

Have students estimate the following quantities and then use a number line to check their answers.

- a) \(\frac{4}{3} \times 6\)  
- b) \(\frac{2}{3} \times 9\)  
- c) \(\frac{4}{5} \times 10\)

PROCESS ASSESSMENT  
7m5, [CN]  
7m1, 7m3, 7m6, [ME, V]
Sample answer: a) students should estimate that the product is between $4 \times 6 = 24$ and $5 \times 6 = 30$ and that the answer will be closer to 24 than 30 because $\frac{1}{3}$ is less than half. Students could reasonably estimate either 25 or 26. To find the exact answer, note that $\frac{1}{3}$ of the distance from 24 to 30 is 26, so $4 \frac{1}{3} \times 6 = 26$.

Comparing the two methods. Have students change the mixed numbers in the problems above to improper fractions and then multiply. Do students get the same answers?

**Example:**

\[
3 \frac{1}{2} \times 6 = \frac{7}{2} \times 6
\]

\[
= 7 \times 6 \div 2
\]

\[
= 7 \times 3
\]

\[
= 21
\]

This is exactly what we found when we used the fact that $3 \frac{1}{2}$ is halfway between 3 and 4.

Multiplication of a whole number and a fraction commutes (i.e., order doesn’t matter). See Investigation 1 on Workbook page 30.

**Process Expectation**

Reflecting on other ways to solve a problem

**Process Expectation**

Communicating

**Process Assessment**

7m2, [R]

**Process Expectation**

Problem-solving, Mental math and estimation

**A strategy for multiplying fractions by fractions.** Challenge students to multiply $3 \times \frac{2}{5} \times \frac{1}{3}$ by using what they have learned in this lesson.

**Answer:** $3 \times \frac{2}{5} \times \frac{1}{3} = \frac{2}{5} \times 3 \times \frac{1}{3}$ since $3 \times \frac{2}{5} = \frac{2}{5} \times 3$. But $3 \times \frac{1}{3} = 1$, so this is $\frac{2}{5} \times 1 = \frac{2}{5}$.

Have students find a) $4 \times \frac{1}{5} \times \frac{3}{4}$ b) $5 \times \frac{2}{3} \times \frac{4}{5}$ c) $10 \times \frac{3}{4} \times \frac{2}{5}$

**Answers:**

a) $\frac{1}{5} \times 4 \times \frac{3}{4} = \frac{1}{5} \times 3 = \frac{3}{5}$

b) $\frac{2}{3} \times 5 \times \frac{4}{5} = \frac{2}{3} \times 4 = \frac{8}{3}$

c) $\frac{3}{4} \times 10 \times \frac{2}{5} = \frac{3}{4} \times 4 = 3$

**Word problems:**

1. If a penny weighs $2 \frac{1}{3}$ g, how much will 100 pennies weigh?

   Express your answer as a mixed fraction in grams.

   **Answer:** $2 \frac{1}{3} \times 100 = \frac{7}{3} \times 100 = \frac{700}{3} = 233 \frac{1}{3}$ g
2. Tina drinks $\frac{5}{6}$ of a bottle of water each day. How many bottles of water does she drink in seven days? ANSWER: $\frac{5}{6} \times 7 = \frac{35}{6} = 5\frac{5}{6}$ bottles

3. a) If one lane of a swimming pool is $\frac{3}{8}$ the length of an Olympic swimming pool, how many lengths of an Olympic swimming pool does Tina swim if she swims the lane five times? 
   ANSWER: $\frac{3}{8} \times 5 = \frac{15}{8} = 1\frac{7}{8}$ lengths of an Olympic swimming pool

   b) An Olympic swimming pool is 50 m long. How many metres did Tina swim? ANSWER: $93\frac{3}{4}$ or 93.75 m

4. The formula for converting temperature in degrees Celsius ($^\circ$C) to temperature in degrees Fahrenheit ($^\circ$F) is 
   $^\circ$F = $^\circ$C + 32.
   a) Convert the following temperatures from $^\circ$C to $^\circ$F:
      i) 15$^\circ$C
      ii) 50$^\circ$C
      iii) 100$^\circ$C
   ANSWERS: i) 59$^\circ$F    ii) 122$^\circ$F    iii) 212$^\circ$F

   b) The water in an Olympic swimming pool must be between 25$^\circ$C and 28$^\circ$C. What is that in degrees Fahrenheit? Round your answer to the nearest whole number. (between 77$^\circ$F and 82$^\circ$F)

   c) i) A recipe says to preheat the oven to 200$^\circ$C. Your oven only shows temperatures in degrees Fahrenheit. What temperature should you set your oven to? (392$^\circ$F)

   ii) The recipe tells you to lower the temperature by 10$^\circ$C partway through the baking time. By how many degrees Fahrenheit should you lower the temperature? Hint: What will the temperature of the oven be after you lower it by 10$^\circ$C? How many degrees Fahrenheit is this temperature? ANSWER: Here are two solutions: First, 10$^\circ$C less than 200$^\circ$C is 190$^\circ$C, so find $9/5(190) + 32 = 374^\circ$F. This means you must reduce the temperature by 392 - 374 = 18$^\circ$F. Second, since 1$^\circ$F is $9/5$ the size of 1$^\circ$C, decrease the temperature by $9/5(10) = 18^\circ$F. [Note that the answer is not $9/5 \times 10 + 32$ because what you want is the difference between two temperatures, and the formula finds equivalent temperatures. If you convert both 200$^\circ$C and 190$^\circ$C to Fahrenheit and then subtract them, you see that the “+ 32” in the formula cancels.]
ACTIVITY

Give students a ruler that has both inches and centimetres. Use inches to multiply whole numbers by halves, fourths, and eighths, and use centimetres to multiply whole numbers by halves, fifths, and tenths.

EXAMPLE: $7 \times \frac{5}{8}$

So $7 \times \frac{5}{8} = 4 \frac{3}{8} = 3 \frac{5}{8}$.

Extensions

1. Find the next two products in the pattern, and justify your answer.
   
   $16 \times 3 = 48$
   $8 \times 3 = 24$
   $4 \times 3 = 12$
   $2 \times 3 = 6$
   $1 \times 3 = 3$

   ANSWERS: $1/2 \times 3 = 3/2$ and $1/4 \times 3 = 3/4$ since the first factor and the product are continuously being divided by 2.

2. Find the missing numbers:
   
   a) $4 \times \frac{1}{6} = 2$
   b) $\square \times \frac{4}{5} = 4 \frac{4}{5}$
   c) $\square \times \frac{3}{7} = 5 \frac{4}{7}$

   d) $5 \times \frac{4}{6} = 4 \frac{1}{2}$
   e) $3 \times \frac{4}{6} = 4 \frac{1}{2}$

   Bonus: Put the same number in both boxes: $\square \times \frac{8}{8} = 4 \frac{1}{2}$

   ANSWERS: a) 2  b) 6  c) 13  d) 5  e) 5

   Bonus: 6 because $4 \frac{1}{2} = 4 \frac{4}{8} = 36/8 = 6 \times 6/8$

PROCESS EXPECTATION

Connecting

3. The force of gravity is different on each body in the solar system (planet, sun, moon, etc.). Your weight depends on which body you’re on. On the Moon, the gravitational pull is only about 1/6 of what it is on Earth, so if you weigh 60 kg on Earth, you would only weigh 10 kg on the Moon (but your mass would still be the same).

<table>
<thead>
<tr>
<th>Body</th>
<th>Sun</th>
<th>Mercury</th>
<th>Venus</th>
<th>Earth</th>
<th>Moon</th>
<th>Mars</th>
<th>Jupiter</th>
<th>Saturn</th>
<th>Uranus</th>
<th>Neptune</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravitational pull as a fraction of Earth’s gravity</td>
<td>$\frac{27}{20}$</td>
<td>$\frac{19}{21}$</td>
<td>$\frac{8}{11}$</td>
<td>$\frac{10}{6}$</td>
<td>$\frac{1}{21}$</td>
<td>$\frac{8}{21}$</td>
<td>$\frac{4}{11}$</td>
<td>$\frac{23}{25}$</td>
<td>$\frac{22}{25}$</td>
<td>$\frac{1}{22}$</td>
</tr>
</tbody>
</table>
a) Which body’s gravitational pull is closest to that of Earth? (Saturn)

b) List the bodies in order from least gravitational pull to highest gravitational pull. (Moon, Mars and Mercury, Uranus, Venus, Saturn, Earth, Neptune, Jupiter, Sun)

c) Scientists have discovered that by raising certain animals from birth in a lab with higher gravitational pull, the animals become stronger. On which of the eight planets would an animal growing up on that planet be the strongest? (Jupiter) The weakest? (Mercury or Mars)

d) Use the choices given beneath each blank to complete the sentence three times:

A ___________________ weighs about 180 kg on ________________.

black rhinoceros (1 000 kg) the Sun
sumo wrestler (150 kg) the Moon
baby (6 kg) Neptune
(black rhinoceros/the Moon; sumo wrestler/Neptune; baby/the Sun)

e) You want to move each of the following objects to a location in the solar system where they will weigh as close to 30 kg as possible. State where you should move each object and justify your answer with calculations.

i) a person weighing 75 kg  
   ii) a bike weighing 12 kg  
   iii) a bottle of water weighing 1 kg  
   iv) a motorcycle weighing 200 kg

ANSWERS:

i) Mercury or Mars, because 75 kg \times \frac{8}{21} \approx 28.6 kg

ii) Jupiter, because 12 kg \times \frac{24}{11} = 12 kg \times \frac{26}{11} \approx 28.4 kg

iii) the Sun, because 1 kg \times \frac{27}{20} = 27 \frac{19}{20} kg

iv) the Moon, because 200 kg \times \frac{1}{6} \approx 33.3 kg
Half of a fraction. Explain to students that just as we can talk about a fraction of a whole number, we can also talk about a fraction of a fraction. Demonstrate finding half of 3/5 by dividing an area model of the fraction into a top half and a bottom half:

\[
\frac{3}{5} \quad \text{of} \quad \frac{3}{5} = \frac{3}{10}
\]

Have students use this method to find half of these fractions:

a) \( \frac{2}{9} \)  

b) \( \frac{5}{7} \)

c) \( \frac{3}{7} \)

d) \( \frac{2}{5} \)

e) \( \frac{5}{6} \)

f) \( \frac{4}{7} \)

Show students another way of dividing the fraction 4/7 in half. Instead of dividing in half from top to bottom, divide in half from left to right (you can do this because 4 is even). Notice that the two methods seem to give different answers:

\[
\frac{1}{2} \quad \text{of} \quad \frac{4}{7} = \frac{4}{14} \quad \text{and} \quad \frac{1}{2} \quad \text{of} \quad \frac{4}{7} = \frac{2}{7}
\]

ASK: Are these answers the same? (yes, they are equivalent) Explain that no matter how you take half of 4/7, the answer should always be the same.

Use a model to find a fraction of a fraction when both numerators equal 1. See Workbook p. 31 Questions 1 and 2.

The formula for multiplying fractions with numerator 1. Students should discover this through examples. The general explanation is as follows:

Notice from the arrays shown in Questions 1–3 on the worksheet that there is always 1 part shaded and the number of parts is the number of rows × the number of columns. So to find \( \frac{1}{a} \times \frac{1}{b} \), draw an array with \( b \) columns and then divide each column into \( a \) parts; the fraction shaded is \( \frac{1}{(a \times b)} \).
The formula for multiplying fractions in general. Again, students should discover this through examples (see Questions 7 and 8 on Workbook p. 32). In this case, to multiply \( \frac{a}{b} \times \frac{c}{d} \), the number of shaded parts is \( a \times c \) and the total number of parts is \( b \times d \). So:

\[
\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}
\]

**Extra practice for Question 10:**

a) \( \frac{2}{5} \times \frac{3}{4} \)  

b) \( \frac{3}{8} \times \frac{4}{15} \)  

c) \( \frac{7}{3} \times \frac{12}{5} \)  

d) \( \frac{4}{5} \times \frac{3}{2} \)  

e) \( \frac{5}{3} \times \frac{6}{25} \)

**ANSWERS:**

a) \( \frac{6}{20} = \frac{3}{10} \)  

b) \( \frac{12}{120} = \frac{1}{10} \)  

c) \( \frac{84}{15} = \frac{28}{5} \)  

d) \( \frac{12}{10} = \frac{6}{5} \)  

e) \( \frac{30}{75} = \frac{2}{5} \)

The reciprocal of a fraction. Show students these pairs of fractions and tell them they are called **reciprocals** of each other:

\[
\frac{3}{4} \quad \text{and} \quad \frac{4}{3}
\]

\[
\frac{2}{5} \quad \text{and} \quad \frac{5}{2}
\]

\[
\frac{3}{7} \quad \text{and} \quad \frac{7}{3}
\]

\[
\frac{5}{6} \quad \text{and} \quad \frac{6}{5}
\]

Have students define what it means to be the reciprocal of a fraction. Have students determine the reciprocal of \( \frac{a}{b} \). (\( \frac{b}{a} \))

**PROCESS EXPECTATION**

**Communicating**

**Multiplying reciprocals.** After students do Question 11, challenge them to explain in their own words why multiplying reciprocals always results in 1. Sample answer: The order in which the numbers are multiplied doesn’t affect the product \( (3 \times 4 = 4 \times 3) \), so

\[
\frac{3}{4} \times \frac{4}{3} = \frac{12}{12} = 1.
\]

In general, \( \frac{a}{b} \times \frac{b}{a} = \frac{ab}{ba} = 1 \) since \( a \times b = b \times a \).

**Using the result of multiplying reciprocals to estimate products of fractions.** Review comparing fractions by creating equivalent fractions with the same denominator (Example: To compare \( \frac{3}{7} \) to \( \frac{2}{5} \), change them to \( \frac{15}{35} \) and \( \frac{14}{35} \)). Then have students decide which of these fractions are greater than \( \frac{3}{4} \):

a) \( \frac{4}{5} \)  

b) \( \frac{9}{11} \)  

c) \( \frac{8}{11} \)  

d) \( \frac{11}{15} \)  

e) \( \frac{13}{16} \)

**ANSWERS:** a), b), and e) are greater than \( \frac{3}{4} \)

Now have students use their answers to the previous question to decide which of the following products are greater than 1 without calculating the products:

a) \( \frac{4}{5} \times \frac{4}{3} \)  

b) \( \frac{9}{11} \times \frac{4}{3} \)  

c) \( \frac{8}{11} \times \frac{4}{3} \)  

d) \( \frac{11}{15} \times \frac{4}{3} \)  

e) \( \frac{13}{16} \times \frac{4}{3} \)

**ANSWERS:** parts a, b, and e are greater than 1 because the first fraction is greater than \( \frac{3}{4} \) and \( \frac{3}{4} \times \frac{4}{3} = 1 \).
Have students check their answers by directly calculating the products.

**Compare multiplying fractions to adding fractions.** Notice that when multiplying fractions, you can just multiply the numerators together and then multiply the denominators—the resulting fraction is the product! **ASK:** Can you do the same when adding fractions—add the numerators together and add the denominators? Is the resulting fraction the sum? (No! This is easy to see when adding fractions that have the same denominator, e.g., \( \frac{3}{11} + \frac{7}{11} = \frac{(3 + 7)}{11} \), not \( \frac{3}{11} + \frac{7}{(11 + 11)} \), which would definitely give the wrong answer.) As another example, the formula for multiplying fractions would give \( \frac{2}{5} + \frac{3}{5} = \frac{7}{8} \), but \( \frac{5}{3} \) is more than 1 so you can’t add to it and get a number less than 1!

Explain to students that people who know how to multiply fractions often make more mistakes when adding fractions than people who don’t because people who know how to multiply fractions try to apply the same principle to adding fractions. (NOTE: This is called **interference** because the new knowledge is interfering with old knowledge. Another example of interference occurs when grade 3 students who have learned multiplication make the mistake of writing \( 3 + 2 = 6 \). A grade 1 student would be much less likely to make that mistake than a grade 3 student.) Explain to students that they have to be careful not to let their new knowledge of multiplying fractions interfere with their old knowledge of adding fractions. Just because the formula works in one context doesn’t mean it works in all contexts.

**Extensions**

1. **Adding reciprocals**

   Have students tell you which fractions are less than 1:
   
   a) \( \frac{3}{5} \)  b) \( \frac{4}{5} \)  c) \( \frac{7}{5} \)  d) \( \frac{8}{11} \)  e) \( \frac{11}{8} \)  f) \( \frac{48}{51} \)  g) \( \frac{51}{48} \)

   **ASK:** How can you tell from a and b if \( \frac{a}{b} \) is less than 1? (if \( a < b \), then \( \frac{a}{b} \) is less than 1; if \( a > b \), then \( \frac{a}{b} \) is more than 1) If \( \frac{a}{b} \) is less than 1, what can you say about \( \frac{b}{a} \)? (if \( \frac{a}{b} \) is less than 1, then a is less than b and b is more than a, so \( \frac{b}{a} > 1 \))

   Write \( \frac{9}{10} \) on the board, and ask students to tell you what whole number this is close to. (1) **ASK:** Is it more than 1 or less than 1? (less) What is its reciprocal? (\( \frac{10}{9} \)) Is 10/9 more than 1 or less than 1? (more than 1) Have students estimate \( \frac{9}{10} + \frac{10}{9} \). What whole number will it be closest to? (1 + 1 = 2) Have students predict (by random guessing) whether the sum will be more or less than 2, then do the calculation to check.

   \[
   \frac{9}{10} + \frac{10}{9} = \frac{181}{90} = 2\frac{1}{90}
   \]

   The sum is more than 2

   Tell students that you want to know how the sum \( \frac{a}{b} + \frac{b}{a} \) compares to 2. Suppose that \( a < b \). Then \( \frac{a}{b} \) is less than 1 and \( \frac{b}{a} \) is more than 1. So we expect \( \frac{a}{b} + \frac{b}{a} \) to be sometimes more than 2 and sometimes less than 2.
Explain that you want a way to estimate whether \( \frac{a}{b} + \frac{b}{a} \) will be more or less than 2, and that you find it easier to look at simpler problems that are similar.

Show students the following problem: Predict whether \( \frac{59}{100} + \frac{42}{100} \) is more or less than 1.

**ASK:** How is this problem similar to predicting whether \( \frac{9}{10} + \frac{10}{9} \) is more or less than 2? (It asks about the sum of two fractions and how it compares to a given whole number) How is it different? (Instead of reciprocal fractions it uses fractions with the same denominator) Is calculating this sum easier or harder than calculating \( \frac{9}{10} + \frac{10}{9} \)? (Easier) Explain that you chose the problem because it is easier.

Now show students the following problem: Predict whether \( 59 + 42 \) is more or less than 100.

**ASK:** How is this problem similar to the problem with \( \frac{59}{100} + \frac{42}{100} \)? Is it easier or harder?

**ASK:** How do you know that \( 59 + 42 \) is close to 100 without actually doing the calculation? (59 is close to 60 and 42 is close to 40, so \( 59 + 42 \) is close to \( 60 + 40 = 100 \)) How do you know that \( 59 + 42 \) will be more than 100 without actually doing the calculation? (Because 42 is more than 40 by more than 59 is less than 60, so \( 59 + 42 \) will be more than \( 40 + 60 = 100 \))

Now return to the original problem: \( \frac{9}{10} + \frac{10}{9} \). **ASK:** By how much is \( \frac{9}{10} \) less than 1? (1/10) By how much is \( \frac{10}{9} \) more than 1? (1/9) Which is greater: 1/10 or 1/9? (1/9) Explain that since the number that is more than 1 is greater than 1 by more than the other number is less than 1, you expect the sum of the two numbers to be more than 2. **ASK:** Is that what we found? (Yes, \( \frac{9}{10} + \frac{10}{9} \) was 2 1/90)

Emphasize that you used the method that you knew for a similar problem with whole numbers and applied it to this problem with fractions.

**BLM Adding Reciprocals** allows students to discover the somewhat surprising result that \( \frac{a}{b} + \frac{b}{a} \) is always more than 2!

2. **These are the ingredients Anna needs to make 12 muffins:**

- \( \frac{3}{4} \) cups flour
- 7 tablespoons butter
- \( \frac{1}{2} \) cup sugar
- 2 teaspoons baking powder
- \( \frac{1}{4} \) teaspoon salt
- 1 small egg
- 1 cup milk
- 2 tablespoons brown sugar
- \( 1 \frac{1}{3} \) teaspoons cinnamon
a) Anna has an 8-muffin tray. What fraction of the ingredients should she use? (2/3, because 8 is 2/3 of 12)

b) Anna can’t use 2/3 of the egg—she needs to use the whole egg. Her egg has a volume of about 3/8 of a cup. How much extra liquid does this create in her muffin mix? (She should use only 2/3 of an egg, and 2/3 of 3/8 = 2/8 = 1/4 of a cup, but she used 3/8 of a cup, so she has 1/8 cup extra liquid.)

c) Anna needs to reduce the amount of milk by the amount of extra egg she used, to keep the total volume of liquid ingredients the same. How much milk should she use? (She should use 2/3 − 1/8 = 13/24 of a cup, or just more than 1/2 a cup.)

Write the complete list of ingredients that Anna should use to make 8 muffins.

ANSWERS:

\[
\begin{align*}
\frac{1}{6} \text{ cup flour} & \quad \frac{13}{24} \text{ cup milk} & \quad \frac{1}{3} \text{ teaspoons of baking powder} \\
\frac{1}{3} \text{ cup sugar} & \quad \frac{8}{9} \text{ teaspoon cinnamon} & \quad 1 \text{ egg} \\
\frac{1}{6} \text{ teaspoon salt} & \quad \frac{4}{3} \text{ tablespoons butter} & \quad \frac{1}{3} \text{ tablespoons brown sugar}
\end{align*}
\]

3. Connect finding fractions of fractions to the Extension in NS7-76, 77. In the Extension, students were finding fractions of fractions of numbers by working backwards in a way (e.g., 3/4 of something is 60—what’s the something?). Now, students can actually multiply the fractions. For example, if 2/3 of 3/4 of 4/5 of 5/6 of 6/7 of a number is 18, then 2/7 of the number is 18. So 2 : 7 = 18 : ?, so ? = 7 × 9 = 63.
Dividing whole numbers by fractions with numerator 1. Use the example in the teaching box on Workbook p. 33 to teach the method, then have students apply it to complete Questions 1 and 2.

Extra practice for Question 2:

a) \(7 \div \frac{1}{3}\)  
b) \(8 \div \frac{1}{5}\)  
c) \(9 \div \frac{1}{6}\)  
d) \(8 \div \frac{1}{8}\)  
e) \(10 \div \frac{1}{17}\)  
f) \(14 \div \frac{1}{5}\)

ANSWERS: a) 21  
b) 40  
c) 54  
d) 64  
e) 170  
f) 70

How does \(a \div \frac{1}{b}\) compare to \(b \div \frac{1}{a}\)? Encourage students to investigate this by substituting various values for \(a\) and \(b\). Students could use a 4-column chart with headings \(a\), \(b\), \(a \div \frac{1}{b}\), and \(b \div \frac{1}{a}\) to record their answers.

Since \(a \div \frac{1}{b} = a \times b\) and \(b \div \frac{1}{a} = b \times a\), students will see that these always have the same answer for any values of \(a\) and \(b\).

Dividing whole numbers by fractions with any numerator. Teach this as in Questions 3–5 on Workbook p. 34.

Extra practice for Question 5:

a) \(20 \div \frac{4}{5}\)  
b) \(24 \div \frac{3}{2}\)  
c) \(15 \div \frac{3}{5}\)  
d) \(18 \div \frac{2}{3}\)  
e) \(10 \div \frac{5}{2}\)

ANSWERS:  
a) 25  
b) 16  
c) 25  
d) 27  
e) 4

For each problem below, have students write a multiplication or division statement and then solve it.

a) John plants a flower in 2 minutes. How long does it take him to plant 30 flowers?

\[(30 \times 2 = 60\text{ minutes})\]
b) Ray plants a flower in \( \frac{2}{3} \) minutes. How long does it take him to plant 30 flowers?

\[
30 \times \frac{2}{3} = 30 \times \frac{5}{3} = 50 \text{ minutes}
\]

**ASK:** How are the two problems above the same? How are they different? (They are asking the same thing, but with different numbers. One uses whole numbers and the other uses a mixed number.) Point out that the one using whole numbers is easier to think about and can be useful to help solve the problem with a mixed number. It makes it clearer that you need to multiply.

Write on the board: a) Lina bought 2 kg of dry lasagne. Each person needs \( \frac{3}{50} \) kg. How many people can she feed?

Tell students you’re going to change the numbers to whole numbers, which are easier to work with, and write: b) Lina bought 6 kg of dry lasagne and each person needs 2 kg. How many people can she feed?

Explain that the amounts of lasagne are no longer reasonable for the context—nobody would eat 2 kg of lasagne!—but that’s not important; you just want to use whole numbers because they are easier to work with.

Point out that \( \frac{3}{50} \) is less than 2, so the whole numbers you chose have the same relationship—2 is less than 6.

**ASK:** Which question is easier—the first or the second? Why? What operation do you need to use—multiplication or division? (division) Look at the first problem: should we divide \( 2 \div \frac{3}{50} \) or \( \frac{3}{50} \div 2 \)? How does looking at the second problem make it easier to decide?

**ANSWERS:**

a) \( 2 \div \frac{3}{50} = 2 \times \frac{50}{3} = 100/3 = 33 \frac{1}{3} \), so she can feed 33 people.

b) \( 6 \div 2 = 3 \), so she can feed 3 people.

Now write on the board: A string 6 m long is divided in pieces of length \( \frac{3}{10} \) m. How many pieces are there?

Have students write an easier problem to help them solve this one, then solve it. **ANSWER:** An easier problem is: A string 6 m long is divided in pieces of length 2 m. How many pieces are there? To solve the easier problem, find \( 6 \div 2 \), so to solve the given problem, find \( 6 \div \frac{3}{10} = 6 \times \frac{10}{3} = 20 \), so there are 20 pieces.

**PROCESS EXPECTATION**

**Using modeling to solve problems.** Write on the board:

4/5 of Helen’s age is 2/3 of Dale’s age.

<table>
<thead>
<tr>
<th>Helen</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Dale</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**ASK:** How does this model show that 4/5 of Helen’s age is 2/3 of Dale’s
age? (because 4/5 of Helen’s bar ends at the same place as 2/3 of Dale’s bar—shade these parts as a visual aid for students) Tell students that you want all the units in each bar to be the same size, so that it is easier to solve problems that ask for their ages.

Helen

Dale

Have students draw models representing Helen’s age and Dale’s age with bars, and with all units in the bars the same size, to show:

a) Helen’s age is 4/5 of Dale’s age.

b) 2/5 of Helen’s age is 2/3 of Dale’s age.

c) 3/5 of Helen’s age is half of Dale’s age.

Ask students to shade the parts that the given information says is the same.

ANSWERS: (putting Helen’s bar on top)

a)  b)  c)

Now demonstrate using the model to solve a problem:

4/5 of Helen’s age is 2/3 of Dale’s age. Dale is 7 years older than Helen. How old are Helen and Dale?

Each unit represents 7 years, because the extra part is 1 unit and Dale is 7 years older than Helen. Helen’s age bar is 5 units long, so Helen is 35 years old and Dale is 42 years old. Have students check this answer against the given information: 4/5 of Helen’s age is 28 and 2/3 of Dale’s age is 28, so this works.

Have students determine Helen’s and Dale’s ages using their models:

a) Helen’s age is 4/5 of Dale’s age. Dale is 2 years older than Helen.

b) 2/5 of Helen’s age is 2/3 of Dale’s age. Helen is 6 years older than Dale.

c) 3/5 of Helen’s age is half of Dale’s age. Dale is 3 years older than Helen.

ANSWERS:  a) Helen is 8 and Dale is 10.

b) Helen is 15 and Dale is 9.

c) Helen is 15 and Dale is 18.
Extension

Carene emptied her piggy bank that holds only pennies. The contents weigh about 1 kg 300 g in total. If each penny weighs $2 \frac{1}{3}$ g, about how much money was in the piggy bank? Round your answer to the nearest half dollar.

\[
\text{ANSWER: } 1300 \div 2 \frac{1}{3} = 1300 \div \frac{7}{3} = 1300 \times \frac{3}{7} = 557 \text{ pennies} = 5.57, \\
\text{so there is about } 5.50
\]

\text{NOTE:} It is essential that students change the mixed number to an improper fraction to divide by a mixed number. One cannot write $1300 \div (2 \frac{1}{3}) = 1300 \div (2 + \frac{1}{3})$ as $1300 \div 2 \div 1300 \div \frac{1}{3}$, because the distributive law only applies to the first term in division, not to the second. Indeed, $1300 \div 2$ is already larger than $1300 \div (2 \frac{1}{3})$ because we are dividing by a smaller number.
NS7-82 Multiplying Decimals by 0.1, 0.01, and 0.001

NS7-83 Multiplying Decimals by Decimals

Workbook pages 36–38

CURRICULUM EXPECTATIONS
Ontario: 6m21; 7m1, 7m2, 7m4, 7m5, 7m19, 7m21, 7m22; 8m9, 8m16, 8m18, 8m24
WNCP: 6N8; 7N2, [ME, PS]

Vocabulary
decimal place
leading digit

Goals
Students will multiply decimals by decimals, using a variety of methods.

PRIOR KNOWLEDGE REQUIRED
Can multiply by 10, 100, and 1 000
Can multiply fractions

MATERIALS
BLM Multiply Decimals by Decimals (p L-109)

Multiplying by 0.1, 0.01, and 0.001. Review moving the decimal point a certain number of places to the left or right to multiply and divide decimals by powers of 10. Then apply this to the decimal 0.1, as in Workbook p. 36 Question 2 a). In this question, students multiply 0.1 by 10, 100, and 1 000 by moving the decimal point in 0.1 one, two, or three places to the right. Now have students look at the answers to Question 2 a) and think about multiplying the same numbers in the other direction: how does the decimal point in 10, 100, and 1 000 move when you multiply by 0.1? (the decimal point in 10 moves one place to the left, and same for 100 and 1 000)

Students can now predict a rule for multiplying by 0.1 and then complete Question 2. Repeat the same line of questioning for multiplying by 0.01 and 0.001, as in Workbook p. 36 Question 3.

Once students see the pattern, show students another way to see the same result. First, remind students that when multiplying whole numbers, such as 4 × 3, they can ask themselves: How much is 4 threes? Similarly, when multiplying whole numbers by 0.1, such as 4 × 0.1, they can ask themselves: How much is 4 tenths? The answer is 0.4. Or, when multiplying 34 × 0.001, they can ask themselves: How much is 34 thousandths? Have students find:

a) 13 × 0.1  b) 27 × 0.1  c) 184 × 0.1  d) 184 × 0.01  e) 184 × 0.001

ANSWERS: a) 1.3  b) 2.7  c) 18.4  d) 1.84  e) 0.184

Point out that this method follows the same rule: to multiply by 0.1, 0.01, or 0.001, move the decimal point either 1, 2, or 3 places to the left. Students can now apply their rules for multiplying by 0.1, 0.01, and 0.001. See Workbook p. 36 Question 4.
Challenge students to predict a rule for multiplying by 0.000 01, then find 18.054 \times 0.000 01.

Introduce the term decimal place. A decimal place is a place value after the decimal point. See Workbook p. 37 Question 1.

The connection between decimal places and the denominator of a decimal fraction. Review decimal fractions, and in particular, converting decimals to decimal fractions. Draw the chart below and have students copy and complete it, to encourage them to notice the relationship between the number of decimal places in a decimal and the number of zeros in the denominator of the corresponding decimal fraction (they are the same). Point this out for students who don’t notice it. Encourage students to choose their own decimals or decimal fractions to place in the last two rows of the chart.

<table>
<thead>
<tr>
<th>Decimal</th>
<th>Number of Decimal Places</th>
<th>Decimal Fraction</th>
<th>Number of Zeros in the Denominator</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.21</td>
<td>2</td>
<td>21/100</td>
<td>2</td>
</tr>
<tr>
<td>0.000 000 3</td>
<td>7</td>
<td>5/100</td>
<td></td>
</tr>
<tr>
<td>0.37</td>
<td></td>
<td>32/10</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.041</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Discuss why the relationship, or pattern, holds. The thousandths digit is the third decimal place after the decimal point. So if the denominator of the decimal fraction is 1 000 (which has three zeros), then the decimal has three decimal places. For example, fifty-two thousandths is 52/1 000 or 0.052.

Note the following connection. Dividing a decimal fraction by 10 adds a zero to the denominator; dividing a decimal by 10 adds a place value. **EXAMPLE:**
Review multiplying by 10, 100, and 1 000. Have students multiply these multiples of 10:

\[
\begin{align*}
10 \times 10 &= 100 \\
10 \times 100 &= 1000 \\
10 \times 1 000 &= 10000 \\
1 000 \times 10 &= 10000 \\
1 000 \times 100 &= 100000 \\
1 000 \times 1 000 &= 1000000
\end{align*}
\]

**ASK:** What is a short way to find the answer? (count the number of zeros in total and put that many zeros in the answer) To guide students who don’t see the pattern, suggest that they write the number of zeros in each factor and in the answer. Remind students that to multiply a whole number by 10, they add a zero to it; so to multiply 10 \times 1 000, simply add a zero to 1 000 (i.e., four zeros instead of three), so 10 \times 1 000 = 10 000. Similarly, to multiply by 100, add two zeros, so 100 \times 1 000 has 5 zeros: 100 000. We are simply adding the number of zeros in the first factor to the number of zeros already there in the second factor.

Now have students solve these problems:

\[
\begin{align*}
100 \times 10 \times 1 000 &= \\
1 000 \times 100 \times 100 \times 10 \times 10 &= \\
1 000 \times 100 \times 100 \times 10 \times 10 &= \\
1 000 \times 100 \times 100 \times 10 \times 10 \times 10 &=
\end{align*}
\]

**Review multiplying decimal fractions.** Review how to multiply a fraction by a fraction, as in NS7-79: first by interpreting a statement such as 1/2 \times 1/4 as “1/2 of 1/4” and using an area model to calculate the products of fractions; then by multiplying first the numerators and then the denominators. Then have students calculate many products of decimal fractions, as on Workbook p. 37 Questions 3, 4, and 5. **EXAMPLE:**

\[
\begin{align*}
\frac{3}{10} \text{ of } \frac{7}{10} &= \frac{21}{100} \\
\frac{3}{10} \times \frac{7}{10} &= \frac{21}{100}
\end{align*}
\]

Write on the board: \(\frac{2}{100} \times \frac{3}{1000}\). **ASK:** How many zeros are in 100? (2) How many zeros are in 1 000? (3) How many will be in their product? (5) How did you get the answer 5 from 2 and 3? (2 \(+\) 3 = 5) Emphasize that the number of zeros in the denominator of the product is the sum of the number of zeros in the denominators of the fractions you are multiplying together.

Have students multiply these decimal fractions:

a) \(\frac{3}{10} \times \frac{2}{1 000}\)  

b) \(\frac{2}{100} \times \frac{4}{1 000}\)  

c) \(\frac{3}{1 000} \times \frac{5}{100}\)  

d) \(\frac{3}{100} \times \frac{2}{1 000} \times \frac{7}{100}\)  

e) \(\frac{2}{10} \times \frac{3}{100} \times \frac{3}{100} \times \frac{7}{1 000}\)
ANSWERS: a) $6/10000$  b) $8/100000$  c) $15/100000$
d) $42/100000000$  e) $126/100000000$

Relate multiplying decimals to multiplying decimal fractions. Write on the board:

\[
\frac{3}{10} \times \frac{2}{1000} = \frac{6}{10000}
\]

Converting each decimal fraction to a decimal, we get $0.3 \times 0.002 = 0.0006$.

To multiply the fractions, multiply the numerators and the denominators separately. Multiplying the denominators is particularly easy because they are all powers of 10—just write a 1 followed by the total number of zeros in the denominators. There are $1 + 3 = 4$ zeros, so the product is $\frac{6}{10000}$.

ASK: If we know $\frac{3}{10} \times \frac{2}{1000} = \frac{6}{10000}$, how can we find $0.3 \times 0.002$, as a decimal? (Convert the fraction $6/10000$ to a decimal, so $0.3 \times 0.002 = 0.0006$)

Now write on the board $0.3 \times 0.005$. Have students convert the decimals to fractions, multiply the decimal fractions, and then convert the answer back to decimals. Summarize the steps that students follow to multiply the fractions as follows:

Step 1: Multiply the fractions as though they are whole numbers by pretending the denominator doesn’t exist: $3 \times 5 = 15$.

Step 2: Add the zeros in the denominators of the fractions to find the number of zeros in the denominator of the product: $1 + 3 = 4$.

Step 3: Write the fraction with numerator from Step 1 and denominator from Step 2: $\frac{15}{10000}$.

Now apply each step to multiplying decimals. In Step 1, instead of treating the fractions as whole number (that is, using only the numerator, without the power of 10 in the denominator), we treat the decimals as whole numbers—we think of $0.3$ as $3$ and $0.005$ as $5$. Since the number of zeros in the denominator of the fraction is equal to the number of decimal places in the decimal, Step 2 says to add the decimal places to find the number of decimal places in the product. Step 3 says to combine the results of Steps 1 and 2.

In summary, to multiply decimals (e.g. $0.3 \times 0.005$) follow these steps:

Step 1: Multiply the decimals as though they were whole numbers:
$3 \times 5 = 15$.

Step 2: Add the decimal places to find the number of decimal places in the product:
$1 + 3 = 4$.
Step 3: Shift the decimal point in the whole number from Step 1 the correct number of decimal places to the left, according to Step 2:

\[ \frac{1.5}{0.005} = 0.3 \]

So \( 0.3 \times 0.005 = 0.0015 \)

PROJECTED EXPECTATION
Representing

Have students multiply decimals using these steps. Then have students convert the decimals to fractions and multiply the fractions. Do they get the same answer both ways?

EXAMPLES: a) \( 2.1 \times 0.04 \)  

b) \( 1.3 \times 0.003 \)  
c) \( 1.02 \times 0.03 \)

Sample answer: a) \( 21 \times 4 = 84 \) and there are 1 + 2 = 3 decimal places in the product, so \( 2.1 \times 0.04 = 0.084\). As fractions, \( \frac{21}{10} \times \frac{4}{100} = \frac{84}{1000} = 0.084\). These are indeed the same.

Practise multiplying decimals. See Workbook pp. 37, 38 Questions 7 and 8.

PROJECTED EXPECTATION
Reflecting on other ways to solve a problem

PROJECTED EXPECTATION
Reflecting on the reasonableness of an answer

Teach students to understand the rules of decimal multiplication through patterns involving repeated division by 10, instead of through multiplying fractions by fractions. See BLM Multiply Decimals by Decimals.

Estimating decimal products to check your answer. Have students calculate \( 8.512 \times 3.807 \) on a calculator. They should get \( 32.405184 \). Ask: Does this answer make sense? What whole number is \( 8.512 \) close to? (9) What is \( 3.807 \) close to? (4) What is \( 9 \times 4 \)? (36) Is our answer close to 36? (yes, fairly close) Is the estimate higher or lower than the actual answer? (higher) Why do you think that is? (because we estimated both numbers as being more than they actually are) Tell students that we can get even closer to the answer if we estimate one as being higher and the other as being lower. For example, try \( 8 \times 4 \) or \( 9 \times 3 \). Which answer is closer to the actual answer? (\( 8 \times 4 \)) Why does this make sense? (for example, because \( 3.807 \) is very close to 4, and \( 8.512 \) is almost as close to 8 as to 9)

Have students round each factor to the nearest whole number to estimate the answer and then tell you where to put the decimal point in the product.

a) \( 3.74 \times 23.1 = 86394 \)  
b) \( 6.014 \times 2.5 = 15035 \)  
c) \( 12.81 \times 69.7324 = 89606134 \)  
d) \( 21 \times 4 = \frac{84}{10} = 0.84 \)

d) \( 199.81247 \times 28.14652 = 56240256831044 \)

d) \( 12.85 \times 69.7324 = 89606134 \)  

e) \( 3.74 \times 23.1 = 86394 \)  

Do part a) as a class: Point out that \( 3.74 \times 23.1 \) is about \( 4 \times 23 = 92 \). Point to various places where the decimal point could be added and Ask: Should I put it here? Why/why not? Where can I place the decimal point so that the number is closest to 92? (86.394) Have students check this on a calculator. Have students do the remaining questions individually.

ANSWERS: b) 15.035  
c) \( 893.272044 \)  
d) \( 5624.0256831044 \)  
e) \( 896.06134 \)
ASK: Which two answers are really close to each other? (the answers to parts c and e) Why does this make sense? (because 12.81 is really close to 12.85 and we are multiplying them by the same number) NOTE: Students who count decimal places will get the wrong answer for part e) because 1 285 × 697 324 is actually 896 061 340, not 89 606 134, so they would be counting from the wrong place! This question was deliberately included to help you assess who is actually using estimation and who is counting decimal places instead. If students want to know why counting decimal places gives the wrong answer, say that you will soon explain why to the whole class.

Rounding to the leading digit is easier than rounding to the nearest whole number. Have students redo the questions above by rounding to the leading digit instead of to the nearest whole numbers. For example, 3.74 × 23.1 becomes 4 × 20 = 80. This will also produce a good estimate and will lead students to the correct location for the decimal point. ASK: Which is easier to multiply—4 × 23 or 4 × 20? Why? (4 × 20 is easier because it requires only single-digit multiplication) Emphasize that when there are two ways of getting the same answer, we might as well pick the easier one.

Estimating to the leading digit is sometimes more accurate than estimating to the nearest whole number. Now ask students to use estimation to add the decimal point to this problem:

7.985 × 0.26 = 2 0 7 6 1

The answer here should be about a quarter (0.25) of 8, or about 2. But rounding each term to the nearest whole number gives 8 × 0 = 0, which suggests placing the decimal point at the beginning, before the 2. This gives 0.20761, a number that is much less than 2. Here, rounding to the nearest whole number gives the wrong answer! Suggest to students that instead of rounding to the nearest whole number they should round to the leading digit, which is the first non-zero digit counting from the left. The leading digit in 7.985 is the ones digit, so rounding to the leading digit still gives 8. Rounding 0.26 to the leading digit, the tenths digit, gives 0.3, so the estimate is 8 × 0.3 = 2.4. The place to put the decimal point so that the number is closest to 2.4 is just after the 2, so we get 2.0761.

Have students practise with these problems:

a) 3.24 × 0.41 = 1 3 2 8 4
b) 5 671.234 56 × 0.023 4 = 1 3 2 7 0 6 8 8 8 7 0 4

ANSWERS: a) 1.328 4  b) 132.706 888 704

Counting decimal places can give the wrong answer! Then show students how counting decimal places gives the right answer for 3.74 × 23.1 but the wrong answer for 6.014 × 2.5 (see questions a) and b) in Estimating decimal products to check your answer, above). In the first case, 3.74 has 2 decimal places and 23.1 has 1 decimal place, so the answer should have 3 decimal places, as indeed it does: 86.394. However,
6.014 has 3 decimal places and 2.5 has 1 decimal place, so counting decimal places gives an answer with 4 decimal places. But the answer has 3 decimal places! What happened? Tell students to multiply the whole numbers 6014 \times 25. How is the answer different from 15 035, which is what was shown above? (the answer is 150 350; it has an extra 0) Point out that when you multiply the whole numbers and then put in 4 decimal places, you do get the right answer: 15.0350. Point out also that when using a calculator, the answer given won’t include the final zero, so to check their calculator’s accuracy, students can’t rely on counting decimal places; they have to rely on estimating instead.

Now have students place the decimal point correctly in these products:

a) 4.388 \times 12.75 = 55 947
b) 73.124 \times 0.396 = 28 957 04

c) 73.124 \times 0.395 = 28 883 98

d) 32.125 \times 8.64 = 27 756

ASK: Are the answers to b) and c) close to each other? (yes, very) Why does that make sense? (the factors are very close to each other, so their products should be too) Do the answers have the same number of decimal places? (no) Emphasize that counting decimal places would give the wrong answer for c), where it looks like there should be 6 decimal places, but there are only 5. In fact, 73 124 \times 395 = 28 883 980 and moving the decimal point 6 places left does give the right answer. In fact, only in b) can you get the correct answer by counting decimal places. If we counted decimal places, the answers to b) and c) would look very far apart.

Emphasize again why it is important for students to estimate their answers. When using a calculator, it is easy to think you pressed the decimal point when in fact the calculator does not register the decimal point as having been pressed. Or, you might enter the decimal point in the wrong place. So it is quite easy to get an incorrect answer when multiplying using a calculator. Also, it is not enough to check that the number of decimal places shown on the calculator is correct. If I am multiplying 3.5 \times 72.4 but accidentally press 3.5 \times 7.24, my calculator will give 25.34 which indeed has 1 + 1 = 2 decimal places. But one can see from the estimate that the answer should be more than 3 \times 70 = 210.

After students do Workbook p. 38 Question 13, encourage them to check their answers on a calculator.

**Word Problems Practice** Have students use a calculator to do these problems.

1. A garden measures 4.7 m \times 3.2 m. What is the area of the garden? (15.04 m²)

2. A car travels at a speed of 50.6 km/h for 2.3 h. Use distance = speed \times time to calculate the distance travelled. (116.38 km)
3. A rectangle has length 7.15 cm and width 1.25 cm. What is its area, to one decimal place? (8.9 cm²)

**Extension**

Multiply, then round your answers to three decimal places.

a) 718.45 × 2.903  

b) 0.008 5 × 2.345  

c) 1.0101 × 9.090 9

**ANSWERS:**
a) 2085.804  

b) 0.020  

c) 9.183
Review the units of measurement: mm, cm, dm and m. Have students bring out their ruler. Have students show you various marks on their rulers—the 6 mm mark, the 12 cm mark, the 2 dm mark. (If necessary, remind students of the relationship between decimetres and centimetres, e.g., 2 dm = 20 cm). Have them draw lines of various lengths: 8 cm, 18 mm, 3 dm. Have students find objects in the classroom that are close to 1 mm wide, 1 cm wide, or 1 dm wide. Then bring out a metre stick and tell students the stick is about 1 m long. **ASK:** Is this stick exactly 1 m long, a little longer than 1 m, or a little shorter than 1 m? (a little longer) How do you know? (the markings don’t start right at the beginning and they don’t go all the way to the end)

Hold up a metre stick and point to various positions on the stick. Ask students to say if the positions are closer to 0 m, 0.25 m, 0.5 m, 0.75 m, or 1 m.

Have students say whether each measurement is closer to 0 m, 0.25 m, 0.5 m, 0.75 m, or 1 m.

a) 10 cm b) 52 cm c) 37 cm d) 82 cm e) 2 cm f) 90 cm

**Review converting metric units to smaller metric units.** Tell students to draw a line in their notebook that is 10 cm long. Have students measure the line in four units: mm, cm, dm, and m. (100 mm, 10 cm, 1 dm, 0.1 m) **ASK:** Which unit is largest: mm, cm, dm, or m? (m) Did you need more of the larger units or fewer of them? (fewer) Which unit is smallest? (mm) Did you

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**Goals**

Students will relate fractions and decimals using units, and will compare the size of different units using decimals and fractions

**PRIOR KNOWLEDGE REQUIRED**

Knows how the sizes of different units of measure (mm, cm, dm, m, km) compare

Can change measurements in larger metric units to measurements in smaller metric units

**MATERIALS**

a ruler for each student

a metre stick

---

**Vocabulary**

mixed measurement

tenth

hundredth

thousandth

millimetre (mm)

centimetre (cm)

decimetre (dm)

metre (m)

kilometre (km)
need more of the smaller units or fewer of them? (more)

Draw a line on the board that is 1 m long. Have students write its measurement in millimetres, centimetres, and decimetres. (1 000 mm, 100 cm, and 10 dm)

Draw this diagram on the board to remind students how the units relate to each other:

\[
\text{m} \times 10 \rightarrow \text{dm} \times 10 \rightarrow \text{cm} \times 10 \rightarrow \text{mm}
\]

Units decrease in size going down and you need 10 times more of each unit as the unit above it. Have students convert:

\[
12 \text{ m} = \underline{\text{______}} \text{ dm} = \underline{\text{______}} \text{ cm} = \underline{\text{______}} \text{ mm}
\]

and then do Workbook p. 40 Question 1.

**Bonus**

\[
987.654 \text{ m} = \underline{\text{______}} \text{ dm} = \underline{\text{______}} \text{ cm} = \underline{\text{______}} \text{ mm}
\]

\[
1 \text{ km} = 1000 \text{ m} \text{ so } 0.987654321 \text{ km} = \underline{\text{______}} \text{ m} = \underline{\text{______}} \text{ dm} = \underline{\text{______}} \text{ cm} = \underline{\text{______}} \text{ mm}
\]

**Converting mixed measurements to the smaller unit.** Ask volunteers to convert, step by step, several measurements in metres and centimetres to centimetres. For example,

\[
2 \text{ m 30 cm} = 2 \text{ m} + 30 \text{ cm} = 200 \text{ cm} + 30 \text{ cm} = 230 \text{ cm}.
\]

Have students convert other mixed measurements to centimetres.

**EXAMPLES:**

a) \(3 \text{ m 78 cm} = \underline{\text{______}} \text{ cm}\)  

b) \(4 \text{ m 64 cm} = \underline{\text{______}} \text{ cm}\)

c) \(9 \text{ m 40 cm} = \underline{\text{______}} \text{ cm}\)

Remind students that \(1 \text{ km} = 1000 \text{ m}\). Then repeat with other mixed measurements such as \(\text{m and dm, cm and mm, dm and mm, and km and m}\). Emphasize that students should first convert the larger unit to the smaller unit and then add to determine the total number of smaller units.

**EXAMPLE:**
4 m 7 dm = 40 dm + 7 dm = 47 dm
4 m 7 cm = 400 cm + 7 cm = 407 cm
4 m 7 mm = 4 000 mm + 7 mm = 4007 mm
4 km 7 m = 4 000 m + 7 m = 4007 m

Have students convert other mixed measurements to the smaller unit:

a) 7 cm 3 mm = _____ mm (73)
b) 8 m 4 dm = _____ dm (84)
c) 5 dm 31 mm = _____ mm (531)
d) 5 km 132 m = _____ m (5 132)
e) 5 km 48 m = _____ m (5 048)

**Bonus**
f) 5 km 354 mm = _____ mm (5 000 354)
g) 7 m 5 dm 4 cm 9 mm = _____ mm (7 549)

**Converting measurements in smaller units to mixed measurements.**

For example, to convert 370 cm to metres and centimetres, start by determining how many hundreds are in 370. This tells you how many metres are in 370 cm (because there are 100 cm in one metre). So, 370 cm = 300 cm + 70 cm = 3 m 70 cm.

145 cm = _____ m _____ cm
354 cm = _____ m _____ cm
789 cm = _____ m _____ cm
18 dm = _____ m _____ dm
542 mm = _____ cm _____ dm
3049 m = _____ km _____ m

**Writing smaller units as fractions and decimals of larger units.** See Workbook page 39 Question 2. For example, there are 100 cm in 1 m, so 1 cm = 1/100 m = 0.01 m. Have the stairway diagram above on the board to guide students.

**What unit does the given unit fit into 10, 100, or 1 000 times?** Instead of providing both units to compare, now provide the smaller unit and the number of times it fits into the larger unit, and have students determine the larger unit. **EXAMPLES:**

a) a cm fits into _____ 10 times (ANSWER: a dm)
b) a mm fits into _____ 1 000 times (ANSWER: a m)
c) a dm fits into _____ 10 times (ANSWER: a m)
d) a mm fits into _____ 100 times (ANSWER: a dm)
e) a mm fits into _____ 10 times (ANSWER: a cm)

Then have students convert their statements above to fractional and decimal statements of the form 1 cm = 1/10 dm and 1 cm = 0.1 dm. See Workbook p. 39 Question 3.

**PROCESS EXPECTATION**

Connecting

**Changing fractional measurements to whole number measurements.**

Write on the board: \( \frac{7}{10} \) m = 7 tenths of a metre. Underline “tenths of a metre”:

\[
\frac{7}{10} \text{ m} = 7 \text{ tenths of a metre}
\]

SAY: A tenth of a metre fits into a metre 10 times. ASK: What unit fits into a metre 10 times? (a dm) Add this to the board to have:

\[
\frac{7}{10} \text{ m} = 7 \text{ tenths of a metre} = 7 \text{ dm}
\]

Have students do questions similar to Workbook p.40 Question 4: What unit will fit into…

a) a metre 1 000 times?
b) a decimetre 10 times?
c) a kilometre 10 000 times?
d) a kilometre 1 000 times?
e) a metre 100 times?
f) a decimetre 100 times?
g) a centimetre 10 times?
h) a kilometre 100 000 times?

Then have students do questions similar to Workbook p.40 Question 5. Students could write out the fraction in words if it helps them. **EXAMPLES:**

a) \( \frac{49}{100} \) dm (= 49 hundredths of a dm = 49 mm).
b) \( \frac{5}{100} \) m (= 5 hundredths of a m = 5 cm)
c) \( \frac{9}{10} \) dm (= 9 tenths of a dm = 9 cm)

**Convert decimal measurements to given units.** Write on the board:

\( 0.7 \) dm = _____ mm. **ASK:** How many millimetres fit into a decimetre? (100) Write on the board: \( 0.7 \) dm = _____ mm = _____ hundredths of a dm. **ASK:** How many hundredths of a decimetre is \( 0.7 \) dm? (0.7 dm = 7 tenths of a dm = 70 hundredths of a dm) So \( 0.7 \) dm = 70 hundredths of a dm = 70 mm.

Have students convert the fractional measurement to the given units:

a) \( 0.71 \) m = _____ mm (0.71 m = _____ thousandths of a metre?

\( 0.71 \) m = 710 mm)
b) 0.65 m = _____ cm (65)

c) 3.2 m = _____ dm (32)

d) 4.7 m = _____ mm (4700)

e) 0.4 cm = _____ mm (4)

f) 0.8 dm = _____ mm (80)

**Converting measurements in larger units to mixed measurements.** Write on the board: \(2.357 \text{ m} = \underline{2} \text{ m} \underline{3} \text{ dm} \underline{5} \text{ cm} \underline{7} \text{ mm}\).

**ASK:**
Which digit shows the whole number of metres? (2) Which digit shows tenths of a metre? (the 3) What unit is a tenth of a metre? (dm) Begin filling in the blanks: \(2.357 \text{ m} = \underline{2} \text{ m} \underline{3} \text{ dm} \underline{5} \text{ cm} \underline{7} \text{ mm}\).

continue filling in the blanks—**ASK:** Which digit shows hundredths of a m? (the 5) What unit is a hundredth of a metre? (cm) Repeat for a thousandth of a metre? (the 7 shows mm) Have students do Workbook p. 41 Questions 8, 9, and 10.

**ACTIVITY**

Measure your shoe. Use this as a benchmark to measure long lengths, such as the width of the classroom, the width of the hallway, or the length of the whole school! (Take steps, toe to heel.)

**Extensions**

1. **Measure Your Stride and Create a Treasure Map**

   Draw a line on the floor. Stand with your toes on the line, as if on a starting line. Walk normally for ten strides (not giant steps!) and mark the place where your front leg’s toes are. Measure the distance you walked with a metre stick and divide it by 10 to get the average length of your stride. Now that you have measured your stride, create a map that shows directions in strides, or paces, to a buried pirate treasure (EXAMPLE: 15 paces north, 3 paces east, and so on). Calculate the actual distance you need to walk to find the treasure.

2. Estimate the width of the school corridor. First, try to compare the length to some familiar object. For example, a minivan is about 5 m long. Will a minivan fit across the school corridor? Then, estimate and measure the width with giant steps or let several students stand across the corridor with arms outstretched. Estimate the remaining length in centimetres. Measure the width of the corridor with a metre stick or measuring tape to check your estimate.
Percent Strips
<table>
<thead>
<tr>
<th>Before Taxes</th>
<th>After Taxes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cashier 1</td>
<td>Cashier 2</td>
</tr>
<tr>
<td>4.99</td>
<td>5.6387</td>
</tr>
<tr>
<td>5.49</td>
<td>6.2037</td>
</tr>
<tr>
<td>5.99</td>
<td>6.7687</td>
</tr>
<tr>
<td>6.49</td>
<td>7.3337</td>
</tr>
<tr>
<td>6.99</td>
<td>7.8987</td>
</tr>
<tr>
<td>7.49</td>
<td>8.4637</td>
</tr>
<tr>
<td>7.99</td>
<td>9.0287</td>
</tr>
<tr>
<td>8.49</td>
<td>9.5937</td>
</tr>
<tr>
<td>8.99</td>
<td>10.1587</td>
</tr>
<tr>
<td>9.49</td>
<td>10.7237</td>
</tr>
<tr>
<td>9.99</td>
<td>11.2887</td>
</tr>
<tr>
<td>10.49</td>
<td>11.8537</td>
</tr>
<tr>
<td>10.99</td>
<td>12.4187</td>
</tr>
<tr>
<td>11.49</td>
<td>12.9837</td>
</tr>
<tr>
<td>11.99</td>
<td>13.5487</td>
</tr>
<tr>
<td>12.49</td>
<td>14.1137</td>
</tr>
<tr>
<td>12.99</td>
<td>14.6787</td>
</tr>
<tr>
<td>13.49</td>
<td>15.2437</td>
</tr>
<tr>
<td>13.99</td>
<td>15.8087</td>
</tr>
<tr>
<td>14.49</td>
<td>16.3737</td>
</tr>
<tr>
<td>14.99</td>
<td>16.9387</td>
</tr>
<tr>
<td>15.49</td>
<td>17.5037</td>
</tr>
<tr>
<td>15.99</td>
<td>18.0687</td>
</tr>
<tr>
<td>16.49</td>
<td>18.6337</td>
</tr>
<tr>
<td>16.99</td>
<td>19.1987</td>
</tr>
<tr>
<td>17.49</td>
<td>19.7637</td>
</tr>
<tr>
<td>17.99</td>
<td>20.3287</td>
</tr>
<tr>
<td>18.49</td>
<td>20.8937</td>
</tr>
<tr>
<td>18.99</td>
<td>21.4587</td>
</tr>
<tr>
<td>19.49</td>
<td>22.0237</td>
</tr>
<tr>
<td>19.99</td>
<td>22.5887</td>
</tr>
<tr>
<td>20.49</td>
<td>23.1537</td>
</tr>
<tr>
<td>20.99</td>
<td>23.7187</td>
</tr>
<tr>
<td>21.49</td>
<td>24.2837</td>
</tr>
<tr>
<td>21.99</td>
<td>24.8487</td>
</tr>
<tr>
<td>22.49</td>
<td>25.4137</td>
</tr>
<tr>
<td>22.99</td>
<td>25.9787</td>
</tr>
<tr>
<td>23.49</td>
<td>26.5437</td>
</tr>
<tr>
<td>23.99</td>
<td>27.1087</td>
</tr>
<tr>
<td>24.49</td>
<td>27.6737</td>
</tr>
<tr>
<td>24.99</td>
<td>28.2387</td>
</tr>
<tr>
<td>25.49</td>
<td>28.8037</td>
</tr>
<tr>
<td>25.99</td>
<td>29.3687</td>
</tr>
<tr>
<td>26.49</td>
<td>29.9337</td>
</tr>
<tr>
<td>26.99</td>
<td>30.4987</td>
</tr>
<tr>
<td>27.49</td>
<td>31.0637</td>
</tr>
<tr>
<td>27.99</td>
<td>31.6287</td>
</tr>
<tr>
<td>28.49</td>
<td>32.1937</td>
</tr>
<tr>
<td>28.99</td>
<td>32.7587</td>
</tr>
<tr>
<td>29.49</td>
<td>33.3237</td>
</tr>
</tbody>
</table>
# Three Types of Percent Problems

## What percent is it?

<table>
<thead>
<tr>
<th>Given</th>
<th>the whole and the part</th>
</tr>
</thead>
<tbody>
<tr>
<td>To find</td>
<td>What percent of the whole is the part?</td>
</tr>
</tbody>
</table>
| Equation          | \[
\frac{\text{part}}{\text{whole}} = \frac{?}{100}
\]
| Examples          | 1. What percent of 50 is 10?  
2. 10 is what percent of 50?  
3. A shirt costs $50.  
   It is $10 off.  
   What percent off is it?  
   \[
\frac{\text{part}}{\text{whole}} = \frac{10}{50} = \frac{?}{100}
\]

## How much is the percent?

<table>
<thead>
<tr>
<th>Given</th>
<th>the whole and a percent, ( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>To find</td>
<td>How much is this percent of the whole?</td>
</tr>
</tbody>
</table>
| Equation          | \[
\frac{?}{\text{whole}} = \frac{P}{100}
\]
| Examples          | 1. What is 30% of 50?  
2. 30% of 50 is what number?  
3. A meal costs $50.  
   Tip and taxes together are 30%.  
   What is the tip and taxes together?  
   \[
\frac{\text{part}}{\text{whole}} = \frac{?}{50} = \frac{30}{100}
\]

## How much is the whole?

<table>
<thead>
<tr>
<th>Given</th>
<th>a part that is ( P% ) of the whole</th>
</tr>
</thead>
<tbody>
<tr>
<td>To find</td>
<td>How much is the whole?</td>
</tr>
</tbody>
</table>
| Equation          | \[
\frac{\text{part}}{?} = \frac{P}{100}
\]
| Examples          | 1. If 5 is 40%, what is the number?  
2. 5 is 40% of what number?  
3. A T-shirt was 40% off.  
   The price was reduced by $5.  
   What was the original price?  
   \[
\frac{\text{part}}{\text{whole}} = \frac{5}{?} = \frac{40}{100}
\]
Multiplying With Counters

a) Place 3 counters on each circle. Then move the counters to as few circles as you can.

\[
4 \times \frac{3}{5} = \text{mixed number} \quad = \text{improper fraction}
\]

b) Place 2 counters on each of 3 circles. Move the counters to as few circles as you can.

\[
3 \times \frac{2}{5} = \text{mixed number} \quad = \text{improper fraction}
\]

c) \[4 \times \frac{2}{5}\]  
d) \[3 \times \frac{4}{5}\]  
e) \[2 \times \frac{3}{5}\]  
f) \[3 \times \frac{3}{5}\]  
g) \[2 \times \frac{4}{5}\]  
h) \[4 \times \frac{4}{5}\]
**Adding Reciprocals**

**INVESTIGATION 1** ► If a fraction \( \frac{a}{b} \) is less than 1, its reciprocal \( \frac{b}{a} \) is more than 1.

How does the sum \( \frac{a}{b} + \frac{b}{a} \) compare to 2?

**A.** Start with a simpler problem: Without adding the whole numbers, decide whether the sum is more than 100 or less than 100.

<table>
<thead>
<tr>
<th>a) 38 + 61</th>
<th>b) 53 + 48</th>
</tr>
</thead>
<tbody>
<tr>
<td>38 is ___ less ___ than 40 by ___</td>
<td>53 is ___ less ___ than 50 by ___</td>
</tr>
<tr>
<td>61 is ___ ___ than 60 by ___</td>
<td>48 is ___ ___ than 50 by ___</td>
</tr>
<tr>
<td>So 38 + 61 is ___ ___ than 40 + 60 = 100</td>
<td>So 53 + 48 is ___ ___ than 50 + 50 = 100</td>
</tr>
</tbody>
</table>

c) 76 + 19
d) 69 + 33

<table>
<thead>
<tr>
<th>c) 76 + 19</th>
<th>d) 69 + 33</th>
</tr>
</thead>
<tbody>
<tr>
<td>76 is ___ ___ than 80 by ___</td>
<td>69 is ___ ___ than ___ by ___</td>
</tr>
<tr>
<td>19 is ___ ___ than 20 by ___</td>
<td>33 is ___ ___ than ___ by ___</td>
</tr>
<tr>
<td>So 76 + 19 is ___ ___ than 80 + 20 = 100</td>
<td>So 69 + 33 is ___ ___ than ___</td>
</tr>
</tbody>
</table>

**B.** Describe what worked for the simpler problems: How did you decide whether the sum is greater or less than 100 without calculating the sum?

**C.** Use the method that worked for the simpler problem for the problem we are investigating.

<table>
<thead>
<tr>
<th>a) ( \frac{5}{6} + \frac{6}{5} ) more or less than 2?</th>
<th>b) ( \frac{5}{7} + \frac{7}{5} ) more or less than 2?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{5}{6} ) is ___ ___ than 1 by ___</td>
<td>( \frac{5}{7} ) is ___ ___ than 1 by ___</td>
</tr>
<tr>
<td>( \frac{6}{5} ) is ___ ___ than 1 by ___</td>
<td>( \frac{7}{5} ) is ___ ___ than 1 by ___</td>
</tr>
</tbody>
</table>

The amount more than 1 is ___ ___ than the amount less than 1, so \( \frac{5}{6} + \frac{6}{5} \) is ___ ___ than \( 1 + 1 = 2. \)

The amount more than 1 is ___ ___ than the amount less than 1, so \( \frac{5}{7} + \frac{7}{5} \) is ___ ___ than \( 1 + 1 = 2. \)
c) Is $\frac{5}{8} + \frac{8}{5}$ more or less than 2?

\[
\frac{5}{8} \text{ is ______ than 1 by _____}
\]
\[
\frac{8}{5} \text{ is ______ than 1 by _____}
\]

The amount more than 1 is ______

than the amount less than 1,

so $\frac{5}{8} + \frac{8}{5}$ is ______ than $1 + 1 = 2$.

d) Is $\frac{8}{11} + \frac{11}{8}$ more or less than 2?

\[
\frac{8}{11} \text{ is ______ than 1 by _____}
\]
\[
\frac{11}{8} \text{ is ______ than 1 by _____}
\]

The amount more than 1 is ______

than the amount less than 1,

so $\frac{8}{11} + \frac{11}{8}$ is ______ than $1 + 1 = 2$.

D. Check your answers by calculating the sums.

a) $\frac{5}{6} + \frac{6}{5}$

b) $\frac{5}{7} + \frac{7}{5}$

c) $\frac{5}{8} + \frac{8}{5}$

d) $\frac{8}{11} + \frac{11}{8}$

BONUS ▶ Generalize the result for any $a < b$: What expression for $x$ works for both equations?

If $\frac{a}{b} + \frac{x}{b} = \frac{b}{b}$ and $\frac{a}{a} + \frac{x}{a} = \frac{b}{a}$ then $x = \underline{\quad}$

How much less than 1 is $\frac{a}{b}$? ______ How much more than 1 is $\frac{b}{a}$? ______

The amount more than 1 is ______ than the amount less than 1. How do you know?

Is $\frac{a}{b} + \frac{b}{a}$ more or less than $1 + 1 = 2$? Explain.
Multiply Decimals by Decimals

Remember: To divide by 10, move the decimal point 1 place left.
If you divide a factor by 10, you have to divide the product by 10 too. If you move the decimal point 1
place left in a factor, you have to move the decimal point 1 place left in the product.

1. Keep dividing by 10 to find the last product.
   a) $3 \times 8 = 24$
   b) $12 \times 5 = 60$
   c) $13 \times 8 = 104$

   
   $0.3 \times 8 = \underline{2.4}$
   $1.2 \times 5 = \underline{6}$
   $1.3 \times 8 = \underline{10.4}$

   $0.03 \times 8 = \underline{0.24}$
   $0.12 \times 5 = \underline{0.6}$
   $0.13 \times 8 = \underline{1.04}$

   $0.03 \times 0.8 = \underline{0.24}$
   $0.012 \times 5 = \underline{0.06}$
   $0.13 \times 0.8 = \underline{0.104}$

   $0.03 \times 0.08 = \underline{0.0024}$
   $0.012 \times 0.5 = \underline{0.006}$
   $0.13 \times 0.08 = \underline{0.0104}$

2. Divide one of the factors by 10, then divide the product by 10.
   a) $9 \times 8 = 72$ so $0.9 \times 8 = \underline{7.2}$
   b) $7 \times 5 = 35$ so $7 \times \underline{1.5}$

   $6 \times 9 = 54$ so $\underline{0.6} \times 9 = \underline{5.4}$
   $3 \times 7 = 21$ so $\underline{0.3} \times \underline{7}$

3. Divide by 10 as many times as you need to.
   a) $7 \times 9 = 63$
   b) $18 \times 5 = 90$

   so $0.07 \times 0.9 = \underline{0.063}$
   so $0.18 \times 0.05 = \underline{0.009}$

   c) $15 \times 12 = 180$
   d) $12 \times 7 = 84$

   so $0.015 \times 0.12 = \underline{0.0018}$
   so $0.012 \times 0.7 = \underline{0.0084}$

4. Look at your answers in Question 3. Complete the chart.

<table>
<thead>
<tr>
<th>Number of Places Decimal Point Moves in First Factor</th>
<th>Number of Places Decimal Point Moves in Second Factor</th>
<th>Number of Places Decimal Point Moves in Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>b)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>d)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. How can you find the number of places you need to move the decimal point in the product from the
   number of places you move the decimal point in the factors?
PS7-4  Searching Systematically

Teach this lesson after: 7.2 Unit 1

Goals:
Students will use systematic search to solve ratio problems and to find numbers satisfying constraints.

Prior Knowledge Required:
Can identify and create equivalent ratios
Can identify and evaluate perfect squares
Can compare and order multi-digit numbers
Can substitute numbers for variables to evaluate algebraic expressions
Can identify and create equivalent fractions (for Problem Bank 6)
Knows that division by zero is not defined (for Problem Bank 7)
Can compare decimals (for Problem Bank 9)
Can compare fractions (for Problem Bank 9)

Vocabulary: ratio

Finding the mystery number by using systematic search. Write on the board:

\[ N \times N \times N \times N \text{ is } 1296. \text{ What number is } N? \]

SAY: A lot of students will not know how to start this problem. A good way to start this kind of problem is to try some numbers for \( N \) and be organized in the way you do it. If you just try random numbers to see if they work, you will probably take a long time to find the answer. But if you look in an organized way and use the information to better your search, you will find the answer a lot sooner. ASK: How can you make sure not to miss any possible answers? (try the numbers in order) SAY: Let’s start by checking the numbers in order. A table is a good way to do this. Draw on the board:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( N \times N \times N \times N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 1 \times 1 \times 1 \times 1 = 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( 2 \times 2 \times 2 \times 2 = 4 \times 2 \times 2 = 8 \times 2 = 16 )</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Have volunteers complete the table on the board. \( 3 \times 3 \times 3 \times 3 = 9 \times 3 \times 3 = 27 \times 3 = 81, 4 \times 4 \times 4 \times 4 = 16 \times 4 \times 4 = 64 \times 4 = 256, 5 \times 5 \times 5 \times 5 = 25 \times 5 \times 5 = 125 \times 5 = 625 \) ASK: Are we getting closer to the answer? (yes) Do you think we will find the answer soon this way? (yes) Why do you think that? (the numbers are getting big quickly) SAY: So, we can continue as we’re doing.
Exercise: Continue the table. Stop when you get the product 1296. What is \( N \)?
Answer: 6

After students finish this exercise, add the next row to the table, as shown below:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( N \times N \times N \times N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 1 \times 1 \times 1 \times 1 = 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( 2 \times 2 \times 2 \times 2 = 4 \times 2 \times 2 = 8 \times 2 = 16 )</td>
</tr>
<tr>
<td>3</td>
<td>( 3 \times 3 \times 3 \times 3 = 9 \times 3 \times 3 = 27 \times 3 = 81 )</td>
</tr>
<tr>
<td>4</td>
<td>( 4 \times 4 \times 4 \times 4 = 16 \times 4 \times 4 = 64 \times 4 = 256 )</td>
</tr>
<tr>
<td>5</td>
<td>( 5 \times 5 \times 5 \times 5 = 25 \times 5 \times 5 = 125 \times 5 = 625 )</td>
</tr>
<tr>
<td>6</td>
<td>( 6 \times 6 \times 6 \times 6 = 36 \times 6 \times 6 = 216 \times 6 = 1296 )</td>
</tr>
</tbody>
</table>

Searching faster by skipping numbers. Write on the board:

If \( N \times N \times N \times N = 1 \, 048 \, 576 \), what is \( N \)?

ASK: Would continuing the table be a good strategy for this question? (no) Why not? (it will take too long to get to the answer) SAY: The answers are getting closer to the answer but not much closer; you still have a long way to go to find the answer. Maybe you need to take bigger steps to find the answer. Instead of trying 1, 2, 3, and so on, maybe we should start with 10, 20, 30, and so on.

Exercises:

a) Complete the table up to 50.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( N \times N \times N \times N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>( 10 \times 10 \times 10 \times 10 = 10 , 000 )</td>
</tr>
<tr>
<td>20</td>
<td>( 20 \times 20 \times 20 \times 20 = 160 , 000 )</td>
</tr>
<tr>
<td>30</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>

b) If \( N \times N \times N \times N = 1 \, 048 \, 576 \), what two 10s is \( N \) between? Explain how you know.

Answers:
a) 810 000, 2 560 000, 6 250 000; b) \( N \) is between 30 and 40 because \( N \times N \times N \times N \) is between 810 000 and 2 560 000

SAY: Now we know that \( N \) is between 30 and 40. Write on the board:

\[
30 \times 30 \times 30 \times 30 = 810 \, 000 \\
N \times N \times N \times N = 1 \, 048 \, 576 \\
40 \times 40 \times 40 \times 40 = 2 \, 560 \, 000
\]

Give students time to think about what their next step will be. Then ask students for strategies to find \( N \). (sample answers: start by guessing 35, the middle number; use the fact that the product is a lot closer to \( 30 \times 30 \times 30 \times 30 \) than it is to \( 40 \times 40 \times 40 \times 40 \), so start at 31 and move up by ones; eliminate possibilities that are odd, because the ones digit is 6) Allow several students to articulate their strategies.
SAY: Even if you don’t think of any strategies, you can always move up by ones until you get the answer because you know the answer is in between 30 and 40. It might take longer than if you use strategies, but you will definitely find the answer. Write on the board:

<table>
<thead>
<tr>
<th>N</th>
<th>N × N × N × N</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>30 × 30 × 30 × 30 = 810 000</td>
</tr>
<tr>
<td>31</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td></td>
</tr>
<tr>
<td>34</td>
<td></td>
</tr>
</tbody>
</table>

Ask a volunteer to use a calculator to complete each row of the table until you get the correct answer. (31 × 31 × 31 × 31 = 923 521 and 32 × 32 × 32 × 32 = 1 048 576, so N is 32)

Exercises:
1. Find N so that N × N × N × N is …
   a) 10 556 001    b) 45 212 176
   **Bonus:** 1 698 181 681. Hint: Move up by hundreds, then by tens, then by ones.
   **Answers:** a) 57, b) 82, Bonus: 203

2. Find N so that N × (N + 1) × (N + 2) is …
   a) 504  b) 24 360  c) 157 410
   **Bonus:** 70 444 584. Hint: Move up by hundreds, then by tens, and then by ones.
   **Answers:** a) 7, b) 28, c) 53, Bonus: 412

**Counting all the whole numbers that satisfy constraints.** Write on the board:

How many perfect squares greater than 0 are less than 1000?

Give students time to think about this question, and then ask for strategies. (list the perfect squares until you get more than 1000; make various guesses until you get close to 1000)

SAY: Instead of starting at 1, I might as well start a bit higher, like 10. ASK: What is 10²? (100)
What is 20²? (400) What is 30²? (900) Write on the board:

10² = 100
20² = 400
30² = 900

ASK: Why do you think I chose 10, 20, and 30? (they are easy numbers to multiply) SAY: We are already at 30² = 900 and I only had to try three numbers instead of 30. ASK: What should I try next? (some students may say 40, and others may say 31; take several guesses) Have students explain why they chose the number they did. (900 is close to 1000, so try 31; the next multiple of 10 after 30 is 40, so try 40) Try both 31 (961) and 40 (1600) if both guesses come up. SAY: You can keep guessing multiples of 10 until you pass 1000, but then you know that the answer is between 30 and 40, so you can start at 31. Have students continue guessing until they find 32² = 1024. SAY: 32 is already too high, so there are 31 perfect squares that are less than 1000.
Exercises:
1. a) How many perfect squares are three-digit numbers, that is, from 100 to 999?
   b) How many perfect squares greater than 0 are less than 800?
   c) How many perfect squares are between 800 and 900, not including 900?
   d) A perfect cube is a number that can be written as $N \times N \times N$ for some whole number $N$.
   How many perfect cubes are three-digit numbers?
   Selected solution: a) there are 31 perfect squares from 1 to 999 and there are 9 perfect squares from 1 to 99, so there are 22 perfect squares from 100 to 999
   Answers: b) 28, c) 1, d) 5

2. How many multiples of 9 are there ...
   a) from 1 to 900?
   b) from 1 to 1000?
   c) from 901 to 1000?
   Answers: a) 100, b) 111, c) 11

Using the order of the whole numbers to check all possibilities when two related quantities change. Write on the board:

Jin is six years older than Sandy.
Ten years from today, their ages will add to 40.
How old is Sandy now?

ASK: What is the youngest age that Sandy can be—how many years old? (0) PROMPT: Have you ever been younger than one year old? (yes) SAY: Sandy can’t be younger than 0 years old, so let’s start at 0 and go up in order. Then we know that we won’t miss any possible ages. Draw on the board:

<table>
<thead>
<tr>
<th>Sandy’s and Jin’s Ages</th>
<th>Their Ages in 10 Years</th>
<th>The Sum of Their Ages in 10 Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 and 6</td>
<td>10 and 16</td>
<td>26</td>
</tr>
<tr>
<td>1 and 7</td>
<td>11 and 17</td>
<td></td>
</tr>
</tbody>
</table>

Have volunteers complete the table until the sum is 40 (ages now are 7 and 13; ages in 10 years are 17 and 23). ASK: So how old is Sandy now? (7 years old)

Exercises: Solve the problem using a table.
   a) Raj is three years older than Vicky. Kim is two years older than Raj. In three years, their ages will add to 38. How old are Raj, Vicky, and Kim right now?
   b) Simon is two years older than Amy. Kate is twice as old as Simon. In two years, their ages will add to 40. How old are Simon, Amy, and Kate right now?
   c) Jake is three years older than Sally. Five years from today, their ages will multiply to 154. How old are Jake and Sally right now?
   Answers: a) Vicky is 7, Raj is 10, and Kim is 12; b) Amy is 7, Simon is 9, and Kate is 18; c) Sally is 6 and Jake is 9
**Using systematic search to solve ratio problems.** Write on the board:

The ratio of adults to children on a bus is 2 : 5.
There are 35 people on the bus.
How many children are on the bus?

SAY: You can make a table to solve this type of problem. Draw on the board:

<table>
<thead>
<tr>
<th>Adults</th>
<th>Children</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

SAY: You start with two adults and five children, and every time that you add two more adults, you add five more children. Then you calculate the total each time until you get 35 because that’s what the question says to look for. Have volunteers add more rows to the table until the total is 35, as shown below:

<table>
<thead>
<tr>
<th>Adults</th>
<th>Children</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>21</td>
</tr>
<tr>
<td>8</td>
<td>20</td>
<td>28</td>
</tr>
<tr>
<td>10</td>
<td>25</td>
<td>35</td>
</tr>
</tbody>
</table>

ASK: So how many children are on the bus? (25)

**Exercises:** Solve the ratio problem using a table.

a) Jack mixed red and blue paint together in the ratio 3 : 4. He used 35 cups of paint altogether. How much red paint did Jack use?

b) Ken used blue and white paint in the ratio 2 : 5. He used 56 cups of paint altogether. How much of each colour of paint did he use?

c) The ratio of girls to boys in a classroom is 5 : 7. There are six more boys than girls. How many girls and how many boys are in the class?

**Answers:** a) 15 cups of red paint, b) 16 cups of blue paint and 40 cups of white paint, c) 15 girls and 21 boys

**Solving problems when the ratio changes.** SAY: Now let’s make it harder. Write on the board:

The ratio of girls to boys is 3 : 4.
Three more girls joined the class.
Now the ratio of girls to boys is 9 : 10.
How many boys are in the class?
Challenge students to come up with a table to draw for this problem. PROMPT: Think about what headings you need. After a minute, ask students for suggested headings. (number of girls, number of boys, new number of girls, new girl : boy ratio) Draw on the board:

<table>
<thead>
<tr>
<th>Number of Girls</th>
<th>Number of Boys</th>
<th>New Number of Girls</th>
<th>New Girl : Boy Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>6</td>
<td>6 : 4</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>9</td>
<td>9 : 8</td>
</tr>
</tbody>
</table>

Have students continue the table until the ratio is 9 : 10. (15 girls and 20 boys becomes 18 girls and 20 boys)

**Exercises:** Draw a table to solve the problem.

a) The ratio of girls to boys in a class is 2 : 3. Then five more girls and two more boys joined the class. Now the ratio is 1 : 1. How many girls are now in the class?

b) The ratio of girls to boys is 5 : 3. Five girls moved to another class. Now the ratio is 5 : 4. How many girls are now in the class?

**Answers:** a) 11, b) 15

**Finding numbers that satisfy constraints.** Write on the board:

Two numbers that add to 12:  
Two numbers that multiply to 35:

SAY: I am trying to find two numbers that add to 12 and that multiply to 35. ASK: Do you think it would be quicker to list all the numbers that add to 12 or all the numbers that multiply to 35? Take all predictions, then have students make both lists.

<table>
<thead>
<tr>
<th>Two numbers that add to 12:</th>
<th>Two numbers that multiply to 35:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 and 11</td>
<td>1 and 35</td>
</tr>
<tr>
<td>2 and 10</td>
<td>5 and 7</td>
</tr>
<tr>
<td>3 and 9</td>
<td></td>
</tr>
<tr>
<td>4 and 8</td>
<td></td>
</tr>
<tr>
<td>5 and 7</td>
<td></td>
</tr>
<tr>
<td>6 and 6</td>
<td></td>
</tr>
</tbody>
</table>

Have students search the first list for two numbers that multiply to 35 (5 and 7). Then have them search the second list for two numbers that add to 12 (5 and 7). ASK: Which list was quicker to search through? (two numbers that multiply to 35) Why was it quicker to search through? (there are only two pairs of numbers to check)

**Exercises:** Find all pairs of numbers that satisfy both properties. Start by predicting which list will be shorter and make that list first.

a) Two numbers that add to 8 and multiply to 15
b) Two numbers that add to 9 and multiply to 14

**Bonus:** Two numbers that add to 30 and multiply to 216

**Answers:** a) 3 and 5, b) 2 and 7, Bonus: 12 and 18
SAY: You can also do the same thing with three numbers. I want to find all ways that three positive integers can multiply to 30. I can start listing the possible smallest numbers in order and then see what the other two numbers have to be based on what they have to multiply to. Draw on the board:

<table>
<thead>
<tr>
<th>Smallest Number</th>
<th>Middle Number</th>
<th>Largest Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>30</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

SAY: If the smallest number is 1, I have to find all the pairs that multiply to 30. If the smallest number is 2, I have to list all the pairs that multiply to 15. I listed 3 and 5 but not 1 and 15.

ASK: Why didn’t I list 1 and 15? (2 wouldn’t be the smallest number) If three numbers multiply to 30, can 3 be the smallest number? (no) Why not? (the other two numbers would have to multiply to 10, so 1 or 2 would have to be there) Can 4 be the smallest number? (no) Why not? (4 is not a factor of 30) Can 5 be the smallest number? (no) Why not? (the other two numbers would have to multiply to 6 so at least one of them would have to be smaller than 5) SAY: These are all the possible ways that three integers can multiply to 30.

**Exercises:** Find all the possible ways that three whole numbers can multiply to …

a) 10 

b) 18 

c) 24 

**Answers:**

a) 1, 1, 10; 1, 2, 5; 
b) 1, 1, 18; 1, 2, 9; 1, 3, 6; 2, 3, 3; 
c) 1, 1, 24; 1, 2, 12; 1, 3, 8; 1, 4, 6; 2, 2, 6; 2, 3, 4

**Problem Bank**

1. The ratio of girls to boys is 5 : 3. There are eight more girls than boys. How many girls and how many boys are there?

**Answer:** 20 girls and 12 boys

2. The ratio of base to height in a parallelogram is 1 : 2. The area is 50 cm². What is the height of the parallelogram?

**Solution:** Make a table for base, height, and area with the ratio 1 : 2 between base and height.

<table>
<thead>
<tr>
<th>Base (cm)</th>
<th>Height (cm)</th>
<th>Area (cm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>32</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>50</td>
</tr>
</tbody>
</table>

The height of the parallelogram is 10 cm.
3. a) The ratio of length to width in a rectangle is 3 : 1. The area is 75 cm². What is the length of the rectangle?
b) The ratio of length to width of a rectangle is 7 : 5 and its perimeter is 48 cm. What is the area?
c) The ratio of length to width of a rectangle is 4 : 3. The perimeter is 84 cm. What is the area?
d) The ratio of length to width in a triangle is 5 : 3. The width is 8 cm shorter than the length. What is the area?

Selected solution: a) Make a table for length, width, and area with the ratio 3 : 1 between length and width:

<table>
<thead>
<tr>
<th>Length (cm)</th>
<th>Width (cm)</th>
<th>Area (cm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>48</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
<td>75</td>
</tr>
</tbody>
</table>

The length of the rectangle is 15 cm.

Answers: b) 140 cm², c) 432 cm², d) 240 cm²

4. a) Ivan has three quarters for every five nickels. He has $5.00 in quarters and nickels. How many quarters and how many nickels does he have?
b) Lynn has three quarters for every two dimes. She has $4.75 in quarters and dimes. How many quarters and how many dimes does she have?
c) Kyle had 6 litres of blue paint and 5 litres of yellow paint. (Use 1 litre = 4 cups.) He made 30 cups of green paint using blue and yellow paint in a ratio of 2 : 1. How much of each colour paint is left over?

Answers: a) 15 quarters and 25 nickels, b) 15 quarters and 10 dimes, c) 4 cups of blue paint and 10 cups of yellow paint

5. a) The ratio of girls to boys is 5 : 3. One fifth of the girls are away for a sports event. What is the new ratio of girls to boys? Does the answer depend on the number of girls or boys in the class or just on the ratio?
b) Sun and Cody have baseball cards in the ratio 4 : 5. Then Sun gave half her cards to Cody. Now what is the ratio of Sun’s cards to Cody’s cards? Does the answer depend on the number of baseball cards they had to start or just on the ratio?

Solutions:
a) Make a table.

<table>
<thead>
<tr>
<th>Girls</th>
<th>Boys</th>
<th>New Number of Girls</th>
<th>New Girl to Boy Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>4</td>
<td>4 : 3</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>8</td>
<td>8 : 6 = 4 : 3</td>
</tr>
<tr>
<td>15</td>
<td>9</td>
<td>12</td>
<td>12 : 9 = 4 : 3</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>16</td>
<td>16 : 12 = 4 : 3</td>
</tr>
</tbody>
</table>

The ratio is always 4 : 3; it doesn’t depend on the number of girls and boys in the class, only on the ratio of girls to boys in the class.
b) Make a table.

<table>
<thead>
<tr>
<th>Sun</th>
<th>Cody</th>
<th>New Number of Sun’s Cards</th>
<th>New Number of Cody’s Cards</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>12</td>
<td>15</td>
<td>6</td>
<td>21</td>
</tr>
<tr>
<td>16</td>
<td>20</td>
<td>8</td>
<td>28</td>
</tr>
</tbody>
</table>

The new ratio of Sun’s Cards : Cody’s Cards is $2 : 7 = 4 : 14 = 6 : 21 = 8 : 28$. It doesn’t depend on the number of baseball cards they had to start, only on the ratio of baseball cards.

6. Find the whole number $A$ if ...

a) \( \frac{A-1}{A+1} = \frac{4}{5} \)

b) \( \frac{A \times A}{A + A} = 4 \)

c) \( \frac{A+2}{(A \times 2) + 1} = \frac{2}{3} \)

d) \( \frac{A+4}{A \times A} = \frac{1}{2} \)

Answers: a) 9, b) 8, c) 4, d) 4

7. Explain why there is no counting number $A$ so that \( \frac{A+1}{A-1} = \frac{5}{7} \). Hint: Try the counting numbers in order.

Answer: All the numbers \((A + 1)/(A - 1)\) are bigger than 1, unless $A$ is 1, in which case the fraction is undefined because you cannot divide by 0. But 5/7 is less than 1, so no such counting number exists.

8. There are six animals (ants with six legs, cats with four legs, and birds with two legs). Altogether, the six animals have 20 legs. There is at least one of each kind of animal. How many of each kind of animal are there? Hint: Use the fact that there is at least one of each kind of animal to make your search faster.

Solution: One of each kind has 12 legs altogether, so taking those away leaves three animals with 8 legs altogether. The only possibility is one cat and two birds, so adding the one of each kind of animal back, the six animals are: one ant, two cats, and three birds.

9. How many counting numbers have a reciprocal between (and including):

a) 0.1 and 0.2   b) 0.2 and 0.3   c) 0.3 and 0.4

d) 0.05 and 0.06 e) $\frac{11}{91}$ and $\frac{58}{91}$

Selected solution: a) the reciprocal of 5 is $1/5 = 0.2$ and the reciprocal of 10 is $1/10 = 0.1$, so every counting number from 5 to 10 has a reciprocal between 0.1 and 0.2, and there are six such numbers

Answers: b) 2, c) 1, d) 4, e) 7
PS7-5  Tape Diagrams

Teach this lesson after: 7.2 Unit 1

Goals:
Students use tape diagrams to solve percent and ratio problems.

Prior Knowledge Required:
Is familiar with linear models
Can calculate the percentage of a number
Can convert a simple fraction to a percentage
Can convert a percentage to a fraction and simplify it

Vocabulary: linear model, part, percent, ratio, tape diagram

Review changing fractions to percentages. Remind students that they can change a fraction to a whole-number percentage when the denominator divides evenly into 100. Write the fraction 7/20 on the board. ASK: Does the denominator divide evenly into 100? (yes) How do you know? (20 × 5 is 100) Have a volunteer find an equivalent fraction with denominator 100 for the fraction 7/20. (35/100) ASK: If 7 out of every 20 students at a school are girls, how many out of every 100 students are girls? (35) What percentage of the students are girls? (35%) Write on the board:

\[
\frac{7}{20} = \frac{35}{100} = 35\%
\]

Exercises: Find the equivalent fraction with denominator 100 and then the equivalent percentage.

a) \(\frac{4}{5}\)  b) \(\frac{1}{2}\)  c) \(\frac{3}{10}\)  d) \(\frac{9}{25}\)  e) \(\frac{5}{50}\)  f) \(\frac{1}{4}\)  Bonus: \(\frac{18}{40}\)

Answers: a) 80/100 = 80%, b) 50/100 = 50%, c) 30/100 = 30%, d) 36/100 = 36%, e) 10/100 = 10%, f) 25/100 = 25%, Bonus: 18/40 = 9/20 = 45/100 = 45%

Review finding percentages of a number. ASK: If you know 10 percent of a number, how can you find 40 percent of that number? (multiply 10% of the number by 4) Tell students that you would like to find 60 percent of 13. ASK: What is 10 percent of 13? (1.3) SAY: Remember that 10 percent of a number is 1/10 of the number, so you have to divide by 10. ASK: If you know that 10 percent of 13 is 1.3, how can you find 60 percent of 13? (1.3 × 6) Review multiplying a decimal number by a whole number with an example: multiply the whole numbers, then put the decimal place in the answer. (13 × 6 = 78, so 1.3 × 6 = 7.8)

Exercises: Find the percentage by finding 10% first.

a) 40% of 25  b) 70% of 30  c) 90% of 7  d) 60% of 3.5

Answers: a) 4 × 2.5 = 10, b) 7 × 3 = 21, c) 9 × 0.7 = 6.3, d) 6 × 0.35 = 2.1
Using tape diagrams to find percentages. Remind students that a percentage is a ratio that compares a number to 100. SAY: Remember that you can use linear models to solve ratio problems. But a percentage is a ratio, so you can use linear models to solve percentage problems too. The linear models we are going to use are called tape diagrams and each part represents the same amount. ASK: How many 10 percents are in 100 percent? (10) SAY: To find 30 percent of 40, I can divide 100 percent into 10 parts. Draw on the board:

```
100% of 40

30% of 40
```

SAY: Each block is 10 percent. ASK: What is 10 percent of 40? (4) Write “4” in each block. ASK: So what is 30 percent of 40? (12) SAY: 10 percent is 4, so 30 percent is three 4s, or 12.

Exercises: Show your answer to the question on a tape diagram.  
a) What is 40 percent of 15?  
b) What is 30 percent of 50?  
c) What is 70 percent of 300?  
d) What is 90 percent of 35?  
**Answers:** a) 6, b) 15, c) 210, d) 31.5

Write on the board:

```
A $500 bike is on sale for $495.       A $20 shirt is on sale for $15.
```

Pointing to the first sentence, ASK: How much money are you saving on the bike? ($5) Pointing to the second sentence, ASK: How much money are you saving on the shirt? ($5) SAY: Both sales are for the same amount, but one of them seems like a lot better deal than the other. ASK: Which one seems like a better deal? ($5 off the shirt) Why is that? ($5 off of $500 doesn’t seem like much, but $5 off of $20 seems like a lot) SAY: The discount in a store is usually given as a percentage of the original price because it’s more meaningful that way.

Exercises: How much money is the discount worth?  
a) a 10 percent discount on a $20 shirt  
b) a 10 percent discount on a $200 bike  
c) a 30 percent discount on a $30 DVD  
d) a 30 percent discount on an $80 sweater  
**Answers:** a) $2, b) $20, c) $9, d) $24

Using tape diagrams for percentages that are not multiples of 10 percent. SAY: These questions are all based on multiples of 10 percent, but some sales are 25 percent off or even 75 percent off. ASK: If you wanted to make a tape diagram with each block representing 25 percent, how many blocks would you need to make the whole 100 percent? (4 blocks)
**Draw on the board:**

<table>
<thead>
<tr>
<th>%</th>
<th>25</th>
<th>25</th>
<th>25</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**SAY:** I drew two parts to the picture: the percentage that each block represents and space for the amount each block represents.

**Exercises:** Draw a tape diagram to show the answer. How much is the discount worth?

a) a 25 percent discount on a $300 bike  
b) a 25 percent discount on a $60 sweater  
c) a 75 percent discount on a $16 T-shirt  

**Answers:** a) $75, b) $15, c) $12

Using the smallest number of parts in the tape diagram that you can. **ASK:** How many 20 percents are in 100? (5) **Draw on the board:**

<table>
<thead>
<tr>
<th>%</th>
<th>10</th>
<th>10</th>
<th>10</th>
<th>10</th>
<th>10</th>
<th>10</th>
<th>10</th>
<th>10</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>%</th>
<th>20</th>
<th>20</th>
<th>20</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**SAY:** 20 percent can be two bars on the first tape diagram or it can be one bar on the second tape diagram. **ASK:** Which tape diagram is faster to draw? (the second one) **SAY:** You might as well draw the tape diagram with the smallest number of parts that you can because it will be faster to draw.

**Exercises:** Draw a tape diagram to show how much the discount is worth. Use the smallest number of parts that you can.

a) a 50 percent discount on a $2.30 granola bar  
b) a 40 percent discount on a $35 blouse  
c) a 25 percent discount on a $40 sweater  
d) a 30 percent discount on a $20 000 car  

**Selected solution:** a) draw a tape diagram with two parts, each part is $1.15, so the discount is worth $1.15  

**Answers:** b) $14, c) $10, d) $6000
Finding the original price from the sale price. SAY: In the previous questions, we found the part (discount) from the whole (original). Explain to students that they can use a tape diagram to find the whole (original price) from the part (discounted price), too. Start with an example.
SAY: After a 20 percent discount, I paid $40 for a T-shirt. I would like to know the original price.
ASK: What percentage is discounted? (20%) SAY: Because I know the discount is 20 percent, I will divide 100 percent into five parts. Draw on the board:

%  20 20 20 20 20

$   Sale price = ? Discount = ?

ASK: How much did I pay after the discount? ($40) Show the sale price on the tape diagram, as shown below:

%  20 20 20 20 20

$   Sale price = $40 Discount = ?

Explain that when you get a 20 percent discount, you actually pay 80 percent of the original price, which is the sale price. Write on the board:

sale price + discount = original price

Point to the diagram and SAY: The sale price is $40. ASK: How many bars on the tape diagram does the $40 represent? (4) What is each part? ($10) How do you know? (40 ÷ 4 = 10) Write “10” in each part of the diagram, as shown below:

%  20 20 20 20 20

$  10 10 10 10 10

Sale price = $40 Discount = ?

SAY: The discount is $10, so the original price was $50. Erase the question mark and write “$50” in its place.
Exercises: a) After a 25% discount, the price is $45. What was the original price?

\[
\begin{array}{c}
100\% = ? \\
\hline
\% & 25 & 25 & 25 & 25 \\
\hline
\$ & 15 & 15 & 15 & 15 \\
\hline
\$45 \\
\end{array}
\]

The original price was $60.

b) After a 40% discount, the price is $30. What was the original price?

\[
\begin{array}{c}
100\% = ? \\
\hline
\% & 20 & 20 & 20 & 20 & 20 \\
\hline
\$ & 10 & 10 & 10 & 10 & 10 \\
\hline
\$30 \\
\end{array}
\]

The original price was $50.

c) After a 20% discount, the price is $56. What was the original price?

\[
\begin{array}{c}
100\% = ? \\
\hline
\% & 20 & 20 & 20 & 20 & 20 \\
\hline
\$ & 14 & 14 & 14 & 14 & 14 \\
\hline
\$56 \\
\end{array}
\]

The original price was $70.

NOTE: In part c), if some students divided 100% into 10 parts, SAY: You can do this, but it is easier to draw if you use parts that are 20 percent instead of 10 percent (see diagram below).
Solving two-step problems using two tape diagrams. Write on the board:

Mike has $80. He spends 60% of it on a sweater. Mike decides to spend 25% of the remaining money on lunch. How much can Mike spend on his lunch?

Point out that the first sentence is similar to the problems you have already done in this lesson. SAY: To solve this problem, we can start with the same type of diagram as earlier. Draw on the board:

ASK: Which part of the diagram shows the money left over? (the last two columns) What is 20 percent of $80? ($16) How do you know? (sample answer: 10% of 80 is 8 and 20% of 80 is $2 \times 8 = 16$) How much does Mike pay for the sweater? ($48$) PROMPT: 20 percent of 80 is 16, and Mike spends 60 percent. ASK: How did you get that? (multiplied 16 × 3) How much is left over? ($32$) Complete the tape diagram, as shown below:

ASK: What is 25 percent of 32? (8) SAY: Mike can spend up to $8 on lunch. Explain to students that they could find 25% of 32 by multiplying 0.25 × 32 or by dividing 32 by 4 even without using the second tape diagram, but tape diagrams make it easier to visualize the problem. The tape diagram makes it easy to see that you can divide by 4 to get 25 percent.
**Exercises:** Luc gives 40% of his 50 stickers to his brother and 60% of the leftover stickers to his friend.

a) How many stickers does Luc’s friend get?
b) Who gets more stickers, Luc’s brother or Luc’s friend?

**Solution:**

<table>
<thead>
<tr>
<th>Total stickers: 50</th>
<th>Stickers left: 30</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>%</td>
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<td>10</td>
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</tbody>
</table>

Brother: 20

Friend: 18

**Answers:**
a) 18; b) Luc’s brother, because he receives 20 stickers, which is two more than Luc’s friend

**Solving problems by working backwards.** Write on the board:

Jasmin has some stickers. She colours 25% of them red and she colours 40% of the remaining stickers green. If Jasmin has 18 stickers left uncoloured, what percentage of the total stickers did she not colour?

SAY: You can draw two tape diagrams—one diagram for each step of the problem—and leave out the information you do not know yet. You can then work backwards to complete the diagrams and determine the answer. Draw on the board:

<table>
<thead>
<tr>
<th>Total stickers: ?</th>
<th>Stickers left: ?</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td>%</td>
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<tr>
<td>25</td>
<td>25</td>
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<tr>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>Red: ?</td>
<td>Stickers left: ?</td>
</tr>
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<td></td>
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</tr>
</tbody>
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</tr>
</tbody>
</table>

SAY: In the diagram on the left, I need to determine the total number of stickers so that I can determine what percentage of the total stickers the 18 uncoloured stickers represent. All parts are unknown, except the percentage. ASK: Can I start with the diagram on the left? (no)

SAY: Look at the diagram on the right. ASK: How many stickers are not coloured? (18) How many stickers does each block represent? (6) How do you know? (18 ÷ 3 = 6) So how many blocks are coloured green? (12) How do you know? (2 × 6 = 12) Erase the question mark beside “Green” and write “12” in its place. Point to the diagram on the right and ASK: How many stickers are represented in this diagram? (30) How do you know? (12 + 18 = 30) Point to the diagram on the left and ASK: So how many are left after Jasmin coloured some red? (30) Erase the question mark beside “Stickers left” in each diagram and write “30” in its place.

Point to the diagram on the left and ASK: How many stickers does each block represent? (10) How many stickers are coloured red? (10) So how many stickers are there in total? (40)
Erase the question marks beside “Red” and “Total stickers” and write “10” and “40” in their places. Fill in the numbers in the tape diagrams. The final picture is shown below.

Erase the question marks beside “Red” and “Total stickers” and write “10” and “40” in their places. Fill in the numbers in the tape diagrams. The final picture is shown below.

SAY: The question asks what percentage of the total stickers Jasmin did not colour. Ask a volunteer to change $18/40$ to an equivalent fraction with denominator 100 to find what percentage of 40 is 18, as shown below:

$$\frac{18}{40} = \frac{9}{20} = \frac{45}{100} = 45\%$$

SAY: 45% of the stickers are not coloured.

Exercise: Matt spends 60% of his money on a book and 75% of the leftover money on a music download. Matt has $6 left after he pays for the book and the music. How much money does Matt start with?
Answer: $60

Problem Bank

1. Jessica reads 15 pages of a book on Saturday. She reads $\frac{2}{5}$ of the remaining pages on Sunday. She still has 36 pages to read. How many pages are in the book?
Answer: 75 pages

2. Alex is 3 years older than Zack. Zack is 7 years older than Evan. All three of their ages add to 62. How old is Zack?
Answer: 22 years old

3. The sum of the angles in a triangle is 180 degrees. The angles are proportional to the numbers 3, 7, and 8. Can you say all the angles are acute? Why?
Solution: Yes, because $3 + 7 + 8 = 18$ and $180/18 = 10$, so the angles are 30, 70, and 80.

4. The ratio of Marko’s money to Kevin’s money is 3 to 4. The ratio of Kevin’s money to Sam’s money is 6 to 5. Marko has $27. How much money do the three friends have altogether?
Answer: $93

5. Tina gives 75% of her beads to her sister and $\frac{2}{5}$ of the remaining beads to her brother.
What fraction of the original number of beads does Tina keep?
Answer: 15% or 3/20
6. Arsham buys art supplies with \(\frac{2}{5}\) of his money and buys a book with \(\frac{3}{8}\) of the rest. Arsham now has $4.80. How much money does he start with?  
**Answer:** $12.80

7. If a store pays $2 for a pen and increases the price by $1 to sell it for $3, the store is marking up the price by 50%. Because the amount of the markup is 50% of the cost, it is called a 50% markup.  
   a) A store pays $300 for 10 shirts. The shirts are marked up by 50%. What price does the store sell one shirt for?  
   
   \[
   \begin{array}{c|c|c|c}
   \% & 50 & 50 & 50 \\
   \hline
   \$ & 100\% = 30 \\
   \end{array}
   \]
   
   Price after markup = ?

   b) What would be the price after a 60% markup?  
   \[
   \begin{array}{c|c|c|c|c|c|c|c|c|c}
   \% & 20 & 20 & 20 & 20 & 20 & 20 & 20 & 20 \\
   \hline
   \$ & 100\% = 30 \\
   \end{array}
   \]
   
   Price after markup = ?

   **Answers:** a) $45, b) $48

8. After a 25% markup, a pair of jeans sells for $45. What was the original cost before the markup?  
   **Solution:**  
   \[
   \begin{array}{c|c|c|c|c|c|c|c|c}
   \hline
   \$ & 9 & 9 & 9 & 9 & 9 \\
   \end{array}
   \]
   The cost was $36.
PS7-6 Using Logical Reasoning I

Teach this lesson after: 7.2 Unit 1

Goals:
Students will solve one- and two-step equations using logic and the properties of operations.

Prior Knowledge Required:
Knows that addition and subtraction undo each other
Knows that multiplication and division undo each other
Can solve equations of the form $ax + b = c$, where $a$, $b$, and $c$ are whole numbers
Knows that integers exist
Can convert between decimals and fractions (for Problem Bank 1)
Can multiply fractions (for Problem Bank 1)
Can add and subtract fractions (for Problem Banks 2, 3)

Vocabulary: coefficient, constant term, equation, variable, variable term

Review showing addition and subtraction equations on a number line. Write on the board:

\[ x + 3 = 9 \]

SAY: This equation says: Start with some number, add 3 to it, and get 9. You can show that on a number line. You start at \( x \), move right three spaces, and you end up at 9. Draw on the board:

\[ \begin{align*}
\text{3} & \quad \text{x} \\
\text{x} & \quad \text{9}
\end{align*} \]

SAY: Both of these number lines show that I can add 3 to \( x \) to get 9, but this second number line is more convenient for bigger numbers. Write on the board:

\[ x + 257 = 314 \]

Have a volunteer show how to express this equation on a number line:

\[ \begin{align*}
\text{257} & \quad \text{x} \\
\text{x} & \quad \text{314}
\end{align*} \]

Now draw the same two number lines with the arrows going the opposite way:

\[ \begin{align*}
\text{3} & \quad \text{x} \\
\text{x} & \quad \text{9}
\end{align*} \]

\[ \begin{align*}
\text{257} & \quad \text{x} \\
\text{x} & \quad \text{314}
\end{align*} \]
SAY: If you can move right three spaces from $x$ to get 9, then you can move left three spaces from 9 to get $x$. Write on the board:

\[
\begin{align*}
    x + 3 &= 9 & 9 - 3 &= x \\
    x + 257 &= 314 & 314 - 257 &= x
\end{align*}
\]

SAY: If you know how to get 9 from $x$, then you know how to get $x$ from 9. This is convenient when it is $x$ that you want to know how to calculate. When you write the equation so that the variable is by itself, the equation is telling you the calculation you need to do to get $x$.

**Exercises:**
1. Write the equation for the number line.
   
   a) \[\begin{array}{c}
       \text{x} \\
       \text{9} \\
       \text{15}
     \end{array}\]  
   b) \[\begin{array}{c}
       \text{x} \\
       \text{9} \\
       \text{15}
     \end{array}\]  
   c) \[\begin{array}{c}
       \text{x} \\
       \text{9} \\
       \text{15}
     \end{array}\]  
   d) \[\begin{array}{c}
       \text{9} \\
       \text{15} \\
       \text{x}
     \end{array}\]  
   e) \[\begin{array}{c}
       \text{9} \\
       \text{x} \\
       \text{15}
     \end{array}\]  
   f) \[\begin{array}{c}
       \text{9} \\
       \text{x} \\
       \text{15}
     \end{array}\]

**Answers:** a) $x + 9 = 15$, b) $9 + 15 = x$, c) $9 + x = 15$, d) $x - 15 = 9$, e) $15 - x = 9$, f) $15 - 9 = x$

2. Draw the number line for the equation.
   a) $x + 32 = 45$  
   b) $19 - x = 3$  
   c) $x - 5 = 8$  
   d) $32 + x = 55$

3. Draw the number line for the equation, reverse the arrow on the number line, and then write the new equation. Solve for $x$.
   a) $x + 6 = 10$  
   b) $45 = x + 5$  
   c) $16 = x - 5$  
   d) $x - 243 = 312$

**Answers:** a) $x = 10 - 6$, so $x = 4$; b) $x = 45 - 5$, so $x = 40$; c) $x = 16 + 5$, so $x = 21$; d) $x = 312 + 243 = 555$

**Noticing a shortcut to solve one-step equations.** Write on the board:

\[
\begin{align*}
    x - 3 &= 7, & x + 3 &= 7, & 3 + x &= 7, \\
    x &= 7 + 3 & x &= 7 - 3 & x &=
\end{align*}
\]

Pointing to the first equation, SAY: To get $x$ by itself, I moved 3 to the other side, but I changed subtracting 3 on one side of the equation to adding 3 on the other side of the equation. Pointing to the second equation, SAY: To get $x$ by itself here, I still moved 3 to the other side, but this time I changed adding 3 on one side of the equation to subtracting 3 on the other side of the equation. You can always move any term to the other side when solving equations to get the variable by itself, but you have to remember to change addition to subtraction and subtraction to addition. Now point to the third equation and ASK: Is 3 being added to $x$ or subtracted from it? (added to it) SAY: So, when I move the 3 to the other side, I have to subtract it from 7. Continue writing on the board:

\[
\begin{align*}
    3 + x &= 7, & x &= 7 - 3
\end{align*}
\]
SAY: It makes sense that these equations mean the same thing because if 3 and \( x \) together make 7, then you can subtract 3 from 7 to get \( x \). ASK: How does it help to have \( x \) by itself? (I can calculate it directly)

**Exercises:** Move the constant term to the other side to get the variable by itself, then solve the equation.

a) \( 38 = 8 + x \)  
b) \( 21 = x - 12 \)  
c) \( 17 + x = 56 \)  
d) \( x - 5 = 14 \)

**Answers:** a) \( 38 - 8 = x \), so \( x = 30 \); b) \( 21 + 12 = x \), so \( x = 33 \); c) \( x = 56 - 17 \), so \( x = 39 \); d) \( x = 14 + 5 \), so \( x = 19 \)

**Review solving one-step multiplication and division equations.** Write on the board:

6 is twice as much as \( x \).

Read the sentence aloud and then ASK: That means \( x \) is what fraction of 6? (half) Continue writing on the board:

6 is twice as much as \( x \), so \( x \) is half as much as 6.

ASK: If 8 is four times as much as \( x \), then \( x \) is what fraction of 8? (one fourth) If 20 is five times as much as \( x \), then \( x \) is what fraction of 20? (one fifth)

**Exercises:** Fill in the blank.

a) 28 is four times as much as \( x \), so \( x \) is _________ as much as 28.  
b) 35 is seven times as much as \( x \), so \( x \) is _________ as much as 35.  
c) 16 is twice as much as \( x \), so \( x \) is _________ as much as 16.  

**Answers:** a) one fourth or one quarter, b) one seventh, c) half

SAY: Six is twice as much as \( x \), so if you multiply \( x \) by 2, you get 6. Have a volunteer show on the board how to write this algebraically as an equation. (\( 2x = 6 \)) SAY: \( x \) is half as much as 6, so you can divide 6 by 2 to get \( x \). Have a volunteer show on the board how to write this algebraically as an equation. (\( 6 ÷ 2 = x \))

SAY: Remember that multiplication and division undo each other just like how addition and subtraction undo each other. So, if you multiply \( x \) by 2 to get 6, then you have to divide 6 by 2 to get \( x \). Write on the board:

\( 7x = 56 \)

SAY: If 56 is seven times as much as \( x \), then \( x \) is one seventh as much as 56. ASK: If you multiply \( x \) by 7 to get 56, what do you have to do to 56 to get \( x \)? (divide by 7) Write on the board:

\[ x = \frac{56}{7} \]
\[ = 8 \]
**Exercises:**
1. Write the correct operation.
   a) 15 is five times \(x\), so \(x\) is 15 ___ 5.
   b) 24 is three times \(x\), so \(x\) is 24 ___ 3.
   **Answers:** a) ÷, b) +

2. Rewrite the equation so that \(x\) is by itself. Then solve the equation.
   a) \(3x = 12\)  
   b) \(4x = 20\)  
   c) \(12x = 120\)
   **Answers:** a) \(x = 12 ÷ 3\), so \(x = 4\); b) \(x = 20 ÷ 4\), so \(x = 5\); c) \(x = 120 ÷ 12\), so \(x = 10\)

**SAY:** Now let’s look at a division equation. Write on the board:

\[
\frac{x}{3} = 4
\]

**SAY:** \(x\) divided by 3 is 4, so 4 is one third as much as \(x\). Write on the board:

4 is one third as much as \(x\), so \(x\) is _________ as much as 4.

Have a volunteer fill in the blank. (three times) **ASK:** If you divide \(x\) by 3 to get 4, what do you have to do to 4 to get \(x\)? (multiply by 3) Write on the board:

\[x = 4 \times 3\], so \(x = 12\)

**Exercises:**
1. Fill in the blank.
   a) 3 is one sixth as much as \(x\), so \(x\) is ________ as much as 3.
   b) 5 is one fourth as much as \(x\), so \(x\) is ________ as much as 5.
   c) 10 is one third as much as \(x\), so \(x\) is ________ as much as 10.
   **Answers:** a) six times, b) four times, c) three times

2. Write the correct operation.
   a) 12 is three times \(x\), so \(x\) is 12 ___ 3.
   b) 12 is \(x\) divided by 3, so \(x\) is 12 ___ 3.
   c) 10 is two times \(x\), so \(x\) is 10 ___ 2.
   d) 10 is \(x\) divided by 2, so \(x\) is 10 ___ 2.
   **Answers:** a) +, b) \times, c) +, d) \times

3. Solve the equation.
   a) \(5x = 40\)  
   b) \(4.8 = 2x\)  
   c) \(12 = 0.2x\)
   d) \(6x = 27\)  
   e) \(\frac{x}{3} = 5\)  
   f) \(\frac{x}{1.2} = 4\)
   **Solutions:** a) \(x = 40 ÷ 5\), so \(x = 8\); b) \(x = 4.8 ÷ 2\), so \(x = 2.4\); c) \(x = 12 ÷ 0.2\), so \(x = 60\); d) \(x = 27 ÷ 6\), so \(x = 4 1/2 = 4.5\); e) \(x = 5 \times 3\), so \(x = 15\); f) \(x = 4 \times 1.2\), so \(x = 4.8\)
SAY: In the multiplication and division equations, the variable is already by itself, but you are just changing it to get the coefficient equal to 1. Point to the two equations on the board (7x = 56 and x/3 = 4) in turn and SAY: In the first equation, the coefficient is 7 and we divided by 7 to make the coefficient equal to 1, and in the second equation, the coefficient is one third and we multiplied by 3 to make the coefficient equal to 1. In both cases, you made the coefficient 1.

**Exercises:** Get x by itself with coefficient 1. Then solve the equation.

a) 43 = 3 + x  
b) x − 32 = 23  
c) x + 44 = 77  
d) 4x = 32  
e) 48 = 2x  
f) 13 = 2x  
g) \frac{x}{3} = 6  
h) \frac{5}{x} = 5

**Solutions:** a) x = 43 − 3 = 40, b) x = 23 + 32 = 55, c) x = 77 − 44 = 33, d) x = 32 ÷ 4 = 8, e) x = 48 ÷ 2 = 24, f) x = 13 + 2 = 6.5, g) x = 6 × 3 = 18, h) x = 5 × 5 = 25

**Solving two-step equations.** Write on the board:

\[ 2x + 1 = 7 \]

SAY: Our goal is still to get the variable by itself. First, let’s get the variable term by itself, then we’ll make the coefficient equal to 1. ASK: What do we have to move to the other side to get the variable term by itself? (the 1) Is the 1 being added or subtracted? (added) So, what do we have to do to bring it to the other side? (subtract it from 7) If students just say, “Subtract it,” ASK: Subtract it from what? (7) Write on the board:

\[ 2x = 7 − 1 \]
\[ = 6 \]

SAY: Now we have to get the coefficient equal to 1. ASK: What is the coefficient right now? (2) So, what do we have to do to get the coefficient equal to 1? (divide by 2) SAY: We have to bring the 2 being multiplied to the other side and divide by it instead. Write on the board:

\[ x = \frac{6}{2} \]
\[ = 3 \]

**NOTE:** Some students might want to start by getting the coefficient equal to 1. The only way to correctly do that is to recognize that \(2x + 1 = 2(x + 1/2)\) and then move the 2 to the other side. If students want to incorrectly write \(x + 1 = 7 \div 2\), explain that in order to bring the 2 to the other side and divide it, it has to be multiplied by everything else on its original side of the equation, not just the \(x\). You can move the 1 to the other side by subtracting it because the 1 is being added to everything else on that side.

**Exercises:** To get the variable term by itself, move the constant term to the right side. Remember to change subtraction to addition and addition to subtraction. Then simplify the right side.

a) \(2x − 3 = 15\)  
b) \(4x − 5 = 7\)  
c) \(3x + 1 = 13\)  
d) \(10x − 1 = 2\)

**Answers:** a) \(2x = 18\), b) \(4x = 12\), c) \(3x = 12\), d) \(10x = 3\)
Explain to students that an equation with all the variable terms on one side and all the constant terms on the other side is easy to solve because, to find \( x \), you just need to divide both sides by the coefficient of \( x \). For example, if \( 2x = 6 \), then \( x = 6 \div 2 = 3 \).

**Exercises:** Solve the equations in the previous exercises by dividing both sides by the coefficient of \( x \).

**Answers:**

a) \( x = 18 \div 2 = 9 \), b) \( x = 12 \div 4 = 3 \), c) \( x = 12 \div 3 = 4 \), d) \( x = 3 \div 10 = 0.3 \)

**Solving two-step equations where the variable term is subtracted.** Write on the board:

\[ 5 - 2x = 3 \]

**SAY:** This equation is different because now the variable term is being subtracted. **ASK:** How can we change it to an equation where the variable term is being added? (move the variable term to the other side) **SAY:** Remember that you want all the constant terms on one side and the variable term on the other. Have a volunteer show how to do this. (\( 5 - 3 = 2x \)) Have another volunteer finish solving the equation. (\( 2 = 2x, 1 = x \)) Write on the board:

\[ 5 - 2x = 3 \text{ and } 5 - 3 = 2x \]

**SAY:** These two equations mean the same thing because in both of them 5 is the total and 3 and \( 2x \) are the parts.

**Exercises:**

1. Solve the equation.
   a) \( 8 - 3x = 2 \)  b) \( 18 - 4x = 2 \)  c) \( 350 - 4x = 70 \)  d) \( 53 = 203 - 3x \)
   **Answers:** a) \( x = 2 \), b) \( x = 4 \), c) \( x = 70 \), d) \( x = 50 \)

2. Does the equation have the same solution as \( 2x + 5 = 17 \)?
   a) \( 17 - 2x = 5 \)  b) \( 17 + 2x = 5 \)  c) \( 5 + 17 = 2x \)  d) \( 17 - 5 = 2x \)
   **Answers:** a) yes, b) no, c) no, d) yes

**Solving (one-variable) equations with several variable and constant terms.** Write on the board:

\[ 3x - 2x + x = 9 - 4 + 7 \]

**SAY:** This looks like there are lots of terms, so it looks complicated, but there is a way to make it easier. **ASK:** What can we do to the right-hand side to make it easier? (calculate it) What is \( 9 - 4 + 7 \)? (\( 12 \)) **SAY:** \( 9 - 4 \) is 5. Add 7 and that’s 12. Write on the board:

\[ 3x - 2x + x = 12 \]

**SAY:** We’re half done. Now let’s try to make the other side easier. Remember that \( 3x \) means three \( x \)’s, so you start with three \( x \)’s. Write on the board:

\[ x + x + x \]
SAY: You need to take away two of them. Erase the first two x’s, as shown below:

\[ x \]

SAY: Then you need to add another one. Continue writing on the board:

\[ x + x \]

ASK: So, what are we left with on the left side? (2x) PROMPT: How many x’s are there? (2) Write on the board:

\[ 2x = 12 \]

Have a volunteer solve the equation. (x = 6)

**Exercises:** Solve the equation.

a) \( 5x + x - 2x = 10 - 3 + 5 \)

b) \( 6x - 2x + x - 4x = 7 - 2 + 4 - 5 \)

c) \( 9x + 3x - 4x + 2x = 8 + 3 - 5 + 14 \)

**Answers:** a) \( x = 3 \), b) \( x = 4 \), c) \( x = 2 \)

**Grouping all the variable terms on one side and all the constant terms on the other side.**

SAY: Now I’m going to make it harder. Write on the board:

\[ 3x - 4 + 2x + 8 = 13 + x - 1 \]

ASK: What makes this equation harder to solve? (the like terms are not already on the same side; the variable terms are not all on the same side) SAY: I want to move all the variable terms to one side and all the constant terms to the other side. Then it will be easy to solve. Let’s move all the variable terms to the left and all the constant terms to the right. Write on the board:

\[ 3x + 2x \]

SAY: These variable terms are already on the left side. Now let’s look at the right side. What other variable terms are there? (just x) Is it added or subtracted? (added) So do we need to add it to the left side or subtract it from the left side? (subtract) Continue writing on the board:

\[ 3x + 2x - x = \]

SAY: Now we have to move the constant terms. Let’s start by writing the terms that are already on the right side. Continue writing on the board:

\[ 3x + 2x - x = 13 - 1 \]
SAY: Now let’s move the constant terms from the left side to the right side. Remember, I have to add the terms that are subtracted and subtract the terms that are added. Continue writing on the board:

\[ 3x + 2x - x = 13 - 1 + 4 - 8 \]

Have a volunteer finish solving the equation. \((4x = 8, x = 2)\)

**Exercises:** Move all the variable terms to the left-hand side and all the constant terms to the right-hand side. Solve the equation.

a) \(3x + 5 - x = 7\)  
  b) \(8x - 4 = 3x + 11\)

  c) \(4x - 7 = 3 - x\)  
  d) \(5x - 8 - 2x = 4\)

  e) \(11x - 5 = 9x + 5\)  
  f) \(5x - 8 = 2x + 1\)

**Answers:** a) \(x = 1\), b) \(x = 3\), c) \(x = 2\), d) \(x = 4\), e) \(x = 5\), f) \(x = 3\)

**Problem Bank**

1. Solve the equation. Hint: Change the decimal to a fraction.

   a) \(x + \frac{1}{2} = 0.6\)  
   b) \(\frac{2}{3}x = 0.4\)

   **Solutions:** a) \(x = 6/10 - 5/10 = 1/10\), b) \(x = 4/10 \times 3/2 = 12/20 = 3/5\)

2. Solve the equation \(2x + \frac{1}{2} = \frac{2}{3}\) two ways. Make sure you get the same answer both ways.

   a) Move the constant term to the other side, subtract the fractions, then divide both sides by 2.
   b) Multiply each term by the LCM of the denominators, then solve the equation with whole numbers.

   **Solutions:** a) \(2x = 2/3 - 1/2 = 1/6\), so \(x = 1/12\); b) \(12x + 3 = 4\), so \(12x = 1\), so \(x = 1/12\)

3. Multiply by the LCM of the denominators to solve the equation.

   a) \(7x - \frac{1}{3} = \frac{1}{4}\)  
   b) \(\frac{1}{2}x + \frac{1}{3} = \frac{7}{3}\)  
   c) \(\frac{2}{3}x - \frac{3}{4} = \frac{1}{2}\)

   **Answers:** a) \(x = 1/12\), b) \(x = 4\), Bonus: \(x = 15/8\)

4. Get \(x\) by itself.

   a) \(x + b = c\)  
   b) \(x - c = d\)  
   c) \(ax = b\)

   **Answers:** a) \(x = c - b\), b) \(x = d + c\), c) \(x = b/a\)

5. Parking Lot A charges a flat fee of $5, plus $2 for every hour. Parking Lot B charges $3 for every hour, without a flat fee. Avril parks at Lot A and Billy parks at Lot B for the same length of time and they pay the same amount.

   a) How long did they park for?  
   b) How much did each person pay?

   **Solutions:** a) Let \(x\) represent the number of hours. Solve \(5 + 2x = 3x\) to get \(x = 5\), so they parked for 5 hours; b) \(3 \times 5 = 15\), or \(5 + 2 \times 5 = 15\), so each person paid $15
6. Lela is 3 cm taller than Tristan, and Tristan is 4 cm shorter than Marla. If Glen is 2 cm taller than Lela, who is tallest? Who is shortest?

**Solution:** Let $x$ represent Lela’s height, then Tristan is $x - 3$ and Marla is $(x - 3) + 4$ or $x + 1$. Glen is 2 cm taller than Lela, so Glen is $x + 2$. Glen is the tallest and Tristan is the shortest.

7. You can solve an equation in which the variable is being subtracted by moving the variable to the other side. Then it will be added.

Example: Solve the equation $8 - x = 3$

Step 1: Move the variable to the other side. $8 = 3 + x$

Step 2: Get $x$ by itself. $8 - 3 = x$

Step 3: Evaluate $x$. $x = 5$

Solve for $x$.

a) $4 = 9 - x$  
   b) $10 - x = 5$  
   c) $13 - x = 7$  
   d) $9 = 17 - x$

**Answers:** a) 5, b) 5, c) 6, d) 8

8. a) Fill in the blank: You can solve an equation in which the variable is being divided by moving the variable to the other side. Then it will be _____________.

   b) Solve the equation $\frac{8}{x} = 2$ in steps.

   Step 1: Move the variable to the other side.
   Step 2: Get $x$ by itself.
   Step 3: Evaluate $x$.

   **Answers:** a) multiplied; b) $8 = 2x$, so $x = 8 \div 2 = 4$

9. Solve $\frac{8}{x} = 2$ by first multiplying both sides of the equation by $x$. Then solve the new equation. Make sure you get the same answer you did in Problem Bank 8. If you did not, find your mistake.

   **Answer:** $8x/x = 2x$, so $8 = 2x$ and $8 \div 2 = x$, so $4 = x$

10. Solve the equation.

   a) $\frac{40}{x} = 5$  
   b) $\frac{0.8}{x} = 4$  
   c) $\frac{5.4}{x} = 9$

   **Answers:** a) $x = 8$, b) $x = 0.2$, c) $x = 0.6$
In this unit, students will determine the relationships between the height, the area of the base, and the volume of right prisms and connect volume to capacity. They will develop and use the formulas for finding the volume of right prisms. Students will also determine surface area of right prisms and solve problems involving volume and surface area of prisms.

Materials: 3-D Shapes
For this unit you will need a variety of 3-D shapes, particularly right and skew prisms. If you do not have a commercial set, collect and use boxes. For example, some chocolate boxes, such as Toblerone boxes, are triangular or hexagonal prisms. You can turn a standard milk carton into a pentagonal prism by cutting off the strip on the top and covering the triangular openings at the top with paper.

You can also have students create a variety of 3-D shapes from the nets on BLM Nets of 3-D Shapes (pp M-21–M-38). Nets are available for the following shapes:

Right prisms with the following bases:
1. Square
2. Regular pentagon
3. Rectangle
4. Regular hexagon
5. Obtuse scalene triangle
6. Right scalene triangles
7. Isosceles trapezoid
8. Parallelogram
9. Irregular hexagon
Skew prisms with the following bases:
10. Parallelogram
11. Rectangle
12. Square
13. Equilateral triangle
Other shapes:
14. Skew rectangular pyramid
15. Right hexagonal pyramid
16. Square antiprism
17. Truncated square pyramid
18. Dodecahedron

Capacity vs. Volume
Volume is the amount of space taken up by a three-dimensional object and capacity is defined as how much a container can hold. One way to distinguish volume from capacity (at least at this level), is to look at the units in which they are measured: volume is measured in linear units cubed—centimetres cubed (cm³), metres cubed (m³), kilometres cubed (km³)—while capacity is measured in millilitres or litres (mL or L).

Meeting Your Curriculum
This unit is for students in Ontario only. Students following the WNCP curriculum will learn this material in Grade 8.
### Goals

Students will find the volume of rectangular prisms.

### Prior Knowledge Required

- Understands the concepts of area and perimeter
- Can find the area of a rectangle
- Knows the formula for the area of a rectangle
- Can multiply or divide decimals
- Knows different units of measurement for length and area

### Vocabulary

- area
- perimeter
- volume
- length
- width
- height
- rectangular prism

#### Units of length, area, and volume.

Review with students the various units used to measure length and area. Point out that one-dimensional objects, like strings and lines, have only one dimension, length, which we measure in centimetres, metres, kilometres, and so on. Objects that have area are two-dimensional; they have length and width, and we measure the area with square units, such as $m^2$ (where the raised 2 reminds us that they are two-dimensional). Remind students how the square metres show up in the calculation of area: $1 \times 1 = 1 \text{ m}^2$. Explain that objects that have length, width, and height are three-dimensional, and we measure them in cubic units, such as $cm^3$. Show students a centimetre cube and point out that its sides are all 1 cm long. Ask students what other measurement units for volume they know and where they have seen them (EXAMPLE: cubic metres; used to measure large volumes, like that of a swimming pool).

Remind students that the third dimension in 3-D figures is called height. Identify the length, width, and height in the prism at left. Students can review the formula for the volume of a rectangular prism using Questions 1 and 2 on Workbook page 42.

Use the terms length, width, and height to label the multiplication statement that gives the volume of the prism at left:

$$3 \text{ cm} \times 2 \text{ cm} \times 4 \text{ cm} = 24 \text{ cm}^3$$

**length width height**

**Ask:** What does the raised three mean? (each side is measured in centimetres, and three sides were multiplied to get the area)

Draw several prisms on the board, mark the height, width, and length (you can use different units for different prisms), and ask students to find the volume. **Sample Problems:**

a) $10 \text{ cm} \times 6 \text{ cm} \times 2 \text{ cm}$

b) $2 \text{ m} \times 3.4 \text{ m} \times 5 \text{ m}$

c) $3 \text{ km} \times 4.6 \text{ km} \times 7.2 \text{ km}$

**Answers:** a) $120 \text{ cm}^3$  
b) $34 \text{ m}^3$  
c) $99.36 \text{ km}^3$

Remind students to include the right units in their answers. Also, remind students that they need to use the same units for each dimension—metres cannot be multiplied by centimetres.
Goals
Students will identify and sketch right prisms.

PRIOR KNOWLEDGE REQUIRED
Can find the volume of a rectangular prism
Can identify right prisms
Can multiply or divide decimals
Is familiar with cubic units of measurement
Can identify right angles

MATERIALS
a variety of 3-D shapes, including right and skew prisms (details below)
modelling clay and toothpicks

Review prisms. Prisms have two identical (congruent) polygonal faces called bases and side faces that are parallelograms. Students might be familiar only with right prisms, whose side faces are rectangles. Present several 3-D shapes one at a time and have students tell whether each shape is a prism or not. To assess students at a glance, you can ask them to answer “yes” and “no” using their thumbs (thumbs up = yes, thumbs down = no) or American Sign Language (shake your fist up and down = yes, touch the thumb with the pointer and the middle finger together = no). Then invite volunteers to place all the prisms base down.

Right prisms and skew prisms. A skeleton of a prism is a model that has only edges and vertices, no faces. Show students how you can make a skeleton of a prism from modelling clay and toothpicks: make two copies of a base, add vertical edges to one of the bases, and attach the other base on top. Place two copies of a skeleton face down on a table and shift the top base of one of them so that the whole prism is tilted to the side. Explain that the prism you have created is called a skew prism. The original prism is called a right prism. Ask students to identify the shape of the side faces of both prisms. (rectangles for right prism, parallelograms for skew prisms). Point out that a rectangle is a special type of parallelogram. Display several prisms (as below) and have students sort them into right prisms and skew prisms as a class. Add several shapes that are not prisms and have students explain why these are not prisms. To help them determine whether or not a shape is a prism, encourage students to turn the shapes around and try different sides as bases. Remind them also that they are looking for a top face that is identical to and sits directly above the bottom face (for a right prism) and a top face that is identical to but shifted from the bottom face (for a skew prism). NOTE: For prisms that have all faces in the shape of a parallelogram, any pair of faces can be taken as bases. Such prisms have 6 faces. Do not include prisms with 6 faces, some of them...
parallellograms and some of them rectangles, in this activity (you will deal with them later).

**Right prisms** – top face directly above the bottom face. Side edges are vertical.

**Skew prisms** – top face shifted from the bottom face. Side edges are not vertical. Side faces are parallelograms.

**Not prisms** – side faces are not parallelograms.

**The angle between the base and the side faces.** Sketch two lines, one vertical and the other horizontal. **ASK:** What is the angle between the lines? (90°, right angle) Place a tall box (such as a large cereal box) on your desk. **SAY:** The sides of the box are vertical and the desk is horizontal. What is the angle between the sides of the box and the desk? (90°, right angle) Hold up prisms one at a time and invite volunteers to stand them on their bases next to the cereal box. Again, exclude prisms with 6 faces, some faces rectangles and some faces parallelograms, such as from **BLM Nets of 3-D Shapes (8, 11).** Prisms with all faces that are parallelograms or all faces that are rectangles can be included. **ASK:** Can you place the prism so that it touches the side of the box with a whole side face? Can you do it for all the side faces? Have volunteers check all the side faces. As a class, sort the prisms into two groups: those that have at least one side face that doesn’t sit flush against the box, and those for which any side face can be placed flush against the box. **ASK:** How would you describe the shapes in each of the two groups? (right prisms and skew prisms) What is the angle between the side faces and the base for right prisms? (a right angle)

**Prisms that have all faces in a shape of a parallelogram.** Present a prism that has 6 faces, all of them parallelograms, e.g., from **BLM Nets of 3-D Shapes (10).** Ask students to identify the shape of the faces. (parallelogram) Remind students that rectangles and squares are special parallelograms. Explain that if a prism has all faces that are parallelograms, you can try different faces as bases. To decide whether this is a right prism, look for a pair of bases such that all side faces can be placed flush against the box. Have students use this method to decide which of the four prisms from **BLM Nets of 3-D Shapes (3, 8, 10, 11)** are right prisms. Students will find that Prism 8 has one pair of bases (the parallelogram faces) that allows all side faces to be placed flush against the box, so it is a right prism. Prism 3 is a right rectangular prism, because any pair of opposite faces is directly above each other, and any pair of opposite faces works as bases.
Prism 10, a parallelepiped with three different pairs of parallelogram faces, does not have a pair of faces that can be placed directly above each other, and no side face will sit flush against the box. Prism 11, a skew rectangular prism, has two faces that are rectangles. Regardless of the choice of bases, no face can be placed flush against the box. Therefore this prism is not a right prism. Indeed, no pair of its faces can be placed directly above each other. Summarize that the two prisms with 4 or 6 faces that are rectangles are right prisms, while the two prisms with 2 or 0 faces that are rectangles are not. Present as a general rule that if at least four of the faces are rectangles, you can take the remaining two faces to be bases and conclude that the prism is a right prism.

Hidden edges. Draw a picture of a cube using dashed lines for the hidden edges. Explain to students that the edges on the back of the shape—the edges we can’t see—are often drawn using dashed lines. The dashed lines and the solid lines might intersect in a picture, but if the point of intersection is not a vertex, there is no real intersection between the edges there.

Sketching cubes. Show your students how they can draw a picture of a cube.

Step 1: Draw a square that will become the front face.

Step 2: Draw another square of the same size, so that the centre of the first square is a vertex of the second square.

Step 3: Join the vertices with lines as shown.

Step 4: Erase parts of the lines that represent hidden edges, to make them dashed lines.

Struggling students will find it useful to draw cubes on dot paper or grid paper.

Sketching right rectangular prisms. Sketch the two diagrams below on the board and ASK: What is the difference between these two shapes and the previous cube? (the length or the width of the shape—the dimension perpendicular to the front face) How is Step 2 performed differently in each drawing? (the vertex of the back face is not at the centre of the front face—it is closer to the respective vertex of the front face in the thinner shape and further from that vertex in the longer shape)
ASK: What would you do in Step 2 to draw a very long rectangular prism? Explain that the vertex of the back face should not sit on the diagonal, so that the edges do not overlap.

![Diagram showing the difference between sitting on the diagonal and not](image)

Add dimensions to one of the prisms you sketched, as shown. ASK: Could these be the dimensions of this prism? What is wrong? Have a student rearrange the dimensions to fit the picture better. Then have students find the volume of the prism.

![Example prism dimensions](image)

Ask students to sketch a rectangular prism, add some dimensions, and swap their sketch with a partner. Have them find the volume of the prisms. Then ask students to sketch a rectangular prism that has a volume of 300 cm³ and compare their answers with a partner. ASK: Did you draw the same prism? Can you draw a different prism with the same volume?

**Sketching other right prisms.** Have students use the same method to draw other prisms, such as triangular or pentagonal prisms.

Show students how to draw a prism standing on a base rather than on one of the rectangular faces. To see the distortion of the base in this position, suggest that your students hold a paper polygon horizontally, slightly below eye level. They should see that the polygon in the base appears shorter and wider from this angle than when you look at it head on. To draw a prism standing on a base, draw the base wider than it is when viewed head on, then draw the second (top) base directly above the bottom base and join the vertices to produce the side faces.

**Extension**

**Sketch a skew prism.** Sketch the first (bottom) base as when drawing a right prism, but sketch the second (top) base so that it is not directly above the first base. Also, add a line perpendicular to the base to show that there is an angle. **EXAMPLE:**
Volume of Triangular Prisms

Volume of Polygonal Prisms

Goals

Students will find the volume and capacity of right prisms.

PRIOR KNOWLEDGE REQUIRED

Can find volume of a rectangular prism
Can identify right prisms
Can identify the base of a prism
Can multiply or divide decimals
Is familiar with cubic units of measurement
Knows the formulas for the areas of a triangle and parallelogram and the connection between them

MATERIALS

prisms made from BLM Nets of 3-D Shapes (6) (p M-26)
details below
boxes of different shapes (not only rectangular)
small liquid measuring cups
centicubes

Review finding the area of parallelograms (by converting them into rectangles) and the area of triangles (by splitting them into two right triangles). Keep the diagrams you draw to illustrate both processes posted for the duration of the lesson.

Volume of rectangular prisms = area of base × height of prism. Draw a rectangular prism on the board, mark the dimensions (say, 2 cm, 3 cm, and 4 cm) and have students find the volume of the prism. Invite volunteers to write the volume of the prism in terms of the area of one of the faces and height, length, or width. Remind students that in the case of a rectangular prism, any pair of faces can be the bases of the prism. The dimension perpendicular to whichever pair of faces are chosen as bases is called height. So if we take, say, the bottom face to be the base, we can rewrite the formula for the volume of the prism as “height of prism × area of base of prism.”

Volume of triangular prisms with a right triangle in the base. Ask students to think about how they could calculate the volume of triangular prisms. The illustrate the process using concrete materials. Ahead of time, photocopy BLM Nets of 3-D Shapes (6) onto two pages of different colours and make two copies of each prism, say, green and blue. Show students two identical prisms (one green, on blue) side by side, and then show how together they make a rectangular prism. What fraction of the volume of the rectangular prism does each triangular prism make? (half)

Have students work through Questions 1–3 on Workbook page 46.

Volume of triangular prisms with a scalene triangle in the base. Join the congruent faces (numbered 1) of the two green prisms together so
that they make a single triangular prism with a scalene obtuse base and show that prism to students. Repeat with the blue prisms. Place the prisms side by side to emphasize that they are identical. Do they have the same volume? (yes) Show how you can make a rectangular prism by splitting the blue prism in two and attaching the smaller blue prisms to the green one (attach the blue face 2 to the green face 2, and the blue face 3 to the green face 3, making a single rectangular blue-and-green prism). What fraction of the rectangular prism is the green prism? (half) Summarize: the volume of a triangular prism with any triangle in the base is half the volume of the rectangular prism with the same height and a base that is twice as large as the base of the triangular prism.

Volume of prisms with a parallelogram in the base. Draw a parallelogram on the board and review with students how they can convert a parallelogram to a rectangle with the same area by cutting off a triangle and shifting it to the other side. Remind students that the area of the parallelogram is \( \text{base} \times \text{height} \). Point out that the word “base” has a different meaning here—the base of a parallelogram is the length of the side to which we draw a perpendicular (but the base of a prism is a face, which can itself be a parallelogram).

Display the two green and two blue prisms again, combining them into a green and a blue prism with a scalene triangle in the base as above, but this time place the prisms so that they stand on the base. Show how you can make a prism with a parallelogram in the base (join face 3 of the combined blue prism to face 3 of the combined green prism in such a way as to make a parallelogram-based prism). Ask students to identify the base and the height of the parallelogram in the base of the prism. (Again, emphasize that the word base refers to two different things here—the length of the side of a parallelogram and the parallelogram itself, which is the base of the prism.)

ASK: How could we convert this parallelogram-based prism into a rectangular prism? Take suggestions, then shift the small blue prism to the other side of the combined blue-and-green prism, so that face 2 is joined to face 2 to obtain a rectangular prism as before.
ASK: What is the length of the new prism? (the base of the parallelogram)
What is its width? (the same as the height of the parallelogram in the base)
What is its height? (the same as the height of the parallelogram-based prism)
What is the volume of this prism? (base of parallelogram × height of parallelogram × height of prism)

What do the first two terms in the product combine to? (area of parallelogram)
Ask students to rewrite the formula for the volume of the parallelogram-based prism using the area of the base of the prism. (area of base × height of prism)

Ask your students to find the volume of a prism with height 7 cm, and a parallelogram in the base that has base 5 cm and height 4 cm. Repeat with more prisms.

For triangular prisms, volume = area of base × height of prism.

Explain that now, when you have a nice formula for a volume of a prism with a parallelogram in the base (area of base × height of prism), you would like to go back to the volume of a triangular prism, to see whether a similar formula would work there. Draw a triangular prism inside a rectangular prism, as shown at left, and ask students what the volume of the triangular prism should be. (half the volume of the rectangular prism)
The volume of the rectangular prism is area of the base × height of the prism. Let’s choose the top face of the rectangular prism to be the base, so that it contains the base of the triangular prism, and both prisms have the same height. So the volume of the triangular prism is:

\[
\frac{1}{2} \text{ of volume of rectangular prism}
\]

\[
= \frac{1}{2} \text{ of area of the base (rectangle) } \times \text{ height of prism}
\]

\[
= \frac{1}{2} \text{ area of rectangle } \times \text{ height of prism}
\]

\[
= \text{ area of triangle } \times \text{ height of prism}
\]

\[
= \text{ area of base of triangular prism } \times \text{ height of prism}
\]

This means that the same formula (area of base × height of prism = volume of prism) works for triangular prisms as well.

EXTRA PRACTICE: Find the volume of these prisms.

Ask students to find the volume of a prism with height 7 cm, and a parallelogram in the base that has base 5 cm and height 4 cm. Repeat with more prisms.
Any polygon can be decomposed into triangles. Ask students to show how to do so for each of the following polygons.

Then have students draw two different polygons and show how to cut them into triangles.

Explain to the students that if they have a prism with a polygon in the base, they can decompose the base into triangles, find the areas of the triangles and add them to get the area of the base of the prism. **ASK:** How can you find the volume of the prism? (by multiplying the area of the base and the height of the prism) Point out that this method allows the students to find the volume of any right prism.

**Capacity.** Explain that the capacity of a container is how much it can hold. Write the term on the board.

Remind students that capacity is measured in litres (L) and millilitres (mL). **ASK:** Where have you seen the prefix “milli” before and what did it mean? (millimetre; one thousandth) How many millilitres are in 1 litre? In 2 litres? In 7 litres? What do you do to change litres to millilitres? (multiply by 1 000)

Write on the board:

\[
1 \text{ metre} = 1 000 \text{ millimetres} \quad 1 \text{ m} = 1 000 \text{ mm} \\
1 \text{ litre} = 1 000 \text{ millilitres} \quad 1 \text{ L} = 1 000 \text{ mL}
\]

Ask students to think of three quantities that are measured in litres and three that are measured in millilitres.

**Connection between mL and cm³.** Hold up a see-through measuring cup with some water in it. Drop a centicube into the cup. Ask your students if they can see how much liquid is displaced by the cube (no, the amount is too small). What can you do to find how much water is displaced by one cube? (One answer: drop 10 cubes in and divide the displacement by 10.) If possible, have all students drop centicubes into measuring cups and measure the displacement. What is the capacity of a cube with volume 1 cm³? (1 mL)

Present a small rectangular box and ask your students how they could measure its capacity. **ASK:** We know the capacity of 1 cm³. What is the capacity of 10 cm³? Of 20 cm³? Invite volunteers to measure the sides of the box and calculate its volume. What is the capacity of the box?

Draw a cube on the board. **SAY:** A cube has a capacity of 1 L. What are the dimensions of the cube? How many millilitres are in 1 L? How do you find the volume of the cube? (You multiply the side by itself 3 times.) Which number is multiplied by itself 3 times to get 1 000? (10) So how long is the side of the cube? (10 cm) Mark the cube sides as 10 cm, and invite volunteers to write both the volume of the cube and the capacity (in mL and L) beside the cube. Remind students that 10 cm = 1 dm, so 1 dm² = 1 L.
Finding capacity of prisms. Draw a box on the board and write its dimensions: 30 cm × 40 cm × 50 cm. **ASK:** What is the capacity of the box? Let your students find the capacity in millilitres first, then ask them to convert it to litres.

Ask students if they can solve the problem another way. (They can convert the dimensions to decimetres and get the result in litres: $3 \times 4 \times 5 = 60$ L.) Did they get the same answer?

Give your students more problems in which they have to find the volume and capacity of a prism. Include some triangular and polygonal prisms.

**EXAMPLE:** Find the volume and the capacity (in L, rounded to 1 decimal place) of a prism with height 12 cm and base as shown.

**SOLUTION:** Split the pentagon into a trapezoid (with bases 33.2 and 21.2 cm and height 20 cm), and a triangle (with base 33.2 cm and height $34.9 - 20 = 14.9$ cm). The area of the trapezoid is 544 cm$^2$, and the area of the triangle will be 247.34 cm$^2$. The area of the base of the prism is then 791.34 cm$^2$, and the volume is 9496.08 cm$^3$. The capacity is then about 9.5 L.

You can also give students boxes in the shape of different prisms (such as chocolate boxes) and have them find the volume and the capacity by taking the necessary measurements themselves.

**EXTRA PRACTICE:**
1. Find the volume and the capacity of the rectangular prisms:
   a) $1 \text{ m} \times 1 \text{ km} \times 1 \text{ m}$  
   b) $5 \text{ cm} \times 0.3 \text{ m} \times 2 \text{ m}$  
   c) $1 \text{ mm} \times 1 \text{ m} \times 1 \text{ km}$

**ANSWERS:**
   a) Volume 1 000 m$^3$, capacity 1 000 000 L  
   b) Volume 0.03 m$^3 = 30 000$ cm$^3$, capacity 30 L  
   c) Volume 1 m$^3$, capacity 1 000 L

2. a) Valerie says that a triangular prism has a volume that is half the volume of a rectangular prism of the same height. Does her statement apply to the prism at left? Explain. Draw a prism that fits her statement.
   b) Find the volume of the triangular prism. (15 cm$^3$)

3. Daniela wants to find the volume of an apple. She puts the apple into a glass box with 600 mL of water. The box has a square base of 10 cm × 10 cm. The water reaches a height of 9.8 cm. What is the volume of the apple? (The volume of the water and apple together is 10 cm × 10 cm × 9.8 cm = 980 cm$^3$, and the volume of the water only is 600 cm$^3$, so the apple has volume of 380 cm$^3$)
Extension

A wealthy king had a treasure chest in the shape of a rectangular prism. He ordered his carpenters to create a larger chest for his treasure.

a) The first carpenter doubled the length of the box and left the width and the height the same. The second carpenter doubled the width of the box and left the length and the height the same. The third carpenter doubled the height of the box and left the length and the width the same. Who made the largest chest for the king’s treasure?

b) The fourth carpenter doubled the length, the width, and the height of the king’s old treasure chest to create his new chest. How many times larger was the new chest than the old one?

c) A fifth carpenter, being jealous of the money the fourth carpenter got, decides to make a chest that has the same volume as the chest of the fourth carpenter. His chest will have the same height but two times the length of the old chest. How many times wider than the old king’s chest must this carpenter make his chest?
ME7-26 Surface Area of Rectangular Prisms

Goals

Students will find the surface area of right rectangular prisms.

Prior Knowledge Required

Can find the area of a rectangle using the correct formula
Can perform basic operations with decimals
Is familiar with square units of measurement

Materials

empty boxes

Introduce surface area. Review area for rectangles, parallelograms, and triangles. Tell students that the surface area of a prism is the sum of the areas of all faces of the shape. Ask your students when they might need to know the surface area of a prism. (Example: to calculate the amount of paper needed to wrap a present)

Find the surface area of a rectangular prism by adding the areas of all faces. Present a rectangular prism. Invite volunteers to measure the sides of the prism. Ask your students to draw the faces of the prism and to mark the dimensions of each face. Have students check that they drew all the faces. How many should be there? (6) Ask students to write a multiplication statement for the area of each face and to add the results for all the faces. Point out to students that because they are measuring area, the measurement units are \( \text{cm}^2 \) and not \( \text{cm}^3 \), even though this is a 3-D shape.

Identify identical faces and use multiplication to find the surface area. Draw a cube on the board and mark the faces as top and bottom, right and left, and front and back. Ask students to name pairs of opposite faces. Ask: Which faces are the same as other faces? Which faces come in pairs? How can we use this to shorten the calculation of the surface area? Have students find the areas of the top, front, and right side faces for several rectangular prisms, and then find the surface area by doubling the area of these three faces first. (Examples: \( 3 \text{ cm} \times 4 \text{ cm} \times 5 \text{ cm}, 4.5 \text{ m} \times 2 \text{ m} \times 3 \text{ m}, 2.3 \text{ km} \times 1.2 \text{ km} \times 4 \text{ km} \)) Students can also measure some empty boxes and find their surface areas.

Bonus: Find the surface area of the prism at left using as few calculations as you can.

Extensions

1. A wealthy King has a treasure chest in the shape of a rectangular prism 30 cm wide, 40 cm long, and 25 cm high. He ordered his carpenters to make a chest that can hold twice as much treasure.
The first carpenter doubled the length of the box and left the width and the height the same. The second carpenter doubled the width of the box and left the length and the height the same. The third carpenter doubled the height of the box and left the length and the width the same. Whose box used the least amount of wood, and so was the least expensive?

**ANSWERS:**

1st carpenter: $2 \times (30 \times 80 + 30 \times 25 + 25 \times 80) = 10 300 \text{ cm}^2$

2nd carpenter: $2 \times (60 \times 40 + 60 \times 25 + 25 \times 40) = 9 800 \text{ cm}^2$

3rd carpenter: $2 \times (30 \times 40 + 30 \times 50 + 50 \times 40) = 9 400 \text{ cm}^2$

The 3rd carpenter made the least expensive chest.

**CHALLENGE:** The three carpenters did not get the job of making the chest for King’s treasure. He ordered the chest from a fourth carpenter, who suggested a chest with a volume that was 101.4% of the box that the king wanted but cost less than the chests of the other carpenters. The fourth carpenter’s chest had a square at the base and was 40 cm high. What were the dimensions of this carpenter’s chest? What was its surface area? Compare the surface area of this chest to that of the best of the three carpenters’ chests.

**ANSWER:**

Volume: 101.4% of $(30 \times 25 \times 40 \times 2) \text{ cm}^3 = 60 840 \text{ cm}^3$.

The height is 40 cm, so the base should be $60 840 \div 40 = 1 521 \text{ cm}^2$, and $1 521 = 39^2$, so the box is $39 \text{ cm} \times 39 \text{ cm} \times 40 \text{ cm}$.

The surface area is $2 \times (39 \times 39 + 39 \times 40 + 39 \times 40) = 9 282 \text{ cm}^2$, 118 cm$^2$ smaller than the box of the 3rd carpenter.

**CONNECTION**

Real world

2. a) One litre of paint covers 7 m$^2$. How much paint would someone need to paint a wall that is 6 m by 3 m? What if the wall has a door that is 2 m high and 80 cm wide? What if the wall also has a window that is a 1 m by 1 m?

b) A room has a closet that is 1 m deep, 2 m wide, and 2.5 m high. The closet has a door 2 m high and 80 cm wide. I want to paint the sides and ceiling of the closet, but not the floor or a door. How much paint do I need?

c) Choose a room in your school or at home. Calculate the amount of paint needed to repaint the room. Consider all surfaces and fixtures, such as doors, windows, closets, electrical outlets, built-in shelves, or ledges. What will you paint and what does not need to be painted? Do you want to use more than one colour? Will you need more than one coat of paint (you will if you are using a dark colour, or painting over a dark colour)?
Identify 3-D shapes from their faces. Draw several shapes (that together are faces of a right prism) on the board and ask your students which 3-D shape they make. If students have trouble identifying the 3-D shape, ask them to circle the base(s) first. Repeat several times.

**EXAMPLES:**

![Diagram of shapes](image)

Making nets for prisms. Hold up a triangular or a pentagonal prism. Ask students which two types of faces it has. How many bases does it have? What is the shape of the side faces? How many side faces does it have? If you wanted to make a net for this prism, the easiest way would be to start with the band of rectangles for side faces (created by rolling a prism and tracing the side faces in turn) and to add the bases. Illustrate this on the board:

![Diagram of net](image)

Another way to make a net for this prism is to start with a base (draw it on the board and write “base” on it), add a side face along each edge of the
base (draw one side face and ask volunteers to draw the rest), and to finish with the second base. Model this method on the board as well:

Various nets for the same prism. This would be a good time to do the Activity below. Afterwards, draw several incorrect examples of nets of triangular prism on the board and ask volunteers to explain why these drawings cannot serve as nets for prisms:

The bases are not the same
The middle face is too short
The bottom base is flipped
A side face is missing

Have students work in pairs, with one student drawing a picture that will not work as a prism net and the other student explaining why the drawing cannot be a prism net. Have students switch roles several times. Then have each pair choose two of their favourite “nets” and swap them with another pair of students. Each group of four can then present one of their drawings and explain why it cannot be a prism net. A more challenging task would be to have students draw pictures that might or might not work as nets, and let the class predict if they are prism nets (students can use thumbs up and thumbs down to show their answer). After each vote, ask students to sketch the net on a sheet of paper. When the voting is done, have different students redraw the nets to scale, cut them out, and fold them to check the prediction.

EXTRA PRACTICE:
How many of each type of face would you need to make this prism?

Sketching nets for prisms with given dimensions. Draw several prisms on the board and write the dimensions beside them. Ask students to sketch the nets for the prisms. Then ask students to mark on the nets which face is which (top, bottom, right, etc.) One by one, go through the faces on the net and have students identify their dimensions. Do the first example as a class and have students do the rest on their own, but present answers on the board. Leave the pictures on the board for future use.
Finding volume and surface area of prisms using nets. Review with students how to find the volume and surface area of prisms and have students find the surface area and the volume for the prisms they sketched nets for above. Then present more sketches of prisms with dimensions and have students find the volume and the surface area of the prisms.

EXAMPLES:

```
2 m
3 m
1 m

1.5 cm
2.3 cm
3.4 cm

5.4 cm
2.4 cm
6.5 cm
7 cm
2 cm
```

ACTIVITY

Give students prisms with faces that are not regular polygons (you can ask students to make prisms from modeling clay first or use nets from BLM Nets of 3-D Shapes (1-9)) and ask them to trace the faces on a piece of paper to create a net. Ask students to cut out the nets they have drawn. Let them cut off faces of the net (one at a time) and re-attach the faces at different places. Will the new net fold into the same prism? Which edges are places where you would want to re-attach the faces and which are not? Students should work with at least two different prisms each. Students will thus explore various ways to create nets for the same solid, rather than memorize a single net shape.

Extension

Have students complete BLM A Net or Not a Net? (p M-39). Students can use tape to join the faces. This exercise emphasizes the fact that it’s very hard to correctly identify nets for skew prisms just by looking at them. You have to cut them out and check! ANSWERS:

```
skew square prism  skew triangular prism  (no)  (no)
```
**ME7-28 Volume and Surface Area**

**Pages 56–57**

**CURRICULUM EXPECTATIONS**
Ontario: 6m42; 7m7, 7m41, 7m42
WNCP: optional, [C, V]; 8SS3, 8SS4

**VOCABULARY**
- volume
- length
- width
- height
- right prism
- parallelogram
- triangle
- triangular prism
- rectangular prism

**Goals**
Students will find the volume and capacity of right prisms.

**PRIOR KNOWLEDGE REQUIRED**
- Can find volume of a rectangular prism
- Can identify right prisms
- Can identify the base of a prism
- Can multiply or divide decimals
- Is familiar with cubic units of measurement
- Knows the formulas for the areas of triangle and parallelogram and the connection between them

**MATERIALS**
- various prisms
- shapes from BLM Nets of 3-D Shapes (1-9) (pp M-21–M-29)

**Review some relevant prior knowledge.** Review the formulas for the areas of triangles, rectangles, and parallelograms. Review finding missing dimensions of shapes when given the area. Review how to find the volume of right prisms (volume = area of the base × height) and the surface area (add areas of all faces; nets can help you keep track of faces). Remind students that writing the units at every step of the calculation helps both to avoid multiplying centimetres by metres and to make sure that you multiply two dimensions for area and three dimensions for volume.

**Finding the volume of rectangular prisms when some of the dimensions are not known.** Present some problems like those in Question 5 on Workbook page 56. Have students identify the missing linear dimension, then find the volume and the surface area of the prisms. EXAMPLES:

1. Missing length = 4 cm
   - Volume = 12 cm$^3$

   Surface area = $2 \times (2 \text{ cm} \times 4 \text{ cm} + 1.5 \text{ cm} \times 4 \text{ cm} + 1.5 \text{ cm} \times 2 \text{ cm})$
   = $2 \times (8 \text{ cm}^2 + 6 \text{ cm}^2 + 3 \text{ cm}^2) = 34 \text{ cm}^2$

2. Missing length = 3.5 m
   - Volume = 14 m$^2 \times 2.1 \text{ m} = 29.4 \text{ m}^3$

   Surface area = $2 \times (2.1 \text{ m} \times 4 \text{ m} + 14 \text{ m}^2 + 3.5 \text{ m} \times 2.1 \text{ m})$
   = $2 \times (8.4 \text{ m}^2 + 14 \text{ m}^2 + 7.35 \text{ m}^2) = 59.5 \text{ m}^2$

**PROCESS ASSESSMENT**
- 7m4, [V]
- Workbook Questions 2, 3

Teacher’s Guide for Workbook 7.2
**Word problems.** Work through the following problems as a class.

1. Pam has a box with a hexagonal base and lid and rectangular sides. The box is 40 cm tall and its base (which is the same as the lid) is shown at left.
   
   a) Sketch the box.
   
   b) Sketch a net for the box and mark the dimensions on the faces.
   
   c) Find the surface area of the box.
   
   d) Pam needs 5 mL of paint for each 100 cm² of area. How much paint will she need to paint the outside of the box?

   **ANSWERS:**
   
   c) The surface area of the box is 5248 cm².
   
   d) 262.4 mL of paint

2. A box without a lid has volume 40 000 cm³. It is 40 cm wide and 50 cm long.
   
   a) What is the height of the box?
   
   b) What are the dimensions of the lid (the missing face)?
   
   c) What is the box’s surface area? (Remember, there is no lid.)
   
   d) The material for the box costs $12.35 per m². How much will the box cost?

   **ANSWERS:**
   
   a) 20 cm
   
   b) 40 cm × 50 cm = 2 000 cm²
   
   c) 40 cm × 50 cm + 2 × (40 cm × 20 cm + 50 cm × 20 cm) = 3 800 cm²
   
   d) 3 800 cm² = 0.38 m² so the cost for the box will be 0.38 × $12.35 which rounds to $4.69.

**Extensions**

1. a) Find the surface area and volume of each right rectangular prism.

   i) 
   
   ![](image)

   Surface area: _______  Volume: _______

   ii) 
   
   ![](image)

   Surface area: _______  Volume: _______

   iii) 
   
   ![](image)

   Surface area: _______  Volume: _______

   b) How does the surface area of a right rectangular prism change when each side length is multiplied by the same amount? How does the volume of a right rectangular prism change when each side length is multiplied by the same amount?

2. Find the surface area and volume of the tissue box shown below. Remember to leave out the opening on the top.
3. You are designing a cereal box for a cereal company. The box needs to have a volume of 2000 cm³. There are many possible boxes you could make with this volume.

a) Verify that these 3 sets of measurements for boxes each have a volume of 2000 cm³.
   
   A: 1 × 1 × 2000  
   B: 2 × 25 × 40  
   C: 5 × 25 × 16

b) Calculate the surface area of each box in a).

c) If the material to make the box costs 25¢ per cm², which box from a) would you recommend?

d) Find three more boxes with the same volume and calculate the surface area of each. Now which box would you recommend?

e) The cereal company wants the front of the box to be at least 20 cm wide and 20 cm high and the depth of the box to be at least 4 cm. Find two boxes satisfying these conditions that each have a volume of 2000 cm³. Which box would you recommend?
Nets of 3-D Shapes (1)
Nets of 3-D Shapes (4)
Nets of 3-D Shapes (6)
Nets of 3-D Shapes (8)
Nets of 3-D Shapes (9)
Nets of 3-D Shapes (10)
Nets of 3-D Shapes (11)
Nets of 3-D Shapes (12)
Nets of 3-D Shapes (13)
Nets of 3-D Shapes (14)
Nets of 3-D Shapes (15)
Nets of 3-D Shapes (17)
Nets of 3-D Shapes (18)
A Net or Not a Net?

Does this picture make a net of a 3-D shape? If yes, what shape? Predict, then cut out the net and fold it to check your prediction.
In this unit, students will review how to draw, read, and interpret bar graphs, double bar graphs, line graphs, and frequency tables. Students will also learn how to draw, read, and interpret relative frequency tables and circle graphs. Students will distinguish between when to use which type of graph.

**Curriculum Differences**
Students in Ontario should cover the entire unit. Students following the WNCP curriculum should cover PDM7-9 to PDM7-13.

**Online Tools**
The following websites are useful sources of data and graphs.

- [www.statcan.gc.ca/kits-trousses/courses-cours/edu05_0017a-eng.htm](http://www.statcan.gc.ca/kits-trousses/courses-cours/edu05_0017a-eng.htm)
  Data, articles, and lessons plans for grades 6 to 8 from Statistics Canada.

  E-Stat is a warehouse of statistics about Canada and Canadians, specially designed for use by the educational community. E-Stat is free to registered educational institutions.

- [www.censusatschool.ca](http://www.censusatschool.ca)
  Census at School is an international online project for students aged 8 to 18. Students complete a brief online survey (questions are non-confidential), analyze their class results, and compare themselves with other students in Canada and in other countries.

- [www.rom.on.ca/ontario/risk.php](http://www.rom.on.ca/ontario/risk.php)
  The Royal Ontario Museum provides accessible information about plants and animals at risk in Ontario.

- [www.cyberschoolbus.un.org](http://www.cyberschoolbus.un.org)
  An educational tool from the United Nations. Use Country at a Glance to find physical and population data for UN Member States and InfoNation to produce graphs that compare data for different countries.

  The CIA World Factbook provides information about countries around the world.

- [www.nhlpa.com](http://www.nhlpa.com)
  The National Hockey League Players’ Association has more than enough data for any conceivable student project.

- [www.ec.gc.ca](http://www.ec.gc.ca)
  Environment Canada has a wealth of statistical data about weather and the environment.
READING BAR GRAPHS.

Have students identify the basic components of a bar graph, such as labels, title, scale and bars, in the graph below (or in another graph of your choice). **ASK:** Can we tell from this bar graph how many immigrants arrived in Canada in 1992? Why not? (No; the data on the horizontal axis is grouped in five-year periods.) Point out that some years, such as 1971, appear twice on the scale. Explain that in this graph, people who entered the country in the first half of the year are included in the first bar, and those who came after July 1st of that year are included in the second bar.

**ASK:** Why do you think the data is presented over periods of five years? What would happen if the graph was made using information for each year separately? How many bars would you need to use? (five times as many bars) Explain that having so many bars would make it harder to isolate information on the graph. **ASK:** Would the scale need to go all the way up to 1400? (no, numbers on the current graph would be split into 5 groups, so the scale would very likely only have to go to 300) Explain that each interval would represent about five times fewer numbers. This would make it easier to read the graph precisely. For example, the first bar (1966–1971) looks like it could represent a number from 840 to 860, whereas if the scale was five times smaller, the number would be from 168 to 172, and we could tell much more accurately from the graph what the actual numbers are. By presenting the data over periods of five years, we are losing some accuracy, but gaining simplicity. By not showing as much information, we can see the information that we are showing much more clearly.

**Goals**

Students will read and interpret bar graphs.

**PRIOR KNOWLEDGE REQUIRED**

Can create and analyze bar graphs based on discrete data.

**VOCABULARY**

bar graph

label

scale

range

axis, axes
Tell students that Samiera found data for the number of immigrants coming to Canada for each of the years from July 1st 2001 to July 1st 2006. She wants to add this data to the bar graph used in the lesson. How many bars will she add? (one—she will need to add the figures because the graph shows five-year periods) What would Samiera do with data for 2007? (nothing—she has to wait until the data for four more years is available to add a bar to this particular graph)

**Drawing conclusions from bar graphs.** **ASK:** During which period was immigration at its highest? At its lowest? How do you know? During which period was immigration almost double what it was in the previous period? Do you need to look at the scale to answer this question? (As long as the scale starts at zero, you can compare the size of the bars without knowing the precise values, that is, without looking at the scale.) During which period was there a drop in the immigration rate of more than 200 000 from the previous period? Do you need to look at the scale to answer this question? Is it important what the range on the scale is and whether the scale starts at zero? (You need to look at the scale, but you do not need to check whether the scale starts at zero)

**Using a frequency table to turn vertical bar graphs into horizontal bar graphs.** Have students transfer the data to a frequency table. **ASK:** Are the figures you obtain from the bar graph precise or rounded? Why? (rounded; the gaps in the scale are too large to show more precise numbers) Have students create a horizontal bar graph using the frequency table.

**EXTRA PRACTICE:**
1. a) Make a bar graph to illustrate the area of the five Great Lakes. Include a title, labels, and appropriate scales.

<table>
<thead>
<tr>
<th>Great Lake</th>
<th>Superior</th>
<th>Michigan</th>
<th>Huron</th>
<th>Erie</th>
<th>Ontario</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area (km²)</td>
<td>82 000</td>
<td>57 000</td>
<td>60 000</td>
<td>26 000</td>
<td>19 000</td>
</tr>
</tbody>
</table>

   b) Make up three questions about your bar graph and answer them.

2. Sally created a table of her activities in a 24-hour day. Draw a bar graph to show her data.

<table>
<thead>
<tr>
<th>Activity</th>
<th>School</th>
<th>Chores</th>
<th>Homework</th>
<th>Sports</th>
<th>Sleeping</th>
<th>Eating</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (h)</td>
<td>7</td>
<td>1</td>
<td>1</td>
<td>1.5</td>
<td>8</td>
<td>2</td>
<td>3.5</td>
</tr>
</tbody>
</table>

After students draw the bar graphs above by hand, have them make the bar graphs using a graphing program, such as Excel. Discuss with students which way of graphing is easier: using a graphing program or drawing by hand. (Drawing by hand may be easier to learn, but once students learn how to enter the commands into the computer, drawing the graph using a graphing program is much less work.) Point out as well that using a graphing program allows you to experiment with changing different properties of the graph, such as the title, the direction of the bars (horizontal or vertical), or the scale, more easily than if you had to re-draw the graph every time you wanted to try something new.
Choosing a scale. Discuss with students whether the scale used in the graph for immigration into Canada (above) would be appropriate for the following data on emigration from Canada. (no, we need a smaller scale)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Emigration</td>
<td>427</td>
<td>358</td>
<td>278</td>
<td>278</td>
<td>213</td>
<td>338</td>
<td>376</td>
</tr>
<tr>
<td>(thousands of people)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

ASK: Which scale would you use? (Best answer: 0, 50,000, 100,000, …, 450,000—the largest value is between 400,000 and 450,000, so this scale would give 10 markings and the values will be reasonably separated) Have students explain their choices.

Present this graph showing emigration from Canada (based on the data above). ASK: Does the graph show the data faithfully? (yes) Does the scale used make it easy to read the values? Why does the graph create an impression that the emigration from Canada increased about three times from the period 1986–1991 to the period 1991–1996?

Emphasize the purpose of marking the vertical axis with a zigzag—it warns the reader that the differences on the vertical axis are exaggerated in order to make reading the data easier.

Why markings should be proportionally spaced. Show your students the bar graph at left. Why does a quick glance make it look as though the number of students with a B was three times the number of students with an A? Why is it important that each interval on the horizontal axis represent the same number of students?

Ask students to identify the incorrect markings on the horizontal axis (5 and 6), and to write those numbers in the right places. Ask students to adjust the length of the bar that reaches the incorrect marks. Then label the marking to the right of the 6 with the number 8. ASK: Is this correct? Why not? Where should the 8 be placed?

Why categories should group the data equally. Put up the following chart and SAY: Katie found some data on the sightings of whales in Bay St. Lawrence, Nova Scotia, in the summer of 2007.
**Date**
- June 1
- June 8
- June 16
- July 3
- July 4
- July 13
- July 21
- Aug 1
- Aug 20

<table>
<thead>
<tr>
<th>Number of Whales Seen</th>
<th>June 1</th>
<th>June 8</th>
<th>June 16</th>
<th>July 3</th>
<th>July 4</th>
<th>July 13</th>
<th>July 21</th>
<th>Aug 1</th>
<th>Aug 20</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>27</td>
<td>34</td>
<td>17</td>
<td>15</td>
<td>21</td>
<td>17</td>
<td>52</td>
<td>31</td>
<td>27</td>
</tr>
</tbody>
</table>

Ask students to draw a bar graph from the data in two ways:

1. Draw a bar graph for each date given and determine the bar’s height by the number of whales seen on that date.

2. Draw a bar for each month and total the number of whales seen in that month to determine the bar’s height.

Discuss which graph provides a clearer representation of the data. Drawing a bar for each month makes it appear that whales are more likely to be seen in July than in August or June, but that may not be true; there is just more data for July than for the other months (4 days vs. 2 and 3 days).

**ACTIVITIES 1–2**

1. **Groups of students (4–5) can look up the starting times for movies at a local movie theatre.** (Different groups can look up start times on different days or at different movie theatres.) Students should tally the results for each hourly interval from 12:00 p.m. to 9 p.m. (12:00–12:59, 1:00–1:59, ..., 8:00–9:00), then make a bar graph using a graphing program, such as Excel. Have students write a report on staffing at the movie theatre. During which time intervals might more staff be needed at the ticket counter, and why? What about at the snack bar, or cleaning the washrooms and theatres? Groups can present their graphs and analysis to the class.

2. **Bar graph posters.** Encourage students to look at bar graphs in books, in magazines, on the Internet, on television (e.g., the weather network) or in brochures (e.g., from financial institutions). Have students record properties that are common to all the bar graphs. Also, how many categories are used in each graph? About how many categories do most bar graphs use? How many markings are on the scale in each graph? About how many markings do most bar graphs use? Have students cut out various bar graphs and make a poster titled Bar Graphs. Keep these posters for PDM7-14.

**Extension**

Determine the values of the other bars on these graphs:

a) ![Graph A](image)

b) ![Graph B](image)

c) ![Graph C](image)
Goals

Students will read and create double bar graphs and use them to compare data.

PRIOR KNOWLEDGE REQUIRED

Can create and analyze bar graphs based on discrete data

Present two graphs of events or quantities that are comparable. The examples below describe population growth from two sources. Explain that **natural population growth** is the difference between the number of births and the number of deaths, and **growth from immigration** is the difference between the number of people who immigrate (move to a country) and the number of people who emigrate (move out of a country).

**Natural Population Growth in Canada**

<table>
<thead>
<tr>
<th>Periods</th>
<th>Numbers of People (thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1966-1971</td>
<td>1200</td>
</tr>
<tr>
<td>1971-1976</td>
<td>1000</td>
</tr>
<tr>
<td>1976-1981</td>
<td>800</td>
</tr>
<tr>
<td>1981-1986</td>
<td>600</td>
</tr>
<tr>
<td>1986-1991</td>
<td>400</td>
</tr>
<tr>
<td>1991-1996</td>
<td>200</td>
</tr>
<tr>
<td>1996-2001</td>
<td>0</td>
</tr>
</tbody>
</table>

**Population Growth from Immigration**

<table>
<thead>
<tr>
<th>Periods</th>
<th>Numbers of People (thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1966-1971</td>
<td>300</td>
</tr>
<tr>
<td>1971-1976</td>
<td>500</td>
</tr>
<tr>
<td>1976-1981</td>
<td>700</td>
</tr>
<tr>
<td>1981-1986</td>
<td>900</td>
</tr>
<tr>
<td>1986-1991</td>
<td>1100</td>
</tr>
<tr>
<td>1991-1996</td>
<td>700</td>
</tr>
<tr>
<td>1996-2001</td>
<td>500</td>
</tr>
</tbody>
</table>
Introduce double bar graphs. Explain that sometimes it is convenient to present two sets of values on the same graph, in order to compare the values. Invite volunteers to transfer both graphs onto the same axes. Point out that the original graphs have the same units on each scale. (If they didn’t, you couldn’t put them together like this.) **ASK:** Which scale should you choose for the new graph? (the scale with the larger range) The graph will look like this:

![Double Bar Graph Example](image)

**Compare the three graphs.** **ASK:** What does the double bar graph have that the other two do not? (a legend or key) During which five-year period was there the largest natural growth in population in Canada? (1966–1971) The smallest? (1996–2001) During which period was there the largest difference between immigration and emigration? (1986–1991) How can you tell from the graph? (This difference is the growth from immigration, so look for the tallest light grey bar.) Is it easier to read the data for each type of population growth from the single graph or from the double bar graph? (single bar graph) Why? (no distraction by the second set of bars)

**Discuss the relationship between the values on the double bar graph.** Emphasize that all the data from each single bar graph remains in the double bar graph—nothing has been lost—but the double bar graph allows you to compare the values more easily. If there are two components to population growth—natural growth (= births minus deaths) and growth from immigration (= immigration minus emigration)—then in which period was the total growth in population the largest? (1986–1991) How can you find the value of the total growth? (add the numbers from the two bars) Which component affected the total growth more in the first 20 years shown on the graph? (natural growth) What happened in the last five years? (growth from immigration overtook natural growth) Can we predict from the graph what the bars for 2001–2006 will look like? (Natural growth is slowly declining. Immigration rates are likely to grow or stay around the same. You might mention that the decrease in natural growth and a growing demand for more skilled workers in parts of Canada is fuelling the increase in the immigration rate.)
Have students complete the Workbook pages. For Question 2, **ASK:** What questions could you ask and answer about the graph? What other things could you compare using a double bar graph?

Tell students that Leslie tracked the height of two seedlings. Have students draw a double bar graph on a computer to show her data.

<table>
<thead>
<tr>
<th>Day</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>4.2</td>
<td>6.5</td>
<td>8.8</td>
<td>11.0</td>
<td>14.1</td>
</tr>
<tr>
<td></td>
<td>1.9</td>
<td>3.6</td>
<td>5.1</td>
<td>6.4</td>
<td>7.9</td>
<td>9.3</td>
</tr>
</tbody>
</table>

**Extensions**

1. This double bar graph shows total population growth and growth from immigration in Canada.

**ASK:** Without looking at the legend, can you tell which bars show the total growth and which show the growth from immigration? (the total growth should always be higher because the natural growth was never negative in Canada) How is this graph different from the one used during the lesson? Compare labels, axis, scale, etc. **SAY:** In the previous graph, it made sense to add the values of the bars to get the total growth in population. Does it make sense to add the values of these bars for the same period? (No; the larger bars alone show the
total growth. The smaller bars are actually parts of the larger bars, so it does not make sense to add them. If you do, some of the people will be counted twice. However, if you subtract the values, you can get the natural growth values.)

2. This stacked bar graph shows the same population data used previously.

![Stacked Bar Graph](image)

Explain that on this bar graph, the bars are stacked one on top of the other. It is easy to read the total growth and the growth from immigration, but it is much harder to read the natural growth. It is also harder to compare which type of growth is larger in any one period (you can do this by measuring the heights of each part of the bars).

3. Discuss with students how dates of birth are generally distributed in a random group of people. Write down the birthdays for the whole class and have students create a bar graph using 4 categories: January–March, April–June, July–September, and October–December. **ASK:** Is there any period where we have substantially more or fewer birthdays? (the answer is usually no)

Students can then find data about the birthdates of players on one of the teams in the Canadian Junior Hockey League and create a bar graph for this data. The data will show that there are many more players with birthdays in the January to March. For a possible explanation, see **Outliers by Malcolm Gladwell.** Gladwell claims that the reason for the anomalous spread of data is that children are divided into teams on the basis of what year they were born, and children born earlier in the year are, on average, bigger and stronger from almost an extra year of development.
Goals

Students will use line graphs to make predictions, determine misleading representations of data, and choose between drawing a bar graph and a line graph for given data.

Prior Knowledge Required

Can read and draw bar graphs

Vocabulary

line graph
scale
labels

Write the following data on the board:

<table>
<thead>
<tr>
<th>Day</th>
<th>Su</th>
<th>M</th>
<th>T</th>
<th>W</th>
<th>Th</th>
<th>F</th>
<th>Sa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature (°C)</td>
<td>26</td>
<td>32</td>
<td>22</td>
<td>16</td>
<td>13</td>
<td>15</td>
<td>13</td>
</tr>
</tbody>
</table>

Have students draw a bar graph of the data in their notebooks. While they are doing so, draw a line graph of the same data on the board. Have students discuss the similarities and differences between the two types of graphs. Tell your students that the graph you drew is called a line graph and ask why they think it is called that. Emphasize that a line graph is made by drawing points instead of bars for each data value and then joining points with a line from one to the next.

Ask:
Which day was the warmest? Which day was the coolest? Which two days are the same temperature? How does the bar graph show this? How does the line graph show this? Does the temperature seem to be increasing over the week or decreasing? How does the line graph show this? If the temperature is increasing, what direction does the line between the points go in—from bottom left to top right or top left to bottom right? What direction does the line between the points go in if the temperature is decreasing? Does the temperature increase more or decrease more over the course of the week? How is the line graph good for showing that?

Have students draw a line graph in their notebooks for the following set of data (suggest the scale: 0, 20, 40, 60, 80, 100, 120, 140):

<table>
<thead>
<tr>
<th>Pairs of Ice Skates Sold</th>
<th>34</th>
<th>29</th>
<th>4</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>8</th>
<th>10</th>
<th>48</th>
<th>53</th>
<th>123</th>
<th>97</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month</td>
<td>J</td>
<td>F</td>
<td>M</td>
<td>A</td>
<td>M</td>
<td>J</td>
<td>A</td>
<td>S</td>
<td>O</td>
<td>N</td>
<td>D</td>
<td></td>
</tr>
</tbody>
</table>

Ask: Which month to which month is there the greatest increase in sales? How does the line graph show this? (from October to November, the line is steepest) Why do you think so many skates are sold in November and December? (Christmas shopping)
**Review reading line graphs.** Draw a line graph on the board for the following data.

<table>
<thead>
<tr>
<th>Rita's Distance from Home (km)</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (min)</td>
<td>0</td>
<td>15</td>
<td>30</td>
<td>45</td>
<td>60</td>
</tr>
</tbody>
</table>

**ASK:** How far from home was Rita after 45 minutes? After 15 minutes? After an hour? After half an hour? Ensure students know how to use a ruler to find the data values. For example, to find the distance from home after 30 minutes, line the ruler up with the point at 30 minutes and see where it meets the vertical axis (the markings on the ruler should be parallel to the vertical axis to guarantee that the ruler goes straight across and not on a diagonal):

![Line Graph Diagram]

Have students predict how far Rita would be from home if she walked for another 15 minutes. **ASK:** What trends do you see in the graph? Is Rita getting closer to home or farther from home? How does the trend you noticed help you predict data outside the data points you recorded? How does the direction the data seems to be going in help you to make predictions?

**Presenting data honestly.** Tina wrote 10 math tests this year, each worth 100 marks. This chart shows her marks.

<table>
<thead>
<tr>
<th>Test</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mark</td>
<td>60</td>
<td>50</td>
<td>65</td>
<td>55</td>
<td>70</td>
<td>75</td>
<td>30</td>
<td>45</td>
<td>80</td>
<td>65</td>
</tr>
</tbody>
</table>

a) Draw an accurate line graph showing all Tina’s marks. Include a title and labels.

b) Draw a line graph showing only tests 1, 3, 6, and 9.

c) What trend does the second graph suggest that the first one does not? If Tina shows her parents the second graph, what will they think? Is Tina being honest if she shows her parents the second graph? Explain.
When to use bar graphs and when to use line graphs. If you want to see trends in the data, it makes more sense to use a line graph than a bar graph, because lines have direction. A line graph is also more useful and appropriate when there is a natural ordering for the data. The following examples illustrate this.

Have students draw two line graphs for the data below: in one graph put the days of the week in order and in the other put the temperatures in increasing order.

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>12</th>
<th>7</th>
<th>10</th>
<th>8</th>
<th>6</th>
<th>7</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day of the Week</td>
<td>Su</td>
<td>M</td>
<td>T</td>
<td>W</td>
<td>Th</td>
<td>F</td>
<td>Sa</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>7</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day of the Week</td>
<td>Sa</td>
<td>Th</td>
<td>M</td>
<td>F</td>
<td>W</td>
<td>T</td>
<td>Su</td>
</tr>
</tbody>
</table>

ASK: Which ordering captures the most about the data? Which ordering is easiest to read on a line graph? How does the second graph make it appear as though the temperature is increasing throughout the week? What trends do you see from the first graph?

Then show students the following sets of data:

<table>
<thead>
<tr>
<th>Time to Run 5 km (min)</th>
<th>34</th>
<th>28</th>
<th>29</th>
<th>36</th>
<th>41</th>
<th>25</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Participants</td>
<td>Sara</td>
<td>Juli</td>
<td>Tom</td>
<td>Mark</td>
<td>Mary</td>
<td>Ron</td>
<td>Tara</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time to Run 5 km (min)</th>
<th>24</th>
<th>25</th>
<th>28</th>
<th>29</th>
<th>34</th>
<th>36</th>
<th>41</th>
</tr>
</thead>
<tbody>
<tr>
<td>Participants</td>
<td>Tara</td>
<td>Ron</td>
<td>Juli</td>
<td>Tom</td>
<td>Sara</td>
<td>Mark</td>
<td>Mary</td>
</tr>
</tbody>
</table>

ASK: Are these the same sets of data? Is there a natural order to put the names of the participants in like there was for days of the week? How does putting the times in order from fastest to slowest make it appear as though there is some sort of trend when really there isn’t? Explain to students that because there is not a natural order to put the names in, it makes more sense to draw a bar graph than a line graph.

It only makes sense to draw a line between one data value and the next when there is a natural “next” data value.

PROCESS EXPECTATION

Selecting tools and strategies

Have students decide in which of the following situations it is more appropriate to draw a bar graph and in which it is more appropriate to draw a line graph.

a) The number of cupcakes each class sold in a school bake sale.
b) The price of buying a math textbook from each company.
c) The price of buying 1 textbook, 10 textbooks, 30 textbooks or 100 textbooks from the same company.
d) The temperature over a given 24-hour time period in New York City.
f) How many winter boots each company sold in February.
g) How many winter boots a particular company sold in each month of the year.

For graphs that they choose to do as line graphs, have students explain why other orderings of the data would make less sense and also to predict any trends they might see in a line graph.

After students finish Workbook page 62 Question 1, tell them that from 2007 through 2010, the wage increased by 75 cents every year. Have students copy and then update the correct graph in that question using the same scales, on grid paper. **ASK:** What part of the graph is the steepest? (from 2007 to 2010) Why? (because the wage in those years is increasing faster than in previous years)

**Trends can be misleading.** Present the data on the average amount of homework for students in each grade.

<table>
<thead>
<tr>
<th>Grade</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average amount of homework (min)</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
</tr>
</tbody>
</table>

Have students draw a line graph. **ASK:** What trend do you see? If the pattern continues, how much homework would an average grade 7 student do every night? An average grade 12 student? Do you think this is reasonable? Discuss the dangers of making predictions too far from the data.

For another example of how it can be misleading to make predictions too far from the actual data values, provide the following data:

<table>
<thead>
<tr>
<th>Minutes spent on math homework per day</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score on math test</td>
<td>75</td>
<td>80</td>
<td>82</td>
<td>89</td>
<td>94</td>
</tr>
</tbody>
</table>

Have students draw a line graph and predict score this student will get if they study for:

a) 15 minutes   b) 45 minutes   c) not at all   d) an hour

Notice that the line graph predicts that if this student studies for an hour they will get more than 100%!
Comparing data and making inferences. ASK: Which company do you think will do better next year? Explain your answer.

![Graph of Sales of Two Companies]

Company A

Company B

ACTIVITIES 1–2

1. Have students record the temperature every day for a week and plot a line graph. Ask students what would happen to the data if they took the temperature at 3 p.m. one day and 10 p.m. the next day. Explain the importance of recording the temperature at the same time every day for the week and then discuss any trends they see in the line graph.

2. **Line graph posters.** Encourage students to look at line graphs in books, in magazines, on the Internet, on television (e.g., the weather network) or in brochures (e.g., from financial institutions). Have students record properties that are common to all the line graphs. About how many markings are on the scale? What do the horizontal labels have in common? (they are usually about time, and always have a natural order to them) Have students cut out various line graphs and make a poster titled Line Graphs. Keep these posters for PDM7-14.
**Goals**
Students will use relative frequency tables to construct circle graphs from circles already divided into 100 equal parts.

**PRIOR KNOWLEDGE REQUIRED**
- Can convert between tallies and numerals
- Can convert between fractions and percents
- Understands when to use the mean, median, or mode

**MATERIALS**
- BLM A Large Circle Graph (p N-40)
- BLM Small Circle Graphs (p N-41)

**Introduce relative frequency tables with fractions.** Tell students that you surveyed 80 students in grades 7 and 8 about their favourite type of movie. You tallied the results as follows:

<table>
<thead>
<tr>
<th>Favourite type of movie</th>
<th>Number of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comedy</td>
<td>12</td>
</tr>
<tr>
<td>Action</td>
<td></td>
</tr>
<tr>
<td>Horror</td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td></td>
</tr>
</tbody>
</table>

Have students use the data to complete this relative frequency table:

<table>
<thead>
<tr>
<th>Type of Movie</th>
<th>Number of People</th>
<th>Fraction of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comedy</td>
<td>12</td>
<td>12/80 = 3/20</td>
</tr>
<tr>
<td>Action</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horror</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Point out that the table shows both the number of times a data value occurs in a set (e.g., comedy is the favourite for 12 people) and the fraction of time each data value occurs (e.g., comedy is the favourite for 12/80 people).

(A frequency table shows only the former.)

Encourage students to add the total numbers and fractions. **ASK:** What should the numbers add to? (80) **Why?** (because you surveyed 80 people)
What should the fractions add to? (1) Why? (because 80 out of 80 people is all of them, so the fraction is 1)

Indeed, the numbers add to $12 + 16 + 32 + 20 = 80$ and the fractions add to $\frac{3}{20} + \frac{1}{5} + \frac{2}{5} + \frac{1}{4} = (3 + 4 + 8 + 5)/20 = 20/20 = 1$.

**Review converting fractions to percents.** Tell students that people often write frequency tables using percents instead of fractions. Then remind students how to convert fractions to percents: change the fraction to a fraction with denominator 100 and then write that fraction as a percent (EXAMPLE: $\frac{3}{20} = \frac{15}{100} = 15\%$).

Have students convert these fractions to percents:

a) $\frac{3}{10}$  
b) $\frac{4}{5}$  
c) $\frac{6}{25}$  
d) $\frac{3}{50}$  
e) $\frac{9}{20}$  
f) $\frac{3}{4}$

Remind students that sometimes the denominator of the fraction does not divide evenly into 100. In some such cases, students can reduce the fraction to make it have a denominator that does divide evenly into 100. Have students do this to convert these fractions to percents:

a) $\frac{8}{40}$  
b) $\frac{2}{8}$  
c) $\frac{18}{75}$  
d) $\frac{14}{70}$  
e) $\frac{9}{15}$  
f) $\frac{36}{48}$

**ANSWERS:** a) 20%  
b) 25%  
c) 24%  
d) 20%  
e) 60%  
f) 75%

**Introduce relative frequency tables with percents.** Have students add another column to the relative frequency table above, with heading Percent of People, and fill it in.

Explain to students what the terms “even strength,” “power play,” and “short handed” mean in hockey, or ask a volunteer to do so. (Even strength means neither team has a penalty, so both teams have the same number of players on the ice. When one team has a penalty, that team has one fewer players than the other team and plays short handed. The team with more players on the ice is said to have a power play.) Then have students complete this relative frequency table:

<table>
<thead>
<tr>
<th>Goals Scored by a Hockey Team</th>
<th>Number</th>
<th>Fraction</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even strength</td>
<td>45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Power play</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Short handed</td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Ask questions about this data:

1. Do you think it is harder for a team to score on the power play or at even strength? (even strength)
2. Why did this team score so many more goals at even strength than on the power play? (they were probably playing at even strength for a much greater portion of each game)
3. These goals were scored over 20 games. What is the average number of goals scored per game? ($60 \div 20 = 3$)

**CONNECTION** Real World
4. How did you know which average—mean, median, or mode—the question above referred to? (it was referring to the mean, because not enough information is given for the other two—we would have to know the number of goals scored per game to figure out the median and the mode)

**Introduce circle graphs.** Demonstrate how to convert the data for the hockey team into a circle graph. Use a transparency of BLM A Large Circle Graph with an overhead projector. Emphasize that the circle is already divided into 100 equal parts, so it is easy to use for a circle graph. Brainstorm a title for the circle graph (e.g., When Goals are Scored), and demonstrate labelling each section “even strength,” “power play,” or “short handed.”

**Using circle graphs to compare data.** Give students two copies of BLM A Large Circle Graph and have students transfer the data used at the beginning of the lesson, about favourite type of movie, to a circle graph. Have students label the circle graph appropriately and title it.

Then tell students that you surveyed a group of 20 students in grades 5 and 6 about their favourite type of movie and found the following results:

<table>
<thead>
<tr>
<th>Type of Movie</th>
<th>Number of People</th>
<th>Fraction of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comedy</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Action</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Horror</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Have students copy and complete this relative frequency table in their notebooks and then convert the data to a circle graph on the second BLM. Students can title this one Another Survey of Favourite Movies.

Have students look at the two circle graphs to compare the data. **ASK:** More people in grades 7 and 8 chose comedy than in grades 5 and 6. Why doesn’t the circle graph show this? Emphasize that the circle graphs compare not the frequencies, but the relative frequencies of data values.

**ASK:** According to this data, in which grades did a greater percentage of people choose horror as their favourite type of movie? How can you tell from the circle graph?

**The total percents must add to 100%**. Tell students that you asked students in 6 different kindergarten classes to name their favourite colour, but you might have made a mistake recording some of the data. Have students translate the data into circle graphs using BLM Small Circle Graphs (they should create six graphs—one graph per class).

<table>
<thead>
<tr>
<th>Class</th>
<th>Red:</th>
<th>Blue:</th>
<th>Yellow:</th>
<th>Other:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class A</td>
<td>35%</td>
<td>25%</td>
<td>10%</td>
<td>35%</td>
</tr>
<tr>
<td>Class B</td>
<td>20%</td>
<td>30%</td>
<td>20%</td>
<td>30%</td>
</tr>
<tr>
<td>Class C</td>
<td>30%</td>
<td>20%</td>
<td>10%</td>
<td>40%</td>
</tr>
<tr>
<td>Class D</td>
<td>15%</td>
<td>40%</td>
<td>15%</td>
<td>40%</td>
</tr>
</tbody>
</table>
Class E: Red: 10%  Blue: 15%  Yellow: 25%  Other: 50%
Class F: Red: 18%  Blue: 15%  Yellow: 26%  Other: 31%

**ASK:** Which data did you have trouble graphing? (data for classes A, D, and F) Why? (because they total either more or less than 100%)

Explain that the percents in a circle graph must always total 100%. This is because when you count all the data, you count 100% of it. Encourage students to use this to check their work for the relative frequency table in Question 4, on Workbook page 65, before completing the circle graph.

**A tip for struggling students.** In Question 3c), Calli surveyed 200 students, so each 1 marking on her graph represents 2 students, whereas Bilal surveyed only 50 students, so 2 markings on his graph represent 1 student. Noticing this can make it easier to fill in the graphs: divide the number of students by 2 to get the number of markings for each section on Calli’s graph, and multiply the number of students by 2 to get the number of markings on Bilal’s graph.

**After students finish Workbook page 65 Question 3.** Have students perform the survey, to determine whose school really is more like theirs, and compare the result to their prediction in part e).

**Circle graphs that look different can still represent the same data.** Draw two circle graphs that show the same data (you can use BLM Large Circle Graph) but rotate one of the graphs 90° before you label it. Discuss how the graphs are different and how they are the same. Emphasize that two circle graphs can represent the same data even though they look different. Then include a circle graph that shows the same data but in a different order, and discuss again how the graphs look different though the data is the same.

**ACTIVITY**

**Circle graph posters.** Encourage students to look for circle graphs in books, in magazines, on the Internet, on TV (e.g., on the weather network) or in brochures (e.g., from financial institutions). Have students record properties that are common to all the circle graphs. How many categories are used in each circle graph? About how many categories do most circle graphs use? Have students cut out various circle graphs and make a poster titled Circle Graphs. Keep these posters for PDM7-14.
Have students draw a circle graph, using BLM Large Circle Graph, with the following data about the percent of people who use each mode of transportation to get to school:

- Bus: 30%
- Bike: 25%
- Walk: 20%
- Car: 15%
- Other: 10%

Then, have students use a protractor to measure the angle of each region (or “pie piece”) to fill in the following chart:

<table>
<thead>
<tr>
<th>Mode of Transportation</th>
<th>Percent</th>
<th>Angle in Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bus</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bike</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Walk</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Car</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The angles in a circle add to 360°. When students are finished the chart, have them add the angles. What do they total? (360°) Why? (because the central angles of a circle add to 360°)

Use the angles to verify percentages of 360°. ASK: The percentage of people biking to school was 25% and the angle you found was 90°.
ASK: Does that make sense? Why? (yes, because 90° is 25% of 360°)
Demonstrate this to students. First, remind students how to find the percentage of a number. For example, to find 25% of 360, write 25% as a fraction (25/100 or 1/4) and then replace "of" with "×." So 25% of 360° = 25/100 × 360° = 360° × 25 ÷ 100 = 90°. Or, write 25% as 1/4 to obtain 1/4 × 360° = 360° ÷ 4 = 90°. A quick explanation of why this works:
To find 1% of 360°, divide 360° by 100. But 25% of 360° is 25 times more than 1% of 360°, so 25% of 360° is 25 × 360° ÷ 100 = 90°.

Have students use this method to verify the angles they measured: Does the percentage of 360° you calculate for each mode of transportation match the measurement?

Drawing circle graphs using a protractor. First mark the centre of a circle you will draw, then draw a circle with radius about 3 cm around that centre point using a compass. Draw a line from the edge of the circle to the centre of the circle (a radius). Demonstrate drawing a circle graph from the data above. Emphasize that this circle is not already divided into 100 equal parts, so students now have to use the angle in the circle to draw the regions.

Have students do questions similar to Workbook page 67 Question 2, but have them draw the circle themselves using a compass. Students should be sure to mark the centre point first, so that they can draw a line from the centre to any point on the circle; this will help them create the first region.

**EXAMPLE:** Survey results: Favourite kind of snack

<table>
<thead>
<tr>
<th>Vegetable</th>
<th>Percent</th>
<th>Angle in Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vegetables</td>
<td>30%</td>
<td></td>
</tr>
<tr>
<td>Crackers</td>
<td>10%</td>
<td></td>
</tr>
<tr>
<td>Chips</td>
<td>25%</td>
<td></td>
</tr>
<tr>
<td>Fruit</td>
<td>15%</td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td>20%</td>
<td></td>
</tr>
</tbody>
</table>

Review writing fractions as percents. See Workbook page 68 Question 3.

Writing fractions with denominator 360. See Workbook page 68 Question 4. Not all fractions can be written this way, but if they can, the corresponding angle in a circle is particularly easy to find—it is just the numerator of the fraction with denominator 360. All fractions that students will find in Workbook page 68 Question 5 can be written this way.

Find the percentage of 360° that a given angle represents. See Workbook page 69 Questions 1–3. For Questions 2 and 3, students will need to measure the angle first and then determine which percentage of 360° the angle represents.

Sometimes the angle in a circle does not correspond to a whole number percent of 360°. Provide the example shown in the box at the top of Workbook page 70.
**PROCESS EXPECTATION**

Technology

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Review changing fractions to decimals. Use long division or estimation, then check your answer on a calculator. See Workbook page 70 Question 4.

**Find the decimal percentage of 360° that a given angle represents.**

Have students round to one decimal place. See Workbook page 70 Question 5.

Show students the steps required to draw a circle graph, when the circle is not already divided into 100 parts.

**EXAMPLE:**

In a grade 7 class, 10 students walk to school, 5 travel by bus, 5 bicycle, and 5 skateboard.

**Step 1:** Find the total number of students. (25)

**Step 2:** Express each piece of data as a fraction of the total (reduce to lowest terms).

\[
\frac{10}{25} = \frac{2}{5} \text{ walk} \quad \frac{5}{25} = \frac{1}{5} \text{ bus} \quad \frac{5}{25} = \frac{1}{5} \text{ bicycle} \quad \frac{5}{25} = \frac{1}{5} \text{ skateboard}
\]

**Step 3:** Change each fraction to an equivalent fraction out of 360.

\[
\frac{2}{5} = \frac{?}{360} \quad \frac{2}{360} \quad \frac{1}{5} = \frac{?}{360} \quad \frac{1}{360}
\]

The angle for the part of the circle graph that represents the students who walked to school should be 144°.

\[
\frac{1}{5} = \frac{72}{360} \quad \frac{1}{5} = \frac{72}{360}
\]

The angle for the parts of the graph that represent the students who bicycled, rode the bus, or skateboarded to school should each be 72°.

**Step 4:** Draw a circle and then draw a radius.

---

**Step 5:** Use a protractor to construct a radius for each of the angles you found in step 3.

**Step 6:** Title and label the circle graph. Include the fraction or percent of the total that each region represents.
How we get to school

- **Walk**: \( \frac{2}{5} = 40\% 
- **Bus**: \( \frac{1}{5} = 20\% 
- **Bike**: \( \frac{1}{5} = 20\% 
- **Skateboard**: \( \frac{1}{5} = 20\% 

Have students do Workbook Question 6. Notice that the total is found to be 100% for part c), but not for part d). Emphasize that this is because rounding makes the results less accurate and hence can make the total appear to be different from 100%.

**Importance of drawing the angles accurately.** Ask students how the following circle graph is misleading. (Students should be able to estimate what the angles should look like. For instance \( \frac{3}{5} \) is greater than \( \frac{1}{2} \), but the part marked H covers less than \( \frac{1}{2} \) the circle. Also \( \frac{1}{5} \) is double \( \frac{1}{10} \) so the parts marked B and S should each cover twice as much area as the part marked O.)

**Favourite sport**

- **H**: Hockey \( \frac{3}{5} \)
- **S**: Soccer \( \frac{1}{5} \)
- **B**: Baseball \( \frac{1}{5} \)
- **O**: Other \( \frac{1}{10} \)

Ask students why it is important to measure each angle accurately when drawing a circle graph.

**Extensions**

1. The following question is based on an actual reasoning mistake seen on a web page. An opinion poll asks people to strongly agree, agree, disagree, or strongly disagree with an opinion. Here’s the graph representing the answers of a group of adults who were surveyed:

What fraction of people surveyed:

- Agree? _____  Disagree? _____
- Strongly agree? _____  Strongly disagree? _____

What fraction of people surveyed either agree or strongly agree? _____

The survey concludes: “Not counting the lunatic 1/10 of people who strongly disagree, only 4/10 of people disagree with us.” Is this correct? Explain.
ANSWER: No, this conclusion is incorrect. Out of 10 people, you would expect 3 to agree, 2 to strongly agree, 4 to disagree, and 1 to strongly disagree. If you aren’t going to count the 1 who strongly disagrees, you have to remove 1 from the total: the fraction of people who disagree is 4/9. This is slightly more than 4/10. As well, the data shows that 50% of people disagree with the opinion. (NOTE: the number of people who disagree is the number of people who disagree to any extent, so the number who “disagree” plus the number who “strongly disagree.”)

ASK: Does the circle graph show this? (yes, but it’s not obvious) Have students redraw the circle graph so that this fact—that 50% of people disagree—is prominent (put the “disagree” and “strongly disagree” sections next to each other).

2. The following data shows the number of deaths due to each recreational activity in one year.

   Boating 80  Swimming 40  Biking 36  Jet skiing 4

   a) Make a frequency table showing the fraction of deaths due to each recreational activity.

   b) Draw a circle graph showing the data.

   c) Can you conclude that biking is more dangerous than jet skiing? Why or why not? What extra information would be relevant? (For example, the number of hours people spend biking is likely significantly larger than the number of hours people spend jet skiing, so even if jet skiing is more dangerous per hour, there could still be a lot more deaths from biking than deaths from jet skiing.)
Using circle graphs to find the mean, median, and mode. Tell students that you surveyed 80 people about the number of cars their family has. Show the data in the first two columns of this relative frequency table and have students fill in the last column.

<table>
<thead>
<tr>
<th>Number of Cars in Family</th>
<th>Number of People</th>
<th>Fraction of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8</td>
<td>1/10</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>1/4</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
<td>2/5</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>1/5</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1/20</td>
</tr>
</tbody>
</table>

ASK: Out of every 20 people who answered the survey, how many families have no cars? (2) Repeat for 1 car (5), 2 cars (8), 3 cars (4), and 4 cars (1). Write down the data for 20 people:

0 0 1 1 1 1 1 2 2 2 2 2 2 2 3 3 3 3 4

Have students find the mean, median, and mode number of cars for these 20 data values. (ANSWERS: mean: 1.85, mode: 2, median: 2)

Then remind students that there were not actually 20 people surveyed, but 80 people. Since these are the data for every 20 people, the actual data set is:

0 0 1 1 1 1 1 2 2 2 2 2 2 2 2 3 3 3 3 4
0 0 1 1 1 1 1 2 2 2 2 2 2 2 3 3 3 3 4
0 0 1 1 1 1 1 2 2 2 2 2 2 2 3 3 3 3 4
0 0 1 1 1 1 2 2 2 2 2 2 2 3 3 3 3 4

Discuss how to find the new mean, mode, and median. The new sum of data values is

\[(0 + 0 + 1 + 1 + 1 + 1 + 1 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 4) \times 4 = \text{old sum} \times 4.\]
The new number of data values is just the old number of data values multiplied by 4, so the new mean is:

\[
\frac{(\text{old sum of data values}) \times 4}{(\text{old number of data values}) \times 4} = \frac{\text{old sum of data values}}{\text{old number of data values}} = \text{old mean}
\]

The new median is also the same: the middle number (2) is still in the middle. Why? Each value now occurs four times as often. Since half the data values were 2 or less and half were 2 or more to start, there are now four times as many that are 2 or less and four times as many that are 2 or more, so the same number of data values are still 2 or less as are 2 or more.

Finally, the new mode is also the same: whichever data value was most common before each one was repeated the same number of times is still the most common!

Tell students that they have just shown that, instead of using the frequencies of each data value to find the mean, median, and mode, they can use the relative frequencies. Even if you survey a million families, you can find the percentage of families with each number of cars, and then find the mean, median, and mode, as though you only surveyed 100 people. Display this circle graph:

**Number of cars per family in Country A**

![Circle graph showing the distribution of cars per family in Country A.]

Measure together the angle for the "0" region. **ASK:** What percentage of families have 0 cars? (Write the fraction 54/360 as a fraction over 100: 54/360 = 15/100, so 15%.) Repeat for 1 car (126°, 35%), 2 cars (90°, 25%), 3 cars (54°, 15%), 4 cars (36°, 10%). Then **ASK:** Out of every 100 families, how many have 0 cars? (15) Repeat for 1 car (35), 2 cars (25), 3 cars (15), 4 cars (10). Tell students to pretend that there are only 100 families in the country. **ASK:** What are the mean, median, and mode for every 100 families? **ANSWERS:**

mean: \[
\frac{15 \times 0 + 35 \times 1 + 25 \times 2 + 15 \times 3 + 10 \times 4}{100} = 1.7
\]

median: Since the circle graph has the data values in order around the circle, the half way point is found by drawing a diameter from before the region for 0 cars—the diameter is exactly between the regions for 1 and 2 cars, so the median is 1.5

mode: the largest region on the circle graph is for 1 car so 1 is the mode

**ASK:** How does this tell you the mean, median, and mode for the whole country? (they are the same, because you can think of the country as divided into many groups of 100 families, all repeating the same data values, i.e., 15 with no cars, 35 with 1 car, and so on)
Explain purchasing power. The incomes given in Question 2 on Workbook page 71 are in Canadian dollars (CAD) and represent the equivalent of what you can buy in a year with that amount. For example, if someone who lives in Prague, in the Czech Republic, could buy as much in Prague as someone in Canada making $30,000 could buy in Canada, we would record their income as $30,000 CAD. Most of the people living in this developing country are making enough money to buy in one year what a Canadian living in Canada could buy for $50. (In fact, this is how the figures are given when the United Nations states the poverty line as $1.25 USD per day. If you can buy in your country as much as someone making $1.25 a day in the US can buy in the US, then you are on the borderline of living in absolute poverty.)

Extensions

1. After students do Workbook page 71 Question 1c), **ASK:** If Tina accidentally divides by 10 instead of by 5, how can she tell that she is wrong?

   **ANSWER:** The average she would get is 6/10 or 0.6. This is closer to 0 than to 2, but there are more families with 2 cars than with 0 cars, so the actual average should be closer to 2 than to 0.

2. Discuss conditions that could affect a country’s average number of cars per family. For example, could a country’s hilliness affect whether people choose to bike or drive? What about how large the country is? Or how densely populated the country is? For example, Canada is likely to have a greater number of cars per household than Denmark or Holland. Students might like to research this.

**CONNECTION**
Social Studies
Compare situations when each type of graph are used. If students made posters in lessons PDM7-6, 8, and 10, bring them out and have students compare when each are used. **ASK:** In which situations are bar graphs used? Repeat for line graphs and circle graphs. **ASK:** When are double bar graphs used? Can you use a double line graph? When would you do so? Could you use circle graphs to compare data? (yes, but you would need two circle graphs side by side)

**Review stem and leaf plots.** Remind students what stems and leaves are, and how they make it easy to order data. Have students order the following from easiest to hardest to find using a stem and leaf plot: mean, mode, median and range. (**ANSWER:** mode, range, median, mean OR range, mode, median, mean are both acceptable answers)

**Review the purpose of each type of graph.** Have students individually do Workbook page 72 Question 1, then take up the answers as a class.

**Review comparing line graphs to bar graphs.** A line graph is used when you are looking for a trend, and when the data can be divided into categories that have a natural order to them. A bar graph is used when the data is divided into categories that do not have a natural order to them. For example, the days of the week or the times of a day have a natural order to them; favourite types of movies do not. **ASK:** Which type of graph should a mall use to determine:

a) The number of people who enter the mall between 9 a.m. and 10 a.m., between 10 a.m. and 11 a.m., and so on, to between 5 p.m. and 6 p.m. (a line graph)

b) The number of people who enter each store in the mall between 9 a.m. and 10 a.m. (a bar graph)

Both bar graphs and line graphs can be used to study frequency of results, although line graphs can also be used to study data where frequency doesn’t make sense (e.g., the temperatures at different times of day).
Include circle graphs in the comparison. Note that circle graphs and bar graphs are both used with categories. Use circle graphs if you are more interested in percentages or relative frequencies, and use bar graphs if you are more interested in absolute frequencies. For example, if you want to know how much one company spends as compared to another, use bar height, but if you want to know if one company spends a greater proportion of its budget on advertising than the other, then comparing heights won’t tell you anything—you need a circle graph with advertising as one of the regions. Have students work through Workbook page 72 Question 2. Then ask students if they would use a circle graph or a bar graph to compare:

a) how many Canadian stamps each of three friends have (bar)
b) what percentage of each friend’s stamp collection is Canadian (circle)
c) the population of various Canadian cities (bar)
d) the percentage of various minority groups in Toronto and Vancouver (circle)

Include stem and leaf plots in the comparison. A stem and leaf plot rearranges the data so that it is in numerical order, making it easy to find the maximum value, minimum value, range, mode, and median. Stem and leaf plots are used to organize numerical quantities related to individual items when the items themselves do not have a natural order to them so that no information is lost by rearranging the data. For example, air temperatures taken at different times of day have a natural order to them, but the boiling temperatures of various liquids do not; water does not come before oil in any natural sense. **ASK**: Would you use a stem and leaf plot to display…

a) running times of different people in a class (yes)
b) running times of one person over the course of a training program (no)

Have students find their own data and then choose the kind of graph they want to use to represent it. (See the Introduction for online sources of data on different topics. Consider cross-curricular connections: Is there any data from another subject that students can graph?) Topics can include:

- prices of hotel rooms in a particular location
- prices of plane tickets to a given destination across companies
- average temperatures and precipitations at different places
- world records or events in a particular sport over a period of time
- information about different countries (e.g., infant mortality rate)
- the spending distribution of different organizations

Students should formulate a question about the data. **EXAMPLES:**

- How have world record times for running the 100-m dash changed over time?
- How many participants in the 100-m dash at the last Olympics would have broken the world record 50 years ago if they ran their times then?
- How has the average number of goals per game in the NHL changed over time?
- How do different charities spend their money?
A census or a sample? Tell students that sometimes they will want to know things about really large groups of people but they may not be able to gather data for everyone in the population. For example, if I want to know how many people in Canada have read *The Wizard of Oz*, it wouldn’t be practical to ask everyone if they’ve read it.

Tell students that you want to find the average shoe size of everyone in Ontario. Ask if you should survey everyone in Ontario or only some people in Ontario, and have students explain their answer. Repeat with various situations. *(Example: If I want to know which books my classmates read last week, should I survey the whole class or only a sample? If I want to know the average shoe size of people on my block, should I survey everyone on the block or only a sample? If I want to know how many people in Canada watched a particular television show, should I ask everyone in Canada or just a sample?)*

A large sample produces better results than a small sample. Bring in a large bowl or jar of beans in two different colours (say red and white). Make 40% of the beans one colour (say, red) and 60% the other colour (say, white), but don’t tell students. Mix beans thoroughly. Tell students that you want to figure out which proportion beans are red and which are white without counting every single bean. Invite students to describe how they might do this.

Then have students choose 10 beans with their eyes closed, and record the results. Invite volunteers to tell how many of each colour they chose. **ASK:** Do you think we can estimate the fraction of red and white beans in the whole bowl based on the fraction in a sample of 10? *(no)* Why not? *(Everyone got different answers, so we wouldn’t know whose answer to take.)* Have students pool their results in pairs and then groups of 4 (pairs pair up). Repeat with groups of 8, 16, then with the whole class. Tell students the exact proportion of beans of each colour in the bowl and **ASK:** Which sample produced the best estimate? How large did our sample need to be before we started getting really close to the actual proportion?

Now tell students that you want to know whether the people in a town are in favour of, or against, a proposal (say, to open a new library). If the red beans represent those in favour and the white beans represent those against, what percentage of people are actually in favour of the proposal? *(40%)*
Ron said that he asked 10 people he met at random and found that 7 people liked the idea of a new library and 3 people didn’t. How did that happen? Did anyone pick 7 red and 3 white beans even though only 40% are red and 60% are white? To make conclusions about a large population, you need to ask a large sample. Is 10 people a large enough sample? Is 300 people better? How much better? This is a good time to do the Activity.

**Bias in a sample.** Tell students that you want to know whether grade 7 students in Ontario prefer action movies or comedies. **ASK:** Can I ask only students in Toronto? Why or why not? Can I ask only boys?

Tell students that a biased sample is not similar to the whole population because some part of the population is not represented. In the above example, “grade 8 students in Ontario” is the whole population. If only boys are surveyed, then girls are not represented. If only Toronto students are surveyed, then people from other cities and towns are not represented. If only public school students are surveyed, then private school students are not represented. However, my question might focus only on a specific population. For example, if I want to know if boys prefer action movies or comedies, it wouldn’t make sense to ask girls.

Tell students that an elementary school (grades 1–8) is planning a games party and wants to decide what games to buy. Would the sample be biased if the school surveyed:

a) all grade 2 students? (yes)

b) all boys (yes)

c) every tenth student, when listed in alphabetical order? (no)

d) 3 people chosen at random from each classroom? (no)

Tell students that a sample that is similar to the whole population is called representative. Finding a representative sample is often the most difficult part of conducting a survey. An apartment building manager would like to reserve the games room once a week for a dance. Discuss the bias if, to decide what kind of music should be played at the dance, the building manager:

a) asks the bridge club. (People in the bridge club are likely to be older.)

b) asks the soccer club. (People in the soccer club are likely to be younger and there might be more males than females or vice versa.)

c) asks the book club. (More women tend to join book clubs than men.)

d) asks the teen movie club. (Only teenagers will be represented.)

e) puts a survey under every tenth door by apartment number. (no bias)

f) lists the names of people living in the building in alphabetical order and picks every tenth person to ask. (no bias)

g) asks people at the playground. (Teenagers are less likely to be represented.)
Even surveying people at the same location, at different times, can produce different biases. **ASK:** How would your samples be different if you surveyed people at the mall on a weekday morning and a weekend morning? If a mall wants to decide whether to rent space to a pet store or a video gaming store, how and when should they conduct a survey? Some points to discuss:

- On a weekday morning, teenagers are definitely excluded from the sample as are many working people. People who go to malls during weekday mornings are either unemployed, retired, or very well off. They might also work part-time or have variable work schedules (i.e., not 9 to 5). University students might also have class schedules that allow them to go to the mall on a weekday morning.

- People who can go to the mall on weekdays might not go on weekends, to avoid the crowds. Thus, any group that was overrepresented on weekday mornings may be underrepresented on the weekend.

- The mall should probably take samples at different times and give them different weightings based on how crowded the mall is. If more customers come during the weekend, the weekend should get a higher weighting. **NOTE:** The mall is interested in biasing their sample to their own customers. If it happens that more older people go to the mall, they want to target older people.

- There are 3 apartment buildings adjacent to the mall. The mall decides to survey every 10th apartment in these 3 buildings instead of asking mall customers at different times of day. Is this a representative sample? (No. Only people living in apartments are represented, and this will bias the sample against people who live in nearby houses. Also, not all the people who live in the apartments necessarily like or even go to this mall, so this survey will not give the mall useful information.)

**The wording of a question can affect the results of the survey.** Discuss the following three cases, where the sample is representative but the results are biased.

1. A town council is thinking of selling a city park and allowing a department store to build in its place. Two groups ask different questions:

   A: Are you in favour of having a new store that will provide jobs for 50 people in our town?

   B: Are you in favour of keeping our parks quiet and peaceful?

   a) Which question do you think was proposed by someone in favour of selling the park? Which was proposed by someone against selling the park?

   b) Write a survey question that is more neutral and does not already suggest an answer.
2. The student council at a school chooses music and orders food for events. Most people enjoy the music, but not the food. Look at the two surveys in the margin.

   a) What is the same about the two surveys?
   b) What is different about the two surveys?
   c) Which survey is more likely to suggest that student council is doing a good job? Why?
   d) How can the order of the questions affect the results of a survey?

3. Teach students that being primed for an answer can affect the results of a survey. Give half the students a list of questions as follows:

   What colour do you add to yellow to make orange?
   What colour do you add to blue to make purple?
   What colour is blood?
   What colour is a stop sign?
   What colour is a strawberry?
   What colour is a poppy?
   What colour of traffic light do you go on?

Have students pair up and ask their partners the list of questions. The correct answer to the last question is green, but because students will be primed to say red from all the previous questions, most students will answer red to the last question as well. You could ask students how many had a partner who answered red to the last question (without saying who their partner was). Emphasize that this is because they were being primed to answer red because of all the previous questions. As soon as they heard the beginning of the question—what colour of traffic light—students would immediately picture a red traffic light.

Tell students that you want to survey two different younger classes (students can actually conduct the surveys if you want) using two different questions (see margin).

**ASK:** How are the questions the same? How are they different? Which question do you think will get more “Yes” answers? Why? (Even though the logical content of the questions is the same, the pictures you get in your mind when answering the two questions are different. When answering Question A, you are likely to picture someone not really paying attention to their friend because they are so interested in the television show: when answering Question B, you may picture two people who are talking with the television on in the background, or two people who are talking about the television show they are watching together. No one will offend the television show by not paying attention to it.)
Extensions

1. **Sample size.** Distribute a page of French text. Put students into the same number of groups as there are paragraphs in the text, and assign a paragraph to each group. Ask students to identify the most common letter in one sentence in their paragraphs (make sure all sentences are used; depending on the length of the text, some sentences may be assigned to more than one student). Is the most common letter the same for all students/sentences? How many letters do students identify as the most common? Which letters do students identify as most common?

Now have each group identify the most common letter in its paragraph. Finally, have students identify the most common letter on the page.

**ASK:** Do you think the most common letter on this page will be the same as the most common letter on another page? in a whole book? in the French language overall? Why is it better to use a large sample size to calculate the most common letter in a language than a small sample (such as a sentence or paragraph)?

2. **Another source of bias.** Tell students that you stood outside a hockey arena and counted the fans who were wearing jerseys of each team playing that evening. The results were: 60% Home and 40% Away. Can you conclude that 40% of the fans at the game supported the Away team? Explain. **PROMPT:** Is a fan who is wearing a jersey more likely to support the home team or the away team? Why? **ANSWER:** A fan of the Away team is more likely to be coming from out of town and hence more likely to make the extra effort to wear their team’s jersey. The fans who wear jerseys are thus a biased sample, more likely to be cheering for the Away team.

**Connection**

**Real World**
Giving choice for survey questions. Conduct a survey with your students by asking them their favourite flavour of ice cream. Do not limit their choices at this point.

Write each answer with a tally, then ask students how many bars will be needed to display the results on a bar graph. How can the question be changed to reduce the number of bars needed to display the results? How can the choices be limited? Should choices be limited to the most popular flavours? Why is it important to offer an “other” choice?

Explain to students that the most popular choices to a survey question are predicted before a survey is conducted. Why is it important to predict the most popular choices? Could the three most popular flavours of ice cream have been predicted?

Have your students predict the most popular choices for the following survey questions:

- What is your favourite colour?
- What is your favourite vegetable?
- What is your favourite fruit?
- What is your favourite animal?

Students may disagree on the choices. Explain that a good way to predict the most popular choices for a survey question is to ask a few people the survey question before asking everyone.

Each person should give a unique answer. Emphasize that the question has to be worded so that each person can give only one answer. Which of the following questions are worded so as to receive only one answer?

a) What is your favourite ice cream flavour?
b) What flavours of ice cream do you like?
c) Who will you vote for in the election?
d) Which of the candidates do you like in the election?
e) What is your favourite colour?
f) Which colours do you like?
When is an “other” category needed? Have your students think about whether or not an “other” category is needed for the following questions:

What is your favourite food group?
- Vegetables and Fruits
- Milk and Alternatives
- Meat and Alternatives
- Grain Products

What is your favourite food?
- Pizza
- Burgers
- Tacos
- Salad

Then ask how they know when an “other” category is needed. Continue with these examples:

- What is your favourite day of the week? (List all seven days)
- What is your favourite day of the week? (List only Friday, Saturday, and Sunday)
- What is your favourite animal? (List horse, cow, dog, pig, cat)
- How many siblings do you have? (List 0, 1, 2, 3, 4 or more)
- Who will you vote for in the election? (List all candidates)

Formulating a good survey question. Have your students explain what they like and don’t like about each of the following proposals for a survey question.

1. Do you like to read for fun?
   - Yes
   - No

2. How many books have you read in the last year for fun? 

3. How often do you like to read for fun?
   - Any chance I get
   - Often
   - Sometimes
   - Not very often
   - Never

4. How many books have you read in the last year for fun?
   - 0
   - 1–5
   - 6–10
   - 11 or more

Emphasize the importance of choosing questions that people will know the answer to (EXAMPLE: some people may not know how many books they read last year), and that provide enough variety in the way someone can answer (EXAMPLE: some people might feel uncomfortable answering Yes or No if they like to read for fun only sometimes). Question 3 is probably the best question since it allows enough variety and although some people may be unsure between two adjacent categories, they will have at least a good idea as to how to answer the question. Furthermore, the number of books may not be the best indication since someone who likes to read may only read a few long books, while someone who doesn’t read very much might read the same number of short books.
Have students make up good survey questions to find out:

a) What kind of games should an elementary school (grades 1–8) have available for an end-of-year party?

b) What kind of snack do students in Ontario like best?

c) How many pets do students in your class have?

If a school can provide only certain kinds of games (EXAMPLE: no computer games will be available), would it make sense to include computer games in the survey? Should the school provide an “other” category or force students to choose from among the games that the school could possibly have available?

Ask students to decide who they would survey in each case. Would it make sense to ask only grade 7 students what kinds of games they like? What problems would this cause?

Tell students that they will be designing their own survey. Have students brainstorm topics for a survey. (EXAMPLE: How do people get to school? How long does it take people to get to school? How big is your family? How long does it take people to get ready for school from the time they get up to the time they arrive—do older children take more or less time than younger children? How tall are people in each grade?)

Ask students to graph their survey results in 2 different ways and to explain the advantages and shortcomings of each representation.

Students will also design and conduct an experiment. Remind students how to double numbers and how to multiply by 5 quickly (multiply by 10 and then halve the result). Tell students that you want to know if they are faster at doubling numbers or multiplying by 5. Give students the following two tests and have students time themselves individually.

<table>
<thead>
<tr>
<th>Doubling Test:</th>
<th>Multiplying by 5 Test:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \times 3 =$</td>
<td>$5 \times 0 =$</td>
</tr>
<tr>
<td>$2 \times 5 =$</td>
<td>$5 \times 1 =$</td>
</tr>
<tr>
<td>$2 \times 4 =$</td>
<td>$5 \times 2 =$</td>
</tr>
<tr>
<td>$2 \times 2 =$</td>
<td>$5 \times 3 =$</td>
</tr>
<tr>
<td>$2 \times 6 =$</td>
<td>$5 \times 4 =$</td>
</tr>
<tr>
<td>$2 \times 1 =$</td>
<td>$5 \times 5 =$</td>
</tr>
<tr>
<td>$2 \times 9 =$</td>
<td>$5 \times 6 =$</td>
</tr>
<tr>
<td>$2 \times 8 =$</td>
<td>$5 \times 7 =$</td>
</tr>
<tr>
<td>$2 \times 0 =$</td>
<td>$5 \times 8 =$</td>
</tr>
<tr>
<td>$2 \times 7 =$</td>
<td>$5 \times 9 =$</td>
</tr>
</tbody>
</table>

Ask students if they think this pair of tests is fair. Then ASK: What about this pair:

| $2 \times 13 =$ | $5 \times 7 =$ |
| $2 \times 9 =$ | $5 \times 4 =$ |
| $2 \times 24 =$ | $5 \times 3 =$ |
Challenge students to make up a pair of tests that they think is fair. Then have students do their own tests and ask them if they are faster at multiplying by 2 or by 5. Have half the class do the multiplying by 2 test first and the other half do the multiplying by 5 test first. Did the order they did the tests in affect their results?

Tell students that sometimes the trickiest part of doing an experiment is making sure that they are really testing for what they want to be testing and that nothing else influences their results.

Tell students that you want to answer the question: Do ice cubes made from the same amount of water but differently shaped containers melt at the same rate?

Discuss the different types of containers that could be used for this experiment and how students would make sure the same amount of water is put into each container. How would they measure the rate of melting? What other equipment would they need? Why is it important to think ahead of time about what kind of equipment they need? Would the experiment be fair if some ice cubes were put in the sun and others in the shade? How would that affect the results of the experiment? Have students predict the results of the experiment—will the container’s shape affect the rate of melting?

**ASK:** What type of graph could you use to display your results? Is there a natural ordering of the containers? (no) Is there a possibility of data in between the data values? (No, not unless students choose their containers in a very structured way, i.e., by increasing the length and decreasing the width, in which case there would be a natural ordering of the containers.)

If time permits, have volunteers bring in the equipment and conduct the experiment with the class. Demonstrate the entire process, from doing the experiment to displaying the results (on a bar graph, since it doesn’t make sense to speak about trends in the shape of the container).

Brainstorm factors that could influence the results of the following experiments:

- **A.** Choose 5 paper rectangles with the same area but different perimeters to build paper airplanes. Which rectangle makes an airplane that flies the farthest?

- **B.** How does adding sugar to strawberries affect how long the strawberries stay fresh?

- **C.** How does adding salt to ice affect the rate at which the ice melts?
In A, the thickness and quality of paper, the environment (such as the presence of wind), the design of the airplanes, and the quality of bending (how precise the creases are) could affect the results, so these should be kept the same for all three models. In B, the initial freshness of the strawberries, how evenly the sugar is spread, the container (metal, plastic, glass, airtight, shape and size), the surrounding temperature, the type of strawberries, and the type of sugar should be the same in each sample to which different amounts of sugar are added. In C, the size and shape of the ice, the type of salt (i.e., sea salt, table salt), the temperature of the surrounding areas, the colour of the plate (white or black plates will absorb the heat differently), and the source of the water should be kept the same as the amount of salt increases.

Note that in all cases, it makes sense to look for trends and so a line graph is most appropriate. In A, put the rectangles in order from smallest to largest perimeter.

**Bias in an experiment.** Discuss the following experiments with students. Which experiments use representative samples, and which use biased samples?

a) • On a weekend, a model airplane club tests two plane designs to see how long they can remain in the air.

• On each day, they test 5 planes of each design. (representative)

• On the first day, they test 10 planes of 1 design and on the second day, they test 10 planes of the other design. (On one day it might be windier, causing the planes to stay in the air longer on that day than on the other day. The conditions are different on the two days, so the sample is biased.)

b) A class is testing two brands of seeds to find out what percent will germinate. They plant:

• 10 seeds from each package

• the 10 largest seeds from each package

**Extensions**

1. Conduct a survey of your class. Have students answer this question: How many people (adults and children) are in your family? Then ask students to predict whether the result for your classroom will be higher or lower than the average size of families with children in Canada. The average is 3.5, and it is likely that your class will have an average family size that is larger than this. Ask students to think about why this might be the case. (One explanation: Children in grade 7 are likely to have more siblings than younger children, just because there has been more time for the family to choose to have another child; a 4-year-old is more likely to be an only child than a 14-year-old.) Students can pursue this reasoning by asking which percentage of children in each grade are
children without siblings (only children). Ask students to predict whether this percentage will increase with age, decrease with age, or stay the same. (The percentage of only children should decrease with age.) What type of graph would show this trend best? Why? (a line graph because line graphs show trends)

2. a) Elections are held in different ways in different countries. Let’s look at four voting systems. In every case, citizens 18 years of age and older can vote. Citizens can choose one candidate or nobody. To vote, citizens need a card sent by the election committee. Cards are used to prevent double voting. Citizens cast their votes when they are alone in a room.

- In country A, citizens choose ballots with the name of one of several candidates and nothing else on them. Nobody can see which ballot is chosen.
- In country B, there are two candidates, the ballots are numbered, and the numbers are recorded on a list with the names of the citizens.
- In country C, there is only one candidate, and the voting is either for or against the candidate. The ballots have the name of the candidate and two options—for or against the candidate—and nothing else on them. Nobody can see what is marked on the ballot.
- In country D, there is only one candidate and the voting is either for or against the candidate. The ballots are numbered and the numbers are recorded on a list with the names of the citizens.

In which countries are the elections biased and how? Explain your thinking.

b) These elections use a census—everyone over a certain age has the right to vote. In practice, however, everyone might not have an equal opportunity to vote. Discuss sources of sample bias (EXAMPLE: if there is no way for people to vote from home, those with mobility problems might be underrepresented; if voting places are all located in urban centres, those from rural areas might be underrepresented).
A Large Circle Graph
Small Circle Graphs
Matching Data to Graphs

Match the data with its graph.

1. Favourite Sport | Frequency | Fraction of Total
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Hockey</td>
<td>42</td>
<td>$\frac{3}{10}$</td>
</tr>
<tr>
<td>Soccer</td>
<td>35</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Baseball</td>
<td>28</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td>Volleyball</td>
<td>28</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td>Other</td>
<td>7</td>
<td>$\frac{1}{20}$</td>
</tr>
</tbody>
</table>

2. Martha’s math test scores (out of 10):

<table>
<thead>
<tr>
<th>Test#</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

3. Month | Jan | Feb | Mar | Apr | May | Jun
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Company A’s home sales</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Company B’s home sales</td>
<td>10</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

4. Class marks on science test:

<table>
<thead>
<tr>
<th>29</th>
<th>84</th>
<th>47</th>
<th>69</th>
<th>71</th>
<th>72</th>
<th>88</th>
<th>64</th>
<th>38</th>
<th>42</th>
</tr>
</thead>
<tbody>
<tr>
<td>56</td>
<td>51</td>
<td>77</td>
<td>79</td>
<td>80</td>
<td>80</td>
<td>43</td>
<td>81</td>
<td>76</td>
<td>77</td>
</tr>
</tbody>
</table>

5. Families are surveyed about how many cars they have:

<table>
<thead>
<tr>
<th>Number of cars</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>12</td>
<td>15</td>
<td>11</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>
Unit 4  Patterns and Algebra

In this unit, students will investigate patterns and their equivalent linear relations, using tables of values, graphs, and formulas. Students will represent patterns and linear relations using a variety of tools and strategies and make connections between representations. They will use and develop their reasoning skills to analyze patterns, formulas, and graphs.

Materials
In many lessons you will need a pre-drawn grid on the board. If such a grid is not already available, you can photocopy BLM Grid Paper (p T-1) onto a transparency and project it onto the board. This will allow you to draw and erase points and lines on the grid without erasing the grid itself.

Meeting Your Curriculum
This unit is core curriculum for both Ontario and WNCP students.
PA7-16 Formulas
Pages 77–80

Goals
Students will create tables of values for linear relations, produce formulas such as \( n + a \) or \( an \) for patterns and tables of values, and predict terms of patterns using the formulas they produced.

PRIOR KNOWLEDGE REQUIRED
Can create and extend a T-table for a pattern
Is familiar with variables
Can extend a linear increasing sequence
Can translate a statement into an algebraic expression

How formulas help with patterns. Draw a simple design, like the one in the margin. ASK: How many pentagons did I use? How many triangles? How many triangles and how many pentagons will I need for two such designs? For three designs? Remind students that they previously (see PA7-3) used T-tables to solve this type of question. Ask students to draw a T-table and to fill it in for five designs. ASK: I want to make 20 such designs. Should I continue the table to check how many pentagons and triangles I need? Can you think of a more efficient way to find the number of pentagons and triangles? How many triangles are needed for one pentagon in the design? (5) What do you do to the number of pentagons to find the number of triangles in any number of designs? (multiply by 5)

Remind students that mathematicians often use letters instead of numbers to represent a changing quantity. For example, they could use \( p \) for the number of pentagons and \( t \) for the number of triangles. We have a verbal rule for the number of triangles used in a design: Multiply the number of pentagons by 5 to get the number of triangles. What algebraic equation does this rule produce? \((5 \times p = t \text{ or } 5p = t)\) Explain that an equation that shows how to calculate one quantity from another is called a formula. Write the term on the board beside the formula itself. The letters that represent numbers are called variables. Point out the connection to the word vary: a variable is a quantity that is able to vary, or change. The number we multiply the variable by is called the coefficient.

Producing a table of values for a formula. Write another formula, such as \( 8 \times s = t \). Explain that \( s \) represents the number of squares in a pattern and \( t \) is the number of triangles, as before. What rule does the formula express? (The number of triangles is 8 times the number of squares, or Multiply the number of squares by 8 to get the number of triangles.) Remind students how to make a table of values that matches the formula by writing the actual number of squares in place of \( s \) and multiplying. Remind students that writing the actual number in place of a variable and finding the value of the expression is called substitution. The result of the multiplication is the number of triangles. Draw a table of values:
# of Squares (s) | Formula \((8 \times s = t)\) | # of Triangles (t)
---|---|---
1 | 5 | 5
2 | 10 | 10
3 | 15 | 15

Ask students to copy the table and to fill in the missing numbers. Then have them add two more rows to the table. **ASK:** Do we need to extend the table to find how many triangles will be needed for 25 squares? (no) How will you find the number of triangles needed for 25 squares? (substitute 25 for \(s\)) Ask students to find the number of triangles for 25 squares. (200)

Let students practise drawing tables for more formulas, such as \(3 \times t = s\) and \(6 \times s = t\) (\(t = \) number of triangles, \(s = \) number of squares).

Ask students to create designs to go with the formulas above.

**Producing a formula of the type \(t = a \times s\) for a T-table.** Tell students that the T-tables below were created using a formula of the same type as above. Now, instead of creating the table for a formula, students will do the opposite: produce a formula for the table. First identify the type of formula. **ASK:** How are all the formulas for the tables below the same? (all formulas are of the sort “number \(\times s = t\)”) How could you find the coefficient for each formula from the table? (e.g., look at the number of triangles in the row with \(s = 1\) or divide the number of triangles in any row by the number of squares) Point out to students that it is essential to check that the formula they produced works for all rows of the table. Include several more-challenging examples, such as the last two below.

<table>
<thead>
<tr>
<th>Squares ((s))</th>
<th>Triangles ((t))</th>
<th>Squares ((s))</th>
<th>Triangles ((t))</th>
<th>Squares ((s))</th>
<th>Triangles ((t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>2</td>
<td>8</td>
<td>2</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>3</td>
<td>12</td>
<td>3</td>
<td>36</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Squares ((s))</th>
<th>Triangles ((t))</th>
<th>Squares ((s))</th>
<th>Triangles ((t))</th>
<th>Squares ((s))</th>
<th>Triangles ((t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>2</td>
<td>8</td>
<td>7</td>
<td>21</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>4</td>
<td>16</td>
<td>9</td>
<td>27</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
<td>6</td>
<td>24</td>
<td>16</td>
<td>48</td>
</tr>
</tbody>
</table>

Students can use the Activity below to practise creating T-tables for multiplicative rules (rules that involve only multiplication) and finding formulas for them.

**Formulas for patterns of the type \(n + a\).** Start with a simple problem: Rose invites some friends to a party. She needs one chair for each friend and one for herself. Can you give Rose a formula or equation for the number of chairs she will need?
Ask students to suggest a variable for the number of friends and a variable
for the number of chairs. Given the number of friends, how do you find the
number of chairs? Ask students to write a formula for the number of chairs
(SAMPLE ANSWER: \(1 + f = c\)). Suggest that students make a T-table
similar to the one they used for multiplicative rules. They should start at 1
and fill the table in for a few rows.

Have students practise writing rules and making T-tables with more such
questions. EXAMPLES:

a) Lily and Rose invite some friends to a party. How many chairs will
they need?

b) Rose, Lily, and Pria invite some friends to a party. How many chairs will
they need? They invited 20 friends. How many chairs will they need?

c) A family invited several friends to a party. The number of chairs they
need is \(6 + f = c\). How many people are in this family? If they invited
10 friends, how many chairs would they need?

Ask your students to write a problem for the formula \(4 + f = c\).

**Tables with input and output.** Explain to students that the number that
you put into a formula in place of a letter is often called the \textit{input}. The result
that the formula provides—the number of chairs, for instance—is called the
\textit{output}. Write these terms on the board and ask volunteers to circle the input
and underline the output in the formulas you have written on the board.

**EXAMPLE:** \(1 + f = c\)

Draw several T-tables like the one in the margin on the board, provide a
rule for each and the input numbers, and ask students to find the output
numbers. Start with simple inputs like 1, 2, 3 or 5, 6, 7 and continue to more
complicated combinations like 6, 10, 14. Provide all types of rules: additive
(add 4 to the input), multiplicative (multiply the input by 5), and subtractive
(subtract 3 from the input).

Suggest that students try a more complicated task: produce a rule and a
formula for a given table. Ask students to think about what was done to
the input to get the output. Remind them to check that the rule works for all
rows. For example, if you look at only the first row in the first table below,
the rule could be \(\text{Input} \times 5\) or \(\text{Input} + 4\), so you need to check the other
rows. Give students several simple tables to work with. EXAMPLES:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

Output = Input + 4  Output = \(\text{Input} \times 2\)  Output = \(\text{Input} - 3\)  Output = \(\text{Input} \times 3\)
Extension

Tell students that a family is having a party. This is the formula for the number of chairs they will need for the party: \( g + 4 = c \). **ASK:** If \( g \) is the number of guests and \( c \) is the total number of chairs needed, how many people are in the family? (4) Point out that any change in the number of guests produces a change in the total number of chairs needed. For example, if there are two guests, \( g = 2 \) and the family will need 6 chairs; if there are three guests, \( g = 3 \) and the family will need 7 chairs; and so on. The number of family members is always 4, and it does not change.

Next, show a different formula for the number of chairs: \( g + f - 1 = c \). Say that \( f \) represents the number of family members, and “\(-1\)” represents a baby in the family who does not need a chair. This time, the number of family members can change, too. What other quantities can change? (the number of guests, the number of chairs) If the family has 10 chairs, how many guests and how many family members could be at this party? (There are different solutions to this problem. Students should find them systematically.)

**PROCESS EXPECTATION**

Looking for a pattern

**PROCESS EXPECTATION**

Organizing data

**ACTIVITY**

Students work in pairs. Each student decides on a formula (such as \( s = 3 \times t \)) and makes a T-table of values for it, with three rows. Students exchange their T-tables. They have to find the formula the T-table was made with and then check each other’s answers. They can also try to produce a design that will go with the formula they found.

**VARIATION:** Students can use the spinner shown and a die to randomize the formulas they produce. Students spin the spinner and roll the die. They write a formula for the rule given by the spinner and the die. For example, if the student spins Multiply and rolls 3, the rule is “Multiply the input by 3” and the formula is “3 \( \times \) Input = Output.”
Introduce ordered pairs using input and output numbers. Draw the following pairs of T-tables on the board.

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**Goals**

Students will treat an input-output pair as an ordered pair, draw graphs from ordered pairs, and determine ordered pairs from graphs.

**PRIOR KNOWLEDGE REQUIRED**

Can create and extend a T-table for a pattern
Is familiar with variables
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Repeat with the second rule in part a). (For input 5, the output is 2) Tell students that the order we say the numbers in—“2 and 5” or “5 and 2”—tells us that the rules are different. Because of this, “2 and 5” and “5 and 2” can be written as different ordered pairs: “2 and 5” is written as (2, 5) and “5 and 2” is written as (5, 2).

**Coordinates as ordered pairs.** **ASK:** Where have you seen ordered pairs before? (coordinate systems, graphs) Draw a coordinate grid and show both (2, 5) and (5, 2). Remind students how the points are plotted: The first coordinate tells us how far to go right of 0, and the second coordinate tells us how far to go up from 0. This is a convention used by mathematicians everywhere, the way > is used to mean more than and < is used to mean less than. Draw several grids on the board, add points, and have students write the coordinates of the points.

Then make a table with headings Ordered Pair, First Number, and Second Number, as in Question 1 on Workbook page 82 and have students fill in such tables for the grids above.

Have students reverse the First Number and Second Number in each table they obtained and draw the new points on a different grid.

Remind students that they can think of inputs and outputs as ordered pairs: The input is the first number and the output is the second number. Have students change given T-tables of inputs and outputs first to ordered pairs and then to points on a graph. See Workbook p, 83, Question 4.

**Extensions**

1. Write the formulas for the T-tables given at the beginning of the lesson. Switch Input and Output in the first formula, then solve for Output (in terms of Input). What do you notice?

   **ANSWER:** The formulas are opposite, just as the rules were opposite. For example, in a) the first formula is Output = Input + 3. If we switch Input and Output, we get Input = Output + 3, and solving for Output gives the formula for the second table: Output = Input − 3.)

2. On a grid on the board, draw the points (1, 4) and (4, 1) using one colour and the points (3, 6) and (6, 3) using a different colour. Ask students to write the ordered pairs and to explain how the pairs of points of each colour are related. (they have the same numbers in different positions; the numbers are transposed or “switched”) Add more pairs of points whose ordered pairs have the same two numbers,
but switched, to help students see the pattern. **EXAMPLES:** (2, 5) and (5, 2), (1, 6) and (6, 1). Have students draw the following sets of points on grids using blue, and the points obtained by switching the numbers in the ordered pairs using red:

a) (5, 8), (5, 7), (5, 6), (5, 5), (5, 4), (5, 3), (5, 2) [points in red: (8, 5), (7, 5), (6, 5), and so on]

b) (0, 10), (1, 13), (2, 14), (5, 15), (8, 14), (9, 13), (10, 10), (9, 7), (8, 6), (5, 5), (2, 6), (1, 7)

c) (3, 1), (4, 3), (5, 5), (6, 7), (7, 9), (8, 11)

d) (1, 2), (2, 3), (3, 4), (4, 5), (5, 6)

**ASK:** What do you notice? What is the relationship between the locations of the red points and the locations of the blue points? (they are reflected in the diagonal from (0, 0) that passes through (1, 1))
**Sequences as Ordered Pairs**

**Graphing Sequences**

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**Goals**

Students will convert sequences to ordered pairs and graph them. They will also investigate how properties of graphs translate to properties of the corresponding sequence.

**PRIOR KNOWLEDGE REQUIRED**

- Can create and extend a T-table for a pattern
- Is familiar with variables
- Can draw points and identify coordinates on a graph (non-negative numbers only)
- Can identify increasing and decreasing sequences
- Can find the gaps in a sequence
- Can produce a sequence using a stepwise rule

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**Introduce term and term number.** Progress as on Workbook page 84 Questions 1 and 2.

**Introduce a sequence as a set of ordered pairs.** Progress as on Workbook page 84 Questions 3–5.

Review the meaning of increasing and decreasing (in the context of sequences). As well, remind students how they used to find the difference between the terms to identify whether a sequence repeats, increases, or decreases, by the same amount or not.

**Graphing sequences.** To graph a sequence, change the sequence to a set of ordered pairs of the form (term number, term) and then plot the ordered pairs on a graph. See Workbook page 85 Question 1.

**EXTRA PRACTICE:** Have students graph the sets of ordered pairs from Workbook page 84 Question 5 and decide whether the points can be joined by a straight line or not. (yes for B and D, no for A and C)

**Graphs of increasing and decreasing sequences.** After students do Workbook page 86 Questions 2 and 3, have them make a conjecture about how the graphs of increasing sequences are different from the graphs of decreasing sequences. Then have students verify their conjecture using the graphs from Workbook page 85 Question 1. Students might find it helpful to circle the increasing sequences from Question 1 with one colour and the decreasing sequences with a different colour. Have students write in their notebooks a sentence about how the graphs of increasing and decreasing sequences are different, and then compare their sentences with that of a partner. Students should try to improve both sentences by making a new sentence. Repeat with groups of four and have students write the resulting sentence as their answer to Workbook page 86 Question 4.
Linear graphs and sequences. Tell students that a sequence is called linear if all the points on its graph can be joined by a straight line. Have students do parts A and B of Investigation 1 on Workbook page 86. Students can draft their answer to C first individually, then with a partner, and finally in a group of four, to continually improve their sentence before writing it in their workbook.

Have students check their conclusion on the sequences in Workbook page 84 Question 5.

**Bonus** for INVESTIGATION 1

Is this sequence linear? 1, 3, 5, 7, then repeat

**ANSWER:** No, but it looks like it for the first 4 terms.

Emphasize that students can now determine properties of graphs by looking at the corresponding sequence of points. Have students summarize what they can say about the graph if the sequence is increasing with the same gap between terms. (The points all lie on the same line and go from bottom left to top right.) Repeat with increasing sequences where the gaps are not all the same (the points are not on the same line, but they all go from bottom left to top right), decreasing sequences where the gaps are all the same, and decreasing sequences where the gaps are not all the same.

**ASK:** Where have you seen pictures that look like the graphs in this lesson? (in line graphs) Pretend the graphs in Question 1 on Workbook page 85 are line graphs, with the horizontal axis being time, and the vertical axis being distance covered. How would you describe the trends for each graph? (The distance increases with time for increasing sequences and decreases with time for decreasing sequences. When the gaps are the same we would say that distance increases or decreases at a constant rate—the object or person whose position the graph describes moves with a constant speed.) So linear sequences increase or decrease at a constant rate.

Have students suggest other relationships that graphs of linear sequences could represent. They can also describe the trends for a relationship of their choice. **EXAMPLES:** recycled material collected, money earned, temperature, average precipitation.

**Matching sequences to graphs.** Before doing Workbook page 87 Question 6, write the following sequences on the board:

i) 2 5 8 11

ii) 12 9 6 3

iii) 4 7 9 13

Show students the graphs below. Explain that the graphs were drawn from the sequences above, but the axes do not start at 0 because we are seeing only part of the coordinate grid, and not the part that starts at 0. **ASK:** Can you still tell which graph belongs to which sequence?
Explain that only the first sequence is both increasing and linear, so it must match the third graph. Only the second sequence is decreasing, so it must match the first graph. The third sequence is increasing but not linear, so matches the second graph.

**Stepwise rules for linear sequences.** Remind students how to produce a sequence given a rule such as “Start at 3. Add 4 each time.” (This is called a “stepwise rule,” but don’t identify it as such yet; just give the example.) Have students work through Investigation 2 on Workbook page 87.

Sequences are linear if their stepwise rule consists only of adding the same number or subtracting the same number. **ASK:** Does it make sense that we can get the same information from the rule as we can from the sequence itself and from the graph? **PROMPT:** Does the rule tell you everything about a sequence? Can you get the sequence from the rule?
**Review line graphs.** Draw a coordinate grid on the board and draw the graph shown in the margin. Explain to students that this graph represents the cost of parking a car in a parking lot. How much would it cost to park the car for 1 hour? For 2 hours? For half an hour?

**ASK:** How much will you pay just to enter the parking lot, even before you park the car there? ($3) Remind students that this amount is called a **flat rate**. How much does each hour of parking cost? ($2) How do you know? Does the hourly rate vary? As a challenge, ask students to give a rule that allows you to calculate the cost of parking (one way to state the rule: $3 flat rate, $2 each additional hour to a maximum of $9) Emphasize that you can park in this lot for more than 3 hours, but you won’t pay more than $9 if you do. As students to write an algebraic expression for the cost of parking for up to 3 hours, using \( t \) for time. \((3 + 2t)\)

Explain that a second parking lot nearby charges $3 per hour with no flat rate and no maximum. What is the mathematical rule for the cost of parking in the second lot? Ask students to write ordered pairs for the time and cost of parking in the second lot and ask them to plot the corresponding graph. Then add the line for the second parking lot to the first graph, above, and **ASK:** Which parking lot will it be cheaper to park in for 2 hours? (second) For 4 hours? (first) How do we see that on the graph? George parked his car in the first lot and Rani parked hers in the second lot at the same time. They each paid the same amount when they left. How long did they each park for? Have students explain the solution.

**SAY:** A third parking lot charges a flat rate but no hourly rate. You pay $12 for parking no matter how long you stay. What would the graph for this parking lot look like? (a horizontal line) Have students draw it. Then have students plot the cost of parking at a fourth parking that charges $1 for any time up to 1 hour, and $6 per hour after that. Which of the four parking lots is the best choice for parking times of 1, 2, 3, and 4 hours? (1 hour: fourth lot, 2 hours: second lot, 3 hours: second or first lot, 4 hours: first lot)
**Bonus** Joe and Zoe left their cars at parking lots 2 and 4 respectively. They parked for the same amount of time and paid the same amount of money. How long did they park?

**SOLUTION:** Let $t$ be the time Joe and Zoe parked.

$$3t = 1 + 6(t - 1)$$

$$3t = 1 + 6t - 6$$

$$3t = 5$$

$$t = \frac{5}{3}$$

Present the following situation and the graph in the margin:

A boat leaves port at 9:00 a.m. and travels at a steady speed. A little while later a man in a kayak leaves his cottage and starts paddling in the same direction as the boat.

**ASK:** How long after the boat left port did the man leave his cottage? (15 minutes) How do you see that on the graph? (the horizontal line becomes a slanted line) At what time did the man start paddling? (9:15) How far from the port is the man’s cottage? (3 km)

**EXTRA PRACTICE:**

a) When did the boat overtake the kayak?

b) How far did the boat travel in the first hour?

c) How long does it take the man in the kayak to travel 1 km?

d) What do you think happened to the man in the kayak 1 hour after the boat left port? (**HINT:** What happens to his line on the graph?) How far did he travel in that hour?

e) Did the boat continue travelling after it met the man or did it stay in the same place for some time? How do you know?

f) The boat docked at another port 9 km away. At what time did this happen?
Review using formulas for linear relations. Tell students that you pay $30 a month for up to 600 text messages plus $2 for each additional text message. **ASK:** How much do each of the first 600 text messages cost? ($30 ÷ 600 = $0.05 = 5¢) If I send 610 text messages this month (i.e., 10 additional text messages), how much will my cellphone bill be? Have students write the expression using the quantities $30, $2, and 10. **ANSWER:** $30 + $2 × 10. **ASK:** Which of these amounts is most likely to change from month to month? (the 10 additional text messages) What do we use to represent an amount that changes? (a variable) What number in the expression $30 + $2 × 10 should we replace with a variable? (10) Why? (because that is what changes) Write on the board:

\[
\text{monthly cost of cellphone} = 30 + 2n, \text{ where } n \text{ is the number of additional text messages above 600}
\]

Remind students that an expression of the sort they have just written can be converted into a verbal rule, such as Multiply the number of text messages above 600 by 2 and add 30.

**ASK:** How much would it cost if I sent 625 text messages? ($30 + $2 × 25 = $30 + $50 = $80) Remind students that replacing a variable with a number in an expression and evaluating it is called substitution. They substituted \( n = 25 \) into the formula for the cost, \( 30 + 2n \).

**Making a T-table from an expression.** Write the expression \( 2n + 30 \) on the board, and show students how you can complete a T-table with headings \( n \) and \( 2n + 30 \) by substituting \( n = 1, n = 2, \) and so on into the expression.
4

Have students finish the T-table up to $n = 5$. Have students make T-tables for various expressions by substituting 1, 2, 3, 4, and 5 for the variable.

**EXTRA PRACTICE:**

a) $3n + 1$  

b) $n + 7$  

c) $2n + 5$  

d) $4n - 2$

**Graphing expressions.** Remind students that we can make a set of ordered pairs from a T-table. Because we can also make a T-table from an expression, we can now make a set of ordered pairs from an expression. Once we have a set of ordered pairs, we can plot the points on a graph, which means that if we are given an expression, we can draw a graph. Follow the steps below to draw a graph from an expression:

**Step 1:** Make a T-table by substituting $n = 1$, $n = 2$, $n = 3$, $n = 4$, and $n = 5$ into the expression.

**Step 2:** Make a set of ordered pairs from the T-table, where the first number is the value for $n$ and the second number is the value of the expression after substituting the first number for $n$.

**Step 3:** Graph the ordered pairs from Step 2.

Have students do Workbook page 90 Question 3. For extra practice, students can graph the expressions above.

**Making a sequence from a formula.** Remind students that a formula is an equation that tells you how to calculate the term from the term number. You would substitute 1 into the formula to get the first term, substitute 2 into the formula to get the second term, and so on.

Demonstrate with the formula in Workbook page 91 Question 1 a):

<table>
<thead>
<tr>
<th>Term Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Term = $2 \times$ Term Number + 1

Model substituting 1 and 2 into the expression and add the results to the table of values, then have students do 3, 4, and 5 individually. Leave the table on the board (to refer to during the discussion below). Point out that the terms now form a sequence.
Have students convert these formulas to sequences:

a) $4 \times \text{Term Number} - 4$

b) $15 - 2 \times \text{Term Number}$

c) Multiply Term Number by 3 and add 5

d) Subtract Term Number from 12

**Bonus** Subtract Term Number multiplied by 3 from 21

**ANSWERS:**

a) 0, 4, 8, 12, 16

b) 13, 11, 9, 7, 5

c) 8, 11, 14, 17, 20

d) 11, 10, 9, 8, 7

**Bonus** 18, 15, 12, 9, 6

Point out that we substitute numbers into the formula in place of Term Number. This means Term Number is a quantity that changes. What do we call a quantity that changes in a formula? (a variable) So Term Number is a variable.

**Converting a formula into a general rule and vice versa.** Tell students that the formula $\text{Term} = 2 \times \text{Term Number} + 1$ can be converted to a verbal rule: Multiply the term number by 2 and add 1. Then ask students to convert several formulas to verbal rules and vice versa. You can use the same examples as above.

**General and stepwise rules.** Write the two types of rules for the sequence 3, 5, 7, 9, 11. (Start at 3 and add 2 each time, Multiply the Term Number by 2 and add 1) Look at the rules and the table of values, side by side, and **ASK:** How are these rules different? (one uses the term number and the other gives directions on how to get the next number from the previous one; one has a variable, Term Number, in it and the other does not) Explain that the rule that uses a variable is called a **general rule.** The rule that tells you how to get the sequence starting from the first term, step by step, is called a **stepwise rule.** Which of the rules is easy to convert to a formula? (the general rule) Why? (it already contains a variable) Emphasize that a general rule is just a verbal form of a formula.

**Producing stepwise rules from sequences.** Rewrite the sequence 3, 5, 7, 9, 11 on the board with enough room to add circles for the gaps, as in Question 1 on Workbook page 91. Demonstrate how to find the rule for the sequence from the gaps. (The gaps are always +2, so the rule is Start at 3 and add 2 each time.) Have students find the stepwise rules for the sequences they found above.

**ANSWERS:**

a) Start at 0, add 4 each time

b) Start at 13, subtract 2 each time

c) Start at 8, add 3 each time

d) Start at 11, subtract 1 each time

**Bonus** Start at 18, subtract 3 each time.

**Connecting sequence properties to the formula.** Which part or parts of the stepwise rule do you see in the formula? What numbers are the same? (the number that you add each time and the number in front of n) Remind students that the number in front of the variable is called the **coefficient** of the variable. **ASK:** Do you think the coefficient of the variable will always be the number that you add each time? Let students share their predictions, and then have them check their predictions on the sequences above and by completing Workbook page 91 Question 1.
SAY: Look at the stepwise rules that tell you to subtract each time instead of to add each time. How are the formulas for those sequences different from the formulas for other sequences? (there is a minus sign in front of the coefficient)

ASK: Where does the gap in the sequence appear in the stepwise rule? Where does it appear in the general rule? How can you find the gap in the sequence from the formula? (The gap is what you add or subtract each time, so it is the coefficient of the variable, i.e., the number that you multiply by the Term Number.)

ASK: How can you find the first term in the sequence from the stepwise rule? (it’s the number you start at) How can you find the first term in the sequence from the formula? PROMPT: What is the term number for the first term? (1, so find the first term by substituting 1 for the term number)

Challenge students to write the rule for the sequence given each formula below, without producing a table of values first:

a) Term = 3 × Term Number + 2
b) Term = 20 − 3 × Term Number
c) Term = 2 × Term Number − 1
d) Term = 11 − 2 × Term number

ANSWERS:

a) The first term is 3(1) + 2 = 5, so start at 5. The coefficient is 3, so add 3 each time. The rule is Start at 5, then add 3 each time.
b) Start at 17, then subtract 3 each time. (Subtract because there is a minus sign in front of the coefficient.)
c) Start at 1, then add 2 each time.
d) Start at 9, then subtract 2 each time.

Repeat for formulas written in terms of $n$. EXAMPLES:

a) $4n - 1$  b) $17 - 2n$  c) $3n + 3$  d) $22 - 3n$

ANSWERS: a) Start at 3, add 4 each time. b) Start at 15, subtract 2 each time. c) Start at 6, add 3 each time. d) Start at 19, subtract 3 each time.

The advantage of the formula over the rule for finding the term for large term numbers. Write the first five terms of a sequence and its corresponding formula. (EXAMPLE: $34 \times \text{Term Number} + 7$; Sequence: 41, 75, 109, 143, 177) Verify that the formula is correct for each of the first five terms. Then challenge students to find the 6th term of the sequence. (211) Ask students for the strategies they used. (EXAMPLES: I found the gap in the sequence and added it to, or subtracted it from, the 5th term. I substituted the term number 6 into the formula.) Have students solve the problem again using whichever strategy they didn’t use the first time, to verify that they get the same answer both ways. Discuss which way was faster. (Using the gap will likely be faster because it only requires adding or subtracting, whereas the formula requires multiplication.) Now tell students...
you want to find the 60th term. **ASK:** Should we keep adding the gap until we find the 60th term, or should we substitute 60 into the formula? Why?

### Extension

For some sequences, it is much easier to produce a stepwise rule than a general rule. For other sequences, a general rule will come more easily than a stepwise rule.

a) Fibonacci numbers are produced using a more complicated stepwise rule. Instead of giving a single number to start with, you have to give the first two terms: Start at 1 and 1. The rule uses the previous two terms instead of just the previous one term: Add the two previous terms to get the next term. Write the first eight terms of the Fibonacci sequence.  
(\text{NOTE:} \text{ A general formula for this sequence requires high school math.})

b) Here is a rule that seems simple: Start at 1 and add the gap each time. But instead of telling you what the gap is, there is a rule to find the gap: Start at 3 (the gap between the first and the second term), and add 2 to the gap each time. Write the first five terms of this sequence. Find a general rule and a formula for the sequence.

c) Find a stepwise rule and a formula for this sequence: 3, 6, 11, 18, 27. Which one is easier to find? (\text{Hint: Use the sequence from part b.})

d) Write the first five terms of the sequence given by this formula:  
\[
\text{Term Number} \times \text{Term Number} \times \text{Term Number} + 5.
\]

Find a stepwise rule for it. (\text{HINT: First find the rule for the gaps in the gaps.})

**ANSWERS:**

a) 1, 1, 2, 3, 5, 8, 13, 21


Formula: \(\text{Term Number} \times \text{Term Number} = (\text{Term Number})^2.\)

General rule: Square the Term Number OR Multiply the Term Number by itself

c) Stepwise rule: Start at 3 and add the gap each time. To find the gap, start at 3 (the gap between the first and the second term), and add 2 to the gap each time.

Formula: \(\text{Term Number} \times \text{Term Number} + 2\)

d) Sequence: 6, 13, 32, 69, 130

Stepwise rule: To find the gaps in the gaps, start at 12 and add 6 each time. To find the gaps, start at 7 (gap between terms 1 and 2) and add the gaps in the gaps. To find the sequence itself, start at 6 and add the gaps you found.
Show students several sequences made of blocks with a multiplicative rule, such as:

ASK: How do we obtain each new figure from the previous one? (by adding two squares and four triangles) Which rule is this rule similar to, a general rule or a stepwise rule? (stepwise) Can you describe how to get a next term in a different way that is similar to a general rule, using the first figure only? (repeat the first figure several times) How many times for each figure? (1 for the first term, 2 for the second term, 3 for the third term, and so on) Point out that the first figure is repeated the number of times that is equal to the figure number.

Have students draw T-tables for the number of triangles and the number of squares, and fill in the numbers for 3 figures in the sequence:

<table>
<thead>
<tr>
<th>Figure Number (f)</th>
<th>Number of Squares (s)</th>
<th>Figure Number (f)</th>
<th>Number of Triangles (t)</th>
<th>Number of Squares (s)</th>
<th>Number of Triangles (t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>2 f</td>
<td>f</td>
<td>4 f</td>
<td>f</td>
<td>2 s</td>
</tr>
</tbody>
</table>

Ask students to write a formula for each table. (If necessary, prompt students to use the general rule they figured out for the whole pattern: the first figure is repeated term number times. There are 2 squares in the first figure, so the nth figure will have 2n squares.) ANSWERS: \( s = 2 \times f, \) \( t = 4 \times f, t = 2 \times s. \)

Direct variation. Remind students of the meaning of the terms input and output. Explain to students that when the rule is “Multiply the input by __,” we say that the output varies directly with the input. So in this pattern, the Number of Squares varies directly with the Figure Number, and the Number of Triangles varies directly with both the Figure Number and the Number of Squares.
Present several tables and have students say whether the output varies directly with the input or not. **EXAMPLES:**

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>8</td>
<td>18</td>
</tr>
<tr>
<td>10</td>
<td>24</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>

**Bonus**

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

Draw a sequence of squares with side lengths 1, 2, 3, and so on. Ask students to find the areas and the perimeters of the squares. Ask them to make a T-table for side length and area and another T-table for side length and perimeter, in order to determine which quantity varies directly with the side length. (perimeter) Ask students to write a formula for the perimeter and for the area of the square.

**Bonus**

a) The number of feet, \( f \), varies directly with the number of people, \( p \). (2 people, 4 feet; 3 people, 6 feet; 4 people, 8 feet; \( f = 2 \times p \)). Does the number of paws vary directly with the number of cats? What is the formula? (\( c = 4p \))

b) A cat has five claws on each front paw and four claws on each back paw. Make a T-table showing the number of cats and the number of claws and another T-table showing the number of paws (add one paw at a time in the same order, e.g., right front, left front, right hind, left hind—then go to the next cat—right front, left front, and so on) and the number of claws. Does the total number of claws vary directly with the number of paws or with the number of cats? (The total number of claws varies directly with the number of cats (\( \text{claws} = 18 \times \text{cats} \)), but not with the number of paws.)

Ask students to draw two sequences of blocks in which each figure is made by adding a fixed number of blocks to the previous figure, such that in one sequence the number of blocks varies directly with the figure number and in the other sequence the number of blocks does not vary directly with the figure number. Students can swap their designs and have their partners identify which design shows direct variation with the figure number and which does not.
EXAMPLE:

The number of blocks varies directly with the figure number.

The number of blocks does not vary directly with the figure number.

Formulas when the number of blocks does not vary directly with the figure number. In the second sequence above, shade the growing towers of blocks in each figure: 3 blocks in Figure 1, 6 in Figure 2, 9 in Figure 3. Ask students to draw two T-tables, one for the figure number and the number of shaded blocks, and the other for the figure number and the total number of blocks. **ASK:** In which T-table does the number of blocks in the output column vary directly with the figure number: the one showing the number of shaded blocks or the one showing the total number of blocks?

Have students produce a formula for the number of shaded blocks and explain their reasoning. Then **ASK:** Does the number of unshaded blocks change from figure to figure? To get the total number of blocks, what do you have to add to the number of shaded blocks? (the number of unshaded blocks) Have students write the formula for the total number of blocks.

**EXTRA PRACTICE:**

A cab charges a flat rate of $4 (you pay this just for using the cab) and $2 for every minute of the ride. Write a formula for the price of a cab ride. How much will you pay for a 4-minute cab ride? For a 5-minute ride?

Question 4 on Workbook page 93 is challenging. (Students will learn this material in depth in the next lesson; use this question as an opportunity for problem solving.) Guide students by suggesting that they look at the problems they did in Question 3 and write the sequences for both the shaded blocks and the total blocks. How are they the same? How are they different? How can they obtain the sequence for the shaded blocks from the sequence for the total number of blocks?

Have students produce formulas for the sequences they drew earlier (the sequences of blocks that do and do not vary directly). They can then swap their designs with partners and produce formulas for their partner’s designs as well.

**Extension**

Find a sequence of rectangles in which area varies directly with length. Does the perimeter vary directly with the length? (**ANSWER:** Use rectangles of the same width, say 5. The area will be $5L$, so it will vary directly with the length ($L$). The perimeter in this case will be $2L + 10$ and will not vary directly with the length.)
The gap between the terms of a sequence is the coefficient in the formula for the sequence. Give each student a pair of dice of different colours. Ask students to roll their dice and write a sequence according to this rule: Multiply the term number by the result on the red die and add the result on the blue die. Students need to write a formula for the rule and produce the first three terms of the sequence. They can record the sequence in a T-table.

Ask students to find the difference between the terms of their sequences (the gap). After they have created several sequences, ask students to identify where the gap appears in each formula. Review with students the fact learned in PA7-23: the coefficient in the formula is the gap in the sequence, so the gap here will be the number rolled on the red die.

The gap in a geometric pattern. Draw or make the following sequence:

Ask students to describe what part of the pattern changes (the shaded part, the vertical stacks) and what part stays the same (unshaded part, the bottom bar). Draw a T-table for the number of blocks in each figure of the sequence as shown.
Ask students to predict the gap between the terms in the output column (Number of Blocks) before you fill in the column. How do they know? (The “gap” between terms in the T-table is the number of new blocks added to the pattern at each stage.) **ASK:** How can you find the number of shaded blocks in each figure, using the figure number? (multiply the figure number by the “gap”) Have students solve Question 2 on Workbook page 94 for practice.

**Formulas for sequences that do not vary directly with the term number.**
Present several sequences (**EXAMPLES:** 7, 10, 13, 16; 5, 9, 13, 17; 12, 19, 26, 33) and have students fill in the first and the third columns in a table with these three columns:

<table>
<thead>
<tr>
<th>Term Number</th>
<th>Term Number × Gap</th>
<th>Term</th>
</tr>
</thead>
</table>

Then ask students to find the gap between the terms and to fill in the middle column of the table. Ask students to compare the numbers in the second and third columns. **ASK:** How can you obtain the numbers in the third column from the numbers in the second column? (by adding the same number) Have students write both a general rule for the sequence (Multiply the term number by ___ and add ___) and a formula (Term Number × ___ + ___).

Compare one of these tables to the table for patterns made from shaded and unshaded blocks above. Which column would the number of shaded blocks correspond to? (Term Number × Gap) What does the gap in this table correspond to? (the number of blocks added each time, the number of vertical stacks) What does the number of unshaded blocks correspond to? (the number added to the second column to get the third column)

Next, present several patterns where the adjustment factor (the number used to get from the second column to the third column) should be subtracted from the product of the term number and the gap. **EXAMPLES:** 1, 4, 7, 10; 3, 7, 11, 15; 4, 11, 18, 25. Again, have students create tables, compare the columns, determine the adjustment factor, and find the general rule and the formula. Ask students also to check by substitution that their formula works for all of the rows of the table.

Let students practise finding rules for various tables, following the steps in the box on Workbook page 95. They can also use the following sequences to create tables:

a) 2, 7, 12, 17  b) 21, 33, 45, 57  c) 2, 23, 44, 65
Applications of pattern rules. Solve the following problem as a class, then have students practise solving similar questions (see Extra Practice below).

Rita builds towers by stacking cubes $5 \text{ cm} \times 5 \text{ cm} \times 5 \text{ cm}$ one on top of the other. Find the formula for the surface area of her tower. What are the height and the surface area of a tower that is 30 cubes tall?

**SOLUTION:** Tower number $n$ has $n$ cubes in it. Each cube face has area 25 cm$^2$. (NOTE: Each time you add a cube to the tower, you are adding 4 cube faces to the total surface area. The area of the top and bottom of the tower are counted only once, at the beginning.)

| Tower number ($n$) | $n \times$ gap | Surface area (cm$^2$)  
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>$6 \times 25 = 150$</td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>$10 \times 25 = 250$</td>
</tr>
<tr>
<td>3</td>
<td>300</td>
<td>$14 \times 25 = 350$</td>
</tr>
</tbody>
</table>

Gap = 100

Formula: $100n + 50 \text{ cm}^2$

A tower 30 cubes tall has $n = 30$. Its height is $5 \times 30 \text{ cm} = 150 \text{ cm}$ and its surface area is $100 \times 30 + 50 = 3050 \text{ cm}^2$.

**EXTRA PRACTICE:**

a) Find the rules for the perimeter and the area of the figures in the following pattern. Use your rules to predict the perimeter and the area of Figure 15.

b) Find a formula for the number of inner line segments. How many inner line segments does the 20th figure in this pattern have?

**ANSWERS:**
a) Perimeter = $4n + 8$. Area = $4n + 3$. For $n = 15$, perimeter is 68, area is 63.
b) Number of inner line segments = $6n + 2$. For $n = 20$, it is 122.

**Extensions**

1. Find the rules for these T-tables. How do the rules relate to each other?
ANSWER: The rule for the first table is Multiply Input by 3 and add 2. The rule for the second table is Subtract 2 from the Input and divide the result by 3. In the second table we are just undoing the operations in the first table since we get back to where we started (Output in the second table = Input in the first table); thus, we have to do the "opposite" operations in the reverse order. Alternatively, some students might notice that another way to formulate the rule for the second table is to divide by 3 and then subtract 2/3.

Indeed, by the distributive property, \((x - 2) \div 3 = x \div 3 - 2 \div 3\).

2. a) In the towers Rita built (see above), how many cubes are in the first tower that has surface area greater than 1 m\(^2\)? (The first tower with surface area greater than 1 m\(^2\) has surface area 10 050 cm\(^2\). So 10 050 = 100n + 50. Solving for n we get \(n = 100\), so the tower would be 100 cubes tall.)

b) How tall is the tower in metres? (5 m)

c) Do you think that Rita can build this tower? Why or why not? (She will need help to build a tower 5 m tall!)
Converting a graph into a sequence. Review finding the gaps in a sequence if you are given a stepwise rule (Start at 3 and add 1 each time), a general rule (Multiply the term number by 2 and add 1), or a formula (Term Number $\times 2 + 1$). Remind students how to produce a sequence from a graph: the horizontal axis shows the term number and the vertical shows the term value, so point (1, 0) mean that the first term in a sequence is 0. Give students several graphs and have them produce a sequence from the graph. **EXAMPLES:**

![Graphs](image_url)

The gap in the sequence on the graph. Have students use the graphs to find the gaps between the terms in the sequences they produced.

**ASK:** Where would you look on the graph to see the difference in the term numbers? (the horizontal axis) Where would you look to see the difference between the term values? (the vertical axis) Where would you look to see the gap in the sequence? (the vertical axis, because the gap is the difference in term values)

Graph several linear increasing sequences and have students identify the gap in each sequence without writing out the sequence itself. Then have them write out the sequences to check their answers.
Present several formulas with different coefficients and ask students to tell which of these formulas could match the graph at left. **EXAMPLE:**

\[ 4n + 2 \quad 3n + 3 \quad 5n + 1 \quad 2n + 4 \]

**ASK:** Why aren’t the other formulas able to match this graph? (The gap in the sequence is 3, so the coefficient of \( n \) must be 3.)

**Using the first term of the sequence to identify the graph.** Ask students to think about how they could decide which of the following formulas with coefficient 3 could be the formula for the sequence in the graph:

\[ 3n + 1 \quad 3n + 2 \quad 3n + 3 \quad 3n - 2 \]

If the idea of looking at the first term does not rise, ask students to think about what the stepwise rule for this sequence could be. Where would they start? What would the stepwise rule for the sequence given by each formula be? Does this information help them decide which formula matches the sequence in the graph?

Finally, present several graphs and their formulas (out of order) and have students match the graphs to the formulas.

**ACTIVITY**

Students will each need grid paper and two dice. Students should roll the dice five times and record the pairs of numbers they get. For each pair of numbers, let \( l \) be the larger number and \( s \) be the smaller number, and write one of two formulas, \( l \times n + s \) or \( l \times n - s \), on separate cards. All formulas should be different, so if the same pair of numbers appears a second time, students should use \( l \times n - s \) if they used \( l \times n + s \) before, and vice versa. If the same pair of numbers appears a third time, students need to roll again. Students should graph the sequences given by these formulas on graph paper, but should not record the formula with the graph. Next, they should swap their formulas and graphs with a partner and match the formulas produced by their partners to the graphs of these formulas.
Review finding values of terms from a graph of a sequence.

Finding the term number given the term value. Have students fill in the table for the graph at left.

<table>
<thead>
<tr>
<th>Term number</th>
<th>1</th>
<th>7</th>
<th>Answers: 4, 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term value</td>
<td>2</td>
<td>4</td>
<td>Answers: 1, 3</td>
</tr>
</tbody>
</table>

If necessary, point out the strategy of starting at the term value on the vertical axis and going right until you reach the graph, and then descending to the horizontal axis to find the term number. Have students extend the line to find out what the next whole-number term value would be. (13th term, 5)

Have students graph these sequences: a) 4, 10, 16, 22; b) 8, 12, 16, 20; c) 4, 7, 10, 13. Alert students to the fact that they will need large grids, because they will need to extend the graphs. Then ask students to use their graphs to find the value of the 10th term of each sequence. To check their answers, students should find formulas for the sequences and find the value of the 10th term using algebra. Which method do students like better? Why?

Next, have students use the graphs to find which term in each sequence equals 40. Then have them substitute the term number they found into the formula to check their answers. (ANSWERS: a) 7th, b) 9th, c) 13th)

As a challenge, have students check whether there is a term in each of the sequences that equals 36. If there is, which term is it? (ANSWERS: a) no, b) 8th, c) no)

Workbook pages 101–104 (PA7-28 Problems and Puzzles) can be used to review all the material studied in this unit.
Extensions

1. a) Use the formula for the sequence, divisibility rules, and factors to explain why 216 cannot be a member of the sequence 4, 10, 16, 22.
   b) Use the formula for the sequence and make an equation to check whether 216 can be a member of these sequences:
   i) 8, 12, 16, 20
   ii) 4, 7, 10, 13

   **ANSWER:**
   a) The formula for the sequence is $6n - 4$. $6n$ is always divisible by 6, but if we subtract 4, the number is never divisible by 6. Since 216 is divisible by 6, it cannot be a term in the sequence.
   b) i) The formula for the sequence is $4n + 4$. If 216 is a member of this sequence for some $n$, $216 = 4n + 4$. Solving for $n$ we get $n = 53$, so 216 is the 53rd term in the sequence. Another way to look at it is to say $4n + 4 = 4(n + 1)$, so if 216 is divisible by 4, it is some term in the sequence. Indeed, $216 = 200 + 16$, both divisible by 4, so 216 is divisible by 4 and is a term in the sequence.
   ii) The formula for the sequence is $3n + 1$. If 216 is a member of this sequence for some $n$, $216 = 3n + 1$. Solving for $n$ we get $n = 71 \frac{2}{3}$, so 216 is not a term in the sequence. Another way to look at it is to say that if 216 is a term in the sequence, then 215 should be divisible by 3. However, $2 + 1 + 5 = 8$ is not divisible by 3, so 215 is not divisible by 3 either. This means 216 is not a term in the sequence.

2. Each arrow on the picture is 1 m long.
   a) Find the areas of the rings, starting with the first ring (inner radius 1 m, outer radius 2 m).
   b) Find a formula for the area of the rings. What is the area of the 50th ring?

   **SOLUTION:**
   a) Ring 1: $4\pi - \pi = 3\pi \text{ m}^2$
   Ring 2: $9\pi - 4\pi = 5\pi \text{ m}^2$
   Ring 3: $16\pi - 9\pi = 7\pi \text{ m}^2$
   b) There can be two formulas for the area of the ring:
      • using the subtraction of areas: $(n + 1)^2\pi - n^2\pi$
      • using the gaps in the pattern: the gaps are $2\pi$, so the formula is $2\pi n + \pi$

   Using both formulas, we get the area of the 50th ring: $51^2\pi - 50^2\pi = 2601\pi - 2500\pi = 101\pi \text{ m}^2$ or $2\pi \times 50 + \pi = 101\pi \text{ m}^2$. 
PS7-7  Looking for a Pattern II

Teach this lesson after: 7.2 Unit 4

Goals:
Students will use algebra to determine the sum of sequences with constant gaps.

Prior Knowledge Required:
Can read and write algebraic expressions
Can determine the formula to get the term from the term number for a pattern with constant gaps
Can solve linear equations
Can add, subtract, and multiply two-digit numbers
Can add fractions with the same denominator (for Problem Bank 11)
Can multiply a fraction by a whole number (for Problem Bank 12)
Can use repeating decimal notation correctly (for Problem Banks 12–15)
Can multiply terminating decimals by 100 (for Problem Banks 13–15)
Can find a given percentage of a given number (for Extended Problem)
Can find what percentage a given number is of another number (for Extended Problem)

Vocabulary: expression, formula, gap, pattern, term, term number

Materials:
BLM Percentage Discounts (pp. O-40–43, see Extended Problem)

Comparing sequences. Write on the board:

3, 6, 9, 12, …

ASK: What is the gap? (3) SAY: Every gap is 3. ASK: What is the next term? (15) How did you get that? (I added 3 to 12) What is the 10th term? (30) PROMPT: These terms are all multiples of 3. ASK: How do you know the 10th term is 30? (3 × 10 = 30) Write on the board:

4, 7, 10, 13, …

ASK: What is the gap? (3) SAY: Every gap is 3, just like the other sequence. Explain to students that to find the 10th term of this sequence, they can compare each term of this sequence with the previous sequence using a number line. Draw on the board:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td></td>
</tr>
</tbody>
</table>

ASK: If the 10th term of the top sequence is 30, what is the 10th term of the bottom sequence? (31) How did you get that? (I added 1) Leave the number lines on the board for the following exercises.
**Exercises:**
a) What is the 100th term of the top sequence?
b) What is the 100th term of the bottom sequence?

**Answers:** a) 300, b) 301

SAY: Both sequences have the same gap, but the top sequence is the multiples of the gap, so it’s easier to work with.

**Exercises:**
1. Decide how the two sequences are related. Then find the 100th term of each sequence.
   a) A. 2, 4, 6, 8, 10, …  
      B. 7, 9, 11, 13, 15, …
   b) A. 4, 8, 12, 16, 20, …  
      B. 3, 7, 11, 15, 19, …
   c) A. 3, 6, 9, 12, 15, …  
      B. 1, 4, 7, 10, 13, …
   d) A. 5, 10, 15, 20, 25, …  
      B. 1, 6, 11, 16, 21, …

**Answers:**
a) A. 200, B. 205; b) A. 400, B. 399; c) A. 300, B. 298; d) A. 500, B. 496

2. Compare with another sequence to find the 100th term. Hint: Find the gap and use the multiples of the gap.
   a) 5, 9, 13, 17, 21, …  
   b) 3, 9, 15, 21, 27, …

**Selected solution:**
a) The gap is 4. The sequence of multiples of 4 is 4, 8, 12, …, and it has 100th term 400, so the given sequence has 100th term 401.

**Answer:** b) 597

**Comparing sequences where one sequence is made by multiplying by the terms in the other sequence.** Write on the board:

<table>
<thead>
<tr>
<th>Term Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>B.</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>

SAY: Here are two sequences. ASK: How are the terms in sequence B obtained from the terms in sequence A? (multiply by the term number) PROMPT: What do I multiply by? SAY: The first terms are the same, the second term is doubled, the third term is multiplied by 3, and in general the \( n \)th term is multiplied by \( n \). ASK: What is the 100th term in sequence A? (101) So, what is the 100th term in sequence B? (100 \( \times \) 101 = 10 100)

**Exercises:**
1. Decide how the three sequences are related. Then find the 100th term of each sequence.
   a) A. 1, 2, 3, 4, 5, …  
      B. 4, 5, 6, 7, 8, …  
      C. 4, 10, 18, 28, 40, …
   b) A. 2, 4, 6, 8, 10, …  
      B. 1, 3, 5, 7, 9, …  
      C. 1, 6, 15, 28, 45, …

**Answers:**
a) A. 100, B. 103, C. 10 300; b) A. 200, B. 199, C. 19 900

2. Create other sequences to help you find the 100th term of the sequence: 4, 14, 30, 52, 80, …

**Solution:** Make the sequence 4, 7, 10, 13, 16, … by dividing each term by the term number, then make the sequence 3, 6, 9, 12, 15, … by subtracting 1 from each term. The sequence 3, 6, 9, 12, 15, … has 100th term 300; the sequence 4, 7, 10, 13, 16, … has 100th term 301, and so the sequence 4, 14, 30, 52, 80, … has 100th term 300.
Comparing sums using variables. Write on the board:

Let \( x = 1 + 2 + 3 + 4 + 5 + 6 + 7 \)
Then: \( 2 + 3 + 4 + 5 + 6 + 7 + 8 = \)
\( 3 + 6 + 9 + 12 + 15 + 18 + 21 = \)
\( 7 + 6 + 5 + 4 + 3 + 2 + 1 = \)
\( 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \)

Have volunteers write simple expressions for the first sum, \( 2 + 3 + 4 + \ldots + 8 \), in terms of \( x \), and explain their reasoning. (it could be \( x + 7 \) because each term is 1 greater and there are seven terms, or it could be \( x + 8 - 1 \) because you are adding 8 but subtracting 1) Ensure both strategies come up. Then repeat for each of the other sums in turn. (3\( x \) because each term is three times the original, \( x \) because you are adding the same numbers but in a different order, \( x + 8 \) because you are just adding 8 to the original sum)

Exercises:
1. Use \( x = 1 + 2 + 3 + 4 + 5 + \ldots + 50 \). Write each sum in terms of \( x \).
   a) \( 1 + 2 + 3 + 4 + 5 + \ldots + 50 = x \)
   b) \( 2 + 4 + 6 + 8 + 10 + \ldots + 100 = \)
   \[
   \begin{align*}
   + 1 + 2 + 3 + 4 + 5 + \ldots + 50 &= x \\
   &- (1 + 1 + 1 + 1 + 1 + \ldots + 1) = \\
   &2 + 4 + 6 + 8 + 10 + \ldots + 100 = \\
   &1 + 3 + 5 + 7 + 9 + \ldots + 99 = \\
   \end{align*}
   \]
   c) \( 1 + 2 + 3 + 4 + 5 + \ldots + 50 = \\
   \[
   + (1 + 2 + 3 + 4 + \ldots + 49) = \\
   + 1 + 3 + 5 + 7 + 9 + \ldots + 99 = \\
   \]
   d) \( 1 + 2 + 3 + 4 + 5 + \ldots + 50 = \\
   \[
   + 1 + 2 + 3 + 4 + 5 + \ldots + 50 = \\
   + 3 + 6 + 9 + 12 + 15 + \ldots + 150 = \\
   \]

Answers: a) \( 2x \), b) \( 2x - 50 \), c) \( x + x - 50 \), d) \( 3x \)

2. Use \( x = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \). Write each sum in terms of \( x \).
   a) \( 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 = \)
   b) \( 10 + 9 + 8 + 7 + \ldots + 1 = \)
   c) \( 5 + 10 + 15 + 20 + \ldots + 50 = \)
   d) \( 1 + 2 + 3 + \ldots + 10 + 11 = \)

Bonus: \( 15 + 20 + 25 + 30 + \ldots + 60 \)

Answers: a) \( x + 20 \) or \( x + 11 + 12 - 1 - 2 \), b) \( x \), c) \( 5x \), d) \( x + 11 \), Bonus: \( 5x + (10 \times 10) \) or \( 5(x + 20) \)

Using algebra to add the numbers from 1 through \( n \). SAY: You can use the fact that adding in any order gets the same answer to find clever ways to add the sum twice. Write on the board:

\[
1 + 2 + 3 + 4 + 5 + 6 + 7 = x \\
+ 7 + 6 + 5 + 4 + 3 + 2 + 1 = x
\]

Have a volunteer add the equations term by term. \( (8 + 8 + 8 + 8 + 8 + 8 = 2x) \) ASK: How many 8s are being added? (7) How do you know? (there are seven numbers from 1 to 7) How does it make it easier to add the numbers from 1 to 7? (by adding it twice, you see that you can just multiply \( 7 \times 8 \), so the original sum must be half of \( 7 \times 8 \)) What is \( 7 \times 8 \)? (56) So, what is the sum of 1 through 7? (28) Write on the board:

\[
2x = 56, \text{ so } x = 28
\]
Exercises: Evaluate \( x \) by first evaluating \( 2x \).

\[ \begin{align*}
a) & \quad 1 + 2 + 3 + \ldots + 10 = x & b) & \quad 1 + 2 + 3 + \ldots + 20 = x \\
& \quad + 10 + 9 + 8 + \ldots + 1 = x & & \quad + 20 + 19 + 18 + \ldots + 1 = x \\
c) & \quad 1 + 2 + 3 + \ldots + 30 = x & d) & \quad 1 + 2 + 3 + \ldots + 40 = x \\
& \quad + 30 + 29 + 28 + \ldots + 1 = x & & \quad + 40 + 39 + 38 + \ldots + 1 = x \\
\end{align*} \]

Answers: a) \( 2x = 10 \times 11 = 110 \), so \( x = (10 \times 11) \div 2 = 55 \); b) 210; c) 465; d) 820

When students finish the exercises, point out how much time they saved by not having to do all that addition. Even with a calculator, adding 30 or 40 numbers would take a long time.

Evaluating sums of consecutive numbers that don’t start at 1. Write on the board:

\[ 21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 \]

Say: I can add the numbers 1 to 20 and I can add the numbers 1 to 30. Ask: How can I add the numbers 21 to 30? (subtract the two sums)

Exercises: Evaluate the sum.

\[ \begin{align*}
a) & \quad 11 + 12 + 13 + \ldots + 20 \\
b) & \quad 21 + 22 + 23 + \ldots + 30 \\
c) & \quad 31 + 32 + 33 + \ldots + 40 \\
\end{align*} \]

Selected solution: a) \((1 + 2 + \ldots + 20) - (1 + 2 + \ldots + 10) = (20 \times 21 \div 2) - (10 \times 11 \div 2) = 210 - 55 = 155\)

Answers: b) 255, c) 355

When students finish the exercises, challenge them to look for a pattern in their answers. (each answer is 100 more than the previous one) Ask: Why does this happen? (each term is 10 more than in the previous sum, so you are adding ten 10s) Write on the board:

\[ \begin{align*}
21 + 22 + 23 + \ldots + 30 &= x \\
10 + 10 + 10 + \ldots + 10 &= 100 \\
31 + 32 + 33 + \ldots + 40 &= x + 100 \\
\end{align*} \]

Say: You are adding 10 to each term and there are ten 10s being added, so you are adding 100 in total. That’s why the answer is always 100 more.

Adding sequences with constant gaps using algebra. Write on the board:

\[ \begin{align*}
1 + 3 + 5 + \ldots + 95 + 97 + 99 &= x \\
99 + 97 + 95 + \ldots + 5 + 3 + 1 &= x \\
100 + 100 + 100 + \ldots + 100 + 100 + 100 &= 2x \\
\end{align*} \]
SAY: It’s easy to see here that you need to add 100 many times, but it is not so obvious how many times you are adding 100. For that, you need to know how many terms are in the sum. But then you need to know the term number for the term 99 in the pattern. Write on the board:

1, 3, 5, 7, …, 99

ASK: What is the gap in this pattern? (2) What is the formula to get the term from the term number? (term number × 2 − 1) Write on the board:

99 = 2 × term number − 1

SAY: Remember, to get the formula for a pattern with constant gap, you can compare the pattern to the sequence of multiples of the gap. Write on the board:

1, 3, 5, 7, …, 99
2, 4, 6, 8, …, 100

SAY: Each term is 1 less than the sequence of multiples, so each term is two times the term number minus 1.

**Exercises:** Write the formula for how to get the term value from the term number.

a) 1, 4, 7, 10, …  
   b) 5, 7, 9, 11, …  
   c) 2, 7, 12, 17, …  
   d) 27, 37, 47, 57, …  

**Answers:** a) 3 × term number − 2, b) 2 × term number + 3, c) 5 × term number − 3, d) 10 × term number + 17

SAY: Once you know the formula, you need to solve an equation. Let’s use \( n \) for the term number. Write on the board:

99 = 2\( n \) − 1

Have a volunteer solve the equation. (\( n = 50 \))

**Exercises:** Which term number has value 127 in each of the previous exercises?

**Answers:** a) 43, b) 62, c) 26, d) 11

Refer students back to the original problem. SAY: Now that we know that there are 50 terms, we know that we are adding 100 fifty times. Write on the board:

\[ 2x = 50 \times 100 = 5000, \text{ so } x = 2500 \]

**Exercises:**

1. Evaluate the sum using algebra.
   a) 1 + 4 + 7 + 10 + … + 127  
   b) 5 + 7 + 9 + 11 + … + 127  
   c) 2 + 7 + 12 + 17 + … + 127  
   d) 27 + 37 + 47 + 57 + … + 127  

**Answers:** a) 128 × 43 + 2 = 2752, b) 62 × 132 + 2 = 4092, c) 129 × 26 + 2 = 1677, d) 154 × 11 + 2 = 847
2. Evaluate the sum.
   a) \(3 + 5 + 7 + 9 + 11 + \ldots + 29\)
   b) \(15 + 16 + 17 + \ldots + 85\)
   c) \(13 + 17 + 21 + 25 + \ldots + 101\)
   **Answers:** a) 224, b) 3550, c) 1311

**Problem Bank**

1. Use the first sum to evaluate the second sum. Explain how you know.
   a) Since \(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36\), what is \(4 + 5 + 6 + 7 + 8 + 9 + 10 + 11\)?
   b) Since \(2 + 3 + 4 + 5 + 6 = 20\), what is \(45 + 51 + 57 + 63 + 69 + 75 + 81\)?
   c) Since \(1 + 2 + 3 + 4 + \ldots + 15 = 120\), what is \(1 + 3 + 5 + 7 + 9 + \ldots + 29\)?
   **Sample answers:** a) \(36 + 8 \times 3 = 36 + 24 = 60\) because I added 3 to each of the eight terms;
   b) \(20 \times 17 = 340\) because I multiplied each term by 17; c) \(120 + (120 - 15) = 225\) because I added term by term the sums \((1 + 2 + 3 + \ldots + 15)\) and \((0 + 1 + 2 + \ldots + 14)\) and the second sum is 15 less than the first sum

2. Evaluate \(46 + 47 + 48 + 49 + \ldots + 60\) in two ways. Make sure you get the same answer both ways.
   a) \[
   \frac{1 + 2 + 3 + \ldots + 45 + 46 + 47 + \ldots + 60}{(1 + 2 + 3 + \ldots + 45)} = \]
   \[
   46 + 47 + \ldots + 60 = \]
   **Answers:** a) 1830 - 1035 = 795, b) 120 + 675 = 795

3. Evaluate \(24 + 26 + 28 + 30 + 32 + 34\) in two ways. Make sure you get the same answer both times.
   a) Use \(12 + 13 + 14 + 15 + 16 + 17\).
   b) Use \(1 + 3 + 5 + 7 + 9 + 11\).
   **Answers:** a) \(24 + 26 + 28 + 30 + 32 + 34 = 2 \times (12 + 13 + 14 + 15 + 16 + 17) = 2 \times 87 = 174\),
   b) \(24 + 26 + 28 + 30 + 32 + 34 = (1 + 3 + 5 + 7 + 9 + 11) + (23 + 23 + 23 + 23 + 23 + 23) = 36 + 6 \times 23 = 36 + 138 = 174\)

4. Use \(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 2.45\) to evaluate \(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\). Explain how you got your answer.
   **Answer:** \(2.45 - 1 = 1.45\)

5. a) Use \(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 1.45\) to evaluate \(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\).
   b) Use \(2.83 \approx 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{9}\) to determine a good estimate for \(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{10}\).
   c) Use \(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 2.45\) to evaluate \(2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6}\).
   **Answers:** a) 0.95, b) 2.93, c) 8.45
6. What is the 50th term of the sequence?
   a) 5, 11, 17, 23, …  
   b) 33, 38, 43, 48, …  
   c) 10, 13, 16, 19, …  
   **Answers:** a) 299, b) 278, c) 157

7. Find the sum of the first 50 terms.
   a) 5, 11, 17, 23, …  
   b) 33, 38, 43, 48, …  
   c) 10, 13, 16, 19, …  
   **Selected solution:** a) \(304 \times 50 \div 2 = 7600\)  
   **Answers:** b) 7775, c) 4175

8. Find the gap.
   a) A sequence with constant gaps has first term 5 and third term 13.  
   b) A sequence with constant gaps has first term 4 and fourth term 13.  
   c) A sequence with constant gaps has first term 2 and fifth term 34.  
   d) A sequence with constant gaps has first term 3 and tenth term 57.  
   **Answers:** a) 4, b) 3, c) 8, d) 6

9. The sum of the numbers from 1 to \(N\) is 153. What is \(N\)?  
   **Answer:** 17

10. The sum of the numbers from 1 to \(N\) is 3570. What is \(N\)?  
    **Answer:** 84

11. You can evaluate infinite sums using algebra too. Use \(x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\)  
   a) Write an expression for \(2x\).
      \[x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\]  
      \[+ x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\]  
      \[2x = \text{___________} \]  
   b) Write an expression for \(2x - 2\).  
   c) Write an equation to show how your answer for part b) compares with \(x\).  
   d) Solve your equation from part c).  
   **Answers:** a) \(2 + 1 + 1/2 + 1/4 + \ldots\), b) \(1 + 1/2 + 1/4 + 1/8 + \ldots\), c) \(2x - 2 = x\), d) \(x = 2\)

12. A repeating decimal is an infinite sum, and you can evaluate it using algebra too.  
    Use \(x = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots\)  
    a) What repeating decimal is \(x\) equal to?  
    b) Write an expression for \(10x\) by multiplying each term by 10.  
    c) Complete the equation: \(10x = x + \text{______}\).  
    d) Solve the equation from part c).  
    e) What fraction is \(x\) equal to?  
    **Answers:** a) \(0.1\); b) \(1 + 1/10 + 1/100 + \ldots\); c) 1; d) \(9x = 1\), so \(x = 1/9\); e) 1/9
13. Use \( x = 0.45 + 0.0045 + 0.000045 + 0.00000045 + \ldots \)
   a) What repeating decimal is \( x \) equal to?
   b) Write an expression for 100\( x \) by multiplying each term by 100.
   c) Complete the equation: 100\( x = x + \ldots \).
   d) Solve the equation from part c).
   e) What fraction is \( x \) equal to?
   **Answers:** a) \( 0.45 \), b) \( 45.45 \), c) 45, d) \( x = \frac{45}{99} = \frac{5}{11} \), e) \( \frac{5}{11} \)

14. a) 5.5555\ldots = 0.5555\ldots + \ldots and 5.5555\ldots = 0.5555\ldots \times \ldots.
    b) If \( x = 0.5555\ldots \), then \( x + \ldots = x \times \ldots \).
    c) Solve your equation from part b). Then write 0.5555\ldots as a fraction.
    **Answers:** a) 5, 10; b) \( x + 5 = 10x \); c) \( x = \frac{5}{9} \)

15. a) 87.8787\ldots = 0.878787\ldots + \ldots and 87.8787\ldots = 0.878787\ldots \times \ldots.
    b) If \( x = 0.8787\ldots \), then \( x + \ldots = x \times \ldots \).
    c) Solve your equation from part b). Then write 0.8787\ldots as a fraction.
    **Answers:** a) 87, 100; b) \( x + 87 = 100x \); c) \( x = \frac{87}{99} = \frac{29}{33} \)

16. How does the grid show the expression?
   a) 1 + 3 + 5 + 7 + 9    b) 5 \times 5
   **Answers:** a) count the squares in each alternating layer and add them, b) there are 5 rows of 5

17. Draw a grid to show that 1 + 3 + 5 + 7 + 9 + 11 = 6 \times 6.
   **Answer:**

18. How does the grid show the expression?
   a) 7 + 10 + 13 + 16    b) 4 \times 23 \div 2
   **Answers:** a) the sum of the grey squares in each row, b) the grey squares are half the 4 \times 23 rectangle

19. Draw a diagram to show that 5 + 9 + 13 + 17 = 4 \times 22 \div 2.
Extended Problem: Percentage Discounts

Materials:
BLM Percentage Discounts (pp. O-40–43)

Extended Problem: Percentage Discounts. Provide students with BLM Percentage Discounts. Tell students that in this extended problem, they will work with percentages of numbers in a real-world context by comparing two types of discounts: a direct percentage discount and one based on requiring the customer to buy one item at full price before getting a discount.

Answers:
1. a) $12; b) $18; c) 60%, sample explanations: 100% − 40% = 60%, or solve the problem “18 is what percent of 30?” to get 60
2. a) $55; b) $12 and $21; c) $33; d) 40%, sample explanations: by the distributive property, 40% of $20 + 40% of $35 is 40% of $55, or solve the problem “22 (the amount off) is what percent of 55?” to get 40
3. a) 100, 60, 90, A; b) 100, 60, 75, A; c) 100, 60, 60, same; d) 100, 60, 50, B
4. a) 100, 90, 10; b) 100, 75, 25; c) 100, 60, 40; d) 100, 50, 50
5. A 50% discount is the greatest percentage discount you can get. This happens when both items are the same price.
6. a) 20% off, b) 10% off, c) 25% off
7. A 25% discount is the greatest percentage discount you can get. This happens when both items are the same price.
Bonus: a) 30% off, b) 15% off, c) x/2% off
Bonus: less; it will cost 98% of the current price in the new year because 0.7 × 1.4 = 0.98
Percentage Discounts (1)

Store A has a sale: Get 40% off all items.

1. Amir buys a sweater from Store A with original price $30.
   a) What is the amount of the discount?

   b) What is the sale price, after the discount?

   c) What percentage of the original price is the sale price? Explain how you know.

2. Grace buys two items from Store A with original prices $20 and $35.
   a) What is the total price before the sale?

   b) What is the sale price of each item?

   c) What is the total price after the sale?

   d) What percentage of the total price did she get? Explain how you know.
### Percentage Discounts (2)

Store A has a sale: Buy any number of items for 40% off each item.

Store B has a sale: Buy one item and get an item of equal or lesser value for free.

3. Which store gives a lower price if you buy two items at the given price?

<table>
<thead>
<tr>
<th>Item 1</th>
<th>Item 2</th>
<th>Total Price before Discount ($)</th>
<th>Total Price after Discount: Store A ($)</th>
<th>Total Price after Discount: Store B ($)</th>
<th>Which Store Has the Lower Price?</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $10</td>
<td>$90</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) $25</td>
<td>$75</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c) $60</td>
<td>$40</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d) $50</td>
<td>$50</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. For each part in Question 3, calculate the percentage discount you would get overall at Store B.

<table>
<thead>
<tr>
<th>Original Total Price ($)</th>
<th>Final Price ($)</th>
<th>Percentage Discount</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>d)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. At Store B, what is the greatest percentage discount you could get on two items? When does this happen?
Percentage Discounts (3)

6. Store C has a sale: Buy one item and get an item of equal or lesser value for half price. What is the percentage discount off the original total price if you buy two items for ...
   a) $30 and $20?
   b) $8 and $32?
   c) $15 and $15?

7. At Store C, what is the greatest discount you could get on two items? When does this happen?
Percentage Discounts (4)

Bonus ►
What is the greatest percentage discount you can get at a store if the store announces the following sale?
Buy one item and get an item of equal or lesser value for …

a) 60% off.

b) 30% off.

c) x% off.

Bonus ►
Rick wants to buy a tennis racket. In the new year, the store selling the racket will increase the price by 40% and then have a 30% discount sale. Will the tennis racket cost more or less in the new year?
PS7-8 Using Logical Reasoning II

Teach this lesson after: 7.2 Unit 4

Goals:
Students will solve problems involving both direct and inverse proportions using logical reasoning. Students will recognize the difference between two quantities being in direct proportional relationship and two quantities being in inverse proportional relationship.

Prior Knowledge Required:
Can solve proportions
Can identify equivalent ratios
Can compare unit rates (for Problem Bank 3)
Can interpret remainders in division problems (for Problem Bank 6)
Can compare fractions (for Extended Problem)
Can interpret remainders in division problems (for Problem Bank 6)
Can divide decimals by whole numbers (for Extended Problem)

Vocabulary: inverse proportional, proportional

Materials:
BLM Making Punch (pp. O-52–55, see Extended Problem)

Identifying proportional quantities. Write on the board:

<table>
<thead>
<tr>
<th>Amount of Juice (mL)</th>
<th>Price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>2</td>
</tr>
<tr>
<td>400</td>
<td>3.50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Amount of Juice (mL)</th>
<th>Price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>1.50</td>
</tr>
<tr>
<td>400</td>
<td>3</td>
</tr>
</tbody>
</table>

SAY: The bigger the size of the juice, the more you pay, but that doesn't mean that twice as much juice costs twice as much money. Point to the first table and ASK: Does 400 mL cost twice as much as 200 mL? (no) Is it more or less than twice the cost? (less) Why might a store charge less than twice as much for twice as much juice? (sample answers: they want to encourage you to buy more from them; packaging costs more on smaller amounts)

Now point to the second table and ASK: Does 400 mL cost twice as much as 200 mL? (yes) SAY: In the second table, the rows are equivalent ratios. When two quantities are always in the same ratio, like they are in the second table, then they are proportional. The amount of juice and the price are not proportional in the first table because they are not always in the same ratio. You can check whether two quantities are proportional by comparing their values in a table. If the rows are equivalent ratios, then the quantities are proportional, which means that whatever number you multiply one quantity by, you have to multiply the other quantity by the same number.
Write on the board:

\[
\begin{array}{cc}
200 : 2 & \neq 400 : 3.5 \\
200 : 1.5 & = 400 : 3
\end{array}
\]

<table>
<thead>
<tr>
<th>Amount of Juice (mL)</th>
<th>Price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>2</td>
</tr>
<tr>
<td>400</td>
<td>3.50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Amount of Juice (mL)</th>
<th>Price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>1.50</td>
</tr>
<tr>
<td>400</td>
<td>3</td>
</tr>
</tbody>
</table>

**Exercises: Are the two quantities proportional?**

a) | Number of Sheets | Price ($) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

b) | Number of T-Shirts | Price ($) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

c) | Distance Run (m) | Time (min) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>5000</td>
<td>35</td>
<td></td>
</tr>
</tbody>
</table>

d) | Distance Driven (km) | Time (hours) |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2</td>
</tr>
<tr>
<td>200</td>
<td>4</td>
</tr>
<tr>
<td>500</td>
<td>10</td>
</tr>
</tbody>
</table>

**Answers:** a) no, b) yes, c) no, d) yes

SAY: To make the same shade of green paint, the blue and yellow paint need to be in the same proportions. But if you want to make different shades of green, then the proportions will change.

**Exercises: Do these mixtures make the same colour of paint?**

a) | Blue Paint (cups) | Yellow Paint (cups) |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
</tr>
</tbody>
</table>

d) | Blue Paint (tbsp) | Yellow Paint (cups) |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>300</td>
<td>15</td>
</tr>
</tbody>
</table>

c) | Blue Paint (tsp) | Yellow Paint (tsp) |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
</tbody>
</table>

d) | Blue Paint (tbsp) | Yellow Paint (cups) |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>300</td>
<td>150</td>
</tr>
</tbody>
</table>

**Answers:** a) no, b) yes, c) no, d) yes

When students finish the exercises, point out that when the amounts are proportional, whatever happens to the numbers on the left (double or triple) happens to the numbers on the right.

**Inverse proportional quantities.** Write on the board:

Four workers can paint three walls a day.

How many walls can eight workers paint a day?
SAY: “A day” means for every one day. Assume that everyone works at the same steady rate. Draw on the board:

<table>
<thead>
<tr>
<th>Workers</th>
<th>Walls Painted</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>?</td>
</tr>
</tbody>
</table>

Point to the table and ASK: Can we solve this problem with a proportion? (yes) Ask a volunteer to find the missing number in the table. (see below)

<table>
<thead>
<tr>
<th>Workers</th>
<th>Walls Painted</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
</tr>
</tbody>
</table>

Write on the board:

Four workers can paint an apartment in three hours. How long will it take eight workers to paint the same apartment?

SAY: Assume that everyone works at the same steady rate. Draw on the board:

<table>
<thead>
<tr>
<th>Workers</th>
<th>Hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>?</td>
</tr>
</tbody>
</table>

Point to the table and ASK: If more people are working, are more hours needed or are fewer hours needed to complete the job? (fewer) Can we solve this problem using a proportion? (no) SAY: If this were proportional, then twice as many workers would need twice as many hours. But they actually need half as many hours because twice as many workers get the job done twice as fast, so it will take only 1.5 hours for eight people to paint the apartment. Replace the question mark in the table with “1.5.”

**Exercises:** Find the missing amount. Is it twice as much or half as much?

a) Workers are painting fences.

i) | Workers | Fences Painted |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

ii) | Workers | Hours Needed to Paint |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>?</td>
</tr>
</tbody>
</table>

b) A car rental is $12 per hour.

i) | Hours | Total Cost ($) |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>?</td>
</tr>
</tbody>
</table>

ii) | Cars Rented | Cost Per Hour ($) |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>?</td>
</tr>
</tbody>
</table>

iii) | People Sharing a Car Rental | Cost Per Person |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>?</td>
</tr>
</tbody>
</table>

**Answers:** a) i) 12 fences, twice as much; ii) 3 hours, half as much; b) i) $24, twice as much; ii) $24, twice as much; iii) $6, half as much
SAY: If the input (left column) increases and the output (right column) decreases, it’s not proportional, so the rows cannot be equivalent ratios. Using part b) iii) from the previous exercises, draw on the board:

<table>
<thead>
<tr>
<th>People Sharing a Car Rental</th>
<th>Cost per Person</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

SAY: If the rows were equivalent ratios, you could multiply the first row by 2 to get the second row. But in fact, while you multiply 1 by 2 to get 2, you have to divide 12 by 2 to get 6. So, the quantities are inverse proportional instead of proportional.

**Exercises:** Decide whether the quantities are proportional or inverse proportional. Then answer the question.

a) the number of workers
   the number of fences they can paint in one hour
   If two workers can paint five fences, how many fences can four workers paint?

b) the number of workers
   the number of hours they need to paint a room
   If three workers need five hours to paint a room, how long would it take nine workers?

c) the number of cars rented
   the total cost per hour
   If one car costs $15 per hour, how much would three cars cost for one hour?

d) the number of people sharing the cost
   the cost per person
   If one car costs $15 per hour and three people share the cost, how much does each person pay to rent the car for one hour?

**Answers:** a) proportional, 10; b) inverse proportional, 5/3 hours; c) proportional, $45; d) inverse proportional, $5

**Using the number of hours for one worker to find the number of hours for any number of workers.** Write on the board:

If it takes six workers five hours to paint a room, how long would it take three workers to paint the room?

<table>
<thead>
<tr>
<th>Workers</th>
<th>Hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>?</td>
</tr>
</tbody>
</table>

Have a volunteer answer the question. (10 hours) ASK: Are these quantities proportional or inverse proportional? (inverse proportional) SAY: You are dividing the number of workers by 2, so you need to multiply the number of hours by 2. That’s because half as many workers need twice as long.
Continue drawing on the board:

<table>
<thead>
<tr>
<th>Workers</th>
<th>Hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>?</td>
</tr>
</tbody>
</table>

Exercises:
1. It takes four workers three hours to fill a hole in the road.
   a) How long would it take one worker to fill the hole?
   b) How long would it take six workers to fill the hole?
   **Answers:** a) 12 hours, b) 2 hours

2. Three people need one hour to clean a house.
   a) How long would one person need to clean the house?
   a) How long would it take five people to clean the house?
   **Answers:** a) 3 hours, b) 3/5 hour or 36 minutes

Problem Bank
1. The areas of two rectangles are equal. The length of rectangle A is twice the length of rectangle B.
   a) How does the width of rectangle A compare with the width of rectangle B?
   b) Are the length and width proportional?
   **Answers:** a) the width of rectangle A is half the width of rectangle B, b) no

2. Can you use a proportion to solve the problem?
   a) Jane is two years old and Yu is five years old. How old will Yu be when Jane is 20 years old?
   b) For every 2 cups of flour, use 1 teaspoon of oil. How many cups of flour would you need for 3 teaspoons of oil?
   c) Three people need one hour to clean a house. How many people would you need if the house needs to be cleaned in 20 minutes?
   **Answers:** a) no, b) yes, c) no

3. In her first 300 games of solitaire, Grace had 50 wins. After another 50 games (350 in total), her computer recorded 70 wins. Is she improving? Explain how you know.
   **Answer:** Yes. In 300 games, the ratio of wins to games was 1 : 6. In 350 games, the ratio of wins to games was 1 : 5, which is a better ratio.

4. Five workers need 40 minutes to paint a fence.
   a) How long would it take one worker to paint the fence?
   b) How long would it take five workers to paint two identical fences?
   c) How many workers can paint three fences in 40 minutes?
   d) Are the quantities proportional?
      i) number of workers and number of minutes required
      ii) number of fences painted and number of minutes required
      iii) number of workers and number of fences painted
   **Answers:** a) 200 minutes or 3 hours and 20 minutes; b) 80 minutes; c) 15 workers; d) i) no, ii) yes, iii) yes
5. In two hours, three people can paint a hall with an area of 60 m². How long would it take for five people to paint a hall with an area of 75 m² at the same rate of speed?

**Answer:** 1.5 hours

6. Five chefs require 20 minutes to prepare 30 appetizers.
   a) How many appetizers can five chefs prepare in 30 minutes?
   b) How many appetizers can five chefs prepare in 50 minutes?
   c) How many chefs can prepare 18 appetizers in 20 minutes?
   d) How many chefs can prepare 48 appetizers in 20 minutes?
   e) How many chefs do you need to hire if you want to prepare 100 appetizers in 20 minutes?

**Selected solution:** e) Solving the proportion \( \frac{100}{30} = \frac{x}{5} \) gives \( 30x = 500 \) or \( x = 50/3 = 16 \ 2/3 \), so you need to hire 17 chefs.

**Answers:** a) 45, b) 75, c) 3, d) 8

7. Four tennis players play a round-robin doubles match (two against two) according to the following rules:
   - They play three sets altogether.
   - Each player plays one set with each partner.
   - Each set has one team winning and one team losing (there are no ties).

Rani says that either one player wins all three sets or one player loses all three sets. Is she right? Explain how you know.

**Answer:** Rani is right. In the first set, one team wins and the other loses. Now consider the two players who won. One of them will win the second set and the other one will lose the second set. So, one player won the first two sets. That player’s partner for the third set will be someone who lost the first two sets. Either that team will win the third set (and the player who won the first two sets will win all three sets) or that team will lose the third set (and the player who lost the first two sets will lose all three sets).
Extended Problem: Making Punch

Materials:
BLM Making Punch (pp. O-52–55)

Extended Problem: Making Punch. Provide students with BLM Making Punch. Tell students that this extended problem involves working with ratios, percentages, and markups.

Answers:
1. Recipe A is 3 : 4, Recipe B is 3 : 8, Recipe C is 1 : 4
3. A costs 11 1/4¢ per cup, B costs 13 1/8¢ per cup, and C costs 13 3/4¢ per cup
4. Yes. This makes sense because using only ginger ale would cost 10¢ per cup and using only cranberry juice would cost 15¢ per cup, so mixing them should cost between 10¢ and 15¢.
5. C. This makes sense because it has the most cranberry taste and cranberry juice is more expensive than ginger ale.
6. A: $1.00, B: $1.15, C: $1.20
7. A: $1.30, B: $1.50, C: $1.56
8. a) A: 270 bottles, B: 675 bottles, C: 405 bottles; b) $463.05; c) $18.52
9. It will taste more like cranberry than Recipe A, but more like ginger ale than Recipe C.
Bonus: Recipe D is 55/111 ginger ale. Recipe E is 1/2 ginger ale. If you mix one recipe of each, you get Recipe F, which is 56/113 ginger ale. Recipe F will taste more like ginger ale than Recipe D, but less so than Recipe E, so 56/113 is between 55/111 and 1/2, so 56/113 > 55/111.
Making Punch (1)

Clara’s CranPunch company makes three kinds of cranberry/ginger ale punch:

**Recipe A:** 3 cups of ginger ale and 1 cup of cranberry juice  
**Recipe B:** 3 cups of ginger ale and 5 cups of cranberry juice  
**Recipe C:** 1 cup of ginger ale and 3 cups of cranberry juice

1. Write the ratio to show the number of cups of ginger ale to the total number of cups of punch for each recipe.
   - **Recipe A:** 
   - **Recipe B:** 
   - **Recipe C:** 

2. Show your reasoning.
   a) Which recipe has the strongest ginger ale taste?

   b) Which recipe has the strongest cranberry taste?

3. The company buys ginger ale for 10¢ per cup and cranberry juice for 15¢ per cup. What is the cost, per cup, for each recipe? Write your answer as a whole or mixed number of cents.
   - **Recipe A:** 
   - **Recipe B:** 
   - **Recipe C:**
4. Are your answers to Question 3 all between 10 and 15? ________
   Why does this make sense?

5. Which recipe is the most expensive, per cup? __________
   Why does this make sense?

6. Clara’s company sells punch in 8-cup bottles. She buys the bottles for 10¢ each. How much does she pay for one bottle of each recipe?
   Recipe A: __________
   Recipe B: __________
   Recipe C: __________

7. To decide how much to sell each bottle for, Clara calculates:
   Total cost of that bottle + 30% of total cost of that bottle
   Then she rounds to the nearest cent. How much does she sell each bottle for?
   Recipe A: __________
   Recipe B: __________
   Recipe C: __________
Making Punch (3)

8. Clara wants to sell 1350 bottles of punch each week. Her customers' preferences are shown below:

<table>
<thead>
<tr>
<th>Recipe</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent of Sales</td>
<td>20</td>
<td>50</td>
<td>30</td>
</tr>
</tbody>
</table>

a) How many bottles of each type of recipe will she need to make?

b) What is Clara's total profit, after subtracting all her expenses?

c) Clara works 25 hours each week. What is her hourly pay, to the nearest cent?

9. Josh made one bottle each of Recipe A and Recipe C. Then he mixed them. Describe the taste of Josh's new recipe by comparing it with each of Recipes A and C.
Making Punch (4)

**Bonus** Without doing any multiplication or division, decide which is greater, \(\frac{55}{111}\) or \(\frac{56}{113}\).

Use the following recipes in your explanation.

**Recipe D:** 55 cups of ginger ale and 56 cups of cranberry juice

**Recipe E:** 1 cup of ginger ale and 1 cup of cranberry juice

Hint: Mix the recipes.
**PS7-9 Using Logical Reasoning III**

**Teach this lesson after:** 7.2 Unit 4

**Goals:**
Students will solve problems involving workers who work at different, but constant, rates.

**Prior Knowledge Required:**
Can add fractions
Can solve worker problems where everyone works at the same rate

**Vocabulary:** rate

**Determining the time to finish a job given the fraction completed in a unit time period.**
Write on the board:

Lily painted a wall. After one hour, she was done \( \frac{1}{3} \) of the wall.

How long would it take her to paint the whole wall?

Read the problem aloud and have a volunteer tell you the answer. (3 hours) Repeat by erasing 1/3 and replacing it with 1/2 (2 hours), 1/4 (4 hours) and 1/10 (10 hours). Next, replace “Lily” with “Zara” and replace 1/2 with 2/3 and SAY: This problem is harder, but I think you can still do it. To guide students, write on the board:

After one hour, Lily was done \( \frac{1}{3} \) of the wall.

After one hour, Zara was done \( \frac{2}{3} \) of the wall.

ASK: Who is working faster, Lily or Zara? (Zara) How much faster? (Zara is working twice as fast) SAY: If Zara can get twice as much done in one hour as Lily can, it would take her half as long to finish the wall. ASK: How long does it take Lily to finish the wall? (3 hours) How long would it take Zara? (half as long as 3 hours, so 1 1/2 hours) PROMPT: What is half of 3?

**Exercises:**
1. How many times as fast can Rick work as John?
   a) John can paint \( \frac{1}{5} \) of the wall in one hour and Rick can paint \( \frac{2}{5} \) of the wall in one hour.
   b) John can paint \( \frac{1}{4} \) of the wall in one hour and Rick can paint \( \frac{3}{4} \) of the wall in one hour.

**Answers:** a) twice as fast, b) three times as fast
2. How many hours would it take the person to paint the whole wall, working alone?
   a) John can paint $\frac{1}{5}$ of the wall in one hour.
   b) Rick can paint $\frac{2}{5}$ of the wall in one hour. Hint: Use your answer to part a).

   **Answers:** a) 5 hours, b) 2 1/2 hours

   SAY: You can answer the same type of question given how much is done in any unit of time.

   **Exercises:**
   1. How long would it take to complete the task?
      a) Amir walks $\frac{1}{9}$ of the distance from home to school in one minute.
      b) Jen walks $\frac{2}{9}$ of the distance from home to school in one minute.
      c) Amir runs $\frac{1}{15}$ of the race in one minute.
      d) Jen runs $\frac{4}{15}$ of the race in one minute.

      **Answers:** a) 9 minutes, b) 9/2 minutes or 4 1/2 minutes, c) 15 minutes, d) 15/4 minutes or 3 3/4 minutes

   2. Write your answers to parts b) and d) of Exercise 1 as improper fractions.
      **Answers:** b) 9/2 minutes, d) 15/4 minutes

   Summarize the answers to Exercises 1 and 2 on the board:

   $\frac{1}{9}$ in 1 minute, so 9 minutes to finish
   $\frac{2}{9}$ in 1 minute, so $\frac{9}{2}$ minutes to finish
   $\frac{1}{15}$ in 1 minute, so 15 minutes to finish
   $\frac{4}{15}$ in 1 minute, so $\frac{15}{4}$ minutes to finish

   **ASK:** Do you see a shortcut to find the amount of time required to finish the job? (look at the fraction of the job completed in one hour or one minute and turn the fraction upside down)
Exercises: How long would it take to complete the task?

a) Liz completes $\frac{3}{8}$ of the multiplication questions in one minute.

b) Cam completes $\frac{4}{7}$ of his homework in one hour.

c) Abella watched $\frac{5}{9}$ of the movie in one hour.

d) Zack walked $\frac{2}{11}$ of the distance in one minute.

Answers: a) 8/3 minutes or 2 2/3 minutes, b) 7/4 hours or 1 3/4 hours, c) 9/5 hours or 1 4/5 hours, d) 11/2 minutes or 5 1/2 minutes

Problems where workers do not work at the same rate as each other. Write on the board:

Tess can paint a wall in three hours, working alone.

Eric can paint the same wall in six hours, working alone.

If they work together, how long will it take them to paint the wall?

SAY: We are used to solving worker problems in which all the workers work at the same rate. Now, Tess and Eric don't work at the same rate. ASK: Will two people take more time or less time than one person working alone? (less time) How is that like the problems we did before with people painting fences? (more people took less time to do the same job) Draw on the board:

![Diagram of a wall with Tess and Eric painting it]

While pointing to their names, SAY: Tess is going to start on this side and Eric is going to start on that side. ASK: How much of the wall will Tess have painted after one hour? (1/3) How do you know? (she takes 3 hours to paint the whole wall, so in 1 hour she can paint 1/3 as much) How much of the wall will Eric have painted after one hour? (1/6) How do you know? (he takes 6 hours to paint the whole wall, so in 1 hour he can paint 1/6 as much)

Exercises: How much of the wall does the person paint in one hour?

a) Tom can paint the wall in four hours.

b) Sharon can paint the wall in seven hours.

Answers: a) 1/4, b) 1/7
SAY: Once you know how much each person paints in one hour, you can find out what fraction of the wall they can paint together in one hour. Colour on the board how much each person will have done after one hour and label each section, as shown below:

![Diagram showing 1/3 and 1/6 of the wall painted]

ASK: What fraction of the wall is painted altogether? (1/2) How did you get that? (added 1/3 and 1/6) SAY: To get the total amount painted, add the amounts they each painted. Write on the board:

\[
\frac{2}{3} \times 1 + \frac{1}{6} \\
\frac{2}{3} \times \frac{1}{3} + \frac{1}{6} \\
\frac{2}{6} + \frac{1}{6} \\
\frac{3}{6} \\
\frac{1}{2}
\]

**Exercises:** Find the total fraction of the wall painted in one hour.

a) Jay painted \(\frac{3}{10}\) of the wall and Megan painted \(\frac{1}{5}\) of the wall.

b) Jay painted \(\frac{1}{8}\) of the wall and Megan painted \(\frac{1}{4}\) of the wall.

c) Jay painted \(\frac{1}{5}\) of the wall and Megan painted \(\frac{2}{15}\) of the wall.

**Answers:** a) 1/2, b) 3/8, c) 1/3

SAY: Remember, you can figure out the amount someone can paint in one hour if you know how long it would take them to paint the whole wall by themselves.

**Exercises:** Find the total fraction of the wall painted in one hour.

a) David can paint the wall in two hours, working alone.
Mandy can paint the wall in three hours, working alone.

b) David can paint the wall in four hours, working alone.
Mandy can paint the wall in three hours, working alone.

c) David can paint the wall in five hours, working alone.
Mandy can paint the wall in two hours, working alone.

**Answers:** a) 5/6, b) 7/12, c) 7/10
SAY: Once you know the total fraction painted in one hour, you can decide how much time they would need to finish the job. ASK: If two people paint 1/2 of the wall in one hour, how long does it take them to paint the whole wall? (two hours) What if they painted 2/3 of the wall in one hour, then how long would it take them to paint the whole wall? (3/2 hours or 1 1/2 hours) How did you get that? (it takes half as long to paint the wall as if they painted 1/3 of the wall in one hour, or I flipped the fraction)

Exercises:
1. Randi can paint a wall in three hours, working alone. Jun can paint the same wall in five hours, working alone.
   a) How much of the wall will Randi have painted after one hour?
   b) How much of the wall will Jun have painted after one hour?
   c) How much of the wall will be painted altogether after one hour?
   d) How long will it take them together to paint the wall?
   Answers: a) 1/3, b) 1/5, c) 8/15, d) 15/8 hours or 1 7/8 hours

2. Randi can paint a wall in three hours, working alone, and so can Ronin. How long will it take them working together?
   Answer: They can each paint 1/3 of the wall in one hour, so working together they can paint 2/3 of the wall in one hour. So, it takes 3/2, or 1 1/2, hours to paint the wall.

3. Look at your answers to Exercises 1 and 2. Does it take Randi and Ronin more or less time than Randi and Jun? How could you have predicted that?
   Answer: less, because Ronin is faster than Jun

SAY: It is always a good idea to go back and check that your answers make sense. ASK: Does it make sense that it takes just under two hours for Randi and Jun to work together? (yes) Why? (You would expect it to take less than three hours because that is how long it takes Randi alone, but more than 1 1/2 hours because that’s how long it would take if Randi and Jun both worked at Randi’s pace. Jun is slower, so it should take longer.)

Exercise: Dory can paint a wall in two hours. Anton can paint the same wall in one hour. How long will it take them, working together, to paint the wall? Check that your answer makes sense.
   Answer: In one hour, Dory can paint half the wall and Anton can paint the whole wall by himself, so in one hour, they can complete 1 1/2 of the job, or 3/2 of the job. So, they can paint the wall together in 2/3 of an hour. This is less time than it takes Anton by himself, but more time than it would take both of them working together at Anton’s speed. That makes sense because Dory is slower than Anton.

Problem Bank
1. A farmer can prune all the apple trees in six days. Another farmer can prune all the apple trees in three days. How long would it take if they worked together to prune all the apple trees?
   Solution: The first farmer can prune 1/6 of the trees per day and the second farmer can prune 1/3 of the trees per day. Together, they can prune 1/6 + 1/3 = 1/2 of the trees per day, so it would take two days to prune all the trees working together.
2. How many times as fast as Luc can Ray work?
   a) Luc can paint \( \frac{1}{5} \) of the wall in one hour. Ray can paint \( \frac{3}{5} \) of the wall in half an hour.
   
   b) Luc can prune \( \frac{2}{5} \) of the trees in one hour. Ray can prune \( \frac{1}{5} \) of the trees in 15 minutes.

   **Answers:** a) 6 times as fast, b) twice as fast

3. Ava can paint a fence by herself in four hours. Jax can paint the same fence in six hours. They work together for two hours and then Ava leaves.
   a) How much of the fence will they be done after one hour?
   b) How much of the fence will they be done after two hours?
   c) How much of the fence will be left for Jax to paint on his own?
   d) How long will it take Jax to complete that amount of the fence, from part c), on his own?

   **Answers:** a) In one hour, Ava paints \( \frac{1}{4} \) of the fence and Jax paints \( \frac{1}{6} \) of the fence. Together, they paint \( \frac{1}{4} + \frac{1}{6} = \frac{5}{12} \) of the fence in one hour; b) In two hours, they paint \( \frac{10}{12} \) or \( \frac{5}{6} \) of the fence; c) when Ava leaves, Jax has \( \frac{1}{6} \) of the fence left to paint; d) It will take Jax another hour.

4. Ansel and Clara work at a factory. Ansel is training Clara to pack bottles. Ansel can pack 20 000 bottles in five hours. Clara can pack 3000 bottles in two hours. How long will it take them to pack 44 000 bottles?

   **Answer:** In one hour, Ansel packs 4000 bottles and Clara packs 1500 bottles, so together they pack 5500 bottles in one hour. So, it will take them \( 44 \, 000 ÷ 5500 = 8 \) hours.

5. Eddy can mow the lawn by himself in three hours. Working together, Eddy and Tina mow the lawn in two hours. How long would it take Tina to mow the lawn by herself?

   **Answer:** Eddy can mow \( \frac{1}{3} \) of the lawn in one hour. In two hours, he completes \( \frac{2}{3} \) of the lawn. So Tina must have done \( \frac{1}{3} \) of the lawn in two hours. So, she would take six hours to mow the lawn by herself.

6. Alice works 20% faster than Braden. That means she can complete the amount of work that Braden can complete, plus 20% of that amount, in the same amount of time. Braden can paint \( \frac{1}{4} \) of a wall in one hour.

   a) What fraction of the wall is 20% of \( \frac{1}{4} \) of the wall?
   b) What fraction of the wall can Alice paint in one hour?
   b) How long will Braden and Alice take to paint the wall if they work together?

   **Answers:** a) \( \frac{1}{20} \), b) \( \frac{3}{10} \), c) \( \frac{20}{11} \) hours or \( 1 \frac{9}{11} \) hours

7. Don and Anne are fixing a bike together. Anne works 30% faster than Don. If they work together, they can complete the job in five hours.
   a) How long would it take Don if he worked alone?
   b) Is your answer more than 10 hours or less than 10 hours? Why does that make sense?
**Answers:** a) If Don does $x$ of the job, Anne does $1.3x$, and together they do $2.3x = 1$ whole job. So, in five hours, Don does $1/2.3 = 10/23$ of the job. In one hour, he does $2/23$ of the job. So, it will take him $23/2$, or $11 1/2$, hours to do the job on his own; b) More than 10 hours, which makes sense because if Don and Anne worked at the same pace and did the job in five hours, it would take each of them 10 hours working alone. But Don is slower than Anne, so it should take him longer than 10 hours.

8. One hose can be used to fill a pool in three hours. Another hose can be used to drain the full pool in four hours.
   a) If the hose used to drain the pool is accidentally left on while the owner is trying to fill the pool, how long will it take to fill the pool?
   b) After three hours, the owner checks on the pool, expecting it to be full. How full is the pool at the three-hour mark?
   c) The owner of the pool immediately turns off the drain hose at the three-hour mark. How much longer will the pool take to fill completely?

   **Answers:** a) in one hour, it will fill $1/3$ of the pool and drain $1/4$ of the pool, for an overall effect of filling $1/3 - 1/4 = 1/12$ of the pool, so it will take 12 hours to fill the pool; b) it will only be $1/4$ full; c) the hose still needs to do $3/4$ of the job that takes three hours, so it needs $9/4$, or $2 1/4$, more hours to finish the job.

**NOTE:** In Problem Bank 9, parts a) to e) are guiding questions to help students solve the problem. Some advanced students may be able to solve it without guidance.

9. Jayden can paint a fence in 10 hours. Karen can paint the fence in 12 hours. They work for three hours and then Shelly comes to help. They finish the job two hours later. How long would it take Shelly to do the job on her own?
   a) How much of the job did Karen and Jayden finish in one hour?
   b) How much of the job did Karen and Jayden finish in five hours?
   c) How much of the job did Shelly do in two hours?
   d) How much of the job did Shelly do in one hour?
   e) How long would it take Shelly to complete the job on her own?

   **Selected solutions:** a) $1/10 + 1/12 = 11/60$, b) $55/60 = 11/12$

   **Answers:** c) $1/12$, d) $1/24$, e) 24 hours
Unit 5  Geometry

In this unit students will construct related lines using angle properties and a variety of tools (compass and straightedge, dynamic geometry software) and strategies. They will learn about congruent and similar shapes, and congruence rules.

**Technology: Geometer’s Sketchpad**

Students are expected to investigate geometric properties of lines, angles, and triangles using dynamic geometry software. Some activities in this unit use a program called Geometer’s Sketchpad and some are instructional—they help you teach students how to use the program. If you are not familiar with Geometer’s Sketchpad, the built-in Help Centre provides explicit instructions for many constructions. Search the Index using phrases such as “How to construct congruent angles” or “How to construct a line segment of given length.”

Please note: Dynamic geometry software is not required to complete the unit.

**Meeting Your Curriculum**

Though the WNCP curriculum does not specifically mention congruence rules for triangles, students are expected to understand this material in Grade 9 and to be able to develop their understanding of it to more rigorous levels. Students are not expected to learn this material in Grade 8. However, expectation 7SS3 of the WNCP curriculum requires students to explain why geometric constructions with a compass and a straightedge work. To understand why the constructions work, students need a working knowledge of congruent triangles and congruence rules. For that reason, we consider lessons G7-13 through G7-15 essential preparation for meeting expectation 7SS3.
G7-13 Congruent Triangles
Pages 105–109

CURRICULUM EXPECTATIONS
Ontario: 7m3, 7m6, 7m7, 7m51
WNCP: essential for 7SS3, [CN, R, V]

VOCABULARY
triangle
acute, obtuse (angle)
congruence, congruent
corresponding angles
corresponding sides
right, isosceles, equilateral
(triangle)

PROCESS ASSESSMENT
7m6, 7m7, [C, V]
Workbook Question 1

Assess prior knowledge. Start the lesson with a diagnostic test as in Workbook page 105 Question 1. Students who describe differences between triangles in terms of size, position, or non-geometric characteristics (e.g., This triangle is pointing down. This triangle is long and thin.) should complete a directed sorting exercise. Give students the remedial BLM Triangles for Sorting and have them cut the triangles out and sort them according to different geometric criteria, such as number of equal sides (or angles), whether a triangle has a right (or obtuse) angle, number of lines of symmetry. Start with sorting by one attribute, then continue to two attributes using Venn diagrams. Then have students test ideas about their triangles. For example, ASK: Can there be a triangle with two right angles? Can you draw a triangle that has no right angles? Can you draw a triangle with all acute angles?

Review the notation for equal sides (equal number of markings) and equal angles (equal number of arcs) in polygons.

Introduce congruent shapes. Explain that congruent shapes have the same size and shape, so if you put one shape on top of the other, they should match exactly. This means that the sides and the angles of congruent shapes have to match exactly as well, and they have to match in order. To illustrate the fact that all sides and angles of two non-congruent shapes can be equal, but just not in the same order, you might use the Activity below. See also Extensions 1 and 2.

In triangles, the order of sides does not matter because there are only three sides. Give your students three straws of different lengths and ask them to mark the ends of the largest straw with different colours. Have students make a triangle with their three straws and trace it on a sheet of

Goals
Students will identify congruent shapes based on information about sides and angles.

PRIOR KNOWLEDGE REQUIRED
Can measure angles and sides of polygons
Is familiar with notation for equal sides and angles
Can name angles and polygons
Is familiar with the symbols for angle and triangle
Can classify triangles

MATERIALS
BLM Triangles for Sorting (p P-40)
straws of different lengths
BLM Two Pentagons (p P-41)
paper. Then ask them to change the order of the straws (e.g., the straw that touched the blue end of the longest straw will now touch the red end). Look at the new triangle. Is it the same triangle or a different one? If students do not see that they created an identical triangle, they can trace the new triangle, cut out the tracing, and compare it to the tracing of the first triangle. Have students combine their straws with a partner’s and try to see whether they can create different shapes using four of their six straws. Does the order you place the straws in matter? (yes) What happens to the triangle if you try to change the order of the sides? (the triangle flips over, is reflected) Explain that in triangles the order of the sides does not matter—if you change the order, you produce a reflection of the first triangle.

**Corresponding sides and angles.** Explain that when we want to check whether shapes are congruent, we pretend we can place the shapes one on top of the other. Which side of one shape will sit on top of which side of the other shape? The sides (or angles) of different shapes that will sit on top of one another are called corresponding sides (angles). If the shapes are congruent, corresponding sides and angles will be equal. Have students identify corresponding sides and angles in triangles as on Workbook pages 105–106, Questions 3 through 6.

Look at pentagons A and B from the Activity below. Point out that when we place these pentagons one on top of the other, we define the order in which we will check the angles and sides. If we place the pentagons so that at least one pair of sides or angles matches, we cannot make all the remaining corresponding sides and angles match. For example, you can match the shapes so that two angles and the side between them correspond (as shown at left), but the other sides adjacent to the angles are not equal, because the order of the sides in each shape is different. When you check for congruency, you either need to try all the possible combinations or show why congruency is impossible. In this case, congruency is impossible because all the right angles are adjacent in pentagon A but not in pentagon B.

**More than one way to find corresponding sides and angles.** Show your students a sheet of paper and have them identify the shape. What properties does it have? (You might use Scribe, Stand, Share to check answers.) Trace the rectangle on the board and label the vertices (say, \(ABCD\)). Then mark each corner of your paper rectangle with a different letter (say, \(EFGH\)). **ASK:** Are these rectangles congruent? (yes) Explain that when you place the rectangle over the tracing, you create correspondence: \(E\) corresponds to vertex \(A\), \(F\) to \(B\), and so on. Have students write down which angles and sides correspond and are equal: \(\angle A = \angle E\), \(AB = EF\), and so on. However, you can place the rectangle onto the tracing in a different way (show how to do so) such that \(E\) corresponds to \(C\), \(F\) corresponds to \(D\) and so on. Have students say which angles and sides are equal in the new way of correspondence. **ASK:** Why were we able to do this? (Since a rectangle has equal angles and two pairs of equal sides, you can place two rectangles one on top of the other in different ways. Another way to say it is that a rectangle is a symmetrical shape; if students use this term
because they are familiar with it from the earlier grades, invite them to explain how symmetry helps here.) To let students practise writing pairs of equal corresponding angles, draw two isosceles congruent triangles and have students mark the equal angles and equal sides, and then write the equalities between corresponding sides and angles in different ways.

**Congruence symbol.** Explain that we have a special symbol we can use to show congruence: \( \cong \). Just as we can write \( AB = CD \) instead of “\( AB \) is equal to \( CD \)”, we can write \( ABCD \cong EFGH \) instead of “\( ABCD \) is congruent to \( EFGH \).”

**Congruence statements.** Explain that the congruence sign means more than the equal sign does. When mathematicians write a congruence statement, such as \( ABCD \cong EFGH \), they agreed to write the statements so that one would be able to say which side is equal to which side and which angle is equal to which angle. Teach students to write congruence statements using the same procedure as on Workbook page 108. After students finish Questions 4 through 7 on Workbook page 106, have them write congruence statements for the triangles in these questions. Then ask students to identify the equal sides and angles in pairs of congruent triangles using Questions 12 to 14 on Workbook pages 108–109.

**Activity**

Give students BLM Two Pentagons. Have students cut out the pentagons and compare the sides and the angles to answer the questions individually. When students are finished, have them signal the number of right angles on each polygon to check the answer to Question a), then ask them to hold up the folded shapes so that they can show that the remaining four angles are all equal. Then ask students to signal their answers simultaneously for Questions c) to h) so that you can check the whole class at the same time. Students can show thumbs up if their answer is “yes” and thumbs down if their answer is “no.” Then pair students up (pair the weakest students with the strongest ones so that the stronger students can coach the weaker ones) and have pairs compare their answers for Question i) and come up with a common answer. Repeat with groups of four and groups of eight and have the groups share their answers with the whole class.

**Questions and Answers:**

a) How many right angles does each pentagon have? (3)

c) Does each side on pentagon A have a side of the same length on pentagon B? (yes)

d) Is there the same number of sides of each length on both pentagons? (yes)

e) If you place one pentagon on top of the other, do they match? (no)

**Process Assessment**

7m3, [R]

Have students explain Tom’s error in Question 15 on Workbook page 109.
Extensions

1. Look at the pentagons on BLM Two Pentagons. Which one has greater area? Take the necessary measurements to check.

   ANSWER: Pentagon A

2. Pentagons similar to those on BLM Two Pentagons can be created by paper folding. Here are the folding instructions.

   Start with a regular sheet of paper. Fold it in two and cut along the fold to create two long and thin rectangles. One rectangle will be enough for both pentagons.

   Pentagon A:

   Create a square from your rectangle by folding the short side down to the long side of the rectangle, and cut the residue off to use for Pentagon B. The result is a square with one diagonal fold.

   Fold the square and unfold it again to create the second diagonal. Fold one of the corners of the square onto the centre of the square. Cut the small triangle off.

   Pentagon B:

   Use the residue (a rectangle) from making Pentagon A. Place Pentagon A on top of the rectangle as shown and mark the length of the “cut off” side of Pentagon A on the short side of the rectangle.

   Place the rectangle with the long side down, so that the mark you’ve made is in the top half of the left side edge. Fold the top left corner of your rectangle down to the bottom edge so that the left side falls
onto the bottom edge, as if you were making a square again. Fold the
residue on your right side over the triangle. Unfold.

Fold the top edge of the rectangle down through the mark, so that the
side edges fold onto themselves, to create a narrower rectangle.

Fold the top right corner of the narrow rectangle diagonally down, so
that it falls over the right vertical fold (it will not reach all the way to the
bottom). Fold the bottom right corner diagonally up so that it falls over
the same vertical fold, and you are done!
Goals
Students will determine the relationship between congruence and both area and perimeter.

PRIOR KNOWLEDGE REQUIRED
Can measure angles and sides of polygons
Is familiar with notation for equal sides and angles
Can name angles and polygons
Is familiar with the symbols for angle and triangle
Can find area of a triangle

MATERIALS
grid paper or geoboards
dynamic geometry software (optional)
BLM Congruence, Area, and Logic (pp P-42–P-43)

Review congruence. Congruent shapes have equal sides and equal angles in the same order. Congruent shapes would match exactly if placed one on top of the other correctly.

Congruence and area for shapes on a square grid. Have students do the Investigation on Workbook page 110 as a warm-up. Then ask them to draw on grid paper two non-congruent shapes with area 6. Then ask them to draw two non-congruent rectangles with the same area. Finally, ask them to draw two non-congruent rectangles with the same perimeter. Students can also use a geoboard for the tasks done on grid paper.

Congruency and area of triangles and parallelograms. Review the formulas for area and perimeter of parallelograms and triangles. If two triangles have the same base and the same height, they have the same area. The same is true for parallelograms. If dynamic geometry software is available, have students use this fact to create triangles and parallelograms with the same area that are not congruent. The Activities below use Geometer’s Sketchpad.

Counter-examples and congruency. Remind students that when a statement is true, it should be true about all cases. For example, look at the statement “When it rains, there are clouds.” This statement is true every time it rains. There cannot be a situation where it rains and there is not a single cloud in the sky. When a statement is false, only one example to show that it is false is enough. For example, look at the statement “All leaves become yellow in the fall.” This statement is about all leaves, and it would take only one counter-example—one example of a leaf that is not yellow in fall—to prove that the statement is not true. For example, some maple leaves become red. There is no need to check all possible leaves. Also,
it does not matter that some leaves become yellow—the statement talks about all leaves, not some leaves. Have students look for a counter-example that shows the statement is false, or a logical explanation that shows the statement is true.

a) All triangles are equilateral.
b) All squares are also rectangles.
c) All squares are congruent.
d) All squares with side 5 cm are congruent.
e) All rectangles with one side 5 cm are congruent.
f) All rectangles with sides 5 cm and 6 cm are congruent.
g) All triangles with base 3 cm and height 4 cm have area 6 cm².
h) All triangles with base 3 cm and height 4 cm are congruent.
i) All triangles with base 3 cm and height 4 cm have perimeter 12 cm.
j) All triangles with base 3 cm and perimeter 12 cm are congruent.

EXTRA PRACTICE: BLM Congruence, Area, and Logic

**ACTIVITIES**

1. **Triangles with the same area.** Students will start with a template that you can prepare ahead of time or find on the JUMP Math website. To prepare the template, create three parallel lines (say, \( j, k, \) and \( l \)) so that the line \( k \) is at the same distance from the lines \( j \) and \( l \). Mark a line segment \( AB \) that you will use as the base of some triangles on line \( l \). Mark a point \( C \) (not directly above the midpoint of \( AB \)) on line \( k \) and create triangle \( ABC \). Ensure that you can move point \( C \) only on the line \( k \). Find the area and the perimeter of \( \triangle ABC \).

   Question to investigate: Can you construct a triangle that has the same area and perimeter as \( \triangle ABC \)?

   a) Choose another point on line \( k \) and label it \( D \). Create \( \triangle ABD \). Find the area of \( \triangle ABD \). Move point \( D \) along line \( k \) while looking at the area. Does the area change? Explain. (Area of \( \triangle ABD = \) Area of \( \triangle ABC \), regardless of the exact position of point \( D \), because the triangles have the same base and height.)

   b) Find the perimeter of \( \triangle ABD \). Move point \( D \) along the line \( k \). Does the perimeter change? Can you make a triangle with perimeter larger than the perimeter of \( \triangle ABC \)? smaller than the perimeter of \( \triangle ABC \)? (yes to both)

   By moving \( D \), try to find a point different from \( C \) so that \( \triangle ABD \) has the same area and perimeter as \( \triangle ABC \). Look at \( \triangle ABC \) and \( \triangle ABD \). Do they look congruent? (yes)

   By moving point \( D \) on line \( k \), can you create a triangle \( ABD \) that will have the same area and the same perimeter as \( \triangle ABC \) and **not** be congruent to \( \triangle ABC \)? Explain. (No. If you move the point \( D \) on the line, the perimeter gets larger the farther point \( D \) is from the midpoint of \( AB \). It only reaches the value of perimeter
Extension

In Activity 1, students obtained a triangle, \( \triangle ABD \), with the same perimeter, base and height (and therefore area) as \( \triangle ABC \). The triangles look congruent and students can use reflection to verify that they are congruent. From previous grades, students should know that reflection does not distort shapes and intuitively understand that a shape and its mirror image are congruent.

Students can follow the steps below to copy one of the triangles and reflect it through the perpendicular bisector of the line segment \( AB \).

a) Construct the midpoint of \( AB \).

b) Construct a perpendicular to \( AB \) through the midpoint.

c) Mark the perpendicular bisector you constructed as a mirror line.

d) Reflect \( \triangle ABC \) through the mirror line. What is the image of \( \triangle ABC \)?

**ANSWER:** The point \( D \) that students found is the mirror image of \( C \) through the perpendicular bisector of \( AB \), and \( \triangle ABC \) is congruent to \( \triangle ABD \).
G7-15 Congruency Rules and Similarity
Pages 111–114

CURRICULUM
EXPECTATIONS
Ontario: 7m2, 7m4, 7m6, 7m7, 7m50, 7m51, 7m53
WNCP: essential for 7SS3, [C, R, V]

VOCABULARY
triangle
corresponding angles
corresponding sides
conjecture
right, isosceles, equilateral (triangle)
congruence rule
SAS (side-angle-side)
ASA (angle-side-angle)
SSS (side-side-side)
SAA (side-angle-side)
congruent
similar

Goals
Students will identify congruent triangles and distinguish between congruent and similar triangles.

PRIOR KNOWLEDGE REQUIRED
Can measure angles and sides of polygons
Is familiar with notation for equal sides and angles
Can name angles and polygons
Is familiar with the symbols for angle, triangle, congruent triangles
Knows that sum of the angles in a triangle is 180°

MATERIALS
protractors
rulers
dynamic geometry software (optional)

Congruence and triangles. Remind students that when checking for congruency we select the order in which we want to check that the sides and angles are equal. This order is called correspondence—we check that corresponding sides and angles are equal. Until now, we checked that all sides and all angles are equal, in order. Explain that today we will be dealing mostly with triangles. They have only 3 sides and 3 angles, so they are the simplest possible polygons, and this might make checking for congruence easier. We will be looking for shortcuts. For example, if we check two pairs of angles, will we need to check the third one? Use the following questions to guide students to the answer: What is the sum of the angles in any triangle? If we know the measures of two angles in a triangle, how can we find the measure of the third angle? (by subtracting the sum of the other two angles from 180°) This means we do not have to check all 6 elements of a pair of triangles (3 sides and 3 angles), we can check only 3 sides and 2 angles. Can we do better?

Conjectures. Before having students do Investigation 1 on Workbook page 111, explain that a conjecture is a statement that you think is true but that hasn’t been proved mathematically. When you see a pattern in some information and wonder if it is always true, you make a conjecture. You need to test your conjecture on some cases to see if it is likely to be true. Mathematicians only take a conjecture to be true if they prove it using logic, because one usually cannot check all possible cases. To see that a conjecture is false, one counter-example is enough.

After doing the Investigation, progress through questions similar to Questions 1 through 8 on Workbook pages 112–113. EXTRA PRACTICE:
1. Which congruence rule tells that the two triangles are congruent? Write the congruence statement.

**a)**

2. Sketch a counter-example to show why each statement is false.

  a) If two triangles have two corresponding angles that are equal, the triangles are congruent.

  b) \( \triangle ABC \) has \( AB = BC = 5 \text{ cm} \). \( \triangle DEF \) has \( DE = EF = 5 \text{ cm} \). Then \( \triangle ABC \cong \triangle DEF \).

If students are having trouble with Workbook page 113 Question 9, have them sketch an acute isosceles triangle with one angle 30° and an obtuse isosceles triangle with one angle 30°. Ask them to label the triangles \( \triangle ABC \) and \( \triangle DEF \), so that \( \angle A \) and \( \angle D \) are both 30°. **ASK:** Can you make one of the triangles larger so that \( AB = DE \)?

Finally, have students do Investigation 2 on Workbook page 114.

**Similar triangles.** Have students use protractors and rulers to draw two triangles as follows:

a) Draw and label any triangle.

b) Measure the angles of your triangle. Is it necessary to measure all three angles or can you find the measure of the third angle when you know the measure of two angles?

c) Choose a side of the triangle you’ve drawn. What are the measures of the angles adjacent to it?

d) Draw a line segment that is longer or shorter than the chosen side. Then draw two angles equal to the angles adjacent to the chosen side. Use the endpoints of the line segment you drew as vertices of your angles.

e) Extend the arms of the angles you drew so that they intersect.

**ASK:** What can you say about the two triangles you constructed? Explain to students that they have just drawn similar triangles—triangles with the same shape but not necessarily the same size. To produce a triangle similar to a given one, you draw a triangle with the same angles. Students will learn more about similar shapes in lesson G7-36 through G7-38.
Remind students that all points on a circle are the same distance from the centre of the circle, and that this distance is called the radius. Remind students that all line segments that connect points on a circle with its centre are also called radii. Explain that an arc is an unbroken part of a circle, the way a line segment is an unbroken part of a line. The centre of the circle is also called the centre of the arc.

**Constructing circles and arcs.** Remind the students how to draw a circle using a compass. Explain that in the next few lessons they will need to draw circles with a radius that is equal to a given line segment, or circles that pass through a given point. Model constructing a circle (with a given centre) through a point as in Workbook page 115 Question 2, and have students practise drawing circles. You can present several dots on a pre-drawn grid (as shown) and ask students first to copy the dots, then to draw three circles with centre O that pass through points A, B, and C in turn.

**Bonus**

Draw a circle with centre A and radius OA. Which points (O, A, B, C) are on the circle? **ANSWER:** O, C.

**Extending arcs.** Mark a point A on the board and draw a short arc centred at A. Mark a point B on the arc. Mark the radius of the arc as 5 cm. Draw a line m so that when the arc is extended it will intersect the line. Tell students you want to find a point C on the line m so that AC = AB. Show students how to extend the arc to find the point C, where the arc intersects the line. **ASK:** What do we know about point C? (it is on the line m and AC = 5 cm) Have students practise extending arcs with questions such as Question 4 on Workbook page 116.

**Copying line segments.** Explain that geometers often need to copy line segments. A ruler is a good tool, but sometimes we need more precision than a ruler can give us. When this is the case, geometers use a compass. Show how to copy line segments to a given line using a compass.
Questions 5 and 6 on Workbook pages 116 and 117 for practice. Then present a more challenging problem: ask your students to draw a line segment on a blank sheet of paper (no gridlines), so that a line segment twice as long will also fit on the page, and have students swap their pages with a partner. Explain that you want students to construct a line segment that is twice as long as the line segment their partners using only a compass and a ruler, but without using the markings on the ruler.

Give students some time to try, and then present the solution. (Students should draw a long line, then copy the segment twice along this line such that the two line segments have a common endpoint.) Then have students create a line segment that is three times as long as the given line segment.

**Bonus**

Construct a line segment that is eight times as long as the given line segment, but use the fewest steps possible.

**SOLUTION:** Construct a line segment that is twice as long as the given one and use its length as the new setting of the compass. Construct a line segment that is twice as long as the new line segment (or four times the given line segment) and repeat setting the compass to the length of the new line segment.

**ACTIVITY**

a) Ask students to draw a line segment using Geometer’s Sketchpad and to measure it. Then ask them to try to move the endpoints so that the length of the segment becomes, say, 3 cm. Is this easy or hard to do? If you drag the line segment around, does its length change? (no) If you move the endpoints around, does the length of the line segment change? (yes) Show students how to draw a line segment of fixed length (using the circle tool). Will moving the endpoints change the length of the line segment now? (no)

b) Ask students to draw a line segment and label it $AB$. Challenge students to draw a line segment $CD$ that is exactly the same length as $AB$. **PROMPT:** Measure the length of $AB$. You can use the length of $AB$ as the parameter when drawing a line segment of fixed length.
Introduce straightedge. Explain to your students that geometry as a science originated in ancient Greece. People there became interested in solving geometric problems not only for measurement purposes, but also as puzzles. One of the types of problems they solved was performing geometric constructions, and in these constructions they used two tools, a compass and a straightedge. A straightedge is anything that has a straight edge, such as a ruler, a compact disc case, or a box. However, when using a ruler as a straightedge you cannot use the marks on it—you can only use it to draw line segments.

Explain that one of the problems that students solved during the last lesson—how to copy line segments—was exactly the sort of problem the ancient Greeks were interested in. Review copying line segments using a compass and a straightedge.

Constructing triangles given three sides. Ask students to draw in their notebooks line segments of length 3 cm, 5 cm, and 6 cm. Explain that you want them to construct a triangle that will have sides precisely equal to these line segments. ASK: Can you do this using only a ruler? Ask students to try doing so: Have students choose one of the line segments and copy it to the middle of a page using their rulers. Ask them to label the line segment AC. ASK: Do you have any information about the angles of the triangle? (no)

Ask students to draw the side AB equal to another of the line segments at the angle of their choice and to use rulers to compare the distance BC in their picture to the length of the third line segment. Are they equal? (most likely no) Ask students to try to draw AB at a different angle and repeat the
measurements. **ASK:** Do you need to make the angle \( BAC \) larger or smaller to get closer to the length of \( BC \)? Let students try again, adjusting their guess depending on whether the distance they obtained is larger or smaller than \( BC \). The resulting picture will look somewhat like the picture at left.

**ASK:** Is any one of the points you drew \((B_1, B_2, B_3, \text{ or } B_4)\) the right distance from \( C \)? (no) How did you decide that none of the points is the right distance from \( C \)? Should you try to draw more line segments equal to the length of the second line segment? Can you ever be sure you drew the second side at the correct angle using this method? (no) Which tool could you use to create all the possible points that are the same distance from \( A \) as the length of \( AB \)? (compass) Why? (A compass produces a circle and all points on the circle are the same distance from the centre. We will get all the points that are at the right distance from \( A \).)

Teach your students to draw a triangle with given side lengths using a compass as shown on page 118 of the Workbook. Have students practise constructing triangles with sides of different lengths using Question 2.

**Drawing isosceles, equilateral, and scalene triangles using a compass and a straightedge.** Review the classification of triangles by the number of equal sides (equilateral, isosceles, scalene) and by the size of angles (acute, right, obtuse). Workbook page 118 Question 3 asks students to classify the triangles they drew in Question 2.

Write several triples of measurements on the board. Tell students these are the side lengths of triangles, and ask students whether they can classify the triangles by looking only at the side lengths. **EXAMPLES:** a) 3 km, 3 km, 3 km; b) 3 m, 3 m, 5 m; c) 3 m, 5 m, 5 m; d) 27 cm, 36 cm, 45 cm. Then ask students to imagine actually constructing these triangles. Which side will they start with? Starting from that side, will they need to set a compass to the same width or to different widths? Ask students to sketch what their constructions will look like.

Have students work through page 119 in the Workbook. Finally, show students more diagrams like those in Question 5 (two overlapping circles with centres marked and joined by a line). **EXAMPLES:**

First, ask students to draw similar diagrams independently. Think aloud to model for students how to do this. **EXAMPLE:** To draw diagram I, I draw and mark a point \( A \). Then I draw a small circle centered at \( A \) ("circle \( A \)"). Now I need to draw circle \( B \) so that it has a larger radius than circle \( A \), but so that it only overlaps with circle \( A \) a little bit. I enlarge the setting of a compass and try to draw an arc that will slightly overlap the first circle. I mark the point where I set the compass the second time \( B \). Then I draw the circle centred at \( B \), using the same setting of the compass. Then I draw line...
segment \( AB \). \textbf{(NOTE:} Tell students that if the two circles in a diagram look like they have equal radii, students can assume that they are indeed equal.\textbf{)}

Now, ask students to predict which, if any, of the diagrams they drew will produce an equilateral triangle if the centres of the circles are joined to one of the intersection points of the circles. Which diagrams produce isosceles triangles? Have students explain their predictions, and then check them. (In the example above, II produces an equilateral triangle and III and IV produce isosceles triangles)

**ACTIVITY**

Students can draw triangles with sides of given measures using Geometer’s Sketchpad. Discuss with students how the method of drawing a triangle with three given side lengths in Geometer’s Sketchpad is similar to the process of using a compass and a ruler. (To draw a line segment of given length in Geometer’s Sketchpad, you use a circle tool, which is similar to using a compass in the pencil-and-paper construction.)

**Extension**

Using a straightedge and compass or dynamic geometry software try to draw two non-congruent polygons with the following side lengths:

a) 2 cm, 2 cm, 3 cm, 3 cm  
b) 1 cm, 2 cm, 3 cm, 4 cm  
c) 3 cm, 3 cm, 3 cm, 3 cm  
d) 2 cm, 2 cm, 3 cm, 3 cm, 4 cm  
e) 2 cm, 2 cm, 2 cm, 2 cm, 2 cm  
f) 2 cm, 2 cm, 2 cm, 3 cm

**PROCESS EXPECTATION**  
Connecting, Technology

**PROCESS EXPECTATION**  
Making and investigating conjectures

When can you draw two non-congruent polygons with the same side lengths? (If the number of sides is more than 3) When can you not do this? (If you have a triangle) Check your guess with two other examples, one where you think it will work and one where you think it will not.
Review congruence rules. Remind students that congruence rules are shortcuts that allow us to check whether or not triangles are congruent by checking only three elements (sides or angles). **ASK:** What three elements could we use? (3 sides, 2 sides and 1 angle, 1 side and 2 angles, 3 angles) Can we use any three elements? (no) Remind students that the order of the elements is important. For example, in the picture in the margin both triangles have angles of 45°, 45°, and 90°, and one of the sides of one triangle is equal to a side of the other triangle, but the triangles are of different size, so they are not congruent. The side that is equal is in a different place with respect to the angles: it is between the 45° angles in the smaller triangle, and between the 90° angle and a 45° angle in the other. As well, when you have 2 sides and an angle, the angle might be either between the sides (side-angle-side) or opposite to one of the sides (side-side-angle). Point out that only one of these situations (side-angle-side) works as a congruence rule. Show the example at left. Here, two triangles have two pairs of equal sides and a pair of equal angles but the triangles are not congruent. Even though the order of the equal elements is the same in both, the triangles are still not congruent. (This is because the equal elements are not in the right order. Side-angle-side is a congruence rule, but side-side-angle is not.) Use Activity 1 below to produce a different pair of triangles with two pairs of equal sides and one pair of equal angles that are not congruent. Students could also do Activity 2 to investigate the congruence rules using Geometer’s Sketchpad.

**Identifying congruent triangles and using a rule to explain why they are congruent.** Have students apply congruence rules to the triangles on Workbook page 120, Questions 1 and 2. Then present several diagrams where the order of equal elements is wrong, and have students explain why they cannot use congruence rules to tell whether the triangles are congruent. **EXAMPLES:**
Identifying congruent triangles and explaining why they are congruent when triangles have a common side. Explain that sometimes you see triangles that share a side. Draw the picture at left to illustrate what you mean. To prove that these triangles are congruent, you might need to use the fact that the common side belongs to both triangles and therefore makes a pair of equal sides. Have students practise identifying which congruence rules prove that triangles are congruent using Question 4 on Workbook page 121. Finally, have students identify congruent triangles in diagrams that will later be used in actual constructions, such as those in Question 5. Here are more diagrams that can be used for additional practice with the same instructions as in Question 5. (There is no point \( P \) in the third diagram; just have students join \( A \) and \( B \) to the intersections of the circles.)

ACTIVITIES

1. Start with a regular sheet of paper. Fold it diagonally and cut along the crease to create two congruent right scalene triangles. Set aside one of the triangles. Fold the other triangle through the right angle such that the fold is perpendicular to the longest side of the triangle. Part of the longest side will fall on top of the other part of the same side. Cut off the part that “sticks out.” This part, which is a scalene obtuse triangle, is the triangle you need. You can discard the isosceles triangle that is left.

Look at the right triangle you set aside at the beginning and the small scalene obtuse triangle. Compare the sides and the angles. Label the pairs of equal sides with the same number. Mark the equal angles. Are the equal angles opposite a pair of corresponding equal sides? (yes) Are the triangles congruent? (no) How many pairs of equal sides do you have? (2) How many pairs of equal angles? (1) Which description identifies the equal elements, in order, in your triangles: side-angle-side (SAS) or side-side-angle (SSA)? Which one of these is an acronym for a congruence rule? (SAS) Is this statement true or false: SSA is a congruence rule. (false) How do the triangles you’ve made help you to decide? (They are a counter-example to this statement.)

2. Divide students into groups of three. Each student in each group should get one page of BLM Congruence Rules on Geometer’s Sketchpad (pp P-44–P-46). Students work on the construction on their page individually and share the results. Students should tell
which elements of the construction could be modified and using which transformation. *(EXAMPLE:)* I could modify triangle $ABC$ any way I wanted by moving any of the vertices. I could only translate triangle $DEF$ by moving the vertex $D$, and I could only rotate the triangle by moving the vertex $E$. When I tried to move vertex $E$, it would only go along a circle, because it was constructed so that $ED$ has fixed length.)

Have students, in their groups of three, match each BLM with the congruence rule it seems to be showing (SSS, SAS, or ASA). For example, page 2 shows SAS by keeping two side lengths and the angle between them constant, which forces the triangle to be fixed.
Goals
Students will investigate the special properties of a median to the unequal side in an isosceles triangle and use them to construct angle bisectors.

Prior Knowledge Required
Can measure and draw line segments with a ruler
Can draw arcs of given radius with a compass
Can name angles and line segments
Is familiar with notation for equal sides, equal angles, and congruent triangles
Can identify congruent triangles using congruence rules
Can identify right angles

Materials
protractors
rulers
grid paper

Introduce median. Explain to students that a line segment that joins a vertex in a triangle to the midpoint of the opposite side is called a median. Ask students to draw three copies each of an isosceles, equilateral, and scalene triangle (9 triangles in total), and then to draw a different median in each one. ASK: Are there triangles where a median splits the triangle into what look like two congruent triangles? (all three medians of an equilateral triangle, and the median to the unequal side of an isosceles triangle) Invite volunteers to sketch these situations on the board. Remind them to mark equal sides. ASK: Are there triangles where a median does not split the triangle into two congruent triangles? Again, invite volunteers to sketch these situations and mark the equal line segments.

Perpendicular to the opposite side and median. Repeat the exercise above with perpendiculars from a vertex to the opposite side. Then ASK: Are there triangles in which the perpendicular from a vertex to the opposite side was also a median? Invite volunteers to sketch these situations on the board. Ask students what they notice. (In an equilateral triangle, a median is also the bisector of the perpendicular side. The same is true in an isosceles triangle when the median is drawn to the unequal side.) Emphasize that the side to which a perpendicular is drawn in an isosceles triangle determines whether or not it is also a median. Ask students to draw two copies of an acute isosceles triangle and to draw in one of them the median to one of the equal sides, and in the other the perpendicular to one of the equal sides passing through the opposite vertex. Do they split the triangles into two congruent triangles?
Review bisectors and perpendicular bisectors. Remind students that when a line segment or an angle is split into two equal parts (two line segments or two angles) we say that it is bisected. The line, ray, or line segment that separates a line segment or angle into two equal parts is called a bisector. Ask students to look at the 9 triangles they have been working with and find one where the median is perpendicular to the opposite side. **ASK:** What do we call lines that are both perpendicular to a line segment and also bisect the same line segment? (perpendicular bisectors)

**If a median is also a perpendicular bisector, the triangle is isosceles.**

Draw the picture at left on the board. Explain that in this triangle one of the medians is also a perpendicular bisector. In what type of triangle have we seen this situation? (isosceles) Point out that students can make a conjecture—a statement that they think is true but that they haven’t yet proved is true. State the conjecture and write it on the board: *If a median in a triangle is also a perpendicular bisector of the opposite side, the triangle is isosceles.*

**ASK:** Do you see any congruent triangles in the picture? Which ones? (\(\triangle ABD\) and \(\triangle CBD\)) How do you know they are congruent? What sides and angles do you know are equal in this pair of triangles? (\(AD = CD\) and \(\angle ADB = \angle CDB = 90^\circ\)) Are there any sides that are common to both triangles? (yes, \(BD\)) Which congruence rule tells you that these triangles are congruent? (SAS) Ask students to write down which other pairs of sides and angles are equal in these congruent triangles. Point out that we now know for a fact that these elements are equal, because we know that the triangles are congruent. **ASK:** What can you tell now about \(\triangle ABC\)? (it is isosceles) Point out that students have now proved the conjecture they made using logic.

**Reverse statement.** Write on the board:

A median is also a perpendicular bisector. The triangle is isosceles.

**SAY:** We proved that one of these statements follows from the other. Draw an arrow pointing from the statement on the left to the statement on the right. **ASK:** Does it work the other way round? Does the first statement follow from the second one? Draw an arrow pointing from the statement on the right to the statement on the left and write the reverse statement: If a triangle is isosceles, then a median is also a perpendicular bisector. Remind students that such pairs of statements are called the reverse of each other. Work with students through Questions 3 and 4 on Workbook page 123.

**In an isosceles triangle, the median to the unequal side is also an angle bisector.** Have students look briefly at the triangles and the medians they drew at the beginning of the lesson. **ASK:** In any of these triangles, does the median also bisect the angle at the vertex it passes through? Ask students to try to formulate a conjecture based on their observations. Ask students to work in pairs to improve their conjectures and come up with a common one. Have pairs for groups of 4 and repeat. Have each group present their conjecture to the class. Then have students solve Workbook page 124 Question 5, and check answers as a class.
Present the three statements below and tell the students they are all true.  
**ASK:** Which one of them have you proved in this lesson? Give students some time to think (without looking for the answer in the workbook), then ask them to hold up the number of fingers equal to the number of the statement that they proved.  
(ANSWER: 2)  
1. In an isosceles triangle $ABC$ with $AB = BC$, a bisector of $\angle B$ is the perpendicular bisector of side $AC$.  
2. In an isosceles triangle $ABC$ with $AB = BC$, a median to $AC$ bisects $\angle B$ and is perpendicular to side $AC$.  
3. In an isosceles triangle $ABC$ with $AB = BC$, a perpendicular bisector of $AC$ bisects $\angle B$.  

**EXTRA PRACTICE:**  
In the triangle $KLM$ the letters got erased.  

$KL = KM$. Measure the sides of the triangle and mark $K, L, M$.  
(NOTE: There are two possible answers. Students can choose either one).  
Mark the diagram to show these two sides are equal.  
a) $Q$ is the midpoint of side _____. $P$ is the midpoint of side ______.  
Both ____ and _____ are medians of $\triangle KLM$. Use a different colour and marks to show this on your diagram.  
b) Which triangles are congruent? _____ and ______  
c) Which of the medians bisects its angle? ______ How do you know?  
d) Phrase the Isosceles Triangle Theorem for $\triangle KLM$:  
   In an isosceles triangle $KLM$ with ____ = ____ the median to the unequal side ____ bisects $\angle$ _____ and is perpendicular to ____.

**Constructing an angle bisector using a ruler.** Have students work the Questions 7 through 10 on Workbook page 124.  
**PROMPTS:**  
Measure the sides of the triangle. What type of triangle is it? What do you know about one of the medians in an isosceles triangle?  

Draw an angle with equal arms on the board. Point out that you drew the angle so that the arms are equal, and that you would like to bisect the angle. Challenge students to think about how to do that.  
**PROMPT:** How can you make an isosceles triangle? (Join the endpoints of the line segments making the angle with a ruler. This line is the unequal side of the isosceles triangle. Find the midpoint of the side and join it to the vertex of the original angle. This is the median to the unequal side of the isosceles triangle, so it is the angle bisector.)  

Ask students to draw a $50^\circ$ angle with each arm $5$ cm long using a protractor and a ruler. Then ask them to bisect this angle using a ruler only, and to check that their construction works using a protractor.
Now draw an angle on the board with clearly unequal arms. Tell students that you want to bisect this angle. **ASK:** How is this angle different from the angle you just bisected? How can we change this problem into a problem you already know how to do? (e.g., extend the shorter arm to the length of the longer one)

**Bonus** How can you check using a ruler only whether \( BD \) bisects \( \angle ABC \)?

**Extension**

Remind students how they reversed statements of the type “All… are…” (**EXAMPLES:** All girls are people with glasses” vs. “All people with glasses are girls”; “All rectangles are squares” vs. “All squares are rectangles”)

Explain that we can rephrase these statements in the form “If…, then…”

For example, if a person is a girl, then this person wears glasses; if a person wears glasses, then this person is a girl. Ask students to convert the following to “If…, then…” statements.

a) All rectangles are squares. (start with: If a shape…)

b) All bananas are tasty fruits.

c) All spiders are animals with eight legs.

d) All decimal fractions are fractions with denominator that is a power of ten.

e) All words that end with “metre” are units of measurement.

Teach students to reverse an “If…, then…” statement. Explain that all they have to do is switch the parts again. For example, the reverse of “If a vehicle is a van, it has four wheels” will be “If a vehicle has four wheels, it is a van.” Have students reverse the “If…, then…” statements they created for a) through e) above. To check that the answers make sense, ask students to reverse the “All… are…” statements, and convert them to “If…, then…” statements. Do students get the same result?

**EXAMPLE:** For c)

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reverse statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>All spiders are animals with eight legs.</td>
<td>All animals with eight legs are spiders.</td>
</tr>
<tr>
<td>If an animal is a spider, then it has eight legs.</td>
<td>If an animal has eight legs, then it is a spider.</td>
</tr>
</tbody>
</table>

Challenge students to do the opposite: convert statements from the form “If…, then…” to the form “All… are…”

f) If a number is a prime number, then it has exactly two factors.

g) If a shape has equal sides, then it is called equilateral.

h) If a triangle has two equal sides, then it has two equal angles.

i) If a shape has two right angles, then it is a right trapezoid.
Then ask students to tell whether each statement above is true or false. Next ask students to reverse the statements, first in the “All… are…” form, then in the “If..., then...” form, and to compare the answers.

Have students look at the statement from the lesson: If a median in a triangle is also a perpendicular bisector, then the triangle is isosceles. Ask students to think how they can convert it to an “All... are...” statement. **ASK:** Why is this statement harder to convert? How are its parts different from the parts of say, statement h)? (Both parts of statement h) talk about properties of a triangle. In this statement, one part talks about the property of a median, the other about the property of a triangle.) Explain that because the median is actually part of a triangle, we can still rephrase the statement. We just have to change the part about the median to something like “triangle that has a median that is also a perpendicular bisector.” Ask students to try to rephrase the “If..., then...” statement about the medians again. (All isosceles triangles are triangles that have a median that is also a perpendicular bisector.) Point out that we can change this sentence to something that sounds better: All isosceles triangles have a median that is also a perpendicular bisector. And we still can reverse the statement: All triangles with medians that are perpendicular bisectors are isosceles.
Remind students that a straightedge is any tool with a straight side. When they are asked to use a straightedge, they can use a ruler, but they cannot use the markings on the ruler. Instead, they can copy line segments with a compass.

**Constructing angle bisectors.** Model the construction of an angle bisector using a compass and a straightedge. Have students practise the construction using Questions 1 through 4 on Workbook pages 125 and 126. Then ask students to draw a large obtuse scalene triangle and to construct angle bisectors to all three angles of the triangle. Assessment tip: If the construction is performed correctly, the bisectors will intersect at the same point. When you have checked students’ work, point out to the students that the bisectors should intersect at the same point, and they can now use it as a self-checking mechanism.

**Why does the construction work?** Do Question 6 on Workbook page 126 together as a class. Have students articulate how the fact that \(\triangle QTU\) and \(\triangle QSU\) are congruent helps them to explain why the construction works. (Since the triangles are congruent, the corresponding angles \(\angle TQU\) and \(\angle SQU\) are equal too. This means \(QU\) splits \(\angle TQS\) into two equal angles, so \(QU\) is the angle bisector of \(\angle TQS\).)
Review constructing triangles with a compass and a straightedge. Review the method briefly, then discuss with students how the diagram for an equilateral triangle is different from the diagrams for constructing other triangles with a compass and a straightedge: since all the sides are equal, the circles have equal radii, and they also pass through the centres of each other. Model the construction of an equilateral triangle on the board or invite volunteers to do so. Have students individually construct an equilateral triangle with a compass and a straightedge.

Review with students the fact that all angles of equilateral triangles are equal. **ASK:** What is the measure of the angles in an equilateral triangle? (60°) How do you know? (The sum of the angles in a triangle is 180°, and the angles are equal, so they are \(180° \div 3 = 60°\).)

**Constructing lines that intersect at an angle of 60° and 30°.** Ask students to use the construction of an equilateral triangle to draw two lines that intersect at an angle of 60°. How could you use this construction to construct a pair of lines that intersect at an angle of 30°? (bisect a 60° angle) Have students perform the construction and check its accuracy using a protractor.

**ACTIVITY**

a) Draw an acute angle \(A\) on a blank sheet of paper (not in your notebook as you will need to fold paper).

b) Draw an angle bisector using a set square.

**Step 1:** Place the set square as shown. Make sure the vertex of the set square is at the vertex of your angle. Draw a line from \(B\) as shown.

**Step 2:** Draw a line from \(C\) as shown. Make sure that the same side of the set square is placed along the arm of your angle and that the vertex of the set square is at the vertex of your angle. Mark \(D\).

**Step 3:** Draw a line through \(A\) and \(D\) as shown.

c) Check your answer by folding the paper along \(AD\). Does line \(AB\) meet line \(AC\)?
d) Draw an obtuse angle and repeat steps b) and c).

e) Right triangles are special: SSA is a congruence rule for them if the angle used is the right angle. Find two right triangles in the picture in Step 3 and formulate the SSA congruence rule for them. (If in triangles $ABD$ and $ACD$ $AB = AC$ and $BD$ is the same in both triangles, and angles $\angle B = \angle D = 90^\circ$, then these triangles are congruent.)

f) How is the construction in b) the same as the construction of an angle bisector using a compass and a straightedge? Why does this construction work? (A set square is used to create points $B$ and $C$ that are at the same distance from $A$. In both constructions you are producing two congruent triangles ($\triangle QTU$ and $\triangle QSU$ in the construction in the workbook and $\triangle ABD$ and $\triangle ACD$ in this construction) with a common side that is the bisector of the angle. In this construction the triangles are congruent because they are both right triangles, and $AB = AC$ and $AD$ is the common side, so the triangles are congruent by SSA. (NOTE: SSA is a congruence rule only in the case when the equal angles are the largest angles in the triangle. For example, if both triangles are right triangles, the rule works.)

Extensions

1. If you bisect an angle, and then bisect each half again, you get 4 equal angles. Trisecting an angle (splitting it into 3 equal parts) is impossible using a compass and a straightedge. However, trisecting a line segment is possible—see the Extension in G7-36.

2. a) Draw a line $AC$ and mark a point $B$ on it. Draw a line segment $BD$ intersecting $AC$.

   b) Using a protractor, draw lines $EB$ and $FB$ so that line $EB$ bisects $\angle ABD$ and line $FB$ bisects $\angle DBC$. (See sample in margin.)

   c) Find $\angle EBF$ without using a protractor and then verify your answer using a protractor.

3. a) Draw an acute angle $ABD$. Extend the arm $BA$ beyond $B$ and mark a point $C$ on the extension. You should get two angles, $\angle DBC$ and $\angle ABD$, that add to $180^\circ$. (See sample in margin.)

   b) Construct the angle bisector $EB$ of $\angle ABD$ and the angle bisector $BF$ of $\angle DBC$ by using a compass and a straightedge.

   c) What is the measure of $\angle EBF$? Predict, then verify with a protractor.

   d) Repeat a), b), and c) for a right angle and an obtuse angle.

   e) Does the measure of $\angle EBF$ depend on the size of $\angle ABD$?
f) To explain your answer in e), let \( x \) be the measure of \( \angle ABE \) and \( y \) be the measure of \( \angle CBF \). Which other angles also have the measures \( x \) and \( y \)? Mark them on the diagram.

g) \[ x + x + y + y = \_\_\_^\circ \]
\[ 2x + 2y = \_\_\_^\circ \]
\[ 2(x + y) = \_\_\_^\circ \]
\[ x + y = \_\_\_^\circ \]

h) What is the measure of \( \angle EBF \) in terms of \( x \) and \( y \)? ______________
How many degrees is that?

i) Find the measure of \( \angle EBF \) by using a protractor.
Review what students know about isosceles triangles: To construct an isosceles triangle with given sides, it is convenient to start with the unequal side and use a compass set to the same width to construct the equal sides. Also, if a median in a triangle is a perpendicular bisector, the triangle is isosceles (Isosceles Triangle Theorem).

Develop the method of constructing perpendicular lines. Tell students that today they will construct perpendicular lines using a compass and a straightedge. Ask students to sketch a line, a point on the line and a perpendicular to this line through the point. Explain that this will be the situation they will be constructing.

Show the table below and ASK: Which diagram has a pair of perpendicular line segments? Which rule or theorem could you use to construct a pair of perpendicular lines? (Isosceles Triangle Theorem)
Draw a line and mark point $M$ on it and ask students to do the same. Tell
students that this will be the point $M$ from the Isosceles Triangle Theorem.

**ASK:** Should it be a vertex of the isosceles triangle? (no, it should be the
midpoint of the unequal side) Ask students to use a compass to create
points $A$ and $C$ on the given line so that $AM = CM$. **SAY:** Now you have
one side of the triangle. What type of triangle are you trying to construct?
(isosceles triangle) Is $AC$ one of the two equal sides? (no)

**ASK:** What do you need to construct now? (point $B$, the third vertex of the
isosceles triangle, so that $AB = CB$) Can you use the same radius that you
used to construct $AM$ and $CM$? Have students try this. Does this create
a triangle? Why not? (the arcs intersect at one point only, point $M$, so no
triangle can be created) Should you use a radius larger or smaller than $AM$
to construct the point $B$? (larger)

Ask students to use a compass to construct a point $B$ so that $AB = CB$ and
to use a straightedge to construct $AB$ and $CB$ to finish the triangle $ABC$.
Finally, ask students to construct $BM$.

**PROCESS ASSESSMENT**

7m2, 7m5, [CN, R]

Present the following explanations and have students signal which is the
best explanation for why $BM$ is perpendicular to $AC$.

1. Because the lines look perpendicular.
2. Because $BM$ is a median of triangle $ABC$.
3. Because $BM$ is a median of an isosceles triangle $ABC$.
4. Because $BM$ is a median to $AC$, which is the unequal side of the
isosceles triangle $ABC$.
5. Because $BM$ is an angle bisector of $\angle ABC$, the angle between the
equal sides $AB$ and $BC$ in the isosceles triangle $ABC$.

**ANSWER:** 4

Practise the construction. Have students practise constructing
perpendicular lines through points on a line or without such points. Have
students also do the Investigation on Workbook page 127.

**EXTRA PRACTICE:**

a) Draw a line segment $AB$. Construct lines perpendicular to $AB$ through
$A$ and through $B$.

b) Set the compass width to the length of $AB$ and construct two circles
with centres $A$ and $B$.

c) Label the points where the perpendicular through $A$ intersects the circle
centred at $A$ as $C$ and $D$. Label the points where the perpendicular
through $B$ intersects the circle centred at $B$ as $E$ and $F$.

d) Join points $C$, $D$, $E$, and $F$ to make a quadrilateral.

e) Name the shape you constructed. What shape is it? Explain.
Bonus

a) Construct a line segment $KL$ with midpoint $O$.

b) Draw two arcs of the same radius centred at $K$ and $L$ so that the arcs intersect twice. Label the intersection points $M$ and $N$.

c) How is this construction similar to the construction of a pair of perpendicular lines? (When you construct a perpendicular line to a given line, you start by constructing a point the way you constructed $M$ or $N$ here.)

d) Construct $MO$ and $NO$. Use what you know about $\angle MOL$ and $\angle NOL$ to explain why $\angle MON$ is a straight angle.

e) Construct a quadrilateral $KMLN$. What type of quadrilateral is that? Explain.
**G7-22 Constructing Parallel Lines**

Page 128

**CURRICULUM EXPECTATIONS**
Ontario: 7m2, 7m46
WNCP: 7SS3, [CN, R, V]

**VOCABULARY**
parallel
perpendicular
triangle

**Goals**

Students will construct parallel lines and explain why the construction works.

**PRIOR KNOWLEDGE REQUIRED**

- Can measure and draw line segments with a ruler
- Can draw arcs of given radius with a compass
- Can name angles and line segments
- Can identify right angles and perpendicular and parallel lines
- Is familiar with notation for equal sides, equal angles, and parallel and perpendicular lines
- Can identify congruent triangles

**MATERIALS**

- compass
- straightedge
- dynamic geometry software (optional)

**Review parallel lines.** Remind students that parallel lines are straight lines that never intersect and are the same distance apart everywhere (where the distance is measured along a perpendicular to both lines). Remind students that when a line intersects one of the parallel lines at a right angle, the other parallel line also intersects the first line at a right angle. Students can review this using a protractor and a pair of parallel lines or dynamic geometry software, such as Geometer’s Sketchpad (see the Activities). They can construct two parallel lines, draw a perpendicular to one of them and measure the angle between the other parallel line and the perpendicular to the first line. Next they can modify the picture to see that the angles do not change.

**Model constructing a line parallel to a given line.** Start by reviewing the construction of a perpendicular line through a point on a given line (see Workbook page 127). Then model constructing a perpendicular to the line you just constructed (that is, a perpendicular to the perpendicular). Does the new line look parallel to the given line? Have students repeat your construction, then draw several perpendiculars to both lines and check the distance between the lines. Are they indeed parallel?

**Why does the construction work?** Work with students through Question 3 on Workbook page 128.
### ACTIVITY

<table>
<thead>
<tr>
<th>PROCESS EXPECTATION</th>
<th>Technology</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>a)</strong> Learn how to draw parallel lines in Geometer’s Sketchpad.</td>
<td></td>
</tr>
<tr>
<td>i) Draw a line. Label it $m$.</td>
<td></td>
</tr>
<tr>
<td>ii) Mark a point $A$ not on line $m$.</td>
<td></td>
</tr>
<tr>
<td>iii) Draw another line $n$ through point $A$.</td>
<td></td>
</tr>
<tr>
<td>iv) Measure the angle between the two lines.</td>
<td></td>
</tr>
<tr>
<td>v) Move the points on $n$ to make lines $m$ and $n$ look parallel. How will you know when you have made the lines parallel? <strong>(PROMPT: What happens to the measure of the angle between the lines? The measure of the angle will become 0.)</strong> Check whether the lines stay parallel when you move any of the points in the picture.</td>
<td></td>
</tr>
</tbody>
</table>

Explain that you need a method to draw parallel lines so that the lines will remain parallel even if the points on the lines are moved. Teach students to draw parallel lines using the parallel line command from the menu. Do these lines stay parallel to the given line even if points are moved around? (yes)

<table>
<thead>
<tr>
<th>PROCESS EXPECTATION</th>
<th>Technology</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>b)</strong> Use Geometer’s Sketchpad to investigate the properties of parallel lines.</td>
<td></td>
</tr>
<tr>
<td>i) Using Geometer’s Sketchpad, draw a line $m$. Mark a point $A$ not on the line $m$.</td>
<td></td>
</tr>
<tr>
<td>ii) Construct a line $p$ perpendicular to $m$ through point $A$. Label the intersection point of $m$ and $p$ as $B$.</td>
<td></td>
</tr>
<tr>
<td>iii) Construct a line $n$ perpendicular to $p$ through point $A$. What can you say about the lines $m$ and $n$? (they are parallel) How is this construction similar to constructing a pair of parallel lines using a compass and a straightedge? How is it different?</td>
<td></td>
</tr>
<tr>
<td>iv) Mark a point $C$ on line $n$. Construct a line $r$ perpendicular to $n$ through point $C$. Label the intersection point of $r$ and $m$ as $D$.</td>
<td></td>
</tr>
<tr>
<td>v) Measure the angle $CDB$. What do you notice? Move the points $A$, $B$, $C$, $D$ around. Does $\angle CDB$ change?</td>
<td></td>
</tr>
<tr>
<td>vi) Using the line segment tool, draw line segments $AB$ and $CD$. Measure their lengths. What do you notice?</td>
<td></td>
</tr>
<tr>
<td>vii) Move $C$ around. Do the lengths of the line segments change?</td>
<td></td>
</tr>
<tr>
<td>viii) Move $A$ around. Do the lengths of the line segments change? Does $AB$ stay equal to $CD$?</td>
<td></td>
</tr>
</tbody>
</table>
**Goals**

Students will construct perpendicular bisectors and lines intersecting at 45° and explain why the constructions work. They will also practise constructing lines that intersect at given angles.

**Curriculum Expectations**

- **Ontario:** 7m1, 7m5, 7m7, 7m46
- **WNCP:** 7SS3, [C, CN, R, V]

**Vocabulary**

- compass
- straightedge
- corresponding angles
- corresponding sides
- congruence rule
- SSS (side-side-side)
- SAS (side-angle-side)
- congruent triangles
- central angle
- bisector
- perpendicular bisector

**Perpendicular bisectors.** Present the following problem:

a) Draw a line segment $PQ$. Using a compass, construct a point $R$ not on $PQ$ so that $PR = QR$. (PROMPT: What type of triangle will $\triangle PQR$ be? (isosceles) How do you construct such triangles?)

b) How can you construct two points like $R$ in a) without changing the settings of the compass? (PROMPT: Sketch the situation on the board so that the line $PQ$ is vertical and ask whether it is important on what side of the line students draw point $R$.)

Ask students to join the two points they have constructed with a line. What special properties does this line appear to have? Why could that be? Prompt students to think about symmetry: the picture they drew is symmetrical, and the line they produced is the line of symmetry, with one arc being the mirror image of the other. From experimenting with Miras in earlier grades students will have intuitive knowledge that the line of symmetry will be the perpendicular bisector of the line that joins the mirror images.

**Prior Knowledge Required**

- Can measure and draw line segments with a ruler
- Can draw arcs of given radius with a compass
- Can construct lines intersecting at 60° and 90° using a compass and a straightedge
- Can construct angle bisectors using a compass and a straightedge
- Can name angles and line segments
- Can identify right angles and perpendicular and parallel lines
- Is familiar with notation for equal sides, equal angles, and parallel and perpendicular lines
- Knows that the sum of the angles in a triangle is 180°
- Can identify central angles
- Knows that the sum of the central angles on a circle is 360°

**Materials**

- compasses
- straightedges
- protractors
Have students practise the construction they have just done by constructing more perpendicular bisectors of line segments. Then work through Questions 2 and 3 on Workbook page 129 to explain why the construction works.

Review central angles. Students need to know what a central angle is to do Question 4 on Workbook page 130. Use the question to reinforce the fact that the sum of the central angles in a circle is 360°.

Constructing lines that intersect at an angle of 45°. Brainstorm with students what angles they know how to construct using a compass and a straightedge. (at minimum, they should mention 90°, 60°, and 30°) Review briefly the construction of each angle. Help students to think of other angles they could construct. For example, if you bisect an angle of 30°, what is the measure of each new angle you get? (15°) If you extend one of the arms of a 60° angle beyond the vertex, what angle do you get? (120°) How can you construct an angle of 150°? (extend one arm of a 30° angle) What angle will you get if you bisect a 150° angle? a 90° angle?

Have students practise the construction of an angle that is 45° by constructing a pair of perpendicular lines and bisecting one of the right angles. Students can also solve the following problem:

a) Jason is constructing a triangle with two angles that are 45° using a compass and a straightedge. He draws a line segment, constructs a perpendicular to it through one of the endpoints, and bisects the angle the line segment and the perpendicular formed. Then he repeats the construction at the other end of the line segment.

Sketch Jason’s construction. (see answer in margin)

What kind of triangle has Jason constructed? Give the most information you can. (right isosceles)

b) Medea is constructing a right isosceles triangle using a compass and a straightedge. She draws a line segment, constructs a perpendicular to it through one of the ends, and measures a line segment equal to the one she started with along the perpendicular using a compass. She joins the endpoints of the two line segments.

Sketch Medea’s construction. (see answer in margin)

What is the measure of the angles in Medea’s triangle? (45°, 45°, 90°)

c) What can you say about Medea’s triangle and Jason’s triangle? Whose method do you like more? (Both triangles are right isosceles triangles. Medea’s method is easier.)

d) Use the construction you prefer to construct a pair of lines that intersect at 45°.
Extension

INVESTIGATION

A. Construct the perpendicular bisectors to each of the sides of \( \triangle ABC \). What do you notice?

B. Call the intersection point of the perpendicular bisectors \( D \). Measure the distances \( AD, BD, CD \). What do you notice?

C. Construct a circle with centre at \( D \) and radius \( AD \). This circle is called the circumcircle of the triangle \( ABC \).

D. Do you think the centre of a triangle’s circumcircle is always inside the triangle?

E. Repeat the construction with a right triangle and an obtuse triangle. What do you notice?

F. Compare your results for part E with the results of other students. Did they have triangles congruent to yours? Did they get the same result you did?

ACTIVITY

a) Construct a circle and cut it out. Draw a scalene triangle \( ABC \) with vertices on the edge of the circle. Fold the circle in half so that \( A \) meets \( B \). Look at the line that the crease in your fold makes. Is it

i) a bisector of angle \( C \)?

ii) a perpendicular bisector of line segment \( AB \)?

iii) a diameter of the circle?

iv) a radius of the circle?

b) Fold the circle in half again, this time making \( A \) meet \( C \). What two properties will the crease fold have? Repeat, making \( B \) meet \( C \).

c) At what point in the circle will all three perpendicular bisectors meet?
Goals

Students will solve problems related to parallel and perpendicular lines.

PRIOR KNOWLEDGE REQUIRED

- Can measure and draw line segments with a ruler
- Can draw arcs of given radius with a compass
- Can identify right angles and perpendicular and parallel lines
- Can name angles and line segments
- Can construct perpendicular and parallel lines using a compass and a straightedge
- Can construct angle bisectors and perpendicular bisectors using a compass and a straightedge
- Is familiar with notation for equal sides, equal angles, and parallel and perpendicular lines
- Knows that the sum of the angles in a triangle is 180°

MATERIALS

- straightedges
- compasses
- protractors

Review congruence rules. Students will be mostly using the side-side-side (SSS) and side-angle-side (SAS) rules during this lesson. Use Question 1 on Workbook page 131 for review. As well, draw several pairs of triangles, as in the margin, and ask students which congruence rule will help them to explain why the triangles are congruent.

Properties of points on the perpendicular bisector of a line segment. Have students investigate the distance from points on the perpendicular bisector to the endpoints of the line segment it bisects. Use Question 2 on Workbook page 131. Explain that points that are the same distance from \( A \) and \( B \) are called equidistant from \( A \) and \( B \). Sketch a line segment \( AB \) and its perpendicular bisector \( CD \) as shown at left. Write on the board:

- Point \( E \) is on \( CD \), the perpendicular bisector of \( AB \)
- Point \( E \) is equidistant from \( A \) and \( B \)

Which statement follows from the other? Ask students to signal which direction they proved in Question 2. Students can point their arms or fingers in the right direction (from left to right). Students can also vote on whether they think the other direction is true as well. Explain that to prove the other direction they need to take a point \( E \) and assume that \( AE = BE \). Will that mean that \( ED \) is perpendicular to \( AB \)? (Point out that since \( D \) is the midpoint of \( AB \), \( ED \) is a bisector of \( AB \), but it is unclear whether it is a perpendicular bisector. So we need to check the angle between the two lines.) Students
can use the Investigation on Workbook page 131 to prove that $ED$ is in fact a perpendicular bisector of $AB$.

Here are some additional problems students can try to solve using the material learned in this unit.

1. a) Miki wants to construct a perpendicular to a line through a point $C$ that is not on the line. He thinks:

   I know how to construct the perpendicular bisector of a line segment, but a line has no bisector.

   I would like to find $A$ and $B$ on the given line so that the line segment $AB$ has its perpendicular bisector passing through $C$.

   I know that all the points on the perpendicular bisector of a line segment $AB$ are __________________ from points $A$ and $B$.

   This means I need to construct points $A$ and $B$ on the given line so that $AC = \text{____}$. 

b) Draw a line and a point $C$ not on the line. Use a compass to construct points $A$ and $B$ on the line so that $AC = BC$.

c) Construct a perpendicular bisector to $AB$. Does it pass through $C$? Explain why this happened. (Point $C$ is equidistant from $A$ and $B$, so it is on the perpendicular bisector to $AB$)

2. How could you use the method for constructing a perpendicular to a line through a point $C$ that is not on the line from Question 1 to draw a line that is parallel to $AB$ and passes through the point $C$?

   Explain why the lines you constructed are parallel.

   **ANSWER:** Construct a line $m$ perpendicular to $AB$. Then construct a line $n$ through $C$ perpendicular to $m$. Lines $AB$ and $n$ are parallel because they are both perpendicular to line $m$.

3. The point $A$ is the centre of arc $BC$.

   a) Construct a perpendicular from $B$ to $AC$.

   b) Construct a perpendicular from $C$ to $AB$.

   c) Call the intersection point of the perpendiculars $D$. Construct the line segment $AD$.

   d) Measure the angles $\angle BAD$ and $\angle CAD$. What do you notice? ($\angle BAD = \angle CAD$)

   e) Construct your own angle with vertex $A$. Construct an arc with centre $A$ intersecting the arms of the angle. Label the intersection points $B$ and $C$. Repeat the construction in a) to d). Did you get the same result? (yes)

   **Bonus** Use symmetry to explain your findings.
Extension

BLM Constructing Regular Polygons with a Compass and a Straightedge (pp P-47–P-49). Students will need to make sketches. Review what this means (students do not have to draw perfect pictures, they just need pictures that convey the important information). For example, a sketch of a regular hexagon can look like the picture at right, but not like the other two.
Triangles for Sorting

A

B

C

D

E

I

G

H

J

K

L

M

N

O
Two Pentagons

a) How many right angles does each pentagon have? _____

b) Fold the pentagons so that you can see that the remaining angles are all equal.

c) Does each side on pentagon A have a side of the same length on pentagon B? _____

d) Is there the same number of sides of each length on both pentagons? _____

e) If you place one pentagon on top of the other, do they match? _____

f) Are they the same shape? _____

g) Can we say that these pentagons have the same sides and angles? _____

h) Are the pentagons congruent? _____

i) What makes the answers to g) and h) different?

__________________________________________________________________________

__________________________________________________________________________

__________________________________________________________________________

__________________________________________________________________________

__________________________________________________________________________
1. a) Find the area and/or the perimeter of the shapes.

   ![Shape A](image1)
   ![Shape B](image2)
   ![Shape C](image3)
   ![Shape D](image4)
   ![Shape E](image5)
   ![Shape F](image6)

   - Area = _____
   - Area = _____
   - Area = _____
   - Area = _____
   - Area = _____
   - Area = _____

   - Perimeter = _____
   - Perimeter = _____
   - Perimeter = _____
   - Perimeter = _____
   - Perimeter = _____

b) The following statements are false. Match the counter-examples from a) to the statements.

   i) Two triangles with the same perimeter are always congruent.
      Counter-example: _____

   ii) Two triangles with the same perimeter always have the same area.
      Counter-example: _____

   iii) Two triangles with the same base and height are always congruent.
      Counter-example: _____

   iv) Two trapezoids with the same bases and height are always congruent.
      Counter-example: _____

   v) Two trapezoids with the same area always have the same perimeter.
      Counter-example: _____
2. a) Find the area and perimeter of each shape.

\[
\begin{array}{cccc}
&A&4&6 \\
&6& & \\
\hline
&\text{Area} &=& \\
&\text{Perimeter} &=& \\
&B&4&5 &6 \\
& & & \\
\hline
&\text{Area} &=& \\
&\text{Perimeter} &=& \\
&C&3&5 &7 \\
& & & \\
\hline
&\text{Area} &=& \\
&\text{Perimeter} &=& \\
&D&3 &8 \\
& & & \\
\hline
&\text{Area} &=& \\
&\text{Perimeter} &=& \\
\end{array}
\]

b) Decide whether each statement is true or false. For each true statement, explain your thinking. For each false statement, find at least one counter-example among the shapes above.

i) Two rectangles with the same area are always congruent.

False. Counter-examples: A and D

ii) Two congruent parallelograms always have the same area.

_____________________________

iii) Two congruent triangles always have the same perimeter.

_____________________________

iv) Two parallelograms with the same base and height always have the same area.

_____________________________

v) Two parallelograms with the same base and height always have the same perimeter.

_____________________________

vi) Two parallelograms with the same sum of base and height have the same area.

_____________________________

3. True or False?

For each true statement, explain your thinking.
For each false statement sketch a counter-example.

a) Two triangles with the same side lengths are always congruent.

b) Two triangles with the same side lengths always have the same area.

c) Two triangles with the same base and height always have the same area.

d) Two triangles with the same base and height always have the same angles.
Congruence Rules on Geometer’s Sketchpad (1)

1. Using the polygon tool, construct a triangle $ABC$. Measure the sides and the angles of your triangle.

2. a) Construct a point $D$. Using a command for constructing circles and the length of $AB$ as the radius, construct a circle with centre $D$. Construct line segment $DE = AB$. Hide the circle.
   
   b) Using a command for constructing circles and the length of $BC$ as the radius, construct a circle with centre $E$.
   
   c) Using a command for constructing circles and the length of $AC$ as the radius, construct a circle with centre $D$.
   
   d) Construct a point that is on both circles. Label it $F$. Use the polygon tool to construct triangle $DEF$. Hide the circles.

3. a) Which sides are equal in $\triangle ABC$ and $\triangle DEF$?
   
   b) Measure the angles of $\triangle DEF$. What can you say about $\triangle ABC$ and $\triangle DEF$?

4. Try to move the vertices of $\triangle DEF$ around.
   
   a) How does your triangle change? Which transformations can you make: rotations, reflections, translations?

   b) Can you move $\triangle DEF$ onto $\triangle ABC$ to check whether they are congruent? Do you need to reflect the triangle to do that?

5. Move $\triangle DEF$ away from $\triangle ABC$. Try to move the vertices of $\triangle ABC$ around.
   
   a) Are the changes you can make in this triangle different from the changes you could make to $\triangle DEF$? Why?

   b) What happens to $\triangle DEF$ when you modify $\triangle ABC$? What can you say about the triangles $\triangle ABC$ and $\triangle DEF$?

6. Are three side lengths enough to determine a unique triangle?
Congruence Rules on Geometer’s Sketchpad (2)

1. Using the polygon tool, construct a triangle $ABC$. Measure the sides and the angles of your triangle.

2. a) Construct a point $D$. Using a command for constructing circles and the length of $AB$ as the radius, construct a circle with centre $D$. Construct line segment $DE = AB$. Hide the circle.
   
b) Using a command for constructing circles and the length of $BC$ as the radius, construct a circle with centre $E$.
   
c) Select $E$ as a centre of rotation. Use a command in the Transformation menu to mark $\angle ABC$ as the angle of rotation. Rotate point $D$ around $E$ by the angle chosen. Construct a ray from $E$ through the image of $D$.
   
d) Construct a point that is on the ray and the circle. Label it $F$. Use the polygon tool to construct triangle $DEF$. Hide the circle and the ray $EF$.

3. a) Which sides and angles are equal in $\triangle ABC$ and $\triangle DEF$?

b) Measure the rest of the sides and angles of $\triangle DEF$. What can you say about $\triangle ABC$ and $\triangle DEF$?

4. Try to move the vertices of $\triangle DEF$ around.
   
a) How does your triangle change? Which transformations can you make: rotations, reflections, translations?

b) Can you move $\triangle DEF$ onto $\triangle ABC$ to check whether they are congruent? Do you need to reflect the triangle to do that?

5. Move $\triangle DEF$ away from $\triangle ABC$. Try to move the vertices of $\triangle ABC$ around.
   
a) Are the changes you can make in this triangle different from the changes you could make to $\triangle DEF$? Why?

b) What happens to $\triangle DEF$ when you modify $\triangle ABC$? What can you say about the triangles $\triangle ABC$ and $\triangle DEF$?

6. Are two sides and the angle between them enough to determine a unique triangle? ______
Congruence Rules on Geometer’s Sketchpad (3)

1. Using the polygon tool, construct a triangle $ABC$. Measure the sides and the angles of your triangle.

2. a) Construct a point $D$. Using a command for constructing circles and the length of $AB$ as the radius, construct a circle with centre $D$. Construct line segment $DE = AB$. Hide the circle.
   
   b) Select $E$ as a centre of rotation. Use a command in the Transformation menu to mark $\angle ABC$ as the angle of rotation. Rotate point $D$ around $E$ by the angle chosen. Construct a ray from $E$ through the image of $D$.
   
   c) Select $D$ as a centre of rotation. Mark $\angle BAC$ as the angle of rotation. Rotate point $E$ around $D$ by the angle chosen. Construct a ray from $D$ through the image of $E$.
   
   d) Construct a point $F$ that is the intersection point of both rays. Use the polygon tool to construct triangle $DEF$. Hide the rays.

3. a) Which sides and angles are equal in $\triangle ABC$ and $\triangle DEF$?

   b) Measure the rest of the sides and angles of $\triangle DEF$. What can you say about $\triangle ABC$ and $\triangle DEF$?

4. Try to move the vertices of $\triangle DEF$ around.
   
   a) How does your triangle change? Which transformations can you make: rotations, reflections, translations?

   b) Can you move $\triangle DEF$ onto $\triangle ABC$ to check whether they are congruent?

5. Move $\triangle DEF$ away from $\triangle ABC$. Try to move the vertices of $\triangle ABC$ around.
   
   a) Are the changes you can make in this triangle different from the changes you could make to $\triangle DEF$? Why?

   b) What happens to $\triangle DEF$ when you modify $\triangle ABC$? What can you say about the triangles $\triangle ABC$ and $\triangle DEF$?

6. Are two angles and a side between them enough to determine a unique triangle? ______
Constructing Regular Polygons with a Compass and a Straightedge (1)

A polygon is called **regular** if all its sides and all its angles are equal.

1. a) Draw a quadrilateral with all sides equal that is not regular. 
   b) Draw a quadrilateral with all angles equal that is not regular.

2. Can you draw a triangle with all sides equal that is not regular? Explain.

3. What triangle is a regular triangle? ________________
   What is the size of its angles? _____

4. What special quadrilateral is a regular quadrilateral? ________________
   What is the size of its angles? _____

5. Construct a regular quadrilateral using a compass and a straightedge.

6. Jason constructed a regular triangle using a compass and a straightedge.
   What is the size of \( \angle ABD \) in his drawing? _____
INVESTIGATION ▶ How could you construct a regular hexagon using a compass and a straightedge? Start by determining the angles between the sides.

A. Sketch a regular hexagon.

B. A regular hexagon has 6 lines of symmetry. They all pass through the same point in the centre of the hexagon. Sketch the lines of symmetry of your hexagon.
   Label the intersection point of the lines of symmetry O.

C. How many non-overlapping angles are around the point O? ______
   What is the total degree measure of all the angles around O? ______
   What is the measure of the angles between the lines of symmetry? ______
   How do you know? ___________________________________________________________

D. The hexagon at right is a regular hexagon. The lines AO, BO, CO, and GO are some of the lines of symmetry of the hexagon.
   Use the information on your sketch to mark the equal line segments and the equal angles on the diagram at right. Include any right angles.
   Use your sketch to determine the size of these angles:
   \( \angle BOG = \) ______  \( \angle AOB = \) ______  \( \angle BGO = \) ______
   Write this information on the diagram at right, too.

E. What is the size of \( \angle OGB \)? ______  How do you know? __________________________________________________________
   What is the size of \( \angle OBC \)? ______
   What is the size of \( \angle ABC \)? ______
   What is the size of the angles between the sides of a regular hexagon? ______
Constructing Regular Polygons with a Compass and a Straightedge (3)

7. Follow the steps below to construct a regular hexagon using a compass and a straightedge.
   a) Sketch a hexagon. Label it $ABCDEF$.
   b) Start constructing $ABCDEF$ by drawing a line segment $AB$. Then construct a line that intersects $AB$ at $120^\circ$ and passes through $B$.
      **HINT:** How could constructing an equilateral triangle help you? See Question 3.
   c) On the line you drew, mark a point $C$ so that $BC = AB$, and $\angle ABC = 120^\circ$.
   d) The angle between $AC$ and $CD$ is $120^\circ$ again. Which of the diagrams at right is better for drawing a hexagon? Use your sketch to decide.
   e) Repeat steps a) and b) with the new line segment. Repeat until you close the polygon. Use your sketch to keep track of what you are doing.

8. How could you use symmetry to construct a regular hexagon using a compass and a straightedge?
   a) Look at the picture of a regular hexagon at right.
      Mark 12 equal line segments and 18 equal angles in the picture.
   b) Draw a circle with the centre at $O$ and the radius equal to a side of the hexagon. Do all the vertices of the hexagon lie on this circle?
Now you will construct your own hexagon. Use the picture above as a sketch.
   c) The hexagon in the sketch consists of 6 equilateral triangles. Construct an equilateral triangle using a compass and a straightedge.
   d) Choose the side of the triangle that will be a side of your hexagon. Mark that side on your sketch.
   e) Where on your triangle is the point $O$? Label it. Label the other two vertices $A$ and $B$. Which sides of the triangle are the future lines of symmetry (diagonals) of the hexagon?
   f) Extend $OA$ and $OB$ beyond $O$. Using a compass, construct a circle as in b).
   g) How can you use symmetry and the circle to construct two more vertices of your hexagon?
   h) How many vertices does a hexagon have? How many more vertices do you have to construct?
   i) Construct the last two vertices of the hexagon. How did you find the vertices? Did you use parallel lines, construction of lines at $120^\circ$, construction of circles with centres $A$ and $B$, or another method?
   j) Using a protractor and a ruler, check that your hexagon is regular.
Unit 6  Number Sense

In this unit students will use integers in context, compare and order integers, add and subtract integers, and make connections between different methods of addition and subtraction.

Because adding and subtracting integers can be counterintuitive for many students, your students will need a lot of practice with this skill—more than can be provided in these lessons. To promote automaticity, we recommend doing problems for a few minutes each day, two or three days a week, until the end of the school year. Students need to perform integer operations automatically in order to be able to solve problems involving integers. At first, display the chart from Workbook page 189, but eventually scaffold students away from requiring the chart.

Materials
Some students might benefit from using integer tiles when adding and subtracting integers. We do not rely on them in our lessons though.
Goals

Students will add sequences of gains and losses.

Prior Knowledge Required

Has an intuitive understanding of gaining and losing money.

Add integers informally using the familiar context of gains and losses. Introduce the plus (+) and minus (−) signs as symbols for gains and losses. Then have students, given a gain and a loss, decide and indicate whether there is a net gain or net loss by using the appropriate sign. Students can signal their answer by forming a plus sign or a minus sign with their arms.

Write a sequence of gains and losses using + and − signs. See Workbook page 132 Question 2.

Translate a sequence of +’s and −’s to gains and losses. EXAMPLE: + 5 − 3 = a gain of $5 and a loss of $3.

Was more gained or lost? Write down a gain followed by a loss or a loss followed by a gain. EXAMPLE: a loss of $3 then a gain of $5. ASK: Was more gained or lost? Repeat with a sequence of +’s and −’s interpreted as gains and losses. EXAMPLE: − 3 + 5. Emphasize that if the larger number has a “+” sign, then more was gained; if the larger number has a “−” sign, then more was lost. Repeat with various examples.

Bonus − 137 + 142

How much was gained or lost overall? See Workbook page 132 Question 4. Have students do several examples, some of which result in no gain or loss. Then ASK: When was there no gain or loss? (when the amount of the gain was the same as the amount of the loss)

Sequences of more than two gains and losses. Write on the board: + 3 + 2 − 4. ASK: How is this question different from the questions we have done so far in this lesson? (there are three gains and losses instead of two) ASK: What is + 3 + 2? (+5) Write: + 3 + 2 − 4 = + 5 − 4. ASK: What is + 5 − 4? (+1) Write:

\[
\begin{align*}
+ 3 + 2 − 4 &= + 5 − 4 \\
&= +1
\end{align*}
\]

Ask students to find the net gain or loss:

a) + 2 − 5 − 4   b) − 3 − 2 − 6   c) + 2 + 5 − 2

d) − 4 + 6 + 2   e) + 3 − 2 + 7

Solution for a): + 2 − 5 − 4 = − 3 − 4 = −7
ANSWERS: b) \(-11\)  c) \(5\)  d) \(4\)  e) \(8\)

Changing the order of gains and losses results in the same net gain or loss. Ask students to solve these problems:

a) \(+3 - 4 + 2\)  b) \(+2 + 3 - 4\)  c) \(-4 + 3 + 2\)  d) \(+2 - 4 + 3\)

**ASK:** What do you notice about your answers? (they are all the same) Why is that the case? (because we added the same gains and losses each time, just in a different order) To emphasize this point, **ASK:** If one week, I gain \($3\), then gain \($2\), then lose \($4\), and another week, I gain \($2\), lose \($4\), then gain \($3\), which week was a better week? (neither, they are the same)

Longer sequences of gains and losses. Show students how to group all the gains (+'s) together and all the losses (−'s) together. Start with a sequence of only three gains and losses, so that students need to group only two gains or two losses. See Workbook page 132 Question 5. Then progress to longer sequences. Suggest that students start by circling all the gains, so that they can more easily see what they have to group together. If students miss some losses, have them use a different colour to circle the losses—this will help ensure that they didn’t miss any losses.

By grouping all the gains and then all the losses, emphasize that we are changing the problem into a problem with just one gain and one loss—and we already know how to do that kind of problem! Emphasize that changing one problem into another that we already know how to do is a strategy that mathematicians use every day.

Cancelling gains and losses. Have students add these gains and losses:

a) \(-3 + 7 + 3\)  b) \(+3 - 3 + 7\)  c) \(+7 + 3 - 3\)  d) \(+3 + 7 - 3\)

**ASK:** What do you notice about all these answers? (they are all the same) Why is this the case? (because all the same numbers are used each time)

Now have students add these gains and losses:

a) \(-4 + 7 + 4\)  b) \(+5 - 5 + 7\)  c) \(+7 + 6 - 6\)  d) \(+2 + 7 - 2\)

**ASK:** What do you notice about all these answers? (they are all the same)

Why is this the case? (because adding the same gain and loss results in no change)

Tell students that when you add the same gain and loss, you can cancel them, because together they add 0 to the result. Have students practise cancelling the same gain and loss, as on Workbook page 133 Question 7; start with a sequence of three gains and losses and then progress to longer sequences of gains and losses. Include examples where more than one pair of gains and losses cancel each other.
Goals
Students will represent, compare, and order integers.

PRIOR KNOWLEDGE REQUIRED
Understands the relationship between how many more (less) than and addition (subtraction)

NOTE: In this lesson, students will see that the minus sign can be used in two different ways: the “−” in 0 − 5 means “subtract,” but the “−” in front of a number (e.g., −5) indicates that the number is less than 0. Just as −5 represents 0 − 5, we can use +5 to represent 0 + 5 = 5, but the signs are being used in a different way. Be aware that this new use of both the plus and minus signs may cause confusion at first, and students will need practise and time to become comfortable using the signs as both symbols of operations and indicators of position.

Using a credit card to have less than $0. Tell students that you don’t have any money, but you want to buy something. It costs $5. Tell students that you know that you will be getting money soon. ASK: How can I pay for it now? (use a credit card) Explain to students that even though you started off with nothing, you have even less now!

Integers. Remind students that to get the number that is 5 less than 8, we can subtract: 8 − 5. ASK: How can you get the number that is 5 less than 0? (subtract: 0 − 5) Repeat for 8 less than 0 (0 − 8), and then 3 more than 0 (0 + 3). Draw this number line on the board:

0 − 5 0 − 4 0 − 3 0 − 2 0 − 1 0 0 + 1 0 + 2 0 + 3 0 + 4 0 + 5

ASK: Does anyone know a shorter way to write these numbers (point to the numbers larger than 0)? When students give the answer, rewrite the number line as follows:

0 − 5 0 − 4 0 − 3 0 − 2 0 − 1 0 1 2 3 4 5

ASK: What number is 3 less than 2? Demonstrate starting at 2 and moving 3 places left, to the marking labelled 0 − 1. Repeat with these questions: What number is 5 less than 3? (0 − 2) What number is 6 less than 1? (0 − 5) What number is 4 more than 0 − 2? (2)

Tell students that you find it cumbersome to always write zero minus something for numbers less than zero. We don’t have to write zero plus something for numbers larger than 0! Is there a shorter way to write the numbers that are zero minus something? Draw the following number line on the board and ask students if this is a good solution:
ASK: What is wrong with this solution? (we can’t tell the difference between 5 more than 0 and 5 less than 0!) Tell students that mathematicians have come up with a shorter way to write numbers less than 0 that allows us to tell the difference between 5 more than 0 and 5 less than 0. **ASK:** Does anyone know how they do it? Explain that, instead of $0 - 5$, we can just write $-5$. The “−” tells us that the number is less than 0, and the 5 tells us by how much. Draw a number line on the board, with integers from −8 to 8.

Tell students that the numbers less than zero are called **negative** numbers; for example, the number that is 5 less than 0 is called **negative 5**. **ASK:** What do you think numbers larger than zero are called? (positive numbers) Tell students that sometimes we want to emphasize the difference between positive and negative numbers. We can write $+5$ for the number that is 5 more than 0, to emphasize that the number is positive. Now draw this number line on the board:

Explain that negative numbers always have the negative sign (−) in front of them, but positive numbers sometimes have the positive sign (+) and sometimes don’t.

**Distinguish between positive and negative numbers.** Draw a chart with the headings “positive” and “negative.” Write various numbers on the board and have students signal which column to put them in (the left or right column), either by pointing to the column or by raising their left or right hand.

**EXAMPLES:** +5  −2  3  +4  −7  7  +2  (+3)  (−4)

**ASK:** Which two numbers are the same? (3 and (+3)) Then discuss where 0 should go. Some students might suggest putting it in both places, others might suggest putting it nowhere. Both are good answers, but mathematicians have decided to call 0 neither positive nor negative.

**Comparing numbers on a number line.** **ASK:** Tom had no money but spent $2 on his credit card. How can we write how much money he has? (−$2) Demonstrate moving 2 places left from 0 on the number line. Then **ASK:** Ahmed has $6 but spent $11 on his credit card. How much money does Ahmed have? (−$5) Demonstrate moving 11 places left from +6 on the number line. **ASK:** Whose situation is better, Tom’s or Ahmed’s? (Tom’s) How can you tell? (−2 is to the right of −5) Explain that because Tom’s amount is more to the right than Ahmed’s amount, Ahmed would have to gain money to have as much as Tom. **ASK:** How much would Ahmed have to gain to have as much as Tom? ($3, because Tom’s amount is 3 places to the right of Ahmed’s) It might help some students to think about temperatures. Which temperature is lower, −2°C or −5°C? (−5°C) How much lower? (3°C —the temperature would have to increase 3°C to get from −5°C to −2°C)
Compare a negative number to a positive number. Referring to the number line from -8 to +8, ask students to circle the smaller integer in each pair by determining which number comes first (is to the left of the other):

a) -3 or +2  
b) -1 or +4  
c) 1 or -3  
d) 8 or -8  
e) -5 or +2

Ask: If one number is positive and the other number is negative, how can you tell which number is smaller? (the negative number is always smaller) How do you know? (the negative number comes before 0 and 0 comes before the positive number, so the negative number comes before the positive number)

Compare two positive numbers and two negative numbers. Emphasize that students already know how to compare two positive numbers. We know that 3 is less than 4 because 3 comes before 4. Ask: But how do you compare two negative numbers? Which comes first, -3 or -4? (-4) Which is greater, -3 or -4? (-3)

Have students compare two positive numbers and then the opposite negative numbers (but do not use this term yet).

Example:

- 2 is __________ than 5  
- -2 is __________ than -5

After doing several such examples, ask: How can you tell by comparing the positive numbers how to compare the negative numbers? (the order is the opposite; since 5 is more than 2, -5 is less than -2) Encourage students to think about which one is further from 0; for positive numbers, the number that is further is larger, but for negative numbers, the number that is further is smaller. Then have students do these without looking at a number line:

a) -3 is __________ than -4  
b) -7 is __________ than -5  
c) -6 is __________ than -9  
d) -2 is __________ than -1

Bonus: -134 is __________ than -135

Opposite integers. Tell students that the same numbers with opposite signs are called opposite integers, so +2 and -2 (or 2 and -2) are opposite integers. Have students write the opposite of each integer:

a) +3  
b) -5  
c) 4  
d) -10  
e) +7  
f) 3

Answers:

a) -3  
b) +5 or 5  
c) -4  
d) +10 or 10  
e) -7  
f) -3

Ask: Which two answers are the same? (a and f) Why? (because +3 and 3 are the same number, just written two different ways)

Emphasize that the number tells how far from 0 a number is and the sign (+ or -) tells in which direction. So opposite integers are each the same distance from 0, just in opposite directions.

Tell students that because 3 is less than 4, you know that -3 is more than -4. Have each student write a statement that generalizes this, using the term “opposite integers.” After each student has a sentence, have students...
work in pairs to write a sentence they both agree is better than their individual sentences. For example: If one number is less than another, its opposite integer is more than the other’s opposite integer.

**The special case of 0.** Remind students that $+3$ means $0 + 3$ and that $-3$ means $0 - 3$. **ASK:** What do you think $+0$ means? ($0 + 0 = 0$) What do you think $-0$ means? ($0 - 0 = 0$) Tell students that 0 is considered to be the opposite of 0 because $+0$ and $-0$ are both 0, and hence are both the same distance from 0.

**The signs for greater than (>) and less than (<).** Remind students how to use these signs. Three possible mnemonics:

1. The side with two ends instead of one end always points to the greater number.
2. Think of the sign as an open mouth wanting to eat a greater number of [insert food students like].
3. Think of the sign as an equal sign that moves: the two ends of the bars that face the smaller number are close together (smaller distance); the two ends that face the larger number are farther apart (larger distance).

Have students do several problems similar to Workbook page 134 Question 4.

Have students find and label several numbers on a number line and then use their answers to list the numbers in order from smallest to largest.

**EXAMPLE:**

- $a - 3$
- $b - 8$
- $c + 7$
- $d - 2$
- $e 5$

**Temperatures as integers.** Ask students where they have seen integers in real life. **EXAMPLES:** gains and losses (money), golf scores, $+/-$ ratings in hockey, temperatures. Mention temperatures if students don’t. Have students decide, for various pairs of temperatures, which one is warmer. Then have students put several temperatures in order from warmest to coldest.

**Extension**

Remind students of the pattern for powers of 10 and tell students that there is such a thing as negative powers as well:

<table>
<thead>
<tr>
<th>3rd power</th>
<th>2nd power</th>
<th>1st power</th>
<th>0th power</th>
<th>1st power</th>
<th>2nd power</th>
<th>3rd power</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 000</td>
<td>100</td>
<td>10</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Challenge students to continue the pattern, by dividing by 10. **ASK:** What is the $-1$st power of 10? What is the $-2$nd power of 10? What is the $-3$rd power of 10? Repeat for powers of 2 and 3.
NS7-88 Adding Integers
NS7-89 Adding Integers on a Number Line

Goals
Students will add integers in different ways.

PRIOR KNOWLEDGE REQUIRED
Understands the relationship between positive integers and how many more than 0
Understands the relationship between negative integers and how many less than 0

Review integers. Draw the number line from 0 – 5 to 0 + 5 from the previous lesson and have students draw and label the number line the more conventional way in their notebooks. Ask: What notation do we use for 0 – 5? Repeat for 0 + 5.

Use integers to represent gains and losses. See Workbook page 135 Question 1.

Using gains and losses to add integers. Progress as on Workbook page 135 Questions 2-4. Explain that we use brackets when adding integers because the signs for adding and subtracting look exactly like the positive and negative signs in front of the integers, and it would be awkward to write and read expressions like +3 – +4 or –5 + +3 + –2 without brackets.

Opposite integers add to 0. First review the definition of an opposite integer—same number but opposite sign. Tell students that when you gain the same amount as you lose, then you end up with nothing, so opposite integers add to 0.

Cancelling opposites to add integers. See Workbook page 135 Question 5, and Workbook page 136 Question 6.

Using sums of +1s and –1s to add integers. See Workbook page 136 Question 7. If any individual students would benefit from using concrete materials, you could give them integer tiles to work with, if you have them.

Adding integers mentally. See Workbook page 136 Question 8. Remind students that if the gain is bigger, the result is positive, and if the loss is bigger, the result is negative.

Using a number line to add integers. See Workbook page 137 Questions 1 and 2.

The investigation on Workbook page 137. Have students predict the result before doing the investigation.

CURRICULUM EXPECTATIONS
Ontario: 7m1, 7m2, 7m5, 7m13, 7m14
WNCP: essential for 7N6, [R, CN]

VOCABULARY
integer
positive
negative
more than (>)
less than (<)
cancel
opposite integer
counter-example

PROCESS ASSESSMENT
[R], 7m1, 7m2
Workbook p 136 Question 9

PROCESS EXPECTATION
Making and investigating conjectures

Teacher’s Guide for Workbook 7.2
In this activity, students use a vertical number line and the context of height above or below sea level to add integers. (You will be able to apply the same model to subtracting integers by using take away in NS7-90.)

Draw a large vertical number line on the board, going from −6 to +6. Cut out a small paper person from white paper, eight small balloons from blue paper to represent helium balloons, and eight small sandbags from brown paper to represents sandbags. Ensure that each paper person, balloon, and sandbag can be easily taped to the board, by attaching tape to their “backs” ahead of time.

Start the paper person at 0 on the number line and explain that the person is at sea level. The person moves up 1 m every time he is given a helium balloon and down 1 m every time he is given a sandbag.

Start by giving the person 3 helium balloons and ASK: Where will the person be now? Have a volunteer move the person accordingly (to +3 on the number line). Then give the person 4 sandbags. SAY: The person now has 3 helium balloons and 4 sandbags. Where will they be? How can we write that with integers? (+3) + (−4) = (−1) Continue in this way, adding helium balloons (positive integers) and sandbags (negative integers) until you have none left to add. Record the addition of integers as you go. EXAMPLE:

Add… Location of person
3 helium balloons +3
4 sandbags −1 because (+3) + (−4) = (−1)
2 helium balloons +1 because (−1) + (2) = (+1)
3 sandbags −2 because (+1) + (−3) = (−2)
1 sandbag −3 because (−2) + (−1) = (−3)
3 helium balloons 0 because (−3) + (+3) = 0

Note that the person now has 8 helium balloons and 8 sandbags. Since 8 helium balloons represents +8 and 8 sandbags represents −8, we can write this as (+8) + (−8) = 0.
What can we do with integers? Tell students that, so far, we know how to compare integers to determine which is larger, order a list of integers, and add integers. Have students predict what else we can do with integers. Encourage students to say subtract, multiply, and divide, although other answers might arise as well (e.g., find the mean of a set of integers). Tell students that we will learn how to subtract integers this year, but not to multiply and divide them (that comes in Grade 8).

What does it mean to subtract integers? Discuss different meanings for subtracting positive numbers. (take away, how many more than, distance between) Tell students that we will see different contexts for subtracting integers. For example, comparing temperatures: If it is $+32°C$ in Florida and $-10°C$ in Alaska, we can ask ourselves how much warmer it is in Florida than in Alaska. The difference between the temperatures is $+32°C - (-10°C)$.

Subtraction is the opposite of addition. Remind students that some actions “undo” others. ASK: If I walk 3 blocks east, how can I get back to where I started? (walk 3 blocks west) Emphasize that it doesn’t matter where you start—if you walk 3 blocks east and then walk 3 blocks west, you always get back where you started. ASK: I add 3 to a number, how can I get back to the number I started with? (subtract 3 from the result) SAY: I start with a mystery number and add 3 to it. The answer is 10. What is the mystery number? (7) How did you get that? (subtracted 3 from 10 because subtracting 3 undoes adding 3) Have students work in pairs: Student 1 thinks of a number. Student 2 tells Student 1 what to add. Student 1 gives the answer and Student 2 has to find the original mystery number. Partners then switch roles.

Review adding and subtracting a positive number on a number line. Draw a number line on the board. Ask your students how they move on
a number line when they start at 7 and then add and subtract 2. Repeat, starting at 5. **ASK:** How do you add 2 to a number on a number line? (move 2 units right) How do you undo moving 2 units right? (move 2 units left) Emphasize that, no matter where you start, if you move 2 units right and then move 2 units left, you always get back to where you started. Then connect this to subtracting 2. Explain that to subtract 2, you have to undo adding 2, which is like undoing a move of 2 units right by moving 2 units left.

**Subtracting positive integers from negative integers.** Write on the board $5 - 2$ and $(−4) - 2$, and draw a number line from $-10$ to 10. First have a volunteer show how to do $5 - 2$ on the number line (move left 2 units starting at 5) and then have another volunteer show how to do $(−4) - 2$ on the number line (move left 2 units starting at $-4$).

Have students practise subtracting more positive integers from negative integers by going left on a number line. **EXAMPLES:**

a) $(−5) - 2$ b) $(−6) - 1$ c) $(−2) - 3$ d) $(−3) - 4$ e) $(−1) - 5$

**ANSWERS:** a) $-7$ b) $-7$ c) $-5$ d) $-7$ e) $-6$

Have students describe how they found their answers. For example, to find $(−5) - 2$, start at $−5$ on a number line and move 2 places left. To place these problems in context, suggest that the expression in part a) is like having a debt of $5$ and then borrowing two more dollars, or like owing $5$ on your credit card and then spending $2$ more on your credit card.

Subtracting a negative number means going right on a number line. Review with the students how adding a positive integer means moving right on a number line, and adding a negative integer means moving left, in the opposite direction. Subtracting a positive integer also means moving left because you are undoing adding a positive integer. Summarize the situation with the following diagram:

$$(-3) + 2 = (-1) \quad (-3) - 2 = (-5)$$

$$(-3) + (-2) = (-5) \quad (-3) - (-2) =$$

Ask students to guess which direction they need to move in when they subtract a negative integer. Ask them to explain their guesses. Emphasize that we go in the opposite direction to the direction we would go in when adding: to add a negative integer, move left, so to subtract the same negative integer, move right the number of places you would move left to add it. This is because subtracting undoes adding.
Ask students to tell how many units right or left they will move from \(-4\) to perform the following additions and subtractions:

\((-4) + (-2) = \) ______ (to add \(-2\), move 2 units left)
\((-4) - (-2) = \) ______ (to subtract \(-2\), move 2 units right, the opposite of how to add \(-2\))
\((-4) + 3 = \) ______ (to add 3, move 3 units right)
\((-4) - 3 = \) ______ (to subtract 3, move 3 units left, the opposite of how to add 3)
\((-4) + (-1) = \) ______ (move 1 unit left)
\((-4) - (-1) = \) ______ (move 1 unit right)

Then have students do all the problems above in their notebooks.

ANSWERS: \(-6, -2, -1, -7, -5, -3\)

**PROCESS EXPECTATION**

Changing into a known problem

**Writing subtraction questions as addition questions.** Write these questions:

a) \((-5) + 2 = \) ______

b) \((-5) - 2 = \) ______

c) \((-5) + (-2) = \) ______

d) \((-5) - (-2) = \) ______

Have students solve them and then describe how to do each question on a number line: Start at \(-5\) and...

a) move 2 units right

b) move 2 units left

c) move 2 units left

d) move 2 units right

**ASK:** Which questions have the same answer? Why? \((-5) + 2\) has the same answer as \((-5) - (-2)\) because you are doing the same thing—moving 2 units right from \(-5\)—to get both answers; also, \((-5) - 2\) has the same answer as \((-5) + (-2)\) because you are doing the same thing—moving 2 units left from \(-5\)—to get both answers.

Repeat the exercises above with these questions:

a) \(5 + 3\)

b) \(5 - 3\)

c) \(5 + (-3)\)

d) \(5 - (-3)\)

Tell students that when faced with a subtraction question, you can always change it into an addition question. Write \(7 - (-4)\). **ASK:** How do you subtract \(-4\) on a number line? (move 4 units right) Why? (because you are undoing adding \(-4\): you have to move left 4 units to add \(-4\), so you have to move right 4 units to subtract \(-4\)) Moving 4 units right is the same as adding what number? \((+4)\) Have a volunteer finish the equation:

\(7 - (-4) = 7 + \) ______ \((+4\) or just \(4)\)

Have students use the same reasoning to answer these questions:

a) \((-8) - 3 = -8 + \) ______ \((\text{answer: } -3)\)

b) \((-6) - (-2) = -6 + \) ______ \((\text{answer: } 2 \text{ or } +2)\)

c) \(5 - (-3) = 5 + \) ______ \((\text{answer: } +3 \text{ or } 3)\)

d) \(9 - 7 = 9 + \) ______ \((\text{answer: } -7)\)
Writing subtraction questions with variables as addition statements.

Have students write these questions as addition statements:

a) \(5 - (-3)\)  
b) \(4 - (-3)\)  
c) \(3 - (-3)\)  
d) \(2 - (-3)\)

e) \(1 - (-3)\)  
f) \((-5) - (-3)\)  
g) \((-3) - (-3)\)  
h) \(x - (-3)\)

Emphasize that subtracting \(-3\) is always like adding 3, no matter what you are subtracting \(-3\) from. Then ask students to turn the following subtractions into additions:

\[
x - (-3) = \_\_\_
\]
\[
y - (-2) = \_\_\_
\]
\[
z - 4 = \_\_\_
\]
\[
a - (-5) = \_\_\_
\]
\[
p - (+1) = \_\_\_
\]

**Bonus** Find \(x - (-5)\) when \(x = 8\).

**Subtraction as “taking away.”** Write on the board: \((-3) - (-2)\). Remind students that subtraction also means “taking away,” so if you write \(-3 = -1 -1 -1\), and \(-2 = -1 -1\), then you can take away two of the \(-1\)s from \(-3\):

\[
\text{take away}
\]
\[
-3 - (-2) = \big(-1\big) -1 -1
\]

Have students solve these problems (some students might benefit from using integer tiles to literally remove the \(-1\)s):

a) \((-8) - (-5)\)  
b) \((-4) - (-3)\)  
c) \((-6) - (-1)\)  
d) \((-7) - (-4)\)

Then write on the board: \((-3) - (-5)\). Have students try to solve this question. **ASK:** What problem do you run into? (there are not enough \(-1\)s to take away: we need to take away five of them, but we only have three) Challenge students to find a way to overcome this problem. Suggest a few problem-solving strategies (**EXAMPLES:** look for a similar problem for ideas, generalize from known examples, look for a pattern). Allow students time to think. Then **ASK:** When have you ever had to subtract and found there’s not enough of something to subtract from? Write on the board:

\[
\begin{array}{c}
83 \\
-46
\end{array}
\]
\[
\begin{array}{c}
86 \\
-43
\end{array}
\]

**ASK:** Do you remember in what grade you first learned how to do this question? (point to the first one) What made that problem harder than this one (point to the second one) **ANSWER:** In this one, there are enough ones to subtract from; in the first problem, there are not. Challenge students to think about how they solved 83 - 46, and to see if they can use a similar strategy to solve \((-3) - (-5)\). Again, allow students time to think.

Remind students that to solve 83 - 46 they had to add ones without changing the original number. **ASK:** How did you do that? (took a ten from the 8 and added ten ones to the 6)
Now bring students’ attention back to the original problem: \((-3) - (-5)\). Write: \(-3 = -1 - 1 - 1\). Tell students that we want to take away five \(-1\)s but there are only three of them! **ASK:** How can we add more \(-1\)s but still keep the original number the same? **PROMPT:** If I add a \(-1\), what do I have to add as well to keep the original number the same? (1) How many \(-1\)s do I need to add? (2) Tell students that you will add two \(-1\)s and two \(+1\)s: \(-3 = -1 - 1 - 1 -1 + 1 + 1\). **ASK:** Can I take away five \(-1\)s now? (yes) Demonstrate doing so:

\[(-3) - (-5) = (\framebox{-1} \framebox{-1} \framebox{-1} \framebox{+1} +1)\]

Have students solve the following problems using this method:

a) \((-2) - (-6)\)  
b) \((+1) - (-2)\)  
c) \((-3) - (-7)\)  
d) \((-4) - (-9)\)  
e) \((+3) - (-2)\)

**EXTRA PRACTICE:** BLM Subtraction as Take Away

**Use patterns to subtract.** For example, to subtract \(4 - (-3)\), have students first solve the sequence of problems shown in the margin.

Then have students continue the pattern in both the question and the answer: The number being subtracted is getting smaller (4, 3, 2, 1, 0, \(-1\), \(-2\), \(-3\), etc.) and the answer is getting bigger (0, 1, 2, 3, 4, 5, 6, 7, etc.). Continue until you reach \(4 - (-3):\)

\[4 - (-1) = 5\]
\[4 - (-2) = 6\]
\[4 - (-3) = 7\]

**ACTIVITIES 1-2**

1. Refer to the Activity in NS7-88/NS7-89, which uses a vertical number line to add integers. Using the same materials, review adding helium balloons and sandbags to the person and writing addition sentences to represent the person’s new location.

Then put the person at sea level (0 m) holding all the helium balloons and sandbags, and take away a sandbag. **ASK:** When I take away a sandbag, what happens to the person? Why? (taking away a sandbag is the opposite of adding a sandbag, so if the person moves down 1 m when you add a sandbag, the person will move up 1 m when you remove a sandbag) Continue in this way, removing helium balloons (subtracting positive integers) and sandbags (subtracting negative integers) until there are no more left to take away. Have students write the integer subtraction that each take-away represents. **EXAMPLE:** Start at sea level (0 m above or below sea level) with 8 helium balloons and 8 sandbags.
### Take away...

<table>
<thead>
<tr>
<th>Location of person</th>
<th>Number Sense 7-90</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 sandbag</td>
<td>+1, and we write this as $0 - (-1) = +1$</td>
</tr>
<tr>
<td>3 helium balloons</td>
<td>−2, and we write this as $(+1) - (+3) = (-2)$</td>
</tr>
<tr>
<td>2 helium balloons</td>
<td>−4, and we write this as $(−2) - (+2) = (−4)$</td>
</tr>
<tr>
<td>5 sandbags</td>
<td>+1, and we write this as $(−4) - (−5) = (+1)$</td>
</tr>
<tr>
<td>3 helium balloons</td>
<td>−2, and we write this as $(+1) - (+3) = (−2)$</td>
</tr>
<tr>
<td>2 sandbags</td>
<td>0, and we write this as $(−2) - (−2) = 0$</td>
</tr>
</tbody>
</table>

### Integer card game for 2 players.

Players will need a deck of cards (remove the face cards, but leave the jokers in). The black cards are positive, the red cards are negative, and jokers are zeros. Deal four cards to each player and lay the rest in a pile face down. The cards of each player are visible to both players.

Players take turns drawing one card from the pile and laying the card face up. Players then look for cards—at least one from the hand of each player—that add to the card in the centre to result in 0. **EXAMPLE:** If the card in the centre is a red (negative) 2, Player 1 can add a black (positive) 6, and Player 2 can add a red (negative) 4. Total: $−2 + 6 + (−4) = 0$. All three cards are discarded.

If players cannot produce a total of 0 with the card that is face up, the player who drew the card adds it to his or her hand and the other player draws a card from the pile. Players must have at least 4 cards in their hands at all times; if a player has less than 4 cards left, he or she takes more from the pile. The object of the game is to get rid of all the cards in the pile and in the players’ hands. When the cards in the pile are all played, players can add the cards in their hands and if the result is 0, they win (as a team). If it is not, they lose. (They will win if they have added the integers correctly.) Encourage students to notice that they simply need all the red cards to add to the same total as all the black cards in order to get an integer sum of zero.

### Extension

**Use opposite integers to subtract.** Instead of adding $−1$s and $+1$s, add a large enough negative integer and its opposite so that you can perform the subtraction.

**EXAMPLE:** $(-3) - (-5) = (-3) + (-2) + (+2) - (-5)$ since adding opposite integers adds nothing

= $(-5) + (+2) - (-5)$

= $(+2)$ since adding and subtracting the same integers adds nothing
**Goals**

Students will subtract positive numbers from positive or negative numbers by using the model of temperature dropping on a thermometer. Students will then use patterns to predict how to subtract negative numbers from positive or negative numbers.

**PRIOR KNOWLEDGE REQUIRED**

Can add integers
Knows that integers that add to 0 have the same number but a different sign
Is familiar with negative temperatures

**MATERIALS**

calculators

**The plan for the lesson.** Explain to students that in the last lesson, they subtracted integers by using a number line. This time, they will subtract integers by using a thermometer model. Then, they will compare the two methods to see if they give the same answer.

**A thermometer as a vertical number line.** Review adding and subtracting on a number line. Then turn the number line from a horizontal to a vertical position. Add by moving up and subtract by moving down. **ASK:** Where have you seen a vertical number line before? (EXAMPLES: a thermometer, the vertical axis on a graph, the scale on a measuring cup) If no one suggests thermometer, remind them. Then tell students that the temperature was 5°C and increased 2°C. **ASK:** What is the temperature now? (7°C) What operation did you use? (addition) Write on the board: 5 + 2 = 7. Then tell students that the temperature was 5°C and dropped 2°C. **ASK:** What is the temperature now? (3°C) What operation did you use? (subtraction) Write on the board: 5 − 2 = 3. **ASK:** If 5 − 2 shows the temperature dropping from 5°C by 2°C, what would 2 − 5 show? (the temperature dropping from 2°C by 5°C) Explain that although we can’t take away 5 objects when we have only 2 objects, temperature can drop by 5°C when it is only 2°C.

Draw a thermometer on the board. Show starting at 5°C and dropping 2°C. Ask a volunteer to show starting at 2°C and dropping 5°C. **ASK:** What is the temperature now? (−3°C) Write on the board: 2 − 5 = −3.

Have students practise subtracting positive integers from smaller positive integers using the thermometer model. Students could copy a vertical thermometer from 5°C to −5°C into their notebooks to help them.

**EXAMPLES:**

3 − 4 = ______
2 − 6 = ______
3 − 5 = ______
4 − 7 = ______
The relationship between $a - b$ and $b - a$. Have students discover this relationship by completing Workbook page 140 Questions 1 and 2; students will subtract only positive numbers from positive numbers. Then explain the relationship as follows: $5 - 2 + 2 - 5 = 5 - 5 = 0$, so $5 - 2$ and $2 - 5$ add to 0. This means that they are opposite integers: they have the same numerical part, but opposite signs.

**Bonus**

$178 - 187 = \underline{\phantom{100}}$  
$234 - 342 = \underline{\phantom{100}}$  
$3456 - 7890 = \underline{\phantom{100}}$

The relationship between $(-a) - b$ and $-b - a$. See Workbook page 140 Questions 3 and 4. Have students subtract only positive numbers from negative numbers. Discuss how this subtraction fits with what they know about dropping temperatures. Whether the temperature was negative numbers. Discuss how this subtraction fits with what they know about dropping temperatures. Whether the temperature was whether it was $-5^\circ$C and dropped $2^\circ$C, the temperature is now $-7^\circ$C. This is just the fact that $2 + 5 = 5 + 2$, or $-2 - 5 = -5 - 2$.

The relationship between $(-a) - b$ and $a + b$. For example, since $3 + 4 = 7$, $(-3) - 4 = -7$ (just change the answer from positive to negative). Show the symmetry of this on a vertical number line. First, start at 3 and move up 4 places, then start at $-3$ and move down 4 places. You are still 7 away from 0, but in opposite directions, so instead of reaching +7, you get to $-7$. Using temperatures, if the temperature starts at $3^\circ$C and increases by $4^\circ$C, it becomes $7^\circ$C; if the temperature starts at $-3^\circ$C and decreases by $4^\circ$C, it becomes $-7^\circ$C.

**Bonus**

$(-372) - 327 = \underline{\phantom{100}}$  
$(-1234) - 3421 = \underline{\phantom{100}}$  
$(-34567) - 67890 = \underline{\phantom{100}}$

Determine $b - (-a)$. Remind students that if they know $5 - 2$, they can find $2 - 5$ just by changing the sign. Tell students that you know that $(-3) - 2 = -5$ because if the temperature is $-3^\circ$C and it drops by $2^\circ$C, then the temperature becomes $-5^\circ$C (show this on a vertical number line). Write $2 - (-3)$ and **ASK**: Can I use the thermometer model to solve this problem? What does it mean for the temperature to drop $-3^\circ$C? (Some students might think of this as an increase of $3^\circ$C. If so, tell them this could be one interpretation, and it will give the same answer, but because dropping a negative number of degrees might not make sense to everyone, you should explain it a different way.) **ASK**: How can I get $2 - (-3)$ from knowing $(-3) - 2$? **PROMPT**: When we found $2 - 5$, how did knowing $5 - 2$ help? Remind students that when they switch the two numbers being subtracted, they just change the sign of the answer: the answer to $5 - 2$ is $+3$, so the answer to $2 - 5$ is $-3$. Have students predict $2 - (-3)$. **ASK**: Since $(-3) - 2 = -5$, what should $2 - (-3)$ be? Write the following on the board:

$8 - 1 = 7$ so $1 - 8 = -7$  
$(-3) - 2 = -5$ so $2 - (-3) = \underline{\phantom{100}}$  

**ASK**: Using the reasoning of the first example, what should go in the blank? (5) Why? (interchanging the numbers in the subtraction changes the sign of the answer, so the answer should be $+5$) Then have students verify this using their calculators. First, ensure that students know how to add and subtract negative numbers on a calculator. The addition and subtraction
buttons should be familiar, but the +/- button that allows students to input a negative number may not be. To find $2 - (-3)$, students should press:

$$
2 \quad - \quad 3 \quad +/- \quad =
$$

Give students more problems to predict and check. **EXAMPLES:**

$$
(−2) − 4 = −6, \text{ so } 4 − (−2) =
$$

$$
(−4) − 2 = −6, \text{ so } 2 − (−4) =
$$

$$
(−1) − 3 = −4, \text{ so } 3 − (−1) =
$$

Then have students make up their own examples to check.

Finally, explain the result as follows: What is $4 − (−2) + (−2) − 4$? Cup your hands around each subtraction statement in the expression: $(4 − (−2)) + ((−2) − 4)$. Then cover the 4s with your hands and explain that you are subtracting and then adding the same number $(−2)$, so they cancel:

$$
4 − (−2) + (−2) − 4 = 4 − 4 = 0 \text{ (Analogy: } 4 − 3 + 3 − 4 = 4 − 4 = 0)
$$

Since $4 − (−2)$ and $(−2) − 4$ add to 0, they are opposite integers. This means that they have the same number, but opposite signs.

**Another way to determine $b − (−a)$.** Now write on the board:

$$
8 − 2 = 6 \text{ so } 8 − 6 = 2
$$

$$
3 − 7 = −4 \text{ so } ________
$$

**ASK:** Using the reasoning of the first example, what should go in the blank? $(3 − (−4) = 7)$

Then have students redo examples of this form using this new method.

That is, to find $3 − (−4) = ______$, solve $3 − ______ = (−4)$, i.e., how much do you have to drop from $3°C$ to get down to $−4°C$? Do students get the same answer?

**Compare subtracting a negative integer to adding a positive integer.**

First, have students subtract $−3$ from various positive numbers by first subtracting the positive numbers from $−3$:

$$
(−3) − 4 = ___ \text{ so } 4 − (−3) = (\text{answer: } 7)
$$

$$
(−3) − 2 = ___ \text{ so } 2 − (−3) = (\text{answer: } 5)
$$

$$
(−3) − 3 = ___ \text{ so } 3 − (−3) = (\text{answer: } 6)
$$

$$
(−3) − 5 = ___ \text{ so } 5 − (−3) = (\text{answer: } 8)
$$

$$
(−3) − 1 = ___ \text{ so } 1 − (−3) = (\text{answer: } 4)
$$

Now erase or cover the first part of each line above, so that students see only the second equation, and add new equations next to them as follows:

$$
4 − (−3) = ___ \quad 4 + ___ = 7
$$

$$
2 − (−3) = ___ \quad 2 + ___ = 5
$$

$$
3 − (−3) = ___ \quad 3 + ___ = 6
$$

$$
5 − (−3) = ___ \quad 5 + ___ = 8
$$

$$
1 − (−3) = ___ \quad 1 + ___ = 4
$$
Have students fill in the blank with the missing integer (the answer is always 3). **ASK:** What do you have to add to get the same answer as subtracting \(-3\)? (add 3)

**SAY:** In these examples we subtracted \(-3\) and saw that the answer is the same as adding \(+3\). Do you think subtracting other negative numbers will be the same as adding the opposite positive numbers? How could we check?

Have students do the \((-a) - b\) mentally to deduce \(b - (-a)\) and fill in the missing number.

**EXAMPLE:** \(3 - (-2) = \boxed{5}\) (think 

\(-2 - 3 = -5\), so \(3 - (-2) = 5\)

Have students suggest examples of their own and perform the subtraction. Then ask students to check whether adding the opposite integers in the same examples would produce the same answer (in the example above, \(3 + 2 = 5\) gives the same answer).

Finally, have students subtract negative integers by adding the opposite positive integer.

**EXAMPLES:**

\[
\begin{align*}
4 - (-2) &= 4 + \boxed{2} = 6 \\
5 - (-3) &= 5 + \boxed{3} = 8 \\
2 - (-5) &= 2 + \boxed{5} = 7
\end{align*}
\]

**Subtracting a negative integer from a negative integer.** Explain that so far in this lesson, we have only subtracted positive numbers from positive or negative numbers, and negative numbers from positive numbers. **ASK:** What kind of problem haven’t we done? (subtract negative numbers from negative numbers) Write: \((-3) - (-5) = \boxed{2}\) **ASK:** What would you predict we can add to \(-3\) to get the same answer? Then write on the board: \((-3) - (-5) = (-3) + \boxed{2}\). Have students use their prediction to fill in the blank and then check on a calculator. Students should press:

\[
\begin{array}{c}
3 \\
+/- \\
- \\
5 \\
+/- \\
= \\
\end{array}
\]

Do they get the same answer as \(-3 + 5 = 2\)?

**Moving up a thermometer to subtract a negative integer.** Explain to students that they can subtract a positive number by imagining moving down a thermometer. They can subtract a negative number by adding the opposite positive number. To do this, they can imagine moving up a thermometer. See Workbook page 141 Question 9.

**Connect the thermometer method to the number line method.** Discuss how the thermometer is like a horizontal number line. Moving up the thermometer is like moving right on a number line. **ASK:** What is moving down the thermometer like? (moving left on the number line) **ASK:** How did you subtract a negative number on a number line? (moved right) What is moving right like? (moving up on a thermometer) Is that how you subtracted a negative number on a thermometer? (yes)
**ACTIVITY**

Repeat the Activity from NS7-90, which uses helium balloons and sandbags as a model, but with a temperature model.

Draw a thermometer and a bathtub full of water on the board, and have ready several small red and blue cardboard circles. The red circles are hot stones that increase the temperature of the water by 1°C; the blue circles are cold stones that decrease the temperature by 1°C. Say the temperature in the tub is 20°C to start and mark that on the thermometer. Adding stones (and removing them!) changes the temperature of the water. Ask students to adjust the thermometer accordingly. **EXAMPLE:** add 2 red stones, temperature rises to 22°C; add 3 blue stones, temperature drops to 19°C; remove one of the blue stones, temperature rises to 20°C.

Now tell students that corn oil has a freezing point of −20°C and that you filled the bathtub with corn oil instead of water. Repeat the exercises above using negative temperatures, starting at −10°C. Be sure to never reach temperatures below −20°C.

If any students previously described dropping temperature by a negative amount as increasing temperature, this is a good place to point out why the students are correct: taking away something that makes the water colder will make the water warmer.

**Extensions**

1. Students can use number lines to solve equations with integers:

   \[ x - (-3) = -6 \quad y - (-2) = -4 \quad z - 4 = -5 \]

   \[ a - (-5) = 3 \quad p - (+1) = -6 \]

   **EXAMPLE:** \( x - (-3) = -5 \). To subtract \(-3\), I have to go right 3 units. I end at \(-5\). So to find the number I started with \(x\), I have to start at \(-5\) and go left 3 units. This means \(x = -8\). Indeed, if I move 3 places right from \(-8\), I end up at \(-5\).

2. The boiling point of hydrogen is −253°C and the boiling point of oxygen is −183°C. Which gas has a lower boiling point, hydrogen or oxygen? How much lower? (Hydrogen has a lower boiling point by 70°C.)
3. Place the integers $-1$, $-2$, and $-3$ in the shapes so that the equation is true.

a) $\bigcirc + \square = \triangle$

b) $\bigcirc - \square = \triangle$

c) $\bigcirc + \square - \triangle = (-2)$

d) $\bigcirc + \square - \triangle = (-4)$

**ANSWERS:**

a) $-1$, $-2$, $-3$ or $-2$, $-1$, $-3$

b) $-3$, $-1$, $-2$ or $-3$, $-2$, $-1$

c) $-3$, $-1$, $-2$ or $-1$, $-3$, $-2$

d) $-2$, $-3$, $-1$ or $-3$, $-2$, $-1$

4. For each equation, put the same integer in all boxes.

a) $(\square - 4) + \square = \square - 6$

b) $\square + \square = \square - 2$

**ANSWERS:**
a) $-1$

b) $-2$
Goals
Students will use different methods to subtract integers and make connections between the methods they use.

Prior Knowledge Required
Understands subtracting positive numbers
Understands that integers with the same number and different sign add to 0
Can compare and order integers
Can write sums of integers as sequences of gains and losses
Understands that variables represent numbers
Can subtract positive integers from positive integers using a thermometer model

Review subtraction for positive numbers as distance apart. Draw a number line from -8 to +8 and ASK: How far apart are 2 and 7? (5 units apart—count the units together) How would we show that using subtraction? (7 - 2 = 5) ASK: Why didn’t you say 2 - 7 instead? Allow students to articulate their answers, then explain that when talking about how far apart things are, we always want a positive answer (there are no negative distances), so we subtract the smaller number from the larger number; 2 is the same distance from 7 as 7 is from 2, but 2 - 7 doesn’t mean the same thing as 7 - 2.

Using distance apart on a number line to subtract a smaller integer from a larger integer. ASK: How far apart are -3 and 4? (7 units apart—count the units together) How could we show that using subtraction? (4 - (-3) = 7) Emphasize that we do not take -3 - 4, because that is subtracting the larger number from the smaller number. Have students use the distance apart to subtract:

a) 5 - (-4)  
  b) 3 - (-5)

Remember: interchanging the numbers being subtracted results in opposite integer answers. Remind students that opposite integers (same number, opposite sign) add to 0. For example, +2 and -2 are opposite integers because they have the same number and opposite sign, but we could also say that they are opposite integers because they add to 0: (+2) + (-2) = 0. Ask students to show, without calculating either 5 - 3 or 3 - 5, that they add to 0. To get them started, write on the board:

5 - 3 + 3 - 5. (ANSWER: 5 - 3 + 3 - 5 = 5 - 5 = 0)
Then have students show, without subtracting, that the following add to 0:

a) $8 - 5$ and $5 - 8$

b) $5 - (-3)$ and $(-3) - 5$

c) $(-2) - 5$ and $5 - (-2)$

d) $(-7) - (-2)$ and $(-2) - (-7)$

**Using opposite integers to subtract a larger number from a smaller number.** Then write on the board: $2 - 7$. **ASK:** Can I solve this by using the distance apart? What is the distance between 2 and 7? (5 units) Is that the answer to $2 - 7$ or to $7 - 2$? (7 - 2) How do you know? (because 7 is more than 2, so the distance between 2 and 7 tells you $7 - 2$, not $2 - 7$) If we know $7 - 2$, can we get $2 - 7$? (yes) How? (change the sign)

Tell students that the distance on a number line to get from the smaller number to the larger number tells you the larger number minus the smaller number. If you want the smaller number minus the larger number, you have to change the sign.

Have students subtract a smaller number from a larger number using the distance apart on a number line:

a) $5 - 3$

b) $5 - (-3)$

c) $4 - (-5)$

d) $(-1) - (-4)$

e) $(-2) - (-8)$

Then have students subtract the larger number from the smaller number by using their answers to the problems above:

f) $3 - 5$

g) $(-3) - 5$

h) $(-5) - 4$

i) $(-4) - (-1)$

j) $(-8) - (-2)$

**Subtracting smaller numbers from larger numbers results in a positive answer; subtracting larger numbers from smaller numbers results in a negative answer.** Tell students to look at the original questions above, a) – e). **ASK:** Are your answers positive or negative? (positive) Why? (because the distance is positive; because we had to move right on a number line; because if we subtract less than we started with, we’ll get a positive answer) Emphasize that when we take away less than we started with, we have something left, so the answer is positive. If we take away more than we started with, we’re left with less than zero. For example, if I have $5 and I spend $7 (on my credit card) I have less than $0, and I need to gain money just to get back to 0.

Write the following problems on the board:

a) $7 - (-2)$

b) $(-3) - 4$

b) $(-3) - (-5)$

d) $(-4) - (-3)$

e) $8 - 9$

f) $(+12) - (+7)$

g) $(+12) - (-3)$

h) $(-5) - (+2)$

Have students circle the larger number in each question. The have students write whether they expect the answer to be positive or negative. Explain to students that if the answer will be positive—if they are subtracting a smaller number from a larger number—they can go ahead and use distance apart on a number line. Have them do this for the questions they said would have a positive answer.

When the answer is negative, find a problem with the opposite positive integer as the answer. **ASK:** For the questions that have a negative answer, switch the two numbers to change the question so that the answer will be the opposite integer. Demonstrate with question b), which is the first...
question that has a negative answer (because 4 is larger than -3). Change the question to 4 - (-3). The answer to this question is the opposite integer to the answer we are looking for. Using distance apart, this question has answer 7, so b) has answer -7.

Then have students use distance apart to answer all the questions they said would be negative by switching the numbers in the question.

Have students circle the larger number and decide whether or not they have to switch the numbers (is the answer positive or negative?) to solve more subtraction questions.

**EXAMPLES:**

a) (-2) - 5  
   b) (-4) - (-1)  
   c) (-8) - (-3)  
   d) (-5) - 3

**ANSWERS:**

a) 5 - (-2) = 7 so (-2) - 5 = -7  
   b) -3  
   c) -5  
   d) -8

Using distance apart and a sign change when necessary to subtract. Write several questions similar to those in Workbook page 142 Question 7 on the board (see example in margin).

Have students start by writing + (if the answer will be positive) or - (if the answer will be negative) in the circle. Then have students write the distance between the two numbers in the blank beside the circle. Emphasize that if the answer is positive, it will just be the distance apart, but if the answer is negative, it will be the opposite of the distance apart—the same number but a different sign. Therefore, writing the distance between the two numbers in the blank and the correct sign in the circle gives the correct answer.

**Distance apart when two numbers have the same sign.** The distance between two negative numbers is the same as the distance between the same two positive numbers, so you can pretend that the numbers are positive and subtract the smaller one from the larger one to get the number in the answer. Then look at the original question to decide whether the answer is positive or negative.

**EXAMPLES:**

a) 53 - 85  
   b) (-13) - (-85)  
   c) (-102) - (-38)  
   d) (-234) - (-57)

**Distance apart when two numbers have different signs.** First divide the problem into two smaller problems—find the difference between the positive number and 0, and then the difference between the negative number and 0; the total difference is the sum of the two differences. For example, to find (-8) - (+5), the difference between the two numbers will be 8 + 5 = 13. Demonstrate this on a number line.

ASK: The difference between -8 and +5 is 13. Is that the answer? (no) Why not? (we are subtracting the larger number from the smaller number, so our answer should be negative) Write: (-8) - (+5) = -13. Have students solve the following problems using this method:

a) (-24) - (-3)  
   b) (+17) - (-28)  
   c) (+324) - (-476)  
   d) (-218) - (+134)
PROCESS EXPECTATION

Looking for a pattern

**Bonus** Have students copy and complete this T-chart in their notebooks:

<table>
<thead>
<tr>
<th>Pair of Numbers</th>
<th>Distance Apart</th>
</tr>
</thead>
<tbody>
<tr>
<td>−1 and +1</td>
<td>2</td>
</tr>
<tr>
<td>−2 and +2</td>
<td></td>
</tr>
<tr>
<td>−3 and +3</td>
<td></td>
</tr>
<tr>
<td>−4 and +4</td>
<td></td>
</tr>
<tr>
<td>−5 and +5</td>
<td></td>
</tr>
<tr>
<td>−6 and +6</td>
<td></td>
</tr>
</tbody>
</table>

Students should extend the pattern and fill in the last two rows themselves. When students are finished, have them decide how far apart these pairs are: −50 and +50, −124 and +124, −31 342 and +31 342. **EXTRA BONUS:** Calculate the distance between: −60 and +61, −349 and +350.

**Different contexts for subtracting integers using distance apart.** There are many contexts in which we can have values above and below zero. For each problem, have students determine what zero represents, write the given information as integers, and then solve the problem:

a) A fish lives 200 m below sea level and a bird flies at 1500 m above sea level, directly above the fish. How far apart are they? *(ANSWER: Think of sea level as 0 m. The bird is at +1 500 m and the fish is at −200 m, so the distance between them is +1 500 − (−200) = 1 700 m.)*

b) Jericho became a city in around 7500 BCE and Canada became a country in 1867 CE. Approximately how many years passed between Jericho becoming a city and Canada becoming a country? *(ANSWER: Think of 0 as switching from BCE to CE. There isn’t actually a year 0, but we only care about the approximate number of years, so we can calculate as though there is a year 0. Then Canada became a country in year +1867 and Jericho became a city in about year −7500. So the number of years that passed is 1 867 − (−7 500) = 9 367, or about 9 000 years.)*

**Subtraction using gains and losses.** Tell students to think of subtracting a positive number as taking away a gain, and subtracting a negative number as taking away a loss. Taking away a gain of $5 is like losing $5, but taking away a loss of $5 is like gaining $5 (for example, if you owe me $5 but I say you don’t have to give it to me, you’re now $5 ahead of where you were yesterday). Draw the chart in the margin (also on Workbook page 143) on the board. Have students use the chart to rewrite sums of integers as sequences of gains and losses.

**Generalizing to variables.** See Workbook page 143 Question 2.

**Using gains and losses to add and subtract integers.** See Workbook page 143 Question 3.
Missing gains and losses. See Workbook page 143 Question 4. If students find it easier to use a number line, allow them to do so.

Missing integers. Write on the board: \((+8) - \underline{\text{_____}} = (+11)\). \textbf{ASK:} Do you need to gain or lose money to get from +8 to +11? (gain) How much? ($3) The question tells us to subtract. What do we have to take away in order to equal a gain of 3? (take away a loss of 3, which means subtract -3). So -3 is the missing integer. Have students do several similar problems using the same type of reasoning. See Workbook page 143, Question 5.

Another strategy for finding missing integers. Have students write two subtraction statements in the same family as the addition statement. Explain what “the same family” means through examples.

\[
\begin{align*}
a) \quad (-3) + 5 &= 2 \\
2 - 5 &= -3 \\
2 - (-3) &= 5 \\
b) \quad (+2) + (-3) &= (-1) \\
c) \quad (-3) + (+5) &= (+2) \\
d) \quad (-4) + (-2) &= (-6) \\
e) \quad (-4) + 7 &= 3 \\
f) \quad 4 + (-7) &= (-3)
\end{align*}
\]

\textbf{PROCESS EXPECTATION} \\
Reflecting on the reasonableness of an answer

Remind students that the question 12 - _____ = 7 has the same answer as 12 - 7 = _____. \textbf{ASK:} What question has the same answer as \((+8) - \underline{\text{_____}} = (+11)\)? \((+8) - (+11)\) or just 8 - 11? After students finish Workbook page 143 Question 5, have them solve the problems using this reasoning, as a self-checking mechanism. Did they get the same answer both ways?

Have students solve _____ + (-4) = -1, and ask how they can use their answer to Question 5c) to check their answer.

\textbf{ACTIVITY}

If you have the equipment, videotape a student walking backwards. Play it, and then rewind it. What do you see when the tape is rewinding? (The student appears to be walking forwards.) Students can think of playing a tape as adding, rewinding the tape as subtracting, walking forwards as a positive integer, and walking backwards as a negative integer. Playing the tape (adding) is the opposite of rewinding (subtracting). This might help some students remember the rule that subtracting a negative integer does the same thing as adding a positive integer.

Extensions

1. How far apart are these pairs:

\[
-4 \text{ and } +4 \quad -3 \text{ and } +5 \quad -2 \text{ and } +6 \quad -1 \text{ and } +7 \quad 0 \text{ and } +8
\]
What do you notice about your answers? Why? Have students find another pair of integers, one positive and one negative, that are also 8 apart.

2. a) How far apart are \( +3 \) and \( -4 \)? How about \( -3 \) and \( +4 \)? Repeat for \( +2 \) and \( -7 \), then \( -2 \) and \( +7 \). What pattern do you notice?

b) How far apart are \( +3 \) and \( +8 \)? How about \( -3 \) and \( -8 \)? Repeat for \( +4 \) and \( +6 \), then \( -4 \) and \( -6 \). What pattern do you notice?

c) Tell students that, to solve \(( -4 ) - ( -1 )\), instead of switching the numbers around to make \(( -1 ) - ( -4 )\), you saw a student solve the same problem this way: switch the signs, so that \(( -4 ) - ( -1 )\) becomes \(( +4 ) - ( +1 )\), solve, and then switch the sign of the answer. **ASK:** Does this method work too? Have students predict whether this method will always give the right answer. Then tell students that you want to investigate whether this works all the time, but you want to make sure you test the prediction for all possible types of questions. Have students brainstorm the kinds of questions you should test (subtract a positive from a negative, a negative from a positive, a positive from a positive, and a negative from a negative). Students should then investigate with their own problems whether the new method will work. (It will.)

3. **Time Zones**

Because of the Earth’s rotation, different parts of the Earth have daylight at different times; when it is night in some places it is day in others. For example, when it is 3 p.m. in one place, it might be 1 a.m. in another place.

Integers are used to show whether the time of day in a particular place is earlier or later than the time in London, England. Athens is \(+2\), which means it is 2 hours later there than in London. Halifax is \(-4\), which means that in Halifax, it is 4 hours earlier than in London.

A circus is traveling the world. Here is a list of the time zones for the cities they travelled to:

- London, England 0
- Athens, Greece \(+2\)
- Moscow, Russia \(+3\)
- Halifax, Canada \(-4\)
- Toronto, Canada \(-5\)
- Vancouver, Canada \(-8\)
- Cape Town, South Africa \(+1\)
- Bangkok, Thailand \(+6\)

a) Each time the circus enters a different time zone, performers need to change the time on their watches. Use a subtraction statement to determine how they should change the time after each move.

i) They travel from Toronto to Moscow. \(+3 - (-5) = +8\)
   *They should change the time to be 8 hours later.*

ii) They travel from Moscow to Athens. \(+2 - (+3) = -1\)
   *They should change the time to be 1 hour earlier.*

iii) They travel from Athens to Halifax.
iv) They travel from Halifax to Cape Town.

v) They travel from Cape Town to Bangkok.

vi) They travel from Bangkok to Vancouver.

vii) They travel from Vancouver back to Toronto.

b) Count the total number of hours they moved their watches earlier and count the total number of hours they moved their watches later. Explain why these numbers should be the same.

c) When flying from Toronto to Moscow, the circus leaves Toronto at 7:00 a.m. (Toronto time) and arrives in Moscow at 11:00 a.m. (Moscow time) the next day.

i) How long did the trip take?

ii) The trip included an 8-hour stopover. How much time did the circus spend in the air?

4. **Plus/Minus Ratings in Hockey**

A hockey player’s plus/minus rating, or +/− rating, is determined as follows: If the player’s team scores while he is on the ice, +1 is added to his rating; if the other team scores while he is on the ice, −1 is added to his rating. Use the statistics for the fictional women’s team below to answer the following questions, or visit the website of the National Hockey League (www.nhl.com/ice/playerstats.htm) to find and substitute up-to-date statistics for your students’ favourite NHL players and teams.

<table>
<thead>
<tr>
<th>Name</th>
<th>Position</th>
<th>Goals</th>
<th>A</th>
<th>PTS</th>
<th>+ / −</th>
</tr>
</thead>
<tbody>
<tr>
<td>Susan</td>
<td>Centre</td>
<td>31</td>
<td>47</td>
<td>78</td>
<td>7</td>
</tr>
<tr>
<td>Miki</td>
<td>Defence</td>
<td>19</td>
<td>49</td>
<td>68</td>
<td>−1</td>
</tr>
<tr>
<td>Tania</td>
<td>Defence</td>
<td>9</td>
<td>58</td>
<td>67</td>
<td>−1</td>
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<tr>
<td>Voula</td>
<td>Left-wing</td>
<td>28</td>
<td>33</td>
<td>61</td>
<td>−12</td>
</tr>
<tr>
<td>Natalie</td>
<td>Centre</td>
<td>17</td>
<td>43</td>
<td>60</td>
<td>−18</td>
</tr>
<tr>
<td>Haley</td>
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<td>18</td>
<td>27</td>
<td>45</td>
<td>−9</td>
</tr>
<tr>
<td>Kate</td>
<td>Centre</td>
<td>11</td>
<td>34</td>
<td>45</td>
<td>0</td>
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<tr>
<td>Colleen</td>
<td>Right-wing</td>
<td>19</td>
<td>19</td>
<td>38</td>
<td>−19</td>
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<tr>
<td>Deepa</td>
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<td>21</td>
<td>17</td>
<td>38</td>
<td>15</td>
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<tr>
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<td>Centre</td>
<td>12</td>
<td>19</td>
<td>31</td>
<td>13</td>
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<tr>
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<td>17</td>
<td>11</td>
<td>28</td>
<td>−6</td>
</tr>
<tr>
<td>Rena</td>
<td>Centre</td>
<td>15</td>
<td>12</td>
<td>27</td>
<td>5</td>
</tr>
<tr>
<td>Maggie</td>
<td>Right-wing</td>
<td>5</td>
<td>11</td>
<td>16</td>
<td>−10</td>
</tr>
<tr>
<td>Joanne</td>
<td>Defence</td>
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<td>6</td>
<td>12</td>
<td>−11</td>
</tr>
<tr>
<td>Michelle</td>
<td>Defence</td>
<td>0</td>
<td>8</td>
<td>8</td>
<td>−5</td>
</tr>
<tr>
<td>Gina</td>
<td>Centre</td>
<td>1</td>
<td>7</td>
<td>8</td>
<td>−15</td>
</tr>
</tbody>
</table>
a) What does a positive +/- rating mean? What does a negative +/- rating mean?

b) How many players had a positive +/- rating? How many players had a negative +/- rating?

c) If the players were listed in order from best +/- rating to worst +/- rating, write down the first five names and the last five names in the order they would occur.

d) Who had the best +/- rating and who had the worst +/- rating? How much did their ratings differ by?

e) You are the team’s coach. In practice, you play Miki, Natalie, Deepa, Maggie, and Joanne against Susan, Tania, Colleen, Yolanda, and Rena. Based on only the sum total of each “team’s” +/- rating, who do you expect to win: Miki’s team or Susan’s?

f) Make up the +/- rating all-star team by choosing the best three forwards (one centre, one right-wing, one left-wing) and the two players in defence.

g) There are no goalies in the list. In fact, goalies do not keep track of +/- ratings. Why is this?

h) For a game that the team won 5–2, a player had a +/- rating of +2. What is the maximum number of goals (by both teams) the player could have been on the ice for? (6; 4 from her own team and 2 from the other team) Another player had a rating of −1 during the same game. What is the maximum number of goals (by both teams) that player could have been on the ice for? (3; 2 from the opposing team and 1 from her own)

5. Tom and Sara played a card game where you win and lose points. Tom lost 3 points the first game, gained 4 points the second game, and lost 5 points the third game.

a) Write down Tom’s sequence of gains and losses. (−3 + 4 − 5)

b) How many points did Tom gain or lose overall? (he lost 4)

c) Sara confesses that she cheated in the last game. They agree to cancel that game. How many points does Tom have now? Write an integer subtraction and solve it. (−4 − (−5) = −4 + 5 = +1)
Subtraction as Take Away

1. Find the number represented by the +1s and −1s by cancelling the same number of +1s and −1s.
   a) \(+1 + 1 + 1 + 1 - 1 - 1 - 1 \) \(-3\)
   b) \(+1 + 1 + 1 + 1 - 1 - 1 - 1\)
   c) \(+1 + 1 + 1 - 1 - 1 - 1 - 1 - 1\)
   d) \(+1 + 1 + 1 + 1 - 1 - 1 - 1 - 1 - 1\)

2. Add +1s or −1s to equal the number on the left.
   a) \(+4 = \) \(-1 - 1 - 1\)
   b) \(-2 = +1 + 1 + 1 + 1\)
   c) \(-3 = \) \(-1 - 1 - 1 - 1 - 1 - 1\)
   d) \(-3 = +1 + 1 + 1\)

To subtract +3 − (−4), first write +3 = +1 + 1 + 1.
To be able to take away −4, you need to have four −1s.
To add four −1s, you need to add four +1s to keep the value at +3.

This adds 0 to +3.

\[+3 = +1 + 1 + 1 + 1 + 1 + 1 - 1 - 1 - 1\]

\[+3 - (−4) = +1 + 1 + 1 + 1 + 1 + 1 - 1 - 1 - 1\]

So \(+3 - (−4) = +7\)

3. Subtract by writing the first number as a sum of +1s and −1s.
   Use enough of each so that you can subtract the second number.
   a) \((+2) - (-1) = \)
   b) \((-3) - (+1) = \)
   c) \((-2) - (-3) = \)
   d) \((+2) - (+3) = \)
   e) \((+3) - (+2) = \)
   f) \((+3) - (-2) = \)
   g) \((-5) - (-2) = \)
   h) \((-4) - (+2) = \)
   i) \((+3) - (+5) = \)
Unit 7  Geometry

Introduction
In this unit, students will identify and plot points in all four quadrants of the Cartesian plane (or coordinate plane), then perform and describe transformations (reflections, rotations, translations, and dilatations) of shapes in the Cartesian plane and in designs and tessellations. Students will deepen their understanding of similarity and congruency.

Materials
In all lessons of this unit you will need a pre-drawn grid on the board. If you do not have such a grid, you can photocopy BLM Grid Paper onto a transparency and project it onto the board. This will allow you to draw and erase shapes and lines on the grid without erasing the grid itself. To save time, students can also use ready-made coordinate grids on BLM Coordinate Grids, but be sure that students can draw the grids themselves.

Meeting Your Curriculum
Lessons G7-27 to G7-33 address WNCP core curriculum expectation 7SS5 (translations, reflections, and rotations in the Cartesian plane). The material covered in these lessons will be studied using a slightly different approach in Grade 8 in Ontario. However, we recommend Ontario teachers teach lessons G7-27, G7-29, and G7-31 at this time, as these will help students to consolidate other concepts learned this year and provide good preparation for other lessons (see details below).

Students following the WNCP curriculum will learn the material of lessons G7-35 to G7-38 in Grades 8 and 9.

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Topic</th>
<th>Ontario</th>
<th>WNCP</th>
</tr>
</thead>
<tbody>
<tr>
<td>G7-25, 26</td>
<td>Coordinate systems</td>
<td>core</td>
<td>core</td>
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<tr>
<td>G7-27</td>
<td>Distances between points on a line</td>
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<td>G7-28</td>
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<td>G7-29</td>
<td>Reflections</td>
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<td>core</td>
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<td>G7-31</td>
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<tr>
<td>G7-36</td>
<td>Similarity</td>
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<td>optional</td>
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<tr>
<td>G7-37, 38</td>
<td>Dilatations</td>
<td>core</td>
<td>optional</td>
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</table>
Introduce coordinates in the first quadrant. Draw a coordinate grid (first quadrant only) on the board. Point out the axes and mention that axes is the plural of axis. The horizontal line is called the x-axis, and the vertical line is called the y-axis. Tell students that the point at which the two axes intersect is called the origin. Label the axes and the origin.

Mark a point such as (4, 3) on the grid and use it to explain how to describe the location of a point on the grid: Go down to the x-axis (show what you mean) and look at the number that is directly below the point. This point has $x = 4$. Then go left to the y-axis and look at the number on the y-axis directly to the left of the point. This point has $y = 3$. Have students identify the $x$ and the $y$ for other points on the grid.

Ask students whether giving $x = 4$ and $y = 3$ can produce any other point on the grid. Have them go to 4 on the x-axis and draw several points that will have $x = 4$. Which of these points will have $y = 3$? Is there more than one point like that? (no) Emphasize that the $x$ and $y$ numbers for a point identify its unique location. They are like an address for the point, and they are called the coordinates of the point.

SAY: Imagine you have to write the coordinates of 100 points. Would you like to write $x = \_\_\_\_$ and $y = \_\_\_\_$ 100 times? What could you do to shorten the notation? Students might suggest making a chart or even writing a pair of numbers, because $x$ comes before $y$ in the alphabet.

Explain to students that mathematicians have made a convention: The place of a point is expressed by two numbers, in brackets. The $x$ is always on the left, and the $y$ is always on the right: $(x, y)$. Have your students rewrite the coordinates of the points they’ve already identified in the new notation. Then ask them to draw a coordinate grid themselves and to plot several points on the grid. Tell students that the $x$-coordinate is often called the first coordinate, because it is written first, and the $y$-coordinate is called the second coordinate.
Students need lots of practice plotting points on a grid. This is a good time to do the Activity.

**EXTRA PRACTICE:** Review the names of special quadrilaterals and types of triangles. Then have students solve the problems below.

1. Graph the vertices $A (1, 2)$, $B (2, 4)$, $C (4, 4)$, $D (5, 2)$. Draw lines to join the vertices. What kind of polygon did you draw? **PROMPTS:**
   - How many pairs of parallel sides does it have? (1, so it is an isosceles trapezoid)
   - How many lines of symmetry does the shape have? (one)

2. On grid paper, draw a coordinate grid. Graph the vertices of the triangle $A (1, 2)$, $B (1, 5)$, $C (4, 2)$. Draw lines to join the vertices. What kind of triangle did you make? (right isosceles)

**Review negative numbers on a number line.** Remind your students that number lines can be extended in both directions.

**Introduce the Cartesian coordinate system.** Draw a coordinate grid on the board (or use the overhead). Make sure the axes are centred on the grid so that all four quadrants are visible. Explain that the grid that was extended to include negative numbers is called a *Cartesian coordinate system*. Label the axes and the origin. Point out that the axes separate the grid into four parts. Explain that these are called *quadrants*, and ask your students which words that they know have the same beginning. (quadrilateral, quadruple)

Point out the connection these words have to the number four. Draw several points in various quadrants and ask your students to tell which quadrant the points are in.

**Points in the first quadrant.** Add numbers to the positive halves of each axis (as you would on a number line) and explain that the $x$-axis labels the vertical grid lines with integers, and the $y$-axis labels the horizontal grid lines with integers. This allows us to extend the coordinate grid in any direction we want—right, left, up, or down. So if a point is on vertical grid line 3 and horizontal grid line 4, it has $x = 3$ and $y = 4$. Have students plot the point $x = 5, y = 4$. Then ask students how they might find the point $(3, 6)$. Remind students that the $x$-coordinate is always the first coordinate. Ask students to plot points such as $(1, 2)$, $(4, 0)$, $(0, 3)$, $(0, 0)$.

**Points in other quadrants, starting from the third quadrant.** Now make a mark at point $(-1, -1)$ and ask your students what the coordinates of this point might be. $(x = -1, y = -1)$ Invite volunteers to add numbers to the negative parts of each axis. Have students find points such as $x = -2$, $y = -3; x = -6, y = -1; x = -1, y = -4$. Continue with points written in this notation: $(-3, -4), (-5, -2)$. Mark a point in the second quadrant, such as $(-3, 4)$, and show students how to find the signs of its coordinates—the point is on the negative side of the $x$-axis and the positive side of the $y$-axis, so the signs of the point will be $(−, +)$. Repeat with a point in the fourth quadrant. Ask students to tell the signs of various points on the coordinate system. Students might mark the signs for each quadrant. Ask your students in which quadrants different points appear. **EXAMPLES:** $x = -2$, $y = 3; x = 6, y = -1; x = 2, y = -5; (-3, 2), (4, -6), (-2, -2)$.  

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**PROCESS EXPECTATION**

**Connecting**
Show students how to find the complete coordinates of a point in any quadrant (number and sign). Then have students determine the coordinates of points in various parts of the grid. For example, ask students to name the coordinate pairs for five points in the second quadrant. What do these points have in common? (they have the same signs) Then do the reverse: give students various coordinates and ask them to mark the points. Start with points whose coordinates have the same numbers and/or same sign, such as (2, 3), (5, 4), (−3, −3); then try slightly harder points, such as (−4, −5); and then try points whose coordinates have the same number but different signs, such as (−3, 3) and (3, −3). Next, give students points such as (−4, 7) or (3, −6). Finally, include some points on the axes, such as (0, 4) or (−8, 0). Students need lots of practice plotting points in different quadrants.

**EXTRA PRACTICE:** Plot the following points and join them. What type of quadrilateral have you drawn?

a) (1, 2), (−1, −1), (2, −3), (4, 0) *(ANSWER: square)*
b) (−2, −2), (−1, 1), (−2, 4), (−3, 1) *(ANSWER: rhombus)*

**Bonus**

Draw the points below on a grid, then join the points in the order you drew them. Join the first point to the last point. What shape did you make? Find the area of the shape. (−3, −2), (−1, 0), (−1, 3), (0, 4), (1, 3), (1, 0), (3, −2), (3, −3), (2, −3), (1, −2), (0, −3), (−1, −2), (−2, −3), (−3, −3) *(ANSWER: rocket, area = 19 units²)*

**ACTIVITY**

Students will each need a pair of dice of different colours. The player rolls the dice and records the results as a pair of coordinates: (the number on the red die, the number on the blue die). He or she plots a point that has this pair of coordinates on grid paper. The player then rolls the dice a second time and obtains a second point in the same way. The player joins the points with a line, then has to draw a rectangle so that the line is a diagonal of the rectangle.

Several rectangles can be drawn from the same diagonal. If the line is neither vertical nor horizontal, the simplest solution is to make the sides of the rectangle horizontal and vertical. In this case, ask your students if they see a pattern in the coordinates of the vertices.

When students have learned to draw points in all four quadrants of the coordinate plane, modify the activity as follows: add two coins, one red and one blue (to match the colours of the dice), with a plus sign on one side and a minus sign on the other side. The red die and coin will determine the x-coordinate, and the blue die and coin will determine the y-coordinate. Alternatively, use a sign spinner like the one at left.

**ADVANCED:** Replace the sign spinner above with a quadrilateral spinner (see margin). The spinner provides the type of quadrilateral that students have to draw starting with a diagonal.
Review coordinate systems. Review with students that the first number in a coordinate pair is the $x$-coordinate (how far you have to go from the origin along the $x$-axis, the horizontal axis, to get to the point), and the second number is the $y$-coordinate. Review the signs of both coordinates in each quadrant.

Number lines made by skip counting. Show students a number line from $-10$ to $10$ that is labelled by $2$s and invite volunteers to show the position of various odd numbers. **EXAMPLES:** $-5$, $7$, $-3$. Then erase the points the volunteers drew and add points yourself (see sample below). Ask students to identify the letters that correspond to various odd numbers or to identify the location of particular letters. **EXAMPLES:** Which letter shows $5$? What are the locations of $C$ and $D$?

![Number line](image)

Repeat with a number line made by counting by $5$s and a number line made by counting by $10$s.

Coordinate systems with scales made by skip counting. On a pre-drawn grid on the board, draw axes and mark them with intervals of $2$. Mark several points in all four quadrants and ask students to find the coordinates of these points. Then ask students to copy the coordinate system that you drew onto grid paper and to mark on their copies points such as $(−4, 6)$, $(−3, −8)$, $(7, 9)$, $(−6, 7)$, $(7, −6)$, $(−3, 5)$, $(−1, 9)$.

Repeat with coordinate systems whose axes are marked with intervals of $5$. Points to plot: $(−15, 6)$, $(−3, −10)$, $(17, 24)$, $(−6, 7)$, $(22, −6)$, $(−23, 5)$, $(−1, 20)$.

Repeat again, this time for coordinate systems with axes marked by $10$s.
PROCESS ASSESSMENT
7m7, [C] Questions 6b), 7c)
7m6, [C, V] Questions 6c), 7d)
7m2, [R] Questions 8, 9

PROCESS EXPECTATION
Connecting

Have students use 1-cm grid paper. Points to plot: (−20, 60), (−30, −45), (25, 35), (−5, 7), (10, −16), (−31, 0), (−1, 43). If possible, use a grid with cells that are 10 cm by 10 cm for this exercise on the board, and prompt students to use the millimetre mark on a ruler to identify the correct place for each point.

Students can practise plotting points in Questions 1 to 5 on Workbook page 149. If students have trouble with Question 3, ask them to measure the sides of the grid squares (1 cm × 1 cm). ASK: If you go from 0 to 5, how many millimetres is that? (10 mm) From 0 to 1? (2 mm) From the x-axis to point A? (22 mm) What coordinate are you measuring when you go up? (y-coordinate, the second coordinate) What is the y-coordinate of point A? (11) Repeat with point B to get that its y-coordinate is 11 as well. Point out that students are actually using a scale. Ask students to look at the problem they just solved. It can be phrased this way: 5 units on this grid are 10 mm. How many units are 22 mm? What does this remind you of? (ratios, rates, proportions; PROMPT: You are comparing things with different units.) Repeat the prompting for point C if necessary.

Investigating coordinates of points on horizontal and vertical lines.
Have students complete Workbook pages 150–151. They will make and investigate conjectures on the common properties of the coordinates of points on the same vertical and horizontal lines, and on the same side of vertical and horizontal lines, in Questions 6 through 9, and apply their findings in Question 10. When students are done, point out that marking the axes with integers is actually marking vertical and horizontal lines with integers. For example, all points on a vertical line through points (1, −1), (1, 2), (1, 5), etc. have the same x-coordinate: x = 1.
Introduce horizontal and vertical distances. Explain to the students that when we measure distance along a horizontal (or vertical) line, we can call it horizontal (or vertical) distance. For points that are on the same horizontal or the same vertical line, these are just regular distance. For points that are not on the same line on a grid, we pretend we can only move horizontally and vertically, and the horizontal and vertical distances are the distances we have to travel in the horizontal and vertical direction respectively. For instance, the horizontal distance between the two points at left is 2 and the vertical distance is 1.

Draw several pairs of points on separate grids, each pair on the same line of a pre-drawn grid, and have students find the vertical or the horizontal distance between pairs of points on the same vertical or horizontal line.

Find horizontal and vertical distances between points in the Cartesian plane. Next, add axes to the grid and mark the axes with both positive and negative numbers. Have students identify the coordinates of the points and find the horizontal and the vertical distances between the points again. Have students look closely at the coordinates of these pairs of points: (3, 0) and (7, 0); (4, 1) and (9, 1); (5, −2) and (1, −2).

ASK: Which coordinates are the same in each pair, the first or the second? (second) Why are the coordinates the same? What does this tell you about the location of the points? (The points are on the same horizontal line; prompt students to think of their findings from the previous lesson.) Which coordinates change? (x-coordinates) How much do they change? How can I find the distance between the points by using the x-coordinates? (subtract the smaller coordinate from the larger)

Remind students that they used distance between two points to subtract integers. Now they can do the opposite: subtract integer coordinates to find
PROCESS ASSESSMENT
7m2, [R, V]
Workbook Questions 5d), 6e), 6f)

Review distances between points in the coordinate plane. Review the connection between distance and integer subtraction as in the box on Workbook page 152. Have students use the subtraction to find the distance between pairs of points. First use pairs of points that students can draw on a grid, then include some points that cannot be drawn on a grid, such as (−457, 81) and (−457, −92).

Review translations on a grid. Use a grid on a transparency and a counter to review how to slide a dot on a grid (physically slide the dot, represented by the counter). Have students first signal the direction in which the dot slid (right or left, up or down), then ask them to hold up the number of fingers equal to the number of units the dot slid. Then ask students to draw a point on a grid, label it A, and draw:

B: the translation of A 2 units up
C: the translation of A 3 units down
D: the translation of A 5 units right
E: the translation of A 1 unit left

Finally try several combinations of these directions (EXAMPLE: 2 units up and 3 units left).

Investigate the change in the coordinates during a translation. Have students work through the Investigation on Workbook page 153. When they are finished, they can solve several problems such as:

A point (5, −4) slides 3 units up and 7 units left. What are the coordinates of the new point? Predict the coordinates of the new point first, then check by drawing the points.

Finally, have students generalize their findings using variables:

A point (x, y) slides 3 units up and 2 units left. What are the coordinates of the new point? Draw a coordinate system and check your prediction for points (3, 3), (4, −2), (−1, 2), and (−3, −4).

ACTIVITY

a) The grids below are treasure maps. The treasure is hidden in one of the squares, and the numbers in other squares are clues to the location of the treasure. The numbers give the sum of the horizontal and vertical distances from the square in which the number is written to the square that holds the treasure. (The distance between adjacent squares is 1.) For example, in the map at left, the treasure is in square B2, and the number in square C4 shows the sum of the vertical and horizontal distances between C4 and B2: 3. Ask students to find the treasure in each map below.

NOTE: In some maps there is not enough information given, and in others the information is redundant. Which ones are these?
b) A game for two: Player 1 draws a 4 × 4 grid as shown and picks a square in which to “bury” the treasure. Player 2 tries to guess the square by giving its coordinates. Each time Player 2 makes a guess, Player 1 writes the distance between the guessed square and the square with the treasure. Players switch roles when the treasure is found.

**ANSWERS:** i) B2, ii) A2, iii) not enough information, could be B2 or A3, iv) A2, v) not enough information, could be B2 or C3, vi) C3, information redundant.
Introduce transformations. Show students two L-shapes made of paper, oriented in different directions, beside each other. Tell students that you want to move them so that they coincide exactly—one on top of the other, facing the same direction. Show this. Then tell the students to pretend that the sheets are actually very heavy and very hot sheets of metal, so you need to program a robot to move them. To write the program you have to divide the process into very simple steps. It is always possible to move a figure into any position in space by using some combination of the following three movements:

- You may slide the figure in a straight line without allowing it to turn at all. This is called translation.
- You may turn the figure around some fixed point, which can be on the figure or not. This is called rotation.
- You may flip the figure over. This is called reflection.

Have students direct you, the robot, in what steps to perform to position the hot sheets of metal.

Explain that mathematicians have proved that any two congruent figures can be made to coincide by some sequence of translations, reflections, and rotations. These three changes to a figure have a common name: they are called transformations. When a point or a shape is changed by a transformation, the resulting point or shape is called the image of the original point or shape. We often add a star (*) or a prime symbol (') to the name of the original point to label the image. For example, the image of point A can be denoted as A'.
How much did the figure slide? Show students the picture in the margin and ask them how far the rectangle slid to the right. Ask for several answers and record them on the board. You may even call a vote.

Students might say the shape moved anywhere between one and seven units right. Take a rectangular block and perform the actual slide, counting the units with the students. The correct answer is 4.

Draw a point at the top right vertex of the rectangle. Is it easier now to see that the translation was 4 points to the right? Why is it easier? (we know how to translate points; this is a problem we have seen before) Point out that the vertex you marked and its image are corresponding points under a translation. When we talk about transformations, we want to know where each point went to under a transformation. Sometimes the correspondence (which point went to which point under the transformation) will be the thing that determines whether we are talking about translation, rotation, or a reflection. We will see such examples later. ASK: When did we meet the word correspondence before? (in congruence) Explain that transformations will be connected to congruence.

Under translation, all points on a shape move the same amount in the same direction. Draw a pair of non-symmetric shapes (e.g., right trapezoids) connected by a slide on a pre-drawn coordinate grid and mark one of the vertices on one of the shapes. Which vertex on the image corresponds to vertex you marked? Have students draw the arrow between the vertex and the image. Ask students to describe how the vertex moved. Repeat with other vertices. Did all the vertices move the same amount horizontally and vertically? (yes) Did they move in the same direction? How do you know? Prompt students to look at the arrows they drew. What do they notice? (the arrows are parallel and they are all the same length) Finally, ask students to verify their observation using coordinates: list the coordinates of the vertices of the original shape and the corresponding vertices of the image, and find the horizontal and the vertical distances between the vertices. Are the distances all the same? (yes)

Describing translation as addition of integers. Remind students that they used distance between the points on a grid to describe a translation. Which operation did they use to find the distance? (subtraction of integers) Tell students that just as they can find the distance between points on a grid using integer subtraction, they can use integer addition to find images of points under translation. Draw a number line from −5 to 5 and ASK: To get from 3 to 5, what do you have to do? (move right two units) How can you describe that in terms of addition? (add 2) So when we describe a translation in terms of addition, we can think of moving right as adding a positive number. What number should I add to 3 if I want to move two units left? (−2) Repeat with a vertical number line. Finally, have students practise describing translations using Questions 7 through 9 on Workbook page 156.
Now present several points on the Cartesian plane and give students translations to perform. **ASK:** When you are moving up or down, which coordinate is affected? (y-coordinate) When you are moving left or right, which coordinate changes? (x-coordinate) Ask students to predict the images of points under these translations and then check the answer by drawing the points, translating them, and checking the coordinates of the image. **EXAMPLES:**

a) Move point (2, 4) 3 units up. (image: (2, 4 + 3) = (2, 7))
b) Move point (2, −4) 2 units down. (image: (2, −4 + (−2) = (2, −6))
c) Move point (2, 4) 3 units right. (image: (2 + 3, 4) = (5, 4))
d) Move point (−2, 4) 5 units right. (image: (−2 + 5, 4) = (3, 4))
e) Move point (−2, 5) 2 units left and 4 units down. (image: (−4, 1))
f) Move point (3, −4) 3 units left and 4 units down. (image: (0, −8))
g) Move point (−2, −3) 6 units up and 3 units left. (image: (−5, 3))

**EXTRA PRACTICE:**

A shape has coordinates A (2, 2), B (5, −3), C (−3, 5), and D (−5, −5). Under a translation, vertex A moved to position (7, 8). Give the coordinates of the other vertices under the translation. **(ANSWER:** B' (10, 3), C' (2, 11), D' (0, 1))

**Bonus**

Abby translated ΔABC 3 units up and 5 units left. The image ΔA'B'C' has vertices (−2, 4), (−1, −1) and (2, 3). What were the coordinates of the vertices of ΔABC? Explain.

**PROCESS ASSESSMENT**

7m2, [R]

**Extension**

a) Draw a coordinate system on grid paper. Draw the line joining each pair of points and find the midpoint of the line segment.

i) (2, 7) (10, 7)  
   ii) (3, 5) (7, 5)  
   iii) (4, −2) (6, −2)  
   iv) (−2, 4) (6, 4)  
   v) (−5, −4) (3, −4)

b) Which coordinate is the same in all three points?

How can you get the coordinates of the midpoint from the coordinates of the endpoints of the line segment?

c) Use the coordinate system you drew to answer the following questions.

i) How long is the line segment between (3, 4) and (3, −4)?

ii) What is the midpoint of the line segment between (3, 4) and (3, −4)?

iii) A horizontal line is 7 units long and starts at (−3, −3). What is the other endpoint?

iv) The midpoint of a line segment is (3, 3). One of its endpoints is (3, −1). What is the other endpoint?
d) Use your grid paper and a ruler to find the midpoint of the line segments joining each pair of points.

   i) (3, 7) (7, 7)  
   ii) (4, 8) (4, 10)  
   iii) (3, 1) (7, 7)  
   iv) (−2, 6) (4, 8)  
   v) (3, 8) (6, 10)

   e) Can you see a pattern for how to determine the midpoint of a line segment if you know the coordinates of both endpoints? Predict the midpoint of each line segment below and then check your answer using grid paper.

   i) (−2, 5) (0, 9)  
   ii) (−3, −1) (1, −7)  
   iii) (−2, 6) (4, 2)  
   iv) (3, −1) (6, 3)  
   v) (7, 5) (−4, 1)

ANSWERS:

   a) i) (6, 7), ii) (5, 5), iii) (5, −2), iv) (2, 4), v) (−1, −4)

   b) y-coordinate; the x-coordinate of the midpoint is the average of the x-coordinates of the endpoints

   c) i) 8 units, ii) (3, 0), iii) (−10, −3) or (4, −3), iv) (3, −7)

   d) i) (5, 7), ii) (4, 9), iii) (5, 4), iv) (1, 7), v) (4.5, 9)

   e) i) (−1, 7), ii) (−1, −4), iii) (1, 4), iv) (4.5, 1), v) (1.5, 3)
Reflecting points through a line.

Review with the students the steps for reflecting a point through a line. Work on a pre-drawn grid and use a grid line as the mirror line. Label the point you are reflecting $A$.

**Step 1:** Draw a line perpendicular to the mirror line through $A$. Label the intersection point $M_A$. Extend the perpendicular beyond the mirror line.

**Step 2:** Measure the distance $AM_A$—the distance from $A$ to the mirror line.

**Step 3:** Construct point $A'$ on the continuation of $AM_A$, so that $A'M_A = AM_A$, and $M_A$ is the midpoint of $AA'$.

Have students practise this construction with several points on a grid. Include a point that is on the mirror line itself. Where should its image be? Ask students to explain their answer. (The image of a point on the mirror line is the point itself, because the distance $AM_A = 0$, so both $M_A$ and $A'$ should be the same point as $A$.)

Reflecting points through the horizontal axis.

Students can practice the same construction on a Cartesian plane using the horizontal axis as the mirror line. When the construction is done, have students identify the coordinates of the original points and of the images. Use Questions 1 to 3 on Workbook page 157.

Ask students to look at the coordinates of points and their images. Which coordinate is the same in the point and in the image? (x-coordinate) Why did that happen? (PROMPT: The original point and the image are on the same line. Is it a horizontal or a vertical line? What do you know about the first coordinate, or the x-coordinate, of points on the same vertical line? (they are the same)) Which coordinate changed? (y-coordinate) How did it change? (the value stays the same, the sign changes) Can you...
predict the coordinates of a point (4, 3) under a reflection through the x-axis? (4, −3)

Reflecting points through a vertical axis. Repeat the above for reflection through the vertical axis. Use Questions 4 to 7 on Workbook page 158 for practice.

**ACTIVITY**

Find a Flip. Students play in pairs. Each pair needs a copy of BLM Find a Flip. Students cut out the cards on the BLM. There are 32 cards in total: four different sets, or suits, of eight congruent shapes, with each suit consisting of four identical shapes (in a row) and their reflections (in the next row).

Game rules: Players shuffle the cards, deal out four cards to each player, and lay one card face up on any square of the game board. If a player has a card that is a reflection of the card on the table over one of its sides, the player can place the card in the same 2 × 2 region of the game board (adjacent to at least one card already in the region) and pick a new card from the deck. The goal is to get rid of all the cards by placing them in groups of four.

If a player does not have a card that is a reflection of any of the cards on the board, the player can place a card with a shape not congruent to any card already on the board (i.e., a card from a new suit) in an empty region. (Since there are four regions and four suits, there will always be an empty region on the board unless all four suits are already in play.) Players cannot place cards so that there are four cards in a row or in a column, or on any of the diagonals, unless the card they place is the last in its region. When a region is completed, all four cards from this region are discarded.

If there are no free regions, and the player cannot place a card, the player takes a card from the deck. If that card cannot be placed, the player misses a turn.

**Extensions**

1. On a blank sheet of non-grid paper, draw a slant line. Mark a point not on the line. Construct a reflection of the point through the line using a compass and a straightedge only. Use the same steps you used to draw a reflection of a point through a mirror line on a grid.

2. If you want to reflect a point through a line so that its x-coordinate doesn’t change, which type of line should you reflect it in? (horizontal)
Investigate distances between points and their mirror images under reflections. Have students do the Investigation on Workbook page 159.

Distinguish between reflections and translations using correspondence. Ask students to plot the following points on the coordinate grid and connect them with line segments to obtain a quadrilateral: $A (-5, 1), B (-4, 3), C (-2, 3), D (-1, 1)$. Ask: What type of a quadrilateral is that? (trapezoid) Ask students to reflect the trapezoid through the $y$-axis and to label the image $A'B'C'D'$, respecting the order of the images. Then ask students to copy $ABCD$ to the new grid and this time to translate it 6 units to the right, labeling the image $A*B*C*D*$. Ask students to compare the images. Is point $(2, 3)$ one of the vertices of the images? Which point? ($D'$ and $A$) Repeat with other vertices. Emphasize what this means: different points of the original trapezoid were moved to the same point by different transformations. The image of the shape is the same in both cases, but the images of various points on the shapes are different.

Show the picture with four triangles at left and ask students to tell which triangle was obtained from $\triangle PQR$ by reflection and which one was obtained by translation. Ask students to identify the mirror lines and the amount by which $\triangle PQR$ was translated. How do they know? Which vertex of each image corresponds to which vertex of the original triangle, $\triangle PQR$?

Ask: Can I take $\triangle UVW$ to $\triangle XYZ$ using one transformation? (no) Which two transformations can I perform to take $\triangle UVW$ to $\triangle XYZ$? Encourage multiple answers. To prompt multiple solutions, ask students to make a copy of this triangle on grid paper, cut it out, and use to physically make the transformation. At least two possible solutions are i) reflect through the $y$-axis, then translate 5 units down and 5 units right, and ii) translate 5 units down and 5 units left, then reflect through the $y$-axis. Have students identify the combinations of transformations that will take one triangle onto the
PROCESS ASSESSMENT

7m7, 7m3, [C, V]
Workbook Question 3c)

Other for all possible pairs of triangles in this picture. If students notice that triangles \(\triangle UVW\) and \(\triangle LKM\) are connected by rotation, accept that answer but ask students to find a pair of reflections or translations to take one triangle onto the other.

Now show the two triangles at left. Ask students what transformation could have taken these triangles onto each other (reflection through the x-axis, translation 4 units down). Ask students to label the vertices of the image according to a reflection and then point to each vertex in turn and ask students to signal which letter it should be. Repeat with a translation.

**ASK:** Why can we get from one of these triangles to the other using only one transformation, but there is more than one answer possible? How is triangle \(ABC\) special? (it is symmetrical, it is an isosceles triangle)

**Extensions**

1. a) Reflect \(\triangle ABC\) through the slant line. Use a protractor or a set square to draw the perpendicular lines.

   b) Repeat part A of the Investigation on Workbook page 159 for this \(\triangle ABC\). What do you notice? **ANSWER:** \(AA' = 2AM_A, BB' = 2BM_B, CC' = 2CM_C\).

2. Draw a triangle with vertices \(A (1, 1), B (4, 2),\) and \(C (3, 3)\). Draw a vertical mirror line through the points \((5, 0)\) and \((5, 5)\). Reflect the triangle in the mirror line. What are the coordinates of the vertices of the image? **ANSWER:** \(A' (9, 1), B' (6, 2),\) and \(C' (7, 3)\).

3. If you want to reflect a point in the x-axis so that the y-coordinate doesn’t change, what does the y-coordinate have to be? \((0)\)
G7-31 Rotations
Pages 162–163

CURRICULUM EXPECTATIONS
Ontario: 7m5, 7m6, optional; 8m52
WNCP: 7SS5, [CN, V]

VOCABULARY
quadrant
x-, y-axis
clockwise
counter-clockwise
rotation
origin
straight angle

PROCESS EXPECTATION
Visualizing

EXAMPLE:

ANSWER: three-quarters turn CW

Goals
Students will rotate points around the origin of the Cartesian plane.

PRIOR KNOWLEDGE REQUIRED
Can identify and draw perpendicular lines and circles
Is familiar with the terms clockwise and counter-clockwise
Knows that the sum of the angles in a circle is 360°
Can identify and perform quarter turns (90°), half turns (180°), and three-quarter turns (270°)
Can find a fraction of a number (e.g., 1/4 of 360° = 90°)

MATERIALS
pre-drawn grid (see Introduction)
BLM Coordinate Grids (p R-48)
BLM Find a Flip (pp R-43–R-44)

Review clockwise and counter-clockwise and describe turns. Review the meaning of “clockwise” and “counter-clockwise” using a large clock or by drawing arrows on the board. If you have a large clock, ask volunteers to rotate the minute hand—clockwise and counter-clockwise—a full turn, half turn, and a quarter turn. You might also ask your students to be the clocks: each student stands with an arm stretched out and turns clockwise (CW) or counter-clockwise (CCW) according to your commands. The outstretched arm clearly draws the fraction of a circle turned and helps to visualize the turn.

Draw several clocks on the board as shown and ask your students to tell you how far and in which direction each hand moved from start to finish. Review the sum of the angles in a circle (360°) and the degree measures that correspond to a quarter turn, half turn, and three-quarter turn. Then draw examples with only one arrow and ask students to turn the arrow:

a) 90° CCW
b) 180° CCW
c) 270° CW
d) 270° CCW
e) 270° CW
f) 180° CW
g) 90° CCW
h) 90° CW

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ASK: Which turns always produce the same result? Why does this happen? (The sum of the angles around the origin is 360°, so a 270° rotation in one direction is the same as a 360° − 270° = 90° rotation in the other direction; 180° = 360° − 180°, so a 180° rotation in either direction produces the same result.)

Perform rotations using a compass and a set square or protractor.
Model the construction of a rotation as in the Example on Workbook page 162. Then ask students to draw a pair of perpendicular lines on grid paper (to serve as the axes) and to mark a point in the first quadrant. Have students rotate the point using a compass and a set square (or protractor), first 90° clockwise, then 90° counter-clockwise. Which quadrant did the point go to? Repeat with points in all other quadrants, then have students do some 180° rotations. (Students can use the grids on BLM Coordinate Grids. If they mark and rotate more than one point per grid, they can use different colours to distinguish between the rotations.) Students should keep these pictures to use in the next lesson. Finally, have students signal the quadrant an image point will be in after rotation. Include some 180° rotations as well. EXAMPLE: Point P is in the third quadrant. I rotate it 90° counter-clockwise. Which quadrant will the image point be in?

ACTIVITY
Let students play Find a Flip as in the Activity in G7-29 with one change: each card placed should be a 90° rotation of the adjacent cards. Players must indicate around which vertex the rotation was made.

After students have played several times, discuss the similarities and differences between this game and the game they played in G7-29. How are the squares completed in the games different? How many cards of each type does each square contain? In which game was it easier to complete a square? Why? (Each suit has 4 cards of one type and 4 cards that are the reflection of the first type. To construct a square in the games with reflections, you need 2 cards of the same type and 2 cards that are reflections. So when the very first card of any suit is placed, you have 4 more cards that are reflections of that card to add to it. In the game with rotations, each region consists of 4 identical cards, so when the very first card of a suit is placed, there are only 3 cards you can add to it. This means it is harder to complete the first region in the game with rotations. There is no difference in completing the second region with the same suit.)

Extension
Using a compass and a straightedge, construct a pair of perpendicular lines to serve as axes. Mark a point in one of the quadrants. Construct the image of the point under a 90° rotation around the origin in any direction using a compass and a straightedge.
Rotating right triangles with one vertex at the origin. Review with the students how they could rotate a right triangle that has one vertex at the origin and one of the shorter sides going along an axis. (Rotate the side going along the axis first, then add the rest of the triangle.) Point out that this is similar to what students did in the previous lessons: if a triangle “points” to quadrant I, then after a 90° clockwise rotation it points into quadrant IV, just as a point from quadrant I is rotated 90° into quadrant IV.

**EXAMPLE:**

Rotating points around the origin of the Cartesian plane. Have students perform several 90° rotations of points around the origin using a compass and a set square or protractor, then ask students to look at the pictures they produced in the previous lesson. Ask them to shade on several pictures a right triangle, as shown, and to rotate the triangle the same way they rotated the point in each picture. Does the image point lay on the image triangle? (yes)
**ASK:** What is easier, to rotate a triangle on a grid or to rotate a point with a compass and a set square? Could we combine these two methods to produce an easy way to rotate a point in the coordinate plane? Have students come up with suggestions. Students may say that they can count the squares between the origin and the point and use that information to rotate the point. For example, if a point is 2 squares down along the y-axis and 4 points right from the origin, its image after a 90° CW rotation will be 2 squares along the x-axis to the right, and 4 squares up. Another way to see this is to draw and rotate the triangle created by tracing out the vertical and horizontal distances to the point from the origin. Discuss with students why drawing and rotating a triangle works: when you rotate the triangle, you rotate each part of it by the same angle. Prompt students to think about how they actually rotate a triangle. What information do they use? Start with \( \triangle AOB \) (see picture). To rotate it, students first draw side \( OB' = OB \), then draw side \( A'B' = AB \) at a right angle to the previous side. This way they produce \( \triangle A'OB' \). How are triangles \( A'OB' \) and \( AOB \) related? (they are congruent) Why, or by which congruence rule? (SAS) Have students write the equalities for the remaining pairs of sides and angles of these triangles. Work with the students through Question 2 on Workbook page 164 to prove that side \( OA \) rotates 90° to side \( OA' \) when students use this method.

**Process Expectation**

Looking at a similar problem for ideas

**Process Assessment**

7m5, 7m7, [C, CN]
Workbook Question 9 c)

**Practise rotating points and shapes.** Have students use the method of rotating a triangle to rotate points in a coordinate plane. See Questions 3–10 on Workbook pages 164–166. Then proceed to rotating shapes by rotating vertices and then joining them to obtain the image of the shape, as in Question 11 on Workbook page 167.
Review transformations learned to date. Ask students to draw a shape with the following vertices on a coordinate grid: (1, 2), (1, 4), (2, 4), (2, 3). Ask students to describe the shape as best they can (right trapezoid with a $45^\circ$ angle). Point out that the shape is “pointing” down and left—the acute angle is at the left bottom vertex, and the slanted side goes right and up from the acute angle. Label this shape A.

Now ask students to perform a number of transformations on this shape to produce other shapes:

- B: Reflect shape A through the $y$-axis.
- C: Reflect shape A through the $x$-axis.
- D: Rotate shape A $90^\circ$ clockwise around the origin.
- E: Rotate shape A $180^\circ$ clockwise around the origin.
- F: Rotate shape A $270^\circ$ clockwise around the origin.
- G: Translate shape A 4 units left and 6 units down.

Ask students to describe the position and orientation of each image the same way they described the position and orientation of shape A. Then ask students to look for the transformations or combinations of transformations that take the other shapes to each other. One way to do so is to think backwards: since all the shapes were made by doing something to shape A, you can take the shape back to shape A. For example, to take shape C to shape D, you can reflect C back to A through the $x$-axis, and then rotate shape A to obtain D. Explain that there is always a different way to describe a transformation and challenge students to find several ways to move between pairs of shapes. If students have trouble doing so, they can cut out a copy of shape A and try the transformations with a prop.

Other ways to produce the same result. You know that you can take shape C to shape D by reflecting C back to A and rotating A to D. Can you take shape C to shape D by doing the operations in reverse order—rotate,
then reflect? Give students thinking time to get the result, and encourage multiple answers. (rotate C 90° counter-clockwise around the origin to get shape X, then reflect X through the x-axis; rotate C clockwise 90° around the origin, and reflect through the y-axis) Since rotation is usually the hardest transformation to visualize, some students might find it useful to work backwards, as suggested in the previous paragraph. They should start with the second operation, the reflection, and then find a suitable rotation to be the initial operation.

Have students practice finding alternate combinations of transformations to get from one shape to the other. Some students might notice that there is a single transformation taking C to D: a reflection through the line passing through the origin and points (1, −1), (2, −2), etc. Encourage students to look for more pairs of shapes that are connected by a single reflection through a slant line. (B and F, C and E, B and D)

**Bonus for Investigation 1:** Draw the translation arrow from part A. Perform R on the arrow. Now draw the translation arrow from part E. What do you notice?

**ACTIVITY**

Have students play another version of Find a Flip (G7-29). A valid placement of the next card could be:

a) a 90° rotation around a common vertex of the card on the table,
b) a reflection of the card on the table over one of its sides, or
c) starting a new region with a card whose shape is not congruent to any of the shapes already on the board.

When a third or a fourth card is added to the region, students should identify the transformations that would take the shape on the new card to all cards in the region.

**Invalid placement**—this shape is a clockwise turn of the shape on the left, but it cannot be obtained from the shape above it by a rotation or reflection over the common side.

**Good placement**—this shape is a counter-clockwise turn of the shape on the left, and a reflection of the shape above over the common side.

Discuss with students what squares you can produce if you have two identical cards and two cards that show a reflection of the shape on...
Extensions

These extensions combine reflections with geometric constructions using a mirror or a Mira. Students will need a protractor to measure angles.

1. a) Draw a straight line and put the mirror across it, at any angle.
   
   i) Look at the angle between the line you drew and the mirror, and the angle between the reflection of the line and the mirror. What do you notice? (the angles are the same)

   ii) Turn the mirror to make the angle between the line you drew and its reflection in the mirror $20^\circ$. How large is the angle between the line and its reflection? ($40^\circ$)

   iii) You want to place a mirror so that the angle between the line and its reflection is $90^\circ$. What angle should there be between the mirror and the line? Try this on grid paper. ($45^\circ$; if you draw the lines along the grid lines, the mirror should be placed diagonally)

   iv) How could you place a mirror so that it is at a right angle to the line without using any tools? **HINT:** What is the degree measure of a right angle? How large is the angle between the line and its reflection when the mirror is at a right angle to the line? ($180^\circ$)

   v) Use a mirror to draw a line perpendicular to the line you drew.

   vi) Mark a point $P$ not on your line. Use a mirror to draw a line perpendicular to the line you drew through the point $P$.

b) Draw a line segment $AB$.

   i) Find the midpoint of $AB$ using a Mira.

   ii) Draw a perpendicular bisector of $AB$ using a Mira.

2. Draw any angle. Place a Mira so that it touches the vertex of the angle and lies between the two arms of the angle. Rotate the Mira around the vertex of the angle until the reflection of one arm of the angle coincides with the other arm of the angle behind the Mira. Using the Mira as a straighedgel, draw a line through the angle (starting at the vertex of the angle). What have you constructed? How do you know?
Goals
Students will use transformations to describe tessellations and designs.

PRIOR KNOWLEDGE REQUIRED
Can plot points and identify coordinates of points in the Cartesian plane
Can perform, identify, and describe rotations, reflections, and translations in the Cartesian plane

MATERIALS
pre-drawn grid (see Introduction)
pictures that show tessellations, including beehive
BLM Find a Flip (pp R-43–R-44)
BLM Triangle Transformation Cards (p R-45)
BLM Triangle Transformations Game Board (p R-46)

Tessellations. Explain to students that a tessellation is a pattern made up of one or more shapes that completely covers a surface, without any gaps or overlaps. One example of a tessellation is a floor tiling with polygons (or other shapes). If you can tile the floor with one shape, this shape is said to tessellate. ASK: Do you know of any shape that tessellates? Your students will almost certainly say squares, but they might also name other shapes, such as rectangles (ask them to draw a picture—there is more than one way to tessellate with rectangles), diamonds, other parallelograms, triangles, or hexagons. Show a picture of a beehive. ASK: Is this a tessellation? If there are any tessellation patterns around the school, mention them as well. Have students do part a) of the Activity.

Tessellate by translating a group of shapes that make a square. Draw the pattern shown at left on the board or on an overhead. ASK: Can you use these L-shapes to tessellate? Ask your students to identify the single transformation that takes each shape onto the others. (rotations of 90° or 180°, clockwise or counter-clockwise) What can you do with all four shapes together to tessellate a surface? How many units do you need to translate the square? In which direction? Lead the students to the idea of making a row of squares by translating the group of four shapes 4 units left (or right) repeatedly, then translating the whole row 4 units up or down repeatedly.

Describing a tessellation on the Cartesian plane using transformations. Draw the partial tessellation shown at left on the board or on an overhead. ASK: Which transformation takes shape A onto shape B? Draw a translation arrow between shapes A and B. Ask your students to describe the translation. (5 down, 2 left) Mark a vertex on shape A and ask your students to identify the vertex of B which is the image of the marked vertex under the translation. Ask students to draw the translation arrow between the new
vertices. **ASK:** What do you notice? (the second translation arrow is the same length and points in the same direction as the first)

Ask students to find a pair of shapes that can be transformed into one another by a reflection (C and D), and ask a volunteer to identify the mirror line (y-axis). Draw a vertical mirror line through the left-hand side of C and **ASK:** What should I do to the image of C after reflection in this mirror line to move it onto shape D? (translate it 8 units right)

Have students find pairs of shapes that can be transformed onto one another by a rotation, and identify the centre of rotation. **SAMPLE ANSWERS:** shape D to E, 90° CCW rotation around (2, 0); shape E to F, 180° rotation around (0, 1); shape G to A, 90° CW rotation around (2, 4).

**Bonus** Shape D can be rotated 90° CCW around a point on shape E, producing shape A. What is the centre of rotation? (1, 1)

Ask students to identify a pair of transformations that would take shape C onto shape E. Encourage students to find multiple answers to this question. **SAMPLE ANSWERS:** reflect C through the y-axis to obtain D, and rotate the image (D) 90° counter-clockwise; reflect C through its right side (vertical line through (−2, 0)) and rotate around (0, 2); rotate C 90° clockwise around (−4, 1), then reflect through the right side of the shape.

**Changes in the orientation of a shape under repeated transformation.** Ask students to describe shape C without using the term “L-shape.” *(EXAMPLE: A rectangle with base 1 and height 3, with an additional 1 × 1 square at the bottom of the right side.)* Then ask students to describe shape D.

**SAY:** I reflect shape C through a vertical line. Where will the additional square be attached now? (at the bottom of the left side) I reflect shape C through a horizontal line. Where will the additional square be attached? (top of the right side) Invite volunteers to check the predictions.

**ASK:** Can I take C onto D using only reflections through the sides? How many reflections will I need? (3) Can you tell where the additional square will be attached after each reflection without actually making the reflections? (bottom left, then bottom right, then bottom left, then bottom right, and so on) I reflected shape C thirty-seven times through vertical lines. Where is the square, on the left or on the right? On the top or on the bottom? (bottom left) I slid shape C in some direction. Where is the square now? On which side, right or left? Top or bottom? (bottom right)

Have students draw a copy of shape C on grid paper and try to rotate it 90° clockwise around different points. Does the additional square point to the same direction every time? (yes, bottom left) Which way will the additional square point if you turn shape C 90° counter-clockwise? (top right)

**Bonus** I slid shape C, turned it 180°, and reflected it through a vertical line. Where is the additional square after all those transformations? (top right)
EXTRA PRACTICE:

a) Describe a series of transformations that could be used to get shape A onto shape C. Give at least two answers.

b) Which transformations can be used to move shape B onto shape D?

**Bonus**
Find a sequence of two transformations that will take shape A onto shape D.

**Describing designs.** Use the cards from BLM Triangle Transformation Cards. Ask students to choose one shape and arrange copies of that shape and its reflection in a hexagon, such that each card is a reflection of each adjacent card through their common side, as shown at left.

Ask students to tell which transformation takes each card onto each other card. Repeat this activity with cards from the BLM Find a Flip, arranging them first into a square (see margin) and then into a 1 × 4 rectangle (below).

Two reflections produced a translation.

Discuss similarities and differences in the designs. In all three cases, the only transformation performed was a reflection. What are the results of two reflections? In the first case, when you start with the bottom-most shape and reflect it twice, you get the shape at the top right; you can take the first shape to the second by a 120° counter-clockwise rotation. In this case, two reflections produced a 120° rotation. What is the angle between the mirror lines? (60°) Now look at any two opposite shapes. What transformation takes one of them onto the other? (a reflection)

Look at two opposite shapes in the square. Which transformation takes one shape to the other? (180° rotation) What is the angle between the mirror lines in this case? (90°) So the result of two reflections in the axes that are perpendicular to each other is a 180° rotation.

Look at the last case. What transformation takes the first shape onto the third shape? (translation) What is the angle between the mirror lines? (0°, the lines are parallel)

Another observation: reflections over intersecting lines produce alternating reflections and rotations. If you number the shapes as if you were going around the hexagon or the square, the pattern of transformations needed to get each next shape from the first shape in the pattern is reflection, rotation, then repeat. However, if the reflections are made in parallel lines, the pattern is reflection, translation, then repeat.

Have students create and describe designs using part b) of the Activity.
a) Divide students into two groups. Students in each group will individually construct a pair of shapes using a ruler and a protractor, and cut out at least 6 copies of each shape.

Group A: An isosceles trapezoid with sides and smaller base 5 cm and angles of 135°, and a square with sides 5 cm.

Group B: A parallelogram with base 5 cm and height 5 cm, with an angle of 45°, and a square with sides 5 cm.

Each student should create at least 3 different tessellations: one using one of the two shapes, one using the other shape, and one using both shapes together. Students should sketch the tessellations in their notebooks. Students with the same pair of shapes will pair up to share their tessellations and then try to come up with another tessellation using one or two of the shapes. Then put students into groups of four to share their tessellations again. Can groups come up with a new tessellation? The goal is to produce as many different tessellations as possible.

b) Ask students to take the trapezoid or parallelogram cards they created above and to draw a very simple asymmetric design on each card (such as a simple tilted flower or even a letter, such as P), so that the cards are exactly the same. Ask students to turn the cards over and trace the design on the back, so that when a card is reflected, the design is reflected as well. Ask students to produce a tessellation or design that involves a reflection. They can sketch the design in their notebooks and describe it in terms of transformations. Students can again share their work in groups of 2 and 4 and try to come up with new tessellations or designs.

Extensions

1. **Working backwards.** To get from shape A to shape B, slide A 3 units left and 4 units up, then rotate 90° CCW, and then reflect it in the x-axis. How can you get from shape B to shape A? Check your prediction.

2. **Triangle Transformations Game**

   Students will play in pairs. Each pair needs **BLM Triangle Transformations Game Board** and a deck of the cards from **BLM Triangle Transformation Cards**. (Students will have 36 cards, of 3 designs.) Players shuffle the deck, deal out 6 cards to each player, and lay one card face up on any triangle of the game board. Players take turns adding cards to the game board so that the new card can be obtained from the adjacent cards in the same region of the board by:

   a) a rotation around a common vertex of a card on the table, or
   b) a reflection of a card on the table over the common side.

**EXAMPLES:**

The middle card is a 60° rotation of the cards on the sides.
If a player cannot add a card to any of the existing regions, and there is a free region, the player can start a new region with a card that has a shape that is not congruent to any of the shapes on the board.

A new card can only be added to a region without being the last card in the region if it does not create a row of 6 cards. If a card is the last card in its region, it must be a rotation or reflection of the cards on either side of it. When the last card in the region is placed, all the cards from the region are removed and discarded.

If there are no free regions, and the player cannot place a card, the player takes a card from the deck. If that card cannot be placed, the player misses a turn. The goal is to get rid of all the cards by placing them in groups of six.

The middle card is a reflection of the card on the left over the common side. The card on the right can be obtained from the middle card by a rotation around the upper vertex of the common side.
Explore tessellation with paper shapes. Present this problem:

Joshua says that he can tile the floor of a bathroom using only regular octagons. Is this correct?

Josef says that he can tile the floor using only regular pentagons. Is this correct?

What problems to Joshua and Josef encounter?

Let students use paper pentagons, hexagons, and octagons to check whether these figures tessellate. Discuss the results. How many shapes of each kind can you place together such that they share a vertex without overlapping? (3 pentagons, 3 hexagons, 2 octagons) Are there any gaps left?

Ask volunteers to sketch tessellations with triangles on the board. Repeat with squares and hexagons.

Tessellate quadrilaterals. Have each student fold a sheet of paper three times, to produce eight layers of paper. Next, students draw a quadrilateral that does not have a line of symmetry (and is not a parallelogram) on the sheet and cut it out, cutting through all eight layers. Students number their quadrilaterals (1–8) and try to arrange them so that they do not have gaps and do not overlap. **ASK:** Does your quadrilateral tessellate? Name the transformations used to obtain Quadrilaterals 2 through 8 from Quadrilateral 1. (Students can repeat this activity with a triangle. **HINT:** Two triangles make a quadrilateral.)

Sum of the angles in a quadrilateral. Remind students how to measure angles with a protractor. Ask students to measure the angles in the
quadrilaterals they created and to add them. What is the sum of the angles in a quadrilateral? Have students fold the quadrilaterals along a diagonal and measure the angles in the triangles created. What is the sum of the angles in a triangle? (180°) Does this fit with the results of measuring angles in the quadrilaterals? Did all students get the same result? Why could that be? Work with students through the proof of the sum of the angles in a quadrilateral as in the box on Workbook page 172.

ASK: What is the degree measure of a straight angle? (180°) Draw a straight angle and ask what the degree measure around the vertex is. There are two straight angles, so there are 360° around the point. Draw a picture of three line segments shaped like the letter Y and ask your students what the sum of the angles around the vertex should be. If the three angles in the Y are the same, what is the measure of each angle?

Have students complete Questions 1–5 on Workbook page 172 individually.

**Sum of the angles in any polygon.** Work through Question 6 on Workbook page 173 as a class to develop a formula for the sum of the angles in any polygon.

**Which regular polygons tessellate?** Review with students what regular polygons are: polygons with equal sides and angles that are all the same. 

ASK: What is a regular quadrilateral? Is a rhombus a regular quadrilateral? Why not? (angles are not equal) Is a rectangle a regular quadrilateral? Why not? (sides are not equal) Can there be a triangle that has equal sides but is not regular? (no, equal sides means equilateral, and equilateral triangles have equal angles) Complete the Investigation on Workbook pages 173–174. Discuss the patterns students notice in the tables and the answers to parts E and F in particular.

**Bonus** Ling wants to create a tessellation using the octagon at left. She thinks she can use a rectangle with sides 2 cm and 4 cm to fill in the gaps between the octagons. Is she correct? (no) What other shape could she use together with this octagon? (a 2-cm × 2-cm square or a 4-cm × 4-cm square)

**Extensions**

1. a) The formula for the measure of one interior angle in a regular polygon with \(n\) sides is: \(180° ÷ n \times (n - 2)\).

   Verify the formula for triangles, squares, pentagons, hexagons, heptagons, and octagons.

   b) Write the following expressions without the brackets:

   \[
   18 ÷ (6 ÷ 3) \quad 24 ÷ (3 \times 8) \\
   24 ÷ (3 \times 8 ÷ 4) \quad 24 ÷ (8 ÷ 4 \times 3)
   \]

   **ANSWERS:**

   \[
   18 ÷ 6 \times 3 \quad 24 ÷ 3 ÷ 8 \quad 24 ÷ 3 ÷ 8 \times 4 \quad 24 ÷ 8 \times 4 ÷ 3
   \]
c) Rewrite the expression for $360^\circ \div (\text{measure of one interior angle})$ so that it has only one pair of brackets, $(n - 2)$. Then show that the expression is equal to $2n \div (n - 2)$.

ANSWER: $360^\circ \div (180^\circ \div n \times (n - 2)) = 360^\circ \div 180^\circ \times n \div (n - 2) = 2 \times n \div (n - 2) = 2n \div (n - 2)$

d) Fill in the table. For which values of $n$ is $2n \div (n - 2)$ a whole number? Use this answer to explain which three types of regular polygons tessellate.

ANSWER: The expression “$360^\circ \div \text{measure of one interior angle}$” gives the number of polygons that meet at a vertex of a tessellation. For the answer to be a whole number, $(n - 2)$ should divide into $2n$. This can happen in the following cases:

i) $n = 3$, $n - 2 = 1$, $2n \div (n - 2) = 6$. Triangles tessellate, 6 triangles meet at a vertex of the tessellation.

ii) $n = 4$, $n - 2 = 2$, $2n \div (n - 2) = 4$. Squares tessellate, 4 squares meet at a vertex of the tessellation.

iii) $n = 6$, $n - 2 = 4$, $2n \div (n - 2) = 3$. Hexagons tessellate, 3 hexagons meet at a vertex.

e) Look at the pattern in the right column. Is it increasing or decreasing? (decreasing)

If $n > 6$, the number in the right column is less than 3. However, it never becomes 2. To see this, look at the following quotients:

\[
\begin{array}{ccccccc}
3 & 4 & 8 & 9 & 6 & 7 & 6 \\
/ & / & / & / & / & / & /
\end{array}
\]

How can you tell if a quotient is larger than 1? (The quotient is larger than 1 when the divisor is larger than the dividend.)

Is $n \div (n - 2)$ larger or smaller than 1? (larger)

If you double a number that is larger than 1, what can you say about it? (it is larger than 2)

Explain why $2n \div (n - 2)$ is always larger than 2.

f) Explain why no other regular polygon tessellates.

SOLUTION: The number $360^\circ \div \text{measure of one interior angle} = 2n \div (n - 2)$ is the number of polygons meeting at a vertex. The value of $2n \div (n - 2)$ is always larger than 2, so it is between 2 and 3 for regular polygons with the number of sides larger than 6. This means that no regular polygons other than triangles, squares, and hexagons will have a whole number of shapes meeting at a vertex of the tessellation, so no other regular polygons tessellate.

2. Look at the pattern in the measure of one angle in regular polygons (look at the last column of the table in part A of the Investigation
on Workbook page 173). Does the pattern increase or decrease? (increase)

Now look at the gaps. Do the gaps increase or decrease? (decrease) What do you think will happen as the number of sides increases? Will the gaps at some point become negative? If the gaps become negative, the pattern will decrease! So maybe there are polygons with a really large number of sides that tessellate! Let’s investigate what happens in polygons with a large number of sides.

Find the measure of interior angles for polygons with 100 and 101 sides, then find the gap between them. Is the gap positive? (Yes. For \( n = 100 \) the formula for the sum of interior angles, \( 180^\circ \times (n - 2) \), gives \( 180^\circ \times 98 = 17,640 \), so each angle is \( 176.4^\circ \). For \( n = 101 \) we get \( 180^\circ \times 99 \div 101 \approx 176.44^\circ \).)

Repeat for polygons with 1000 and 1001 sides. Round the answers to 4 decimal places. (ANSWER: \( 179.6400^\circ \) and \( 179.6404^\circ \) respectively)

As a matter of fact, the gap becomes smaller and smaller but never reaches zero, and the measure of interior angle increases each time, approaching but never reaching \( 180^\circ \). From a geometric point of view, when the number of sides increases, a regular polygon looks more and more like a circle, though it never becomes a circle. Circles do not tessellate, and no polygons with more than six sides can tessellate either.

3. Create your own tessellating pattern. Choose a tessellation made by polygons using slides only (no rotations or reflections!). Cut a piece from the tessellating polygon along a line of your choice, then slide the piece to the opposite side and re-attach it. Repeat several times, as you wish. Create a picture on the shape that you’ve got and create a tessellating pattern with that picture. EXAMPLE:

Step 1: Take a tessellation with parallelograms.

Step 2: Cut off a piece, slide it to the opposite side of the parallelogram, and re-attach it. Repeat several times.

Step 3: Add a drawing to your shape.

Step 4: Arrange copies of your shape in a tessellating pattern!

4. Project idea: Tessellations in the work of M. C. Escher.

Many works of M. C. Escher contain mathematical ideas. The drawings in the series Regular Divisions of the Plane show many interesting tessellations. For example, Regular Division of the Plane with Birds shows a tessellation using copies of two different shapes, and Regular Division of the Plane with Horsemen shows a tessellation with many copies of the same shape. The most interesting tessellations for the
following exercise are the pictures of reptiles; there is more than one, and they are mathematically different.

One possible source for artwork is the official M. C. Escher website: www.mcescher.com. You will find symmetry drawings in the Picture Gallery.

Find several designs by M. C. Escher that show tessellations.

a) Which of the designs are made from many copies of one shape? Which are made from more than one different shape? Sort the designs.

b) Choose a design that is made from many copies of the same single shape. Identify a point on the picture where more than two shapes meet. Label it \( P \). How many shapes meet at \( P \)?

c) Label the shapes around \( P \) (use A, B,\ldots). Which transformation or combination of transformations will take shape A onto each of the other shapes?

d) Using tracing paper, copy one shape from the design (say, shape A). Go along the perimeter of the shape. Can you find another point on the picture where more than two shapes meet? How many points like that can you find along the perimeter of shape A? Mark all these points on your tracing.

e) Check each of the points you marked on the tracing. How many shapes meet at each of these points? Write the answer beside each point. Is it the same number of shapes at each point? Does the answer surprise you? Why?

f) Choose a point (different from \( P \)) where more than two shapes meet. If possible, choose a point where a different number of shapes meet than at \( P \). Repeat c) with the point you chose.

5. In this unit, students have seen examples of tessellations of a plane. However, anything that is covered with many copies of the same shape (or several shapes) is a tessellation. What examples of 3-D tessellations have students seen?

Bring in a soccer ball. Soccer balls show tessellation of a sphere. What shapes are used to tessellate a sphere in a soccer ball? (pentagons and hexagons) How many shapes meet at each vertex? (three: two hexagons and a pentagon)

Bring in a 3-D puzzle in the shape of a ball. Ask students to disregard the indentations and the “ears” of the puzzle pieces, and think of them as triangles. Ask students to look at the corners of the pieces—the vertices of the tessellation. How many pieces meet at each vertex? Is it the same number at each vertex? (no, and depending on the puzzle, it might be 6, 4, 3, and 5)
Produce similar shapes with a flashlight. Start the lesson with a demonstration. You will need a darkened room with a single source of light, such as a small flashlight. Show students several paper shapes. Hold a shape (say, a right-angled scalene triangle) parallel to the floor below the light to create a clear, sharp shadow. Have students trace the shadow on the floor with masking tape. (The shapes should be larger than the source of light to produce a clear, sharp shadow.)

Have students compare the shape and its shadow. ASK: What is the shape of the shadow? Are the triangles of the same kind? Invite volunteers to measure the sides and the angles of both triangles and record them in a table. If one side of the shadow is, say, two times larger than the corresponding side of the paper triangle, what happens with the other sides? (If you hold the triangle parallel to the floor, the other sides of the shadow should also be twice as large as the corresponding sides of the paper triangle.) What do students notice about the angle measures in both triangles? (They are the same.) Repeat with the triangle at a different height (to create a shadow of a different size).

Define similar shapes. Explain to students that similar shapes have the same shape but not necessarily the same size. ASK: What are shapes that have the same size and shape called? (congruent) When we talked about congruency, what did we say about the sides and the angles of the shapes? (sides are equal and angles are equal) Remind students about the pentagons they saw in lesson G7-13 (see BLM Two Pentagons; show the pentagons if possible): the pentagons had equal sides and equal angles, but they did not have the same shape. What was wrong with them? (the order of the equal sides was different) Point out that order is also important in similarity. What term do we use to say which sides and which angles should be compared? (corresponding sides/angles) So two shapes...
are similar if their corresponding angles are equal and their corresponding sides are proportional.

**Review the meaning of proportion.** Remind students that proportion is equality between ratios. In the case of proportional sides, this means that the ratios of the sides are equal. Draw two right scalene triangles, one having sides twice as large as the other, and label the vertices as shown. Explain that in these two triangles, the proportion between the sides looks like this:

\[ AB : DE = AC : DF = BC : EF = 2 : 1 \]

What does this tell us about the ratio of the sides \( AB : DE \)? (\( AB : DE = 2 : 1 \)) Which side is larger? How many times larger? (\( AB \) is twice as large as \( DE \)) Repeat with the other pairs of sides.

**ASK:** Which angles do we need to check to make sure these triangles are similar? Point out that just as we need to preserve the order of the sides, we need to keep the order of the angles. We might start with the largest angles, and then continue in order. Invite volunteers to measure the angles and to write the equalities.

**Checking for similarity.** Ask students to copy the following pairs of triangles to grid paper and to say which sides and angles they are going to compare in order to check if the triangles are similar. Then ask them to compare the sides and the angles.

Repeat with rectangles 4 \( \times \) 8 and 6 \( \times \) 12, then 5 \( \times \) 6 and 6 \( \times \) 7. **ASK:** Do we need to check the equality between the angles for the rectangles? Why not? (all the angles in the rectangles are 90°, so they are already equal) Repeat with the shapes at left.

Emphasize that one way to prove that two shapes are not similar is to show that a pair of corresponding sides in the two shapes is not in the same ratio as another corresponding pair of sides. These shapes are not similar because the side marked with an X in B is 3 times longer than the corresponding side in A, whereas the side marked with a circle in B is only 2 times longer than the corresponding side in A (it should be 3 times longer).
Using similarity to find missing sides. Draw two rectangles and tell students that they are similar. Give the dimensions of two sides of one rectangle, and one side of the other, and have students find the length of the missing side using proportion. **EXAMPLES:**

a) Width of A: 3 cm, length of A: 7 cm. Width of B: 9 cm. What is the length of B?

b) Width of A: 3 cm, length of A: 8 cm. Length of B: 16 cm. What is the width of B?

c) Width of A: 6 cm, length of A: 8 cm. Width of B: 3 cm. What is the length of B?

d) Width of A: 4 cm, length of A: 12 cm. Length of B: 3 cm. What is the width of B?

Repeat with triangles. This time, label the missing sides with variables and have students write the ratios between the corresponding sides. Have students find the missing sides. **EXAMPLE:** \( \frac{x}{5} = \frac{60}{12} = \frac{y}{13} \), so \( x = 25 \) and \( y = 65 \).

Finding missing sides when the order of sides is not given. Present the following problem:

Sarah built two similar triangles with toothpicks. One has sides 3 toothpicks, 4 toothpicks, and 6 toothpicks, and the other has one side of 15 toothpicks and another side of 30 toothpicks. What is the length of the third side of the second triangle?

Ask students to make a sketch and to label the triangles. Which sides are given in the second triangle? Could it be the smallest side and the middle side? Ask students to write the proportion, assuming that this is the correspondence. Is that a correct proportion? (no, 3 : 15 is not 4 : 30) Which sides are given? (the smallest and the largest sides, because 3 : 15 = 6 : 30) Have students find the third side. **ANSWER:** 20 toothpicks

Have students solve a harder problem:

Sarah built two similar triangles. One has sides 9 cm, 12 cm, and 16 cm, and the other has one side 36 cm and another side 48 cm. Steven said the third side of the second triangle is 64 cm long. Sarah measured the third side and said that the length of the third side is a whole number of centimetres, but Steven is wrong. How can that be?

**ANSWER:** Sarah did not say which side corresponds to which. Steven thought that the sides given in the second triangle corresponded to the first two sides of the first triangle, in which case the lengths are multiplied by four and the length of the third side has to be 16 cm \( \times 4 = 64 \) cm. However, if the sides in the second triangle correspond to the second and the third sides of the first, the lengths are multiplied by three and the missing side, corresponding to the first side, is 9 cm \( \times 3 = 27 \) cm.)
**Similarity in real life.** Ask students to find examples of similar shapes in real life. **EXAMPLES:** shades produced by finger bunnies on a wall, overhead screen projector (shapes on transparency are small, shapes on the screen are large), toys that grow when you put them in water, photos developed from negatives, scale drawings.

**ACTIVITIES**

1. Students can measure various dimensions of a dry toy that grows in water (length, circumference of various parts, and so on) and compare them with the dimensions of the larger, wet toy to check whether the toys are similar.

2. Give students a set of paper rectangles that can be divided in three sets of at least three similar rectangles. (Sample can be found on **BLM Rectangles**, p R-47.) Ask students to draw diagonals on all of the rectangles. Let students measure the sides of the rectangles and divide all rectangles into sets, so that each set contains only similar rectangles. Ask students to order the rectangles in each set from smallest to largest and to arrange the rectangles in each set into a pile, with the largest rectangle on the bottom and the smallest on the top, so that the rectangles in each pile share a common vertex.

   What do students notice about the diagonals of the similar rectangles? (the diagonals coincide; the opposite vertex of each smaller rectangle lays on the diagonal of a larger rectangle) Does this hold for diagonals of non-similar rectangles? (no) A diagonal in a rectangle divides the right angle into two angles. What can students say about these angles in similar rectangles? (The angles between the diagonals and the corresponding sides of the similar rectangles are equal.)

3. Prepare pairs of shapes, some similar and some not. Tape the larger shape in the pair to the board and place the smaller shape on the overhead. Ask students to move the overhead projector until the image is the same size as the shape on the board. If the image coincides with the shape on the board, what does this say about the shape on the board and the shape on the projector?

**Connection**

Visual Arts

**Extension**

Rita wants to estimate the height of a tree growing in the middle of a park. Rita uses a pencil and a measuring tape. Rita holds the pencil vertically in her outstretched hand so that the point of the pencil is level with the top of the tree. Rita moves far enough away from the tree so that the tree appears smaller than the pencil.
Rita holds the pencil so that her fingers are level with the bottom of the tree and the point of the pencil is still level with the top of the tree. She compares the size of the pencil and the tree. For example, the tree appears to be as long as three quarters of the pencil. Rita measures the pencil and finds that three quarters of the pencil is 15 cm. She also measures her arm and finds that her arm is 45 cm. Rita says that the image of the tree is three times shorter than her arm. After that, Rita measures the distance between herself and the tree with giant steps. She says that the tree is three times shorter than the distance to it. So she divides the distance by 3 and obtains the height of the tree in giant steps. (A giant step is close to 1 m.) Can you explain why Rita’s method works?

**EXPLANATION:** Rita uses two similar right-angled triangles. The smaller triangle has Rita’s eye, fingers, and the point of the pencil as vertices. The larger triangle’s vertices are Rita’s eye and the bottom and the top of the tree. The triangles are similar because they have the same angles (see the picture). The ratio between the corresponding sides is the same in both triangles.

As an activity, students can use Rita’s method to measure objects outside the class.
Review ratios. Draw several line segments on a pre-drawn grid on the board and have students find the ratios between the line segments. Include several pairs of line segments that are on the same line and have the same endpoint, as in Question 1 on Workbook page 177. Remind students what ratios mean: if \( a : b = 3 : 1 \), this means that \( a \) is three times larger than \( b \). Then give students several proportions of lengths of line segments and have them find the missing length. **EXAMPLES:**

\[
\begin{align*}
OA : OA' &= 4 : 1, OA = 8 \text{ cm. How long is } OA' \text{?} \\
OB : OB' &= 4 : 1, OB' = 8 \text{ cm. How long is } OB' \text{?} \\
OA : OA' &= 2 : 3, OA = 12 \text{ cm. How long is } OA' \text{?} \\
OC : OC' &= 4 : 5, OC' = 12.5 \text{ cm. How long is } OC' \text{?}
\end{align*}
\]

Drawing line segments with a given ratio of lengths. Give students several line segments and ratios, and have them construct line segments so that the ratio of the new segment to the given segment is the given ratio. **EXAMPLE:** Draw line segment \( PQ = 6 \text{ cm.} \) Construct line segment \( PR \) (as part of \( PQ \)) so that \( PQ : PR = 3 : 4 \).

Horizontal and vertical distances. Remind students that when two points are on the same horizontal (or vertical) line, we called the distance between these two points horizontal (or vertical) distance. Explain that when two points are not on the same line, we can still find horizontal and vertical distances between the two points by going along the grid lines. Think of the translation you need to do to move from one point to the other. The distance you need to move up or down is the **vertical distance**, and the distance you need to move left or right is the **horizontal distance**. Have students find horizontal and vertical distances between pairs of points on the grid.

Give students several questions similar to those in Question 5a) on Workbook page 177. Students need lots of practice finding points on the same line as a given pair of points, with a given ratio between horizontal and vertical lengths. Then have students find the ratios of the actual lengths of line segments, as in Question 5b). Is this ratio the same as the ratios between the horizontal distances? the vertical distances? (yes)
Have students use their findings to draw line segments on the same line with a common endpoint with given ratio of lengths, such as in Question 6 on Workbook page 177.

**Dilatations.** Have students perform a dilatation of the triangle in Question 7 on Workbook page 178. Then introduce the term *dilatation*—a transformation that enlarges or reduces a shape, similar to what an overhead projector does. The point through which all of the lines in the construction are drawn and from which all distances are measured is called the *centre of dilatation*. For the overhead projector, the centre of dilatation is the source of light. Work with students through the Investigation on Workbook page 178.

**Scale factor.** Explain to students that the ratio of the *new* distance (from a point on the new shape to the centre of dilatation) to the *old* distance (from a point on the original shape to the centre of dilatation) is called the *scale factor*. We often regard a scale factor as a unit ratio, in which case we can have a scale factor of 2 (the new points are twice as far from the centre of dilatation) or 1/3 (the new points are three times as close to the centre of dilatation). Have students perform a variety of reductions and enlargements, as in Questions 9–11 on Workbook page 179. Then ask students to look at the shapes they produced and to check the scale factors. What are the scale factors for enlargements? (greater than 1) What are the scale factors for reductions? (less than 1) Will a dilatation with a scale factor of 0.75 produce a reduction or an enlargement? (reduction) A dilatation with a scale factor 1.4? (enlargement)

**The ratio between sides of shapes is equal to the scale factor.** Draw the picture at left on the board and ask students to copy it. Then ask them to perform a dilatation with the scale factor 2 (or 2 : 1) of the line segment $AB$ using $O$ as the centre of dilatation. Ask students to look at two triangles they constructed, $\triangle OAB$ and $\triangle O'A'B'$. Ask students what they know about these triangles. Do they know something about the ratios of some of the sides? ($OA' : OA = OB' : OB = 2 : 1$) Do they know something about the angles? (The angles are equal. Students are likely to see that both triangles are right-angled.) **ASK:** Which angles coincide? ($\angle AOB = \angle A'O'B'$) What do we know about the remaining pair of angles? Ask students to verify that the angles are indeed equal by measuring them, and to check the ratio $A'B' : AB$. What do students notice? (the ratio is $2 : 1$, which is the same as the scale factor) What can students say about the triangles? (the triangles are similar)

Explain to students that when a shape is dilated, the ratio of the sides of the new shape to the sides of the old shape is the same for all sides, and this ratio is equal to the scale factor. This means students can use the ratio of the sides to find the scale factor of the dilatation performed on a pair of shapes.

**Combinations of transformations.** Remind students that rotations, reflections, and translations all produce shapes congruent to the original. **SAY:** A shape and its image under rotation are congruent. Then ask
students to consider this statement: If I have two congruent shapes, there is a rotation taking one into the other. Is this true or false? Ask students to draw a counter-example to the statement. How do they know the shapes are congruent? How do they know that this pair of shapes is not connected by a rotation? Repeat with “reflection” and “translation” instead of “rotation.” Then **ASK:** Does dilatation produce congruent shapes? How are the shapes differing by a dilatation connected? (they are similar) Ask students if the following statement is true or false: If two shapes are similar, there is a dilatation taking one onto the other. (false) Again, ask students to draw a counter-example and explain why this is a counter-example.

**Extension**

a) On a pre-drawn grid, mark points O and P not on the same horizontal or vertical line, so that the horizontal distance \(h\) between O and P is 6 and the vertical distance \(v\) is 2. Draw the line \(OP\). Note that \(v : h = 2 : 6 = 1 : 3\).

b) Mark grid point Q on the line segment OP. Find the horizontal and the vertical distances from O to Q. Then find the ratios of the horizontal distance to \(h\) (defined above) and of the vertical distance to \(v\). What do you notice? (the ratios are equal)

Repeat with point R on the extension of OP beyond P.

c) Draw a horizontal line through O and label it OT. Draw a reflection of the line OP through OT. Mark two grid points M and N on the mirror image of OP and repeat b) for them. What do you notice?

d) Mark a point A not on the line OP and not on its mirror image. Find the horizontal and the vertical distances from O to A. Then find the ratios of the horizontal distance to \(h\) and of the vertical distance to \(v\). Are they the same?

e) We know that \(v : h = 2 : 6 = 1 : 3\). Find the ratios of the vertical distance to the horizontal distance from the points above to O (see table). Which ratios are the same? Which point has a different ratio?

<table>
<thead>
<tr>
<th>point</th>
<th>vertical distance to O : horizontal distance to O</th>
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<tbody>
<tr>
<td>P</td>
<td>(2 : 6 = 1 : 3)</td>
</tr>
<tr>
<td>Q</td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td></td>
</tr>
</tbody>
</table>

f) The ratio \(v : h\) depends on the angle TOP. Look at angles TOQ, TOR, TOM, TON, TOA. What do you notice? \(\angle TOP = \angle TOQ = \angle TOR = \angle TOM = \angle TON \neq \angle TOA\)
Find a Flip (1)
Find a Flip (2)
Triangle Transformation Cards
Triangle Transformations Game Board
Rectangles
Regular Polygons

- Triangle
- Square
- Octagon
- Pentagon
PS7-10  Choosing Strategies

Teach this lesson after: 7.2 Unit 7

Goals:
Students will use the problem-solving strategies learned so far to solve problems, choosing which strategy to use as appropriate.

Prior Knowledge Required:
Can add integers (for Problem Banks 4, 11, 12)
Can multiply integers (for Problem Bank 18)
Can find the average of numbers (for Problem Bank 19)
Can solve equations that have linear expressions on both sides (for Problem Banks 7, 21)
Can plot coordinate points in the four quadrants (for Problem Bank 24)
Can determine amounts given the percent increase (for Problem Bank 26)
Can divide multiples of 100 by 25 (for Extended Problem)
Can multiply multi-digit numbers (for Extended Problem)
Can solve linear equations of the form $ax + b = c$, where $a$, $b$, and $c$ are whole numbers (for Extended Problem)
Can recognize bias in a sample (for Extended Problem)
Can multiply whole numbers by decimals (for Extended Problem)
Can calculate the circumference of a circle given its diameter using $\pi = 3.14$ (for Extended Problem)

Vocabulary: average, consecutive, factor, negative, opposite, percent, perfect square, positive, power, ratio, regular, sequence, square of a number, surface area

Materials:
disks and rods, or labelled cubes (optional, see Problem Bank 36)
BLM Hundreds Chart (p. R-62, see Problem Bank 40)
BLM Amusement Park (pp. R-64–67, see Extended Problem)

NOTE: The following Problem Bank questions reflect a selection of the problem-solving strategies used in the problem-solving lessons for Grade 7. Students will need to choose among all the strategies they have learned this year to solve these problems.

Problem Bank
1. A sequence starts with 2, 3, 1. Find the next five terms in the pattern if the sequence continues as follows.
a) After the first three terms, the next term is found by adding the previous three terms and subtracting the result from 10 (so the next term is $10 - 6 = 4$).
b) After the first three terms, the next term is found by multiplying the previous three terms and then dividing 24 by the result.
Answers: a) 4, 2, 3, 1, 4; b) 4, 2, 3, 1, 4
2. The sequences you found in Problem Bank 1 look like repeating patterns. Explain how you know that they will continue to repeat.

**Sample answer:** You are doing the same operations to the same numbers, so you will get the same answer.

3. Find the 100th term of each pattern from Problem Bank 1.

**Answers:** a) 1, b) 1

4. A sequence is made by the rule “After the third term, each term is the opposite of the sum of the previous three terms.” What is the 100th term in the sequence?
   a) The sequence starts with 2, 3, 4.
   b) The sequence starts with 1, 2, −6.
   c) The sequence starts with 0.83, 3.4, −0.5.
   d) The sequence starts with \(a, b, c\).

**Answers:** a) −9, b) 3, c) −3.73, d) \(-(a + b + c)\)

**NOTE:** Students solved Problem Banks 5 to 7 below in PS7-4: Searching Systematically. Now they can solve the same problems by using algebra instead.

5. The ratio of length to width in a rectangle is 4 : 3. The perimeter is 84 cm. What is the length of the rectangle?

**Answer:** 24 cm

6. The ratio of girls to boys is 5 : 3. One fifth of the girls go away for a sports event. What is the new ratio of girls to boys?

**Answer:** 4 : 3

7. a) The ratio of girls to boys in a class is 2 : 3. If there are \(2x\) girls, how many boys are there?
   b) Five more girls and two more boys joined the class. Write an expression for the new number of girls and for the new number of boys.
   c) Now the ratio of girls to boys is 1 : 1. Write an equation that shows this. How many girls are now in the class?

**Answers:** a) \(3x\); b) \(2x + 5\) girls and \(3x + 2\) boys; c) \(2x + 5 = 3x + 2\), so \(5 - 2 = 3x - 2x\), so \(x = 3\), so there are now 11 girls in the class

8. Solve each problem in two ways. Make sure you get the same answer using both ways.
   A. The value of my dimes is twice the value of my quarters. How many times as many dimes as quarters do I have?
   B. The value of my dimes is \(\frac{4}{5}\) the value of my quarters. How many times as many dimes as quarters do I have?
      a) Try different numbers of quarters and see how many dimes you have.
      b) Use algebra.
### Answers:

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<thead>
<tr>
<th></th>
<th>Value of Quarters (¢)</th>
<th>Value of Dimes (¢)</th>
<th>Number of Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) A.</td>
<td>1</td>
<td>25</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>75</td>
<td>150</td>
</tr>
</tbody>
</table>

So, I have five times as many dimes as quarters.

<table>
<thead>
<tr>
<th></th>
<th>Value of Quarters (¢)</th>
<th>Value of Dimes (¢)</th>
<th>Number of Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.</td>
<td>1</td>
<td>25</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>50</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>75</td>
<td>60</td>
</tr>
</tbody>
</table>

So, I have twice as many dimes as quarters.

b) A. If there are $x$ quarters, they are worth 25¢ $x$ cents. The dimes are worth twice as much, so they are worth 50¢ $x$ cents. So, there are $5x$ dimes. The number of dimes is five times the number of quarters.

B. If there are $x$ quarters, they are worth 25¢ $x$ cents. The dimes are worth $4/5$ of 25¢, or 20¢ $x$ cents. So, there are $2x$ dimes. The number of dimes is twice the number of quarters.

9. A website provides estimated biking times from one point to another. Nora takes 8 minutes to bike to school, but the website tells her that biking to school along that same route will take only 5 minutes. The same website tells her that biking to a friend’s place will take only 20 minutes. Use this information to estimate how long it will actually take Nora to bike to her friend’s place.

**Answer:** 32 minutes

10. A Canadian dollar is worth 80% of an American dollar.

a) What percent of a Canadian dollar is an American dollar?

b) An American tourist visits Canada and makes a purchase that costs CAN$30. He pays with an American $50 bill. How much Canadian change should he expect back?

**Answers:** a) 125%; b) US$50 = CAN$50 × 1.25 = CAN$62.50, so he should expect CAN$32.50 back.

11. What is the sum of integers $1 - 2 + 3 - 4 + 5 - 6 + \ldots + 49 - 50$?

**Answer:** $-25$, because there are 25 groups that total $-1$: $(1 - 2) + (3 - 4) + \ldots + (49 - 50)$

12. The sum of the numbers from $-20$ to $N$ is 90. What is $N$?

**Answer:** 24

13. a) Draw a grid to show why the sum of the first $n$ odd numbers is equal to $n^2$. What is the sum of the first 50 odd numbers?

b) How much greater than the sum of the first 50 odd numbers is the sum of the first 50 even numbers?

c) What is the sum of the first 50 even numbers?

d) What is the sum of the first 100 numbers?

**Answers:** a) $50^2 = 2500$; b) 50, since each of the first 50 even numbers is 1 greater than the corresponding odd number; c) $2500 + 50 = 2550$; d) $2500 + 2550 = 5050$
14. Use $1 + 2 + 3 + 4 + \ldots + 9 = 45$ to evaluate the sum $1.1 + 2.2 + 3.3 + 4.4 + \ldots + 9.9$.

**Answer:** 49.5

15. Use $48 + 49 + 50 + 51 + 52 = 250$ to evaluate the sum $48\frac{1}{3} + 49\frac{1}{3} + 50\frac{1}{3} + 51\frac{1}{3} + 52\frac{1}{3}$.

**Answer:** 251 2/3

16. How much greater than $\frac{2003 + 25}{25}$ is $\frac{2003}{25} + 25$?

**Answer:** 24

17. A regular hexagon and a regular triangle have the same perimeter. How do their areas compare?

**Solution:** To have the same perimeter, the triangle must have side lengths twice those of the hexagon because the hexagon has twice as many sides. Each of the small equilateral triangles shown below has the same side length and area:

So, the area of the triangle is 4/6, or 2/3, of the area of the hexagon.

18. The product of the numbers from −10 to $N$ is positive. What can you say about $N$?

**Answer:** $N$ is negative and odd; it is −1, −3, −5, −7, or −9.

19. a) The average of two numbers is 30 and the average of three other numbers is 40. What is the average of all five numbers?

b) The average of two numbers is $x$ and the average of three other numbers is $x + 10$. What is the average of all five numbers?

c) How can you get your answer in part a) from your answer in part b)?

**Answers:** a) the sum of all five numbers is $60 + 120 = 180$, so their average is 36; b) the sum of the two numbers is $2x$ and the sum of the three other numbers is $3x + 30$, so the sum of all five numbers is $5x + 30$ and their average is $x + 6$; c) substituting $x = 30$ gets the average $30 + 6 = 36$

20. a) How many rectangles have whole-number side lengths and perimeter 102?

b) How many rectangles have whole-number side lengths and perimeter 103?

**Solutions:** a) Begin by making a table for the first few rectangles with whole-number side lengths:

<table>
<thead>
<tr>
<th>Perimeter</th>
<th>Number of Rectangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1 (1 by 1)</td>
</tr>
<tr>
<td>6</td>
<td>1 (1 by 2)</td>
</tr>
<tr>
<td>8</td>
<td>2 (2 by 2 and 1 by 3)</td>
</tr>
<tr>
<td>10</td>
<td>2 (2 by 3 and 1 by 4)</td>
</tr>
<tr>
<td>12</td>
<td>3 (3 by 3, 2 by 4, and 1 by 5)</td>
</tr>
<tr>
<td>14</td>
<td>3 (3 by 4, 2 by 5, and 1 by 6)</td>
</tr>
</tbody>
</table>

From the first few terms, the number of rectangles is the quotient when dividing by 4 (ignoring the remainder). So, for perimeter 102, the number of rectangles is 25 (they are 1 by 50, 2 by 49, ..., 25 by 26); b) none, because the perimeter has to be even, since it is $2 \times (\text{length} + \text{width})$
21. At the end of a tennis lesson, each student picks up five balls, leaving four for the instructor to pick up. The next day, the instructor brings the same number of tennis balls, but two fewer players show up. At the end of that lesson, each student picks up seven balls, leaving only two for the instructor to pick up. How many tennis balls did the instructor bring each day?

**Solution:** Let $x$ be the number of students in the lesson on the first day. Then the number of balls on the first day is $5x + 4$ and the number of balls on the second day is $7(x - 2) + 2$. These are equal because the instructor brings the same number of balls both days, so $5x + 4 = 7(x - 2) + 2$, so $5x + 4 = 7x - 14 + 2$, so $4 + 14 - 2 = 7x - 5x$, so $16 = 2x$, so $x = 8$. There are eight students in the class on the first day, so the number of balls is $44 = 5 \times 8 + 4$. As confirmation, this is also equal to $7 \times 6 + 2$.

22. How many factors does 1 000 000 000 have?

**Solution:** Look for a pattern: 10 has 4 factors, 100 has 9 factors, 1000 has 16 factors. These numbers are the perfect squares: add 1 to the number of zeros in the number and multiply the result by itself. There are 9 zeros in 1 000 000 000, so the number of factors is $10 \times 10 = 100$.

23. If $4x : 5y = 3 : 2$, what is $x : y$?

**Answer:** $x : 5y = 3/4 : 2 = 3 : 8$, so $x : y = 3 : 8/5 = 15 : 8$

24. A path continues spiralling, as shown below. Each arrow shows 1 unit along the $x$- or $y$-axis. What is the length of the path ... a) from (0, 0) to (11, 0)?   b) from (0, 0) to (0, −20)?

**Answers:** a) 121 units ($11 \times 11$ units), b) 1580 units

25. What is the length of the thick line path from A to B?

**Solution:** $9 + 9 + 19 = 37$, so 37 m

26. One "cube" in an ice cube tray has a rectangular 3 cm by 5 cm base and is 2.2 cm tall. Water expands 10% when it freezes. If the ice is exactly level with the top of the tray when it’s frozen, how deep was the water in the tray to start with?

**Answer:** 2 cm
27. A cube is made of 27 smaller cubes. The cube is painted red. How many smaller cubes have ...
   a) exactly one face painted red?
   b) exactly two faces painted red?
   c) exactly three faces painted red?
   d) no faces painted red?
   e) more than three faces painted red?
   **Answers:** a) 6, b) 12, c) 8, d) 1, e) 0

28. Three cubes are stacked as shown. The side lengths are 3 cm, 2 cm, and 1 cm. What is the total surface area of the new shape?

   **Solution:** The surface area of the three cubes if they weren't stacked is \(6(1^2) + 6(2^2) + 6(3^2) = 84\). Subtract \(2(1)^2 + 2(2)^2 = 10\) for the four hidden areas. The total surface area is 74 cm².

29. Rob divides 45 by a number and gets a remainder of 9. What could he have divided by?
   **Solution:** \(45 = a \times b + 9\), so \(36 = a \times b\). So, \(a\) is a factor of 36 and \(a\) is at least 10, since dividing by it gets a remainder of 9. Rob could have divided by 12, 18, or 36.

30. Design a cereal box that has a capacity of 2.5 L and side lengths that are each a whole number of centimetres. Make it have reasonable side lengths for a cereal box.
   **Sample answer:** 5 cm by 20 cm by 25 cm

31. a) Find a four-digit number so that each number tells how many times the digit above it occurs in the number.

   
   
   
   

   b) Repeat part a) for ...
   i) a five-digit number

   
   
   
   
   
   ii) a seven-digit number

   
   
   
   
   
   iii) an eight-digit number

   
   
   
   

   iv) a nine-digit number

   
   
   
   

   v) a ten-digit number

   
   
   
   

   c) Calculate the sum of the digits for each part of a) and b). Why could you have predicted this?
Answers:
a) 1210
b) i) 21 200, ii) 3 211 000, iii) 42 101 000, iv) 521 001 000, v) 6 210 001 000
c) The sum of the digits in each part is a) 4, b) i) 5, ii) 7, iii) 8, iv) 9, v) 10. I could have predicted this because each digit tells how many times the number above it occurs, so the sum of the digits has to tell how many times all the digits occur together, and that must be the number of digits in the number.

32. You have a 3 L pail, a 5 L pail, an 11 L pail, and a 20 L pail. There is a river nearby.
a) How can you measure 29 L?
b) How can you measure 7 L?
Answers:
a) Fill the 20 L pail, the 11 L pail, and the 3 L pail to get a total of 34 L. Pour water from, say, the 20 L pail to fill the 5 L pail; this leaves 15 L in the 20 L pail, which together with the water in the 11 L and 3 L pails makes 29 L.
b) Fill the 20 L and the 3 L pails. Pour from the 20 L pail to fill the 5 L and 11 L pails; this leaves 4 L in the 20 L pail, which together with the water in the 3 L pail makes 7 L.

33. You have a 2 L pail, a 5 L pail, and there is a river nearby.
a) How can you make the 5 L pail have exactly 4 L in it?
b) When the 5 L pail has exactly 4 L in it, how can you make the 2 L pail have 1 L in it?
Answers:
a) Fill the 2 L pail and pour the contents into the 5 L pail, twice
b) Fill the 2 L pail and pour the contents into the 5 L pail until the 5 L pail is full, leaving 1 L in the 2 L pail.

34. You have a 3 L pail, a 7 L pail, and there is a river nearby.
a) Suppose you want to put 5 L in the 7 L pail. How would it help to have 1 L already in the 3 L pail?
b) How could you put 1 L in the 3 L pail so that you can then do the steps from part a) to put 5 L in the 7 L pail?
Answers:
a) Fill the 7 L pail completely, then pour as much as you can into the 3 L pail to fill it up. If there is already 1 L in the 3 L pail, you would have poured 2 L from the 7 L pail, leaving 5 L in the 7 L pail
b) Fill the 7 L pail completely, then pour as much as you can into the 3 L pail. Empty the 3 L pail and again, pour as much as you can from the 7 L pail into the 3 L pail. There is now 1 L in the 7 L pail. Empty the 3 L pail, and pour the 1 L from the 7 L pail into the 3 L pail.
35. How can you measure 16 L by using a 7 L pail and a 22 L pail? There is a river nearby
Solution: In the diagrams below, the bigger pail represents the 22 L pail and the smaller pail
represents the 7 L pail. The numbers represent the amount of water in the pail (in L).

Step 1: Fill the 22 L pail from the river.
Step 2: Fill the 7 L pail from the 22 L pail.
Step 3: Empty the 7 L pail into the river.
Step 4: Fill the 7 L pail from the 22 L pail.
Step 5: Empty the 7 L pail into the river.
Step 6: Fill the 7 L pail from the 22 L pail.
Step 7: Empty the 7 L pail into the river.
Step 8: Pour the 1 L from the 22 L pail into
the 7 L pail.
Step 9: Fill the 22 L pail from the river.
Step 10: Fill the 7 L pail from the 22 L pail.

Now you have measured 16 L.

36. The Tower of Hanoi problem. There are three rods and some disks. All the disks are
different sizes and they are stacked on one rod from smallest to largest. The goal is to move the
stack to another rod by moving only one disk at a time from the top of one stack to the top of
another stack. No disk can be placed on top of a smaller disk. Solve the problem for the given
number of disks, using the smallest number of moves you can, and complete the table.

NOTE: The problem is easier to do using concrete materials. If you don’t have disks of different
sizes, use cubes and tape numbers to them in order from 1 to the number of “disks.” The rule
then becomes: “No cube can be placed on top of a smaller numbered cube.” You will have to
imagine the rods.

<table>
<thead>
<tr>
<th>Number of Disks</th>
<th>Number of Moves Required to Solve the Puzzle</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Predict how many moves would be required if you started with eight disks.
Answer: \(2^8 - 1\) or 255 moves would be required.
37. The sequence 1, 1, 2, 3, 5, 8, 13, 21, 34 was made by the rule “Add each two consecutive terms to get the next term.” If the 30th term is 832 040, what is the product of the 29th and 31st terms? Hint: Make a table with these headings: Term Number, Term Value, and Product of Surrounding Terms.

**Solution:** Starting with the second term, compare the square of each term with the product of the two surrounding it.

<table>
<thead>
<tr>
<th>Term Number</th>
<th>Term Value</th>
<th>Product of Surrounding Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>N/A</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1 × 2 = 2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1 × 3 = 3</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2 × 5 = 10</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>3 × 8 = 24</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>5 × 13 = 65</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>8 × 21 = 168</td>
</tr>
<tr>
<td>8</td>
<td>21</td>
<td>13 × 34 = 442</td>
</tr>
</tbody>
</table>

Notice that the products are close to the square of the term value, so add a column for the square of the term value. When the term number is even, the product is one more than the square, so the product of the 29th and 31st terms is $832\,040^2 + 1 = 692\,290\,561\,601$

38. Write the numbers 1 to 100 in a circle. Starting by circling 1 and going in order, circle every second number that is not yet circled. What is the last number left uncircled? Hint: Make a pattern and be persistent; you will need to try many numbers before you see a pattern.

**Solution:** Start by following the instructions when the circle has only 1, then 1 and 2, then 1, 2, and 3, and so on. The tables below show the pattern (the numbers in the left column show how many numbers are in the circle and the numbers in the right column show the last uncircled number).

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
<th>9</th>
<th>2</th>
<th>17</th>
<th>2</th>
<th>25</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>10</td>
<td>4</td>
<td>18</td>
<td>4</td>
<td>26</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>11</td>
<td>6</td>
<td>19</td>
<td>6</td>
<td>27</td>
<td>22</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>12</td>
<td>8</td>
<td>20</td>
<td>8</td>
<td>28</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>13</td>
<td>10</td>
<td>21</td>
<td>10</td>
<td>29</td>
<td>26</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>14</td>
<td>12</td>
<td>22</td>
<td>12</td>
<td>30</td>
<td>28</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>15</td>
<td>14</td>
<td>23</td>
<td>14</td>
<td>31</td>
<td>30</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>16</td>
<td>16</td>
<td>24</td>
<td>16</td>
<td>32</td>
<td>32</td>
</tr>
</tbody>
</table>

Every power of 2 is the last uncircled number in its own circle, and then the pattern starts over at 2. Continue the table starting at the last power of 2 that is less than 100:

<table>
<thead>
<tr>
<th>64</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>2</td>
</tr>
<tr>
<td>66</td>
<td>4</td>
</tr>
<tr>
<td>67</td>
<td>6</td>
</tr>
<tr>
<td>68</td>
<td>8</td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>?</td>
</tr>
</tbody>
</table>
There are 100 − 64 = 36 numbers from 65 to 100, so when the circle is made from the numbers 1 to 100, the last uncircled number is the 36th term in the sequence 2, 4, 6, ..., which is 2 × 36 = 72.

39. Bowl A has five spoonfuls of red paint and two spoonfuls of white paint. Bowl B has one spoonful of red paint and one spoonful of white paint. All the spoons are the same size.
   a) Which bowl has paint that is darker red? Explain how you know using fractions.
   b) If you pour the contents of Bowl B into the contents of Bowl A, will it make the paint in Bowl A darker or lighter red?
   c) What is the new fraction of red paint in the bowl in part b)? Is that fraction greater than or less than \( \frac{5}{7} \)? How can you tell without doing any calculations?
   d) Is \( \frac{35}{69} \) greater than or less than one half?
   e) Without doing any calculations, use your answer to part d) to decide if \( \frac{36}{71} \) is greater than or less than \( \frac{35}{69} \).

   Answers: a) Bowl A is darker red because the fraction of red paint is \( \frac{5}{7} \), while the fraction of red paint in bowl B is \( \frac{1}{2} \), and \( \frac{5}{7} \) is greater than \( \frac{1}{2} \); b) lighter red, because you are adding paint that is lighter red to bowl A; c) \( \frac{6}{9} \), or \( \frac{2}{3} \), which is less than \( \frac{5}{7} \) because making the paint lighter red requires making the fraction of red paint smaller; d) more than one half; e) \( \frac{36}{71} \) is less than \( \frac{35}{69} \), because it’s like you added paint that is only \( \frac{1}{2} \) red, so paint that is \( \frac{36}{71} \) red is lighter red than paint that is \( \frac{35}{69} \) red.

40. A two-digit number is divided by the sum of its digits. What two-digit number will result in the largest remainder? Solve this problem in steps.
   a) Start by dividing the two-digit numbers by the sum of their digits, in order:
      \[
      \begin{align*}
      10 & \div 1 = \_ \_ \ R \_ \_ \\
      11 & \div 2 = \_ \_ \ R \_ \_ \\
      12 & \div \_ \_ = \_ \_ \ R \_ \_ \\
      13 & \div \_ \_ = \_ \_ \ R \_ \_ \\
      14 & \div \_ \_ = \_ \_ \ R \_ \_ \\
      \end{align*}
      
   b) Is the strategy from part a) a good strategy to continue? Why or why not?
   c) On a hundreds chart (e.g., from BLM Hundreds Chart), calculate the sum of the digits of all the two-digit numbers. Write them on the hundreds chart squares.
   d) What is the largest sum of digits a two-digit number can have?
   e) The remainder must be smaller than the divisor, which is the sum of the digits. Make a table starting with the largest sum of the digits.

<table>
<thead>
<tr>
<th>Sum of Digits</th>
<th>Two-Digit Number</th>
<th>Division</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td></td>
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<tr>
<td>16</td>
<td></td>
<td></td>
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<tr>
<td>16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
f) What is the largest remainder you found in part e)?
g) All the two-digit numbers not in the table so far have sum of digits at most 15. Can you get a higher remainder than the one you found in part c)? Explain.

Selected answers:
a) 10 ÷ 1 = 10 R 0, 11 ÷ 2 = 5 R 1, 12 ÷ 3 = 4 R 0, 13 ÷ 4 = 3 R 1, 14 ÷ 5 = 2 R 4
b) no, because the remainder can't be bigger than the sum of the digits, so we should start with numbers that have bigger sums of digits
d) 18
e) 

<table>
<thead>
<tr>
<th>Sum of Digits</th>
<th>Two-Digit Number</th>
<th>Division</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>99</td>
<td>99 ÷ 18 = 5 R 9</td>
</tr>
<tr>
<td>17</td>
<td>98</td>
<td>98 ÷ 17 = 5 R 13</td>
</tr>
<tr>
<td>17</td>
<td>89</td>
<td>89 ÷ 17 = 5 R 4</td>
</tr>
<tr>
<td>16</td>
<td>97</td>
<td>97 ÷ 16 = 6 R 1</td>
</tr>
<tr>
<td>16</td>
<td>88</td>
<td>88 ÷ 16 = 5 R 8</td>
</tr>
<tr>
<td>16</td>
<td>79</td>
<td>79 ÷ 16 = 4 R 15</td>
</tr>
</tbody>
</table>

f) 15
g) no, the highest remainder you can get when dividing by 15 is 14, so 15 is the largest remainder possible

41. Evan divides a three-digit number by the sum of its digits. What is the largest possible remainder he can get?

Solution: Make a table starting with as large as possible sum of digits.

<table>
<thead>
<tr>
<th>Number</th>
<th>Sum of Digits</th>
<th>Number ÷ Sum of Digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>999</td>
<td>27</td>
<td>37 R 0</td>
</tr>
<tr>
<td>998</td>
<td>26</td>
<td>38 R 10</td>
</tr>
<tr>
<td>989</td>
<td>26</td>
<td>38 R 1</td>
</tr>
<tr>
<td>899</td>
<td>26</td>
<td>34 R 15</td>
</tr>
<tr>
<td>988</td>
<td>25</td>
<td>39 R 13</td>
</tr>
<tr>
<td>898</td>
<td>25</td>
<td>35 R 23</td>
</tr>
<tr>
<td>889</td>
<td>25</td>
<td>35 R 14</td>
</tr>
<tr>
<td>799</td>
<td>25</td>
<td>31 R 24</td>
</tr>
<tr>
<td>979</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>997</td>
<td>25</td>
<td></td>
</tr>
</tbody>
</table>

The largest possible remainder is 24, since dividing by 25 or less cannot result in a remainder that is larger than 24, so we can stop here at 799 ÷ 25 = 31 R 24.
Extended Problem: Amusement Park

Materials:
BLM Amusement Park (pp. R-64–67)

Extended Problem: Amusement Park. Provide students with BLM Amusement Park. Tell students that in this extended problem they will work with equations, bias in surveys, and circumference of circles.

Answers:
1. 800
2. a) 24, b) 23, c) 18 400
3. a) 180 + 15x, b) $630, c) $180, d) 8 hours
4. a) both estimates will affect the amount of money the owner will make: more people means more money, and a greater ratio of adults to children also means more money
   b) i) both estimates will be biased because it’s likely more people will come overall, but also more children will come because they are free every day instead of just on weekends; ii) the ratio of adults to children should not be biased, but the total number of people will be more than average
   c) $26 692
5. 301.44 metres

Bonus: a) 3, 5, 7, 9, 10, 12, 14, 17, 19, 24; b) 2, 5, 10, sample explanation: The lowest-scoring region, obtained three times, gives a total of 6, so the lowest-scoring region must be 2. The highest-scoring region, obtained three times, gives a total of 30, so the highest-scoring region must be 10. To get the second lowest scoring total, use 9 = 2 + 2 + (second lowest scoring region), so the second lowest scoring region is 5.
Amusement Park (1)

You work at an amusement park. The Ferris wheel ...

• allows one complete rotation per ride.
• has 20 cabins that each carry 40 people.
• takes 20 minutes to make one complete rotation and stops for 5 minutes between rides.

1. How many people can go on each ride?

2. The Ferris wheel runs from 10 a.m. to 8 p.m.
   a) At most, how many Ferris wheel rides can the park offer each day?
   b) The Ferris wheel sometimes stops for longer periods when maintenance is needed or if someone needs assistance getting on or off. Subtract one ride from your answer in part a) to create a more realistic maximum number of rides.
   c) On one summer day, the Ferris wheel was constantly full. Using your answer from part b), how many people rode the Ferris wheel that day?
Amusement Park (2)

3. You work at the amusement park and are on call for six 5-hour shifts a week. You only need to go in if they need you, but they pay you $30 for each shift that you are on call, plus $15 for each hour that you work.

   a) How much money would you make in a week if you worked $x$ hours?

   b) What is the most money you could make in a week?

   c) What is the least money you could make in a week?

   d) How many hours would you need to work to make $300 in one week?
Amusement Park (3)

4. An adult ticket costs $20 and a child ticket costs $12. The owner of the Ferris wheel wants you to estimate ...
   - the ratio of adults to children, on average.
   - the number of people who ride the Ferris wheel, on average.
   a) Why might the owner be interested in each estimate?

b) You want to pick 30 days in the year and check how many adult tickets and how many child tickets were sold. Explain how bias might be introduced in each sample below. Would the estimate be biased?
   i) All 30 days are during summer vacation.
   ii) All 30 days are bright, sunny days throughout different times of the year.

c) You picked 30 days randomly. The average number of adult tickets sold was 956 and the average number of child tickets sold was 631. How much money can you expect the Ferris wheel to bring in each day?

5. The bottom of the Ferris wheel is 4 metres above the ground. The top of the Ferris wheel is 100 metres above the ground. What is the circumference of the Ferris wheel? Use 3.14 for $\pi$. 

Amusement Park (4)

**Bonus** A game at the amusement park allows a player to toss three beanbags aimed at a target, with points as shown below:

![Target Diagram]

a) If all three beanbags hit the target, what are all the possible scores a player can get?

b) On a similar target, the three scoring regions are not labelled, but the possible scores, assuming all beanbags hit the target, are 6, 9, 12, 14, 15, 17, 20, 22, 25, and 30. What points are assigned to each scoring region? Explain how you know.
In this unit students will express probability as fractions, ratios and percents, find sample space for probability experiments involving compound events, and find probability of compound events using a variety of methods. Students will also conduct probability experiments and determine experimental probability of various events and compare it to theoretical probability. They will also learn about applications of probability in real life.

Meeting your curriculum
This section is core curriculum for students following either Ontario or WNCP curriculum.
Goals

Students will determine the theoretical probability of simple events and represent it different ways.

PRIOR KNOWLEDGE REQUIRED

- Understands ratios, percents, and fractions
- Can convert between ratios, fractions, and percents
- Has seen experiments with spinners, dice, and marbles

Measuring likelihood. Show students two pencils of different lengths. Ask students how they could determine which pencil is longer. Then present two measurements that cannot be compared directly, such as the length of a ruler and the circumference of a cup. (Students might suggest using a measuring tape to compare them indirectly.) Ask students how they could compare the weight of two objects, say a book and a cup, or the temperature in two different places. Point out that in all cases students tried to attach a number to the characteristic or quantity and to compare the numbers. They used different tools—a measuring tape, a scale, or a thermometer—to get a number, that is, a measurement. ASK: What would you do to compare the likelihood of two events, such as the likelihood of rolling 8 on a pair of dice and the likelihood that your favourite hockey team wins 5 to 3 in its next game? Is there a tool to measure likelihood? (no) Explain that probability is the branch of mathematics that studies the likelihood of events and expresses this likelihood in numbers. The measure of likelihood of an event is called probability.

Outcomes. Hold up a die and ask students to predict what will happen when you roll it. ASK: Can it land on a vertex? On an edge? No, the die will land on one of its sides. Ask students to predict which number you will roll. Then roll the die (more than once, if necessary) to show that while it does land on a side, it does not necessarily land on the number students predicted. Explain that the possible results of rolling the die are called outcomes, and to predict the future students must learn to identify which outcomes of various actions are more likely to happen and which are not. But first, they must learn to identify outcomes correctly.

Hold up a coin and ASK: What are the possible outcomes of tossing a coin? How many outcomes are there? Show a spinner with three equal but differently coloured regions and a set of marbles of different colours. What are the possible outcomes of spinning the spinner or picking a marble with your eyes closed? Ask students to identify the possible outcomes of a soccer game. (3 outcomes: team A wins, team B wins, a draw)
Equally likely outcomes. Draw two spinners as in the margin and ask students how they are the same and how they are different. Are you more likely to spin blue on one of the spinners than on the other? (no) Are you more likely to spin yellow on one of the spinners than on the other? (no) Why? (the spinners have identical colouring, the blue region is just split in two on one of the spinners) Ask students to identify the outcomes of each spinner. Point out that the spinner with three regions has three different outcomes, one for each region. On this spinner there are two different ways to spin blue.

Ask students to draw two spinners, each with six possible outcomes, such that one spinner has equally likely outcomes and the other has outcomes that are not equally likely.

Events. Explain that when you describe a specific outcome or set of outcomes, such as rolling a 6, rolling an even number, tossing a head, or spinning blue, you identify an event. Ask students to identify all nine possible outcomes in this situation: Rina and Edith play Rock, Paper, Scissors. (You can use the Scribe, Stand, Share strategy to check students’ answers.) Then ASK: Is “Rina wins” on the list of outcomes? Why is “Rina wins” an event and not an outcome? (because Rina can win in more than one way) Which outcomes belong to this event? (Rina has rock, Edith has scissors; Rina has paper, Edith has rock; Rina has scissors, Edith has paper.)

Probability of events with equally likely outcomes. Explain that in mathematics, when the outcomes are equally likely, we describe the probability of an event as a fraction or a ratio:

$$P(\text{Event } A) = \frac{\text{Number of outcomes that suit } A}{\text{Number of all outcomes}}$$

If the outcomes are not equally likely, this formula does not work, and we need to change the problem so that the outcomes will be equally likely (but we will deal with that later). For the blue and yellow spinner with three regions, above, there are three possible and equally likely outcomes, so the probability of spinning yellow is $\frac{1}{3}$ or $1 : 3$ and the probability of spinning blue is $\frac{2}{3}$ or $2 : 3$.

ASK: What fraction of the spinner is coloured yellow? What fraction of the spinner is coloured blue? What do you notice? (the fraction of the spinner that is a particular colour is the probability of spinning that colour)

Have students find the probability of events that consist of equally likely outcomes, as in Questions 7 and 8 on Workbook page 183.

EXTRA PRACTICE:

Find the probability of drawing one marble of each colour separately from this collection of 24 marbles: 12 red, 8 blue, 3 yellow, and 1 white. What is the probability of drawing a marble that is not yellow? What is the probability of drawing a marble of a colour that is on the Canadian flag?
When events are not equally likely. Return to the two blue and yellow spinners shown earlier in the lesson. Ask students how they could find the probability of spinning blue or spinning yellow on the spinner with two unequal regions. PROMPT: Is the probability of spinning either blue or yellow different for the two spinners? (No, so we can use the spinner with equal regions to determine the probability for the spinner with unequal regions.)

Present the spinner at left and ask students how they could find the probability of spinning each colour on this spinner. ASK: Are the outcomes equally likely? (no) What could you do to make the outcomes equally likely? (split the large region into 4 equal regions that are the same as the other, smaller regions) Invite volunteers to do so and to find the probability of spinning each colour.

Have students practise finding probabilities of events with outcomes that are not equally likely, as in Question 9 on Workbook page 183. Then have students decide whether the outcomes of certain experiment are equally likely or not and find the probability of the events.

EXTRA PRACTICE:
Find the probability that a dart lands on each colour. Assume that a dart always lands on the board, with the same probability of landing in regions of equal area.

Expressing probability in different ways. Remind students that decimals, fractions, percents, and ratios are all numbers, and students have learned to convert between them. Use Workbook page 185 Question 1 to check students’ ability to identify various representations of the same number and Question 2 on the same page to check that students remember how to convert between representations.

Have students read the definition of the probability of an event in the box on Workbook page 183. ASK: What is larger, the number of all possible outcomes or the number of outcomes that suit an event? Is the fraction that you get more than 1 or less than 1? (less) Can it be 1? (yes) What do we call events with probability 1? (certain) Can it be 0? (yes) What do we call events with probability 0? (impossible) Can it be less than 0? (no) Ask students to give examples of an event with probability 1 and an event with probability 0. Use Scribe, Stand, Share to check the answers. NOTE: If students have trouble answering the questions using the general formula, have them look at several specific examples, then look at the general formula again.

Applications of probability. Ask students to think about when people use probability or probability-related knowledge in real life. Possible examples: playing the lottery, gambling, weather forecasting (e.g., probability of precipitation). Knowledge of probability also applies to any game that depends on chance, such as most card games. Probability is also used by scientists studying population, plant growth, genetics, and more.
Using probability to decode messages.

The Roman emperor Julius Caesar developed a way to encode messages known as the Caesar cipher. His method was to shift the alphabet and replace each letter by its shifted letter. A shift of 3 letters would be:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
D E F G H I J K L M N O P Q R S T U V W X Y Z A B C

Look at the following message: What’s gone with that boy, I wonder?
Using the shift of 3 letters, the message would begin: Zkdw'v jrqh

a) Finish encoding the message.
b) Tally the occurrence of each letter in the original message and in the encoded text.

c) Which letters occur most often in the original message? Which letters occur most often in the encoded message? How are these letters related?
d) In the English language, the letter occurring most often is almost always “e,” especially for large samples of the language.

Use a Caesarean shift cipher encoder (find one on the Web by searching the keywords “Caesarean shift cipher encoder”) to decode and encode messages.

If you encode a message with a shift of 3, you will need to decode the message using a shift of 26 − 3 = 23 to get back your original message. What shift would you use to decode a message that was encoded using a shift of 7? A shift of 16? A shift of 24?

e) To decode the following message (which you can copy from the electronic version of the lesson plan), first determine which letter occurs most often by using an online letter frequency counter (search for “online letter frequency counter”).

"ZUS!"
Tu gtyckx.
"ZUS!"
Tu gtyckx.
"Cngz’y mutk cozn zngz hue, O cutjkk? Eua ZUS!"
Tu gtyckx.
Znk urj rgje varrkj nkx yvkizgirky juct gtt ruuqkj ubkx znks ghuaz znk xuus; znkt ynk vaz znks av gtt ruuqkj uaz atjkk znks. Ynk ykrjus ux tbbkx ruuqkj znxuamn znks lux yu ysgrg g znotm gy g hue; znk
Which letter do you think represents the letter “e”? Why? (k, because that is the most common letter)

What shift do you think was used? (6-shift) What shift do you think will decode the message? (20-shift) Check your answer using an online Caesarian shift cipher encoder. (The decoded message is the beginning of The Adventures of Tom Sawyer by Mark Twain.)
**Goals**

Students will determine sample space of compound events using tree diagrams.

**PRIOR KNOWLEDGE REQUIRED**

- Can describe events as likely, unlikely, very likely, very unlikely, equally likely, certain, or impossible
- Can identify the outcomes of spinning spinners, rolling dice, and drawing marbles

**Introduce tree diagrams.** Explain to students that mathematicians often use tree diagrams when they have to make choices and want to keep track of all the possible combinations. For example, Katie has 3 pairs of mittens and 2 hats. How many different outfits can she wear? Show students how Katie can build a tree diagram to keep track of her choices.

The first two branches lead to the first choice: white hat or blue hat. Once she’s chosen a hat, Katie can choose from 3 pairs of mittens: green, white, or blue. Each path along the diagram is a different outfit. For example, the highlighted path shows the combination of a white hat and blue mittens.

The number of endpoints, or “leaves,” in a tree diagram is the total number of possible combinations. In this case, Katie has a total of 6 different outfits.

Expand the tree diagram by adding a third choice: a scarf. Katie has 2 scarves, a white scarf and a green scarf. First she chooses the hat, then the mittens, and then the scarf. Add 2 new leaves to each endpoint in the current diagram to show Katie’s third choice. **ASK:** How many different combinations does Katie have? (12) How many of these combinations have 3 colours? (3) How many of the combinations have only 1 colour? (1) Two colours? (12 − 1 − 3 = 8) Let students count the different paths to find the answers.

Explain to students that tree diagrams make it easy to see all the different outcomes of an action or series of actions, such as putting together an outfit. When you know the total number of outcomes, you can quickly identify the number of suitable outcomes for a particular problem or question. **ASK:** Is it likely or unlikely that Katie will wear an outfit that has two colours? (likely) Is it likely or unlikely that Katie’s outfit will be all white? (unlikely)

Practise finding all possible outcomes of combined experiments using Questions 1–8 on Workbook pages 186–187.

**EXTRA PRACTICE:**

1. Draw a tree diagram to show the results of tossing a pair of regular dice. Is it likely or unlikely that one of the numbers is twice as large as the other?
2. A restaurant offers 3 main courses (chicken, fish, vegetarian) and 5 desserts (cake, ice cream, cookies, fruit, pie). Draw a tree diagram to show all the possible dining combinations.

**Combinations of experiments.** Explain that we can think of choosing an outfit or dish as performing one or more experiments. Experiments can be associated with different events and have many outcomes. **EXAMPLE:**

- experiment: roll 2 dice
- event: roll a total of 6
- outcomes: 3 3, 2 4, 4 2, 1 5, 5 1

Experiments can be performed separately or combined. For example, we can roll a die and spin a spinner at the same time; one outcome for this combination of experiments is rolling 3 and spinning red. In the first extra practice question, above, rolling each die is a separate experiment (even though we may have rolled the dice at the same time). Choosing each article of clothing when you choose an outfit is also a separate experiment, and we can use tree diagrams to find all the outcomes for the combination of experiments. The choices at each level are the outcomes of each individual experiment.

**The total number of outcomes.** Ask students to complete a chart like the one in Question 9 on Workbook page 187 for all the tree diagrams they made during the lesson. **ASK:** How can you find the total number of outcomes for a combination of experiments (the total number of paths in the diagram) from the number of outcomes for each individual experiment, which is the number of branches at each level? (multiply them) Why do you need to multiply the number of branches at each level? (Each choice at one level has the same number of choices at the next level.)

**EXAMPLES:**

1. Rolling 2 dice with 12 faces each. \(12 \times 12 = 144\) outcomes
2. Pulling a card from a deck of 52 cards and spinning a 3-part spinner. \((52 \times 3 = 156\) outcomes)
3. Rolling 3 regular dice. \((6 \times 6 \times 6 = 216\) outcomes)

**Bonus** Tossing 10 coins \((2 \times 2 \times \ldots \times 2\) (ten times) \(= 1024\) outcomes)

**Extension**

The colour of a flower is determined by its genes. A red rose, for example, has two R genes, a white rose has two r genes, and a pink rose has one R gene and one r gene. If two pink roses crossbreed, the “child rose” can be either red, white, or pink. The possible results are:
a) When two pink roses crossbreed, what is the probability the resulting rose will be pink?

b) When a pink rose crossbreeds with a red rose, what is the probability that the resulting rose will be red? Pink? White?

c) When a pink rose crossbreeds with a white rose, what is the probability that the resulting rose will be pink? Red? White?

d) What happens when a red rose crossbreeds with a white rose?
PDM7-20  Counting Combinations
Pages 188–189

CURRICULUM EXPECTATIONS
Ontario: 7m2, 7m85
WNCP: 7SP5, [C, CN, R, V, T]

VOCABULARY
probability
outcome
event
tree diagram

Goals
Students will determine the sample space (all possible outcomes) of compound events using charts and organized lists.

PRIOR KNOWLEDGE REQUIRED
Can describe events as likely, unlikely, very likely, very unlikely, equally likely, certain, or impossible
Can identify outcomes of spinning spinners, rolling dice, drawing marbles

Introduce charts. Remind students that in the previous lesson they used tree diagrams to keep track of all possible outcomes of several experiments. Ask students to think about using a tree diagram to figure out all possible outcomes of two experiments with many outcomes, such as rolling a 20-sided die and a 6-sided die. How would using a tree diagram in this case be inconvenient? (it is hard to draw 20 branches, the tree diagram will take up a lot of space) Tell students that another way to keep track of the outcomes for two or more experiments is to use a chart.

Show students the spinners at left. Draw a two-column chart with the headings Colour Spinner and Number Spinner.

<table>
<thead>
<tr>
<th>Colour Spinner</th>
<th>Number Spinner</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>1</td>
</tr>
<tr>
<td>Red</td>
<td>2</td>
</tr>
<tr>
<td>Red</td>
<td>3</td>
</tr>
</tbody>
</table>

Spin the colour spinner and list its outcome in the appropriate column. SAY: I am going to spin the second spinner now. What are the possible outcomes? Invite volunteers to list all the possible outcomes of spinning the second spinner and to fill in the colour you spun on the colour spinner next to each one. If you spun “red” on the colour spinner, you chart will look like the chart in the margin.

SAY: I am going to spin the colour spinner again. Suppose I get a different colour. What are the possible outcomes of the number spinner? Are they different from the previous spin? (No, we have the same three outcomes: 1, 2, 3.) Continue filling in the table for different outcomes. How many outcomes are there in total? (12)

ASK: How many times did we write each outcome of the first spinner? (3) Why? (one for each possible outcome of the second spinner) Write on the board: There are:

- 3 outcomes for spinning red
- 3 outcomes for spinning green
- 3 outcomes for spinning blue
- 3 outcomes for spinning yellow
- 12 outcomes altogether

ASK: If the second spinner had 5 numbers, how many times would you write each outcome of the first spinner? (5)
**EXTRA PRACTICE:**
Draw a chart showing all the outcomes of spinning the two spinners at left. When students finish, **ASK:** How many outcomes did you find? (15)

Write on the board:

- 5 outcomes for spinning red
- 5 outcomes for spinning green
- + 5 outcomes for spinning blue
- 15 outcomes altogether

Ask students to count the number of outcomes for the pair of spinners at left, without drawing a chart:

- ___ outcomes for spinning red
- ___ outcomes for spinning green
- ___ outcomes for spinning blue
- + ___ outcomes altogether

Write on the board:

Number of outcomes = 5 + 5 + 5 + 5 = 4 × 5.

**ASK:** How did we know there are five outcomes for each colour? (there are 5 outcomes for the second spinner) How did we know to write the 5 four times? (we write it once for each outcome of the first spinner) **SUMMARIZE:**

\[
\text{# of outcomes} = (\text{# of outcomes of the 1st spinner}) \times (\text{# of outcomes of the 2nd spinner})
\]

**Practise finding the sample space.** Review the steps for systematically listing all of the different outcomes (the sample space) for two experiments, as on Workbook page 188. Have students create more charts to answer questions such as these:

a) Find all possible outcomes of rolling a regular die and spinning a spinner with three colours.

b) Find all possible outcomes of rolling two dice, one with eight sides marked 1 to 8, and another with four vertices marked 1 to 4.

c) Find all possible outcomes of flipping two coins.

Find all possible outcomes of Rock, Paper, Scissors for two players. (Remind students that “I win” is not an outcome, it is an event! It can happen in three different ways: I have rock, you have scissors; I have scissors, you have paper; I have paper, you have rock.)

Have students decide whether specific events related to the charts they drew are likely or unlikely. **EXAMPLES:** Rolling 3 in b) (unlikely, 2 outcomes out of 32), flipping at least one head in c) (likely, 3 out of 4), John winning when John and Bob play Rock, Paper, Scissors (unlikely, 3 out of 9)
Compound events. Show your students three cards, two red (R) and one black (B). Explain that you are planning two probability experiments. For Experiment A, you will draw a card, write down the colour, and put the card back in. Then you will draw another card and write down the colour. The result of your experiment is two colours. Discuss with students what the possible results of the experiment are. (RR, RB, BR, BB) Write the rules for Experiment A on the board and the possible results for it.

For Experiment B, you will draw a card, write down the colour, and leave it out of the deck. Then you will draw another card and write down the colour. The result of your experiment is two colours. Discuss with students what the possible results of the experiment are. (RR, RB, BR) Can you get two black cards? Why not? (there is only one black card, so if you take it out on the first draw, you only can get a red card on the second draw) Write the rules and results for Experiment B on the board as well.

Explain to students that these experiments are combinations of two separate experiments: two drawings of a card from a deck. Events that are combinations of other events are called compound events. Today students will learn to find the probability of compound events.

Independent events. Explain that the two events in a compound event are called independent events if the results of the experiments do not depend on each other. For example, if I roll a die two times, the number I roll the first time doesn’t affect the number I will roll the second time; the die rolls are independent. Are the draws of cards in Experiments A and B independent? Have students signal the answer for each experiment separately, then ask them to explain their thinking. (Yes for A and no for B; in B, the card you draw first affects the possible results for the card you draw second, but...
Present several compound events and have students identify the sub-events in each. Then ask students to decide whether the events within each compound event are independent or not. Students can again signal the answers and explain their thinking afterwards. **EXAMPLES:**

a) Rolling two dice. (independent)

b) Roll a regular die. If the number is even, roll it again. If it is odd, roll a 12-sided die. (dependent)

c) Draw two cards from a deck, one after the other, without replacing. (dependent)

d) Draw two cards from a deck, one after the other, replacing the card in the deck after the first draw. (independent)

**Bonus**

Use 3 regular dice of different colours (say red, white, and blue). Roll the red die and record the number. If you roll a number that is less than 4, roll the white die and record the number. If you roll a number that is 4 or more, roll the blue die and record the number. (Since you do not record the colour and the dice are identical, the results of the second roll do not depend on the results of the first roll. The events are independent.)

Perform Experiments A and B to see whether the probability of drawing two red cards is different. Divide students into two groups, one group to perform Experiment A and the other to perform Experiment B. Have students work in pairs: one partner mixes the cards, the other draws and records the answers. Partners switch roles after each pair of draws. Have students perform each experiment 30 times, tally the results, and graph them using, say, a circle graph. Then ask pairs to move into groups of four and combine their results, so that each student makes a graph reflecting 60 trials of the same experiment. Which fraction of the results is RR (red, red)? Compare the results of the two experiments. Did RR come out more with one group? What does this mean? (the probability of RR is larger in one of the experiments)

**Introduce experimental probability.** Explain that the experimental probability of an event A is a ratio, often expressed as a fraction:

\[
\text{Exp P(} \text{Event A)} = \frac{\text{Number of times A happened}}{\text{Number of experiments performed}}
\]

Have students find the Exp(RR) for the experiments they performed with a partner and for the combined results in their group of four. Finally, combine the numerical results of all students in the class for each of Experiment A and Experiment B and have students find the Exp(RR) for both.

**Find the theoretical probability of both experiments.** Ask students to draw tree diagrams to show the possible outcomes for each experiment (see margin). Ask students to circle the leaves for all the paths that produce two red cards. What is the theoretical probability of RR in each experiment? (4 : 9 for Experiment A, 1 : 3 for Experiment B)
Compare theoretical and experimental probabilities. Ask students to look at the three values for the experimental probability of RR and the theoretical probability of RR. Which experimental probability is closer to the theoretical probability? Check the answer for each group of four at least. Explain to students that in general, the more trials that are performed, the closer the experimental probability becomes to the theoretical probability.

Students can practise calculating the probabilities of compound events using tree diagrams or lists of outcomes or charts. Encourage students to write the answers using fractions, ratios, and percents.

1. Jennifer rolls two dice, one red and one blue. She uses the result of the red die as the denominator of a fraction and the result of the blue die as the numerator. List all possible combinations and find
   a) the probability of rolling a fraction already reduced to lowest terms.
   b) the probability of rolling a fraction greater than 1.
   c) the probability of rolling a fraction less than 1.

2. Samantha flips 3 coins: 2 dimes and 1 nickel. Use a tree diagram to find
   a) the probability of flipping 2 heads and 1 tail.
   b) the probability of flipping 2 tails and 1 head. Compare this answer to your answer in a).
   c) the probability of flipping tails on 2 coins with a total value of 15 cents.

PROCESS EXPECTATION

Making and investigating conjectures

**ACTIVITY**

Each student receives the game board below and a pair of dice. Play the game as follows: Eleven runners are entered in a race. Each runner gets a number from 2 to 12. Runners move one place forward every time their number is the total number rolled on a pair of dice. Which number wins the race? Have students predict the answer before they roll the dice 20 times and move the runners accordingly. Was their prediction correct?

<table>
<thead>
<tr>
<th>2</th>
<th>3</th>
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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
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</tbody>
</table>

After performing the experiment, have students make a chart to show all possible outcomes and to find the theoretical probabilities of each result. Discuss the results of the experiment as a class. What was the most common number to win in the class? (most likely 7) Was 7 always the winner? (most likely not) Explain why the actual winner wasn’t always the expected winner. (experimental probability is not the same as theoretical probability)
Extension

What is the probability of winning Lotto 6/49?

In the lottery game Lotto 6/49, balls numbered 1 to 49 are randomly mixed in a machine. Six balls are then chosen at random from the machine. Balls are not put back, so the same number cannot be chosen twice. Players pick their own numbers on lottery tickets, hoping to match those that come out of the machine. Even if more tickets are sold than the number of different combinations possible, it’s possible that no one wins because many people might have picked the same numbers. The probability of winning doesn’t depend on how many tickets are sold, but the amount you win does. This is the opposite of a raffle, where the probability of winning does depend on the number of tickets sold (the more tickets sold, the lower your probability of winning).

a) If you pick 6 different numbers from 1 to 49 in Lotto 6/49, how do you think a game called Lotto 2/7 would work? (pick 2 different numbers from 1 to 7)

b) The only possible combination in Lotto 2/2 is 1 2. Lotto 2/3 has 3 possible combinations: 1 2, 1 3, 2 3. Use an organized list to find the possible combinations for:
   - Lotto 2/4
   - Lotto 2/5
   - Lotto 2/6
   - Lotto 2/7

HINT: Start by listing all the combinations that start with 1, then the remaining combinations that start with 2, and so on.

c) Complete the following chart:

<table>
<thead>
<tr>
<th>$N$</th>
<th>Lotto</th>
<th>Number of combinations</th>
<th>Probability of winning</th>
<th>$N \times (N - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2/2</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2/3</td>
<td>3</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>2/4</td>
<td>6</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>2/5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>7</td>
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</tr>
</tbody>
</table>

d) By comparing the last two columns of the chart in c), write down a rule for the probability of winning the Lotto 2/$N$ in terms of $N$ (where $N$ is the numbers you have to choose from).

e) Find the probability of winning:
   - i) Lotto 2/10  
   - ii) Lotto 2/49

f) Mathematicians proved that the chances of winning Lottos 2/49, 3/49, and 4/49 are as shown at left. Extend the pattern to predict the chances of winning Lotto 6/49.

g) Check www.lotterycanada.com to find the odds of winning the Lotto 6/49. Does your prediction match what is published on the website? If not, check your calculation.
Students can now try to answer one of the following questions and write a concluding report describing their position.

i) Check the website of the National Lightning Safety Institute for information on how many people are struck by lightning annually in the US.

Find the population of the US. What is the probability of being struck by lightning in the US?

What is larger: the probability of winning Lotto 6/49 or the probability of being struck by lightning? About how many times larger? Does the answer surprise you? Why?

Do you see or hear more stories about people struck by lightning or about people who have won Lotto 6/49? Why do you think that happens?

ii) What does it cost to play one combination of numbers in Lotto 6/49? How much money do you need to buy all possible combinations so as to ensure that you win?

Assume it takes 5 seconds to mark each combination on a lottery ticket. How much time will it take you to mark all the possible combinations? Assuming you spend 12 hours a day marking lottery tickets, how many days will it take? Will you be done in 3 days? In a week?

Assume you have enough time and money to buy all the possible combinations, guaranteeing that you will win. Other people can buy tickets too, and some of them might win as well. If the jackpot is $30 000 000, would you buy all the possible combinations? Why or why not?

iii) Bill buys a lottery ticket every week for 25 years. How many tickets does he buy in total?

Bill’s chances of winning the big jackpot at least once in 25 years are about 1 : 10 000. Assuming the price of a lottery ticket does not change, how much money does he spend on lottery tickets in 25 years?

Joanna saves $2 per week for 25 years. She can invest the money different ways. Use the Internet to find the interest rate for any three ways Joanna could invest her money. Use an online compound interest calculator to see how much money she will have in total after 25 years, depending on the way she invested her money.

Would you rather try to win a lottery or invest your money?
Probability as the average result. Show your students the spinner at left and ask them which fraction of it is coloured in each colour. What are the possible outcomes of spinning this spinner? How many outcomes are there? (3) What is the probability of spinning each of the colours? Are the chances of spinning blue the same as the chances of spinning red? Why? What about the chances of spinning green and the chances of spinning blue?

SAY: I am going to spin the spinner 12 times. How many times do you expect me to spin blue? Why? Write the calculation on the board:

\[
\frac{1}{3} \text{ of } 12 = 4 \text{ (12 ÷ 3 = 4 times)}
\]

Remind students that actual outcomes usually differ from expected outcomes. You might not get blue 4 times every time you make 12 spins, but 4 is the most likely number of times you will spin blue. Remind students that when we say “the chances of spinning blue are 1 out of 3,” we mean that on average 1 out of every 3 spins is blue. What does this mean? Suppose 4 people each spin this spinner 3 times. They write the number of times they got blue. We expect the mean of this set of data to be 1.

Have students each spin the spinner 3 times and write down the number of times they spun blue. Then put students into groups of four and ask them to find the mean of the number of times they spun blue. (A group of four will have, say, these results: 1, 1, 1, 0. The mean is 0.75, which means the group spun blue, on average, less than 1 out of 3 times.) Is the experimental mean the same as the expected mean? Combine the results of the whole class. Is the class’s mean closer to 1? Why?

Expected outcome of \( n \) experiments. Review with students how to find a fraction of a set and a fraction of a number, as in NS7-22. Show the spinner at left and ASK: Which fraction of this spinner is blue? What are the chances of spinning blue? (3 out of 4, 3/4, 75%) What is the probability of spinning
blue on this spinner? If I spin the spinner 20 times, how many times should I expect to get blue? (3/4 of 20, 15 times)

Have students practise calculating expected outcomes with these and similar questions:

a) If you flip a coin 16 times, how many times do you expect to get a tail?

b) Hong wants to know how many times he is likely to spin green if he spins the spinner with three equal parts above 24 times. He knows that 1/3 of 24 is 8 (24 ÷ 3 = 8). How can he use this information to find how many times he is likely to spin green?

c) If you roll a regular die 18 times, how many times do you expect to get a 4? To get a 1?

d) How many times would you expect to spin blue if you spin the spinner above 50 times? How many times would you expect to spin green?

e) If you roll a die 30 times, how many times do you expect to roll an even number? How many times do you expect to roll either 4 or 6?

f) You flip two coins, a nickel and a dime, 12 times. List the possible outcomes. How many times do you expect to get a head and a tail?

g) Jack and Jill play Rock, Paper, Scissors 18 times. List all possible outcomes of the game. (HINT: “Jack has rock, Jill has paper” is different from “Jack has paper, Jill has rock”! Why?) How many times do you expect to see a draw? What is the probability of a draw?

**Using probability to predict population.** Explain that probability concepts are often used in science. Present the following situation and work through the questions with the class.

Scientists want to estimate the number of fish in a lake. They catch 100 fish, tag them, and release them back into the lake. A week later, they catch 200 fish. Forty of them are tagged. The scientists assume that the percentage of tagged fish in the second catch is the same as the percentage of tagged fish in the total population of fish (all the fish in the lake).

a) Look at the second catch. What is the probability that a fish chosen at random is tagged? (40 out of 200, or 20%)

b) We know there are 100 tagged fish in the lake. What fraction of the total population do those 100 fish form? (1/5)

c) Estimate the number of fish in the lake. (500)

d) Why would scientists be interested in counting the number of fish in a lake?

**EXTRA PRACTICE:**
A government committee is choosing a program to save endangered frogs. There are two programs with similar costs that can be put into action, in two different areas. Program A can work only in Area A, and Program B will...
work only in Area B. There is funding for only one program. Without any
intervention, 90% of the frogs in each area will perish. The committee is
interested in saving as many frogs as possible. Which program should
be implemented?

a) In Area A, 70% of 600 endangered frogs can be saved by the proposed
program. In area B, 2/7 of 1700 endangered frogs can be saved.

b) In Area A, 70% of 600 endangered frogs can be saved by the proposed
program. In area B, 2/7 of 2000 endangered frogs can be saved.

**ANSWERS:**

a) If Program A is implemented, about \(420 + 170 = 590\) frogs will survive
(420 is the 70% of 600 frogs that will be saved in Area A, and 170 is
the 10% that will survive with no intervention in Area B). If Program B is
implemented, about \(60 + 485 = 545\) frogs will survive. Program A gives
better results.

b) Program B gives better results.

**Extensions**

1. You have 50 coins with a total value of $1.00. If you lose one coin what
is the chance it is a quarter?

2. Sally asks her classmates when their birthdays are. There are 366
possible outcomes.

   a) Is the outcome August 31 equally probable to, more probable than,
   or less probable than the outcome February 29? Explain
   your answer.

   b) February 29 is one outcome out of 366. Is the probability of being
   born on February 29 equal to \(1/366\)? Why or why not?

   c) Find the probability of being born on February 29. **HINT:** How
   many days are there in any four consecutive years? How many
   of those days are February 29?

   d) Assume that there are 6 000 000 people in the Greater Toronto
   Area (GTA). How many people in the GTA would you expect to
   have a birthday on February 29?

3. Look at the following question.

   In what year did women in Canada gain the right to vote?

   a) 1915   b) 1916   c) 1917   d) 1918   e) 1919

   a) If you guess randomly, what is the probability of answering
correctly?

   b) On a test of 30 similar questions, how many questions would you
   expect to guess correctly? Incorrectly? Assume you answer each
   question blindly.
c) You get 4 points for each correct answer and −1 for each incorrect answer. What do you expect your final score to be?

4. Tom buys a lottery ticket twice every week for 25 years. How many tickets does he buy in total? Assuming the price of lottery tickets does not change, how much money does he spend on lottery tickets in 25 years?

Check the website http://www.lotterycanada.com to see the odds of winning each type of prize. About how many times during these years could Tom win each prize?

For each prize, use the payout information from the last week to see how much money Tom could win during all these years. Did he get all the money he spent back in prizes?

Write an opinion piece on whether it is worth Tom’s money to buy lottery tickets.

Partial answers: In 25 years, Tom bought 2600 tickets at a total cost of $5200.

<table>
<thead>
<tr>
<th>Numbers matched</th>
<th>Probability of winning</th>
<th># of expected wins</th>
<th>Sample prize size</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 out of 6</td>
<td>1 : 13 983 816</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5 out of 6</td>
<td>1 : 2 330 636</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5 out of 6 + Bonus</td>
<td>1 : 55 492</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4 out of 6</td>
<td>1 : 1 033</td>
<td>3</td>
<td>$75.40</td>
</tr>
<tr>
<td>3 out of 6</td>
<td>1 : 57</td>
<td>46</td>
<td>$10</td>
</tr>
<tr>
<td>2 out of 6 + Bonus</td>
<td>1 : 81</td>
<td>32</td>
<td>$5</td>
</tr>
</tbody>
</table>

**Total Tom’s winnings** $846.2

**NOTE:** Many people understand that their chances of winning the big jackpot in a lottery are very low, but they still play in the hopes of winning “something.” The calculations above show that this “something” is far less than what people may spend to buy tickets.
Number Sense – AP Book 7, Part 2: Unit 1

AP Book NS7-55

page 1

1. A. 0.4
B. 0.004
C. 0.44
D. 0.04
E. 0.044

2. a) \( \frac{5}{10} = 0.5 \)
b) \( \frac{1 \times 2}{5 \times 2} = \frac{2}{10} = 0.2 \)
c) \( \frac{1 \times 25}{4 \times 25} = \frac{25}{100} = 0.25 \)
d) \( \frac{3 \times 25}{4 \times 25} = \frac{75}{100} = 0.75 \)
e) \( \frac{6 \times 4}{25 \times 4} = \frac{24}{100} = 0.24 \)
f) \( \frac{17 \times 2}{50 \times 2} = \frac{34}{100} = 0.34 \)

3. Teacher to check.

4. a) \( \frac{35}{100} - \frac{7}{20} \)
b) \( \frac{25}{100} - \frac{1}{4} \)
c) \( \frac{6}{10} - \frac{3}{5} \)
d) \( \frac{40}{100} - \frac{2}{5} \)
e) \( \frac{48}{100} - \frac{12}{25} \)

5. a) i) Numerators start at 1 and increase by 1 each time.
Denominators start at 5 and increase by 5 each time.

ii) Numerators start at 3 and increase by 3 each time.
Denominators start at 4 and increase by 4 each time.

b) i) 0.2
ii) 0.75

6. a) Circle:
\[
\begin{array}{c|c|c|c|c}
2 & 12 & 8 & 10 & \text{15} \\
15 & 125 & 100 & \text{1000} & \text{100} \end{array}
\]
b) All the fractions circled in part a) can be written in lower terms because the common factors in the numerator and denominator will cancel out.

5. b) \( \frac{1}{5} = 0.2 \)
\( \frac{21}{5} = 21 \times 0.2 = 4.2 \)
\( \frac{1}{2} = 0.5 \)
\( \frac{13}{2} = 13 \times 0.5 = 6.5 \)

6. b) \( 0.5 + 0.2 = 0.7 = \frac{7}{10} \)
\( 0.5 + 0.8 = 1.3 = \frac{13}{10} \)
\( 1.5 + 0.75 = 2.25 \)
\( = \frac{225}{100} = \frac{9}{4} \)

AP Book NS7-56

page 2

1. a) 3
b) \( \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3 \times \frac{1}{4} \)
c) \( \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{4}{5} \)

2. a) \( \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{4}{5} \)
\( = 0.2 + 0.2 + 0.2 + 0.2 = 0.8 \)
\( \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 0.25 + 0.25 + 0.25 = 0.75 \)
\( \frac{1}{2} + \frac{1}{2} \)
\( = 0.5 + 0.5 + 0.5 = 1.5 \)

3. b) \( 3 \times \frac{1}{4} = 3 \times 0.25 = 0.75 \)
\( 3 \times \frac{1}{5} = 3 \times 0.2 = 0.6 \)
\( 5 \times \frac{1}{4} = 5 \times 0.25 = 1.25 \)

4. a) Start at 0.05 and add 0.05 each time.
b) 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55
\( \ldots \frac{11}{20} = 0.55 \)
c) \( \frac{11}{20} = 11 \times 0.05 = 0.55 \)

AP Book NS7-57

page 3

1. a) 4
b) 1 + 3
c) 1 + 6

2. a) Since 24 = 3 \times 8,
\( 24 + 2 = (3 \times 8) + 2 = 3 \times (8 + 2) \)
\( b) \) Since 3 = 3 \times 1,
\( 3 + 8 = (3 \times 1) + 8 = 3 \times (1 + 8) \)
\( c) \) 3 + 8 = 3 \times \frac{8}{1} = 3 \times \frac{1}{8} \)
\( d) \) 3 \times \times \frac{3}{8} \times \frac{3}{8} \)

3. a) \( \frac{1}{5} = 0.2 \)
b) \( \frac{2}{5} = 2 + 0.5 = 0.4 \)
c) \( \frac{3}{5} = 3 + 6 = 0.6 \)

4. a) \( \frac{4}{10} = 4 + 10 = 0.4 \)
b) \( \frac{7}{2} = 7 + 2 = 3.5 \)
c) \( \frac{9}{4} = 9 + 4 = 2.25 \)

5. Check not shown.

6. a) \( \frac{125}{1000} = 0.125 \)
b) \( \frac{0.25}{10} = 0.025 \)
c) \( \frac{3}{8} = 0.375 \)
\( \frac{7}{8} = 0.875 \)

AP Book NS7-58

page 4

1. a) 0.33333333
b) 0.03333333

c) 0.00033333

d) 0.52525252

2. Circle:
0.222222222..., 0.512512512...

3. a) 0.5
b) 2.34

c) 5.237

d) 57.77

e) 8.162

f) 0.9705

4. a) \( \approx 0.333; = 0.3 \)
b) \( \approx 0.666; = 0.6 \)
Number Sense – AP Book 7, Part 2: Unit 1 (continued)

5. a) \(0.166666; 0.1\overline{6}\) 
   b) \(0.444444; 0.4\overline{4}\) 
   c) \(0.090909; 0.0\overline{9}\) 
   d) \(0.416666; 0.4\overline{16}\)
6. A. \(0.3\) 
   B. \(0.5\) 
   C. \(0.6\) 
   D. \(0.7\)
7. \[
\begin{array}{cccc}
\frac{2}{7} & 0.3 & 0.29 & 0.286 \\
\frac{5}{13} & 0.4 & 0.38 & 0.385 \\
\end{array}
\]
8. a) < 
   b) > 
   c) > 
9. a) \(0.4, 0.42, 0.42\overline{2}\) 
   b) \(0.3, 0.16, 0.1\overline{6}, 0.1\overline{6}\) 
   c) \(0.387, 0.3\overline{87}, 0.387, 0.3\overline{87}\) 
   d) \(0.546, 0.5\overline{46}, 0.5\overline{46}, 0.5\overline{46}\) 
   e) \(0.383, 0.3\overline{83}, 0.3\overline{83}, 0.3\overline{83}\) 
   f) \(0.786, 0.7\overline{86}, 0.7\overline{86}, 0.7\overline{86}\)
10. a) \(\frac{1}{9} = 0.111\ldots = 0.\overline{1}\) 
   b) \(\frac{2}{9} = 0.222\ldots = 0.\overline{2}\) 
   c) \(\frac{3}{9} = 0.333\ldots = 0.\overline{3}\) 
   d) \(\frac{4}{9} = 0.444\ldots = 0.\overline{4}\) 
   e) \(\frac{5}{9} = 0.\overline{5}\) 
   f) \(\frac{6}{9} = 0.\overline{6}\) 
   g) \(\frac{7}{9} = 0.7, 0.\overline{7}\) 
   h) \(\frac{8}{9} = 0.\overline{8}\) 
   i) \(\frac{9}{9} = 0.9\) or 1

AP Book NS7-59 page 6
1. a) 0.2 
   b) 0.75; closest to 0.7 
   c) 0.2; closest to 0.25 
   d) 0.4; closest to 0.42

2. a) \(0.800, 0.700, 0.667; \) 
   : closest to \(\frac{2}{3}\) 
   b) \(0.143, 0.125, 0.111; \) 
   : closest to \(\frac{1}{8}\) 
   c) \(3.500, 3.333, 2.667; \) 
   : closest to \(\frac{10}{3}\)

3. Using long division, we can see that: 
   \[
\frac{5}{6} = 0.833; \frac{13}{17} = 0.764; \\
\frac{56}{73} = 0.767; \frac{4}{5} = 0.8; \\
\frac{5}{5} = 1.0 \\
\]
   b) \(\frac{5}{36}; \frac{3}{8}; \frac{89}{121}; \frac{9}{11}\)

4. a) i) \(57 \div 60 < \frac{1}{10}\) 
   : \(0.95 < \frac{3}{5}\) 
   ii) \(80 \div 83 < \frac{1}{10}\) 
   : \(0.96 < \frac{4}{5}\) 
   iii) \(111 \div 200 < \frac{1}{10}\) 
   : \(0.557 < \frac{2}{3}\) 
   b) i) \(0.57 < \frac{3}{5}\) 
   ii) \(0.8 < \frac{2}{3}\) 
   iii) \(0.37 < \frac{2}{3}\) 
   c) Teacher to check.
5. a) \(\frac{23}{45}\) 
   b) \(0.28\overline{6}\)
6. Answers may vary. 
   Sample solutions:
   a) \(\frac{1}{2}\) 
   b) \(\frac{1}{3}\) 
   c) \(\frac{3}{10}\) 
   d) \(\frac{2}{5}\) 
   e) \(\frac{3}{5}\) 
   f) \(\frac{1}{10}\)

7. a) \(\frac{8}{13} = 0.615384\) 
   b) \(\frac{9}{11} = 0.81\) 
   c) \(\frac{5}{3} = 0.16\) 
   d) \(\frac{3}{17} = 0.17647\) 
   e) \(\frac{89}{121} = 0.72641\) 
   f) \(\frac{3}{8} = 0.375\)

AP Book NS7-60 page 7

INVESTIGATION
A. Answers will vary but the decimal expansions will all terminate.
B. They can always be written as some integer divided by some power of 10, i.e., as a decimal fraction.
C. a) terminates: 
   \[
\frac{5}{8} = 0.625 = \frac{625}{1000} \\
\] 
   b) repeats 
   c) repeats 
   d) repeats 
   e) repeats 
   f) terminates: 
   \[
\frac{13}{2000} = 0.0065 \\
\] 
   100 = 2 \times 5 \times 5 
   1000 = 2 \times 5 \times 5 \times 2 
   E. Answers will vary but this type of fraction will always terminate.
F. \(\frac{1}{6}\) and \(\frac{5}{6}\) are already in simplest form; 
   \[
\frac{2}{6} = \frac{1}{3} = \frac{1}{3} \times \frac{2}{2} = \frac{2}{3} \\
\] 
   Only \(\frac{3}{8}\) terminates since it's the only one that reduces to a fraction whose denominator is a multiple of 2s &/or 5s.
b) i) \( \frac{1}{9} \div \frac{2}{9} = \frac{3}{9} = 0.3 \)
  
ii) \( \frac{3}{5} + \frac{4}{7} = \frac{7}{11} \approx 0.7 \)
  
iii) \( \frac{1}{4} + \frac{4}{8} = \frac{8}{9} \approx 0.8 \)
  
4. a) i) \( 0.25 \)
   
ii) \( 0.33 \)
   
iii) \( 0.5 \)
  
4. a) i) \( 0.25 \)
   
ii) \( 0.33 \)
   
iii) \( 0.5 \)
   
4. a) i) \( \frac{1}{4} + \frac{1}{3} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12} \approx 0.583 \)
  
ii) \( \frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{1}{12} = 0.083 \)
  
iv) \( \frac{1}{2} + \frac{3}{8} = \frac{4}{8} + \frac{3}{8} = \frac{7}{8} \approx 0.875 \)
  
5. a) \( \frac{1}{2} + \frac{1}{2} = \frac{3}{2} = 1.5 \)

6. a) \( \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \approx 0.667 \)

AP Book NS7-62

1. a) \( \frac{1}{11} = 0.09, \frac{2}{11} = 0.18 \)
   
\( \frac{3}{11} = 0.27, \frac{4}{11} = 0.36 \)
  
\( \frac{5}{11} = 0.45, \frac{6}{11} = 0.55 \)
  
\( \frac{7}{11} = 0.63, \frac{8}{11} = 0.73 \)
  
\( \frac{9}{11} = 0.81, \frac{10}{11} = 0.91 \)
  
\( \frac{11}{11} = 1 \)

b) \( \frac{0.09 \times 5}{11} \times 5 = \frac{5}{11} \approx 0.45 \)

AP Book NS7-63

2. a) Teacher to check.
  
b) \( 0.17, 0.1717, 0.171717, 0.17 \)
  
c) \( 17 \times 0.1 = 17 \times 0.11 = 1 \)
  
d) \( 0.45, 0.4545, 0.454545, 0.45 \)
  
e) \( 0.09 \times 5 = \frac{5}{11} \)

3. a) \( 0.25 \)
  
b) \( 0.38 \)
  
c) \( 0.97 \)
  
d) \( 0.86 \)
  
e) \( 0.07 \)
  
f) \( 0.04 \)

4. a) \( 0.33 \approx 0.33 \)
  
b) \( 0.8 \)
  
c) \( 0.8 \)
  
d) \( 0.5 \)
  
e) \( 0.5 \)
  
f) \( 0.44 \)

5. a) \( 5 \times 0.25 = 1.25 \)
  
b) \( 5 \times 0.38 = 1.9 \)
  
c) \( 5 \times 0.97 = 4.85 \)
  
d) \( 5 \times 0.86 = 4.3 \)
  
e) \( 5 \times 0.07 = 0.35 \)
  
f) \( 5 \times 0.04 = 0.2 \)

6. a) \( 1 \)
  
b) \( 2 \)
  
c) \( 3 \)
  
d) \( 4 \)
  
e) \( 5 \)
  
f) \( 6 \)

1. a) \( 7 \)
  
b) \( 23 \)
  
c) \( 5 \)
  
d) \( 441 \)
  
e) \( 652 \)
  
f) \( 98 \)
  
g) \( 5 \)
  
h) \( 461 \)
  
i) \( 38 \)
  
j) \( 61 \)

2. a) \( 254,4444444 \)
  
b) \( 266,6666666 \)
  
c) \( 2,491919191 \)
  
d) \( 32,32 \)
  
e) \( 0.0032 \)
  
f) \( 5436,136 \)
  
g) \( 0.0341 \)
  
h) \( 0.007432 \)
  
i) \( 364,324 \)
  
j) \( 2,4337 \)

Answer Keys for AP Book 7.2
Number Sense – AP Book 7, Part 2: Unit 1 (continued)

AP Book NS7-64

page 11

1. a) 30 : 100, 30%
   b) 47 : 100, 47%
2. a) 20%
   b) 63%
   c) 5%
   d) 55%
3. a) 30 : 100
   b) 12 : 100
   c) 25 : 100
   d) 34 : 100
4. a) \( \frac{50}{100} = 50\% \)
   b) \( \frac{10}{100} = 10\% \)
5. a) 40%
   b) 28%
   c) 43%
   d) 5%
   e) 10%
6. a) \( \frac{11}{100} = 11\% \)
   b) \( \frac{89}{100} = 89\% \)
   c) \( \frac{9}{100} = 9\% \)
   d) \( \frac{75}{100} = 75\% \)
   e) \( \frac{100}{100} = 100\% \)
7. Teacher to check drawing.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Fraction} & \frac{63}{100} & \frac{45}{100} & \frac{4}{100} \\
\hline
\text{Percent} & 63\% & 45\% & 4\% \\
\hline
\end{array}
\]

AP Book NS7-65

page 12

1. a) Red: 10%
   Blue: 40%,
   \( 10 + 40 = 50 \)
   \( \therefore 50\% \) of the grid is coloured.
2. a) \( \frac{30}{100} + \frac{20}{100} + \frac{50}{100} = 50\% \)
   b) \( \frac{10}{100} + \frac{50}{100} + \frac{60}{100} = 60\% \)
3. a) \( \frac{50 - 25}{100} = \frac{25}{100} = 25\% \)
   b) \( \frac{70 - 30}{100} = \frac{40}{100} = 40\% \)
4. a) \( \frac{20}{100} = 20\% \)
   b) \( \frac{3}{10} = \frac{30}{100} = 30\% \)
   c) \( \frac{7}{10} = \frac{70}{100} = 70\% \)
   d) \( \frac{23}{100} = 23\% \)
   e) \( \frac{57}{100} = 57\% \)
   f) \( \frac{8}{100} = 8\% \)
5. a) \( \frac{10}{100} = 1\% \)
   b) \( \frac{25 \text{ cents}}{100} = 25\% \)
   c) \( \frac{52}{100} = 52\% \)
   d) \( \frac{22}{100} = 22\% \)
   e) \( \frac{60}{100} = 60\% \)
   f) \( \frac{100}{100} = 100\% \)
   g) \( \frac{100}{100} = 100\% \)
   h) \( \frac{0}{100} = 0\% \)
   i) \( \frac{70}{100} = 70\% \)
6. a) \( \frac{50}{100} = 50\% \)
   b) \( \frac{10}{100} = 10\% \)
   c) \( \frac{150}{100} = 150\% \)
   d) \( \frac{225}{100} = 225\% \)
   e) \( \frac{100}{100} = 100\% \)
   f) \( \frac{0}{100} = 0\% \)
   g) \( \frac{1}{100} = 1\% \)
   h) \( \frac{10}{100} = 10\% \)
   i) \( \frac{1}{100} = 1\% \)
   j) \( \frac{2}{100} = 2\% \)
7. Teacher to check drawing.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Fraction} & \frac{63}{100} & \frac{45}{100} & \frac{4}{100} \\
\hline
\text{Percent} & 63\% & 45\% & 4\% \\
\hline
\end{array}
\]

AP Book NS7-66

page 13

1. a) 
   b) 
   c) 
   d) 
2. 70% white, 30% grey
3. a) Students should shade 8 dots.
   b) 20%
   c) 25% or 25 marbles
5. a) \( \frac{90}{100} = 0.90 \)
   b) \( \frac{35}{100} = 0.35 \)
   c) \( \frac{22}{100} = 0.22 \)
   d) \( \frac{6}{100} = 0.06 \)
   e) \( \frac{52}{100} = 0.52 \)
   f) \( \frac{2}{100} = 0.02 \)
   g) \( \frac{60}{100} = 0.60 \)
   h) \( \frac{100}{100} = 1.00 \)
6. a) 0.25
   b) 0.75
   c) 0.13
   d) 0.40
   e) 0.07
7. a) 20%
   b) 63%
   c) 5%
   d) 55%
8. a) 40%
   b) 60%
   c) 30%
   d) 10%
   e) 80%
   f) 72%
   g) 20%
   h) 45%
   i) 6%
   j) 88%
9. a) 38%
   b) 93%
   c) 38%
   d) 10%
10. \( \frac{6}{10} = 60\% \) jazz, \( 40\% \) rock

AP Book NS7-67

page 14

1. a) \( \frac{80}{100} = 80\% \)
   b) \( \frac{15}{100} = 15\% \)
   c) \( \frac{32}{100} = 32\% \)
2. 40%
3. a) \( \frac{1}{5} = \frac{20}{100} = 20\% \)
   b) \( \frac{1}{2} = \frac{50}{100} = 50\% \)
   c) \( \frac{1}{4} = \frac{25}{100} = 25\% \)
   d) \( \frac{75}{100} = 75\% \)
4. a) 42%
   b) \( \frac{1}{3} = \frac{0.33}{100} = 0.33 \)
   c) \( \frac{2}{3} = \frac{0.67}{100} = 0.67 \)
   d) \( \frac{2}{9} = \frac{0.22}{100} = 0.22 \)
e) $\frac{5}{6} = 0.83 = 0.83 = 83%$

f) $\frac{1}{7} = \frac{0.142857}{0.14} = 14%$

**AP Book NS7-68**

**page 15**

1. a) 50%
   
   b) 25%
   
   c) 20%
   
   d) 50%
   
   e) 75%
   
   f) 50%
   
   g) 60%
   
2. Teacher to check.

3. $\frac{4}{20} = \frac{20}{100} = 0.20 = 20%$

4. 0%, 40%, 50%, 70%, 100%

5. Teacher to check.

6. Teacher to check.

7. $\frac{20}{50} = 0.40$

8. $\frac{4}{20} = 20%, 80%$

9. Teacher to check.

**AP Book NS7-69**

**page 16**

1. | Fraction | Decimal | Percentage |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{4}$</td>
<td>0.25</td>
<td>25%</td>
</tr>
<tr>
<td>$\frac{7}{20}$</td>
<td>0.35</td>
<td>35%</td>
</tr>
<tr>
<td>$\frac{3}{20}$</td>
<td>0.15</td>
<td>15%</td>
</tr>
<tr>
<td>$\frac{2}{5}$</td>
<td>0.40</td>
<td>40%</td>
</tr>
<tr>
<td>$\frac{3}{5}$</td>
<td>0.60</td>
<td>60%</td>
</tr>
<tr>
<td>$\frac{6}{15}$</td>
<td>0.40</td>
<td>40%</td>
</tr>
<tr>
<td>$\frac{23}{25}$</td>
<td>0.92</td>
<td>92%</td>
</tr>
<tr>
<td>$\frac{3}{4}$</td>
<td>0.75</td>
<td>75%</td>
</tr>
<tr>
<td>$\frac{11}{20}$</td>
<td>0.55</td>
<td>55%</td>
</tr>
</tbody>
</table>

2. b) $\frac{50}{100} < \frac{53}{100}$
   
   : $\frac{1}{2} < 53%$

   c) $\frac{25}{100} > \frac{23}{100}$
   
   : $\frac{1}{4} > 23%$

   d) $\frac{75}{100} > \frac{70}{100}$
   
   : $\frac{3}{4} > 70%$

   e) $\frac{40}{100} > \frac{32}{100}$
   
   : $\frac{2}{5} > 32%$

   f) $\frac{27}{100} < \frac{62}{100}$
   
   : $0.27 < 62%$

   g) $\frac{2}{10} < \frac{11}{100}$
   
   : $0.02 < 11%$

   h) $\frac{1}{10} = \frac{10}{100}$
   
   : $\frac{1}{10} = 10%$

   i) $\frac{76}{100} < \frac{93}{100}$
   
   : $\frac{19}{25} < 93%$

   j) $\frac{23}{50} = \frac{46}{100}$
   
   : $\frac{23}{50} = 46%$

   k) $\frac{90}{100} > \frac{10}{100}$
   
   : $0.9 > 10%$

   l) $\frac{55}{100} > \frac{19}{100}$
   
   : $\frac{11}{25} > 19%$

3. a) $\frac{3}{5} = 0.6, 42% = 0.42$
   
   : $0.42, 0.6, 0.73$

   b) $\frac{1}{2} = 0.5, 80% = 0.8$
   
   : $0.5, 0.73, 0.8$

   c) $\frac{1}{4} = 0.25, 15% = 0.15$
   
   : $0.09, 0.15, 0.25$

4. a) Drama
   
   b) Pepperoni

**AP Book NS7-70**

**page 17**

1. a) 0.7
   
   b) 1
   
   c) 3.5
   
   d) 21
   
   e) 0.64
   
   f) 5.06

2. a) 0.1
   
   b) 0.39
   
   c) 0.405
   
   d) 0.674
   
   e) 0.009
   
   f) 6.008

3. a) 1.5
   
   3 \times 1.5 = 4.5
   
   b) 10% of 24 = 2.4
   
   5 \times 2.4 = 12
   
   c) 10% of 7.8 = 0.78
   
   2 \times 0.78 = 1.56
   
   d) 10% of 75 = 7.5
   
   4 \times 7.5 = 30
   
   e) 10% of 86 = 8.6
   
   9 \times 8.6 = 77.4
   
   f) 10% of 0.5 = 0.05
   
   8 \times 0.05 = 0.4

4. 5% 10% 20% 30% 40% 50% 60% 70% 80% 90% 100%

   a) 3
   
   b) 0.01 \times 2000 = 20
   
   c) 0.01 \times 15 = 0.15
   
   d) 0.01 \times 60 = 0.6

5. a) 4
   
   b) 6
   
   c) 24

6. a) 32
   
   b) 1
   
   c) 6.6
   
   d) 0.08
   
   e) 3.15

**AP Book NS7-71**

**page 19**

1. a) $\frac{2}{4} \times \frac{4}{4} = \frac{2}{5}$
   
   b) $\frac{1}{9} \times \frac{2}{18} = \frac{1}{8}$

2. a) 5.29
   
   b) 3.9
   
   c) 3.9
   
   d) 37.12
   
   e) 37.12
3. a) $\frac{15}{20} = \frac{3}{4}$
   
   $\therefore 40 \times 35 = 1400$
   
   $1400 + 100 = 14$
   
   $\therefore 35\%$ of 40 is 14

   b) 18 : 20 = 9 : 10
   
   $\frac{18}{20} = \frac{9}{10}$

   c) $? : 16 = 3 : 4$
   
   $\frac{2}{16} = \frac{3}{4}$

   d) 20 : 30 = 2 : 3
   
   $\frac{20}{30} = \frac{2}{3}$

   e) $\frac{3}{8} = \frac{9}{24}$

   f) $\frac{2}{3} = \frac{12}{18}$

   g) $\frac{13}{20} = \frac{65}{100}$

   h) $\frac{5}{9} = \frac{40}{72}$

   4. a) $\frac{15}{20} = \frac{3}{4}$
   
   $15 \div 100 = 14$
   
   $\therefore 35\%$ of 40 is 14

   b) Yes

   c) $\frac{1}{2}$ : half of 40

   Teacher to check explanation.

   4. The answer is 15 for both since $(30 \times 50) \div 100 = (50 \times 30) \div 100$.

   AP Book NS7-72

   page 20

   1. b) $\frac{5}{10}$ are shaded
   
   $\frac{1}{2}$ are shaded
   
   5 is $\frac{1}{2}$ of 10
   
   $5 : 10 = 1 : 2$

   c) $\frac{4}{12}$ are shaded
   
   $\frac{1}{3}$ are shaded
   
   4 is $\frac{1}{3}$ of 12
   
   $4 : 12 = 1 : 3$

   d) $\frac{6}{8}$ are shaded
   
   $\frac{3}{4}$ are shaded
   
   6 is $\frac{3}{4}$ of 8
   
   $6 : 8 = 3 : 4$

   2. b) $6 : 10 = 3 : 5$
   
   c) $2 : 8 = 1 : 4$

   3. a) 15 : 20 = 3 : 4
   
   $15 \div 20 = \frac{3}{4}$

   b) 18 : 20 = 9 : 10
   
   $18 \div 20 = \frac{9}{10}$

   c) $? : 16 = 3 : 4$
   
   $\frac{2}{16} = \frac{3}{4}$

   d) 20 : 30 = 2 : 3
   
   $\frac{20}{30} = \frac{2}{3}$

   e) $\frac{3}{8} = \frac{9}{24}$

   f) $\frac{2}{3} = \frac{12}{18}$

   g) $\frac{13}{20} = \frac{65}{100}$

   h) $\frac{5}{9} = \frac{40}{72}$

   4. The answer is 15 for both since $(30 \times 50) \div 100 = (50 \times 30) \div 100$.

   5. b) $\frac{5}{10}$ are shaded
   
   $\frac{1}{2}$ are shaded
   
   5 is $\frac{1}{2}$ of 10
   
   $5 : 10 = 1 : 2$

   c) $\frac{4}{12}$ are shaded
   
   $\frac{1}{3}$ are shaded
   
   4 is $\frac{1}{3}$ of 12
   
   $4 : 12 = 1 : 3$

   d) $\frac{6}{8}$ are shaded
   
   $\frac{3}{4}$ are shaded
   
   6 is $\frac{3}{4}$ of 8
   
   $6 : 8 = 3 : 4$

   2. b) $6 : 10 = 3 : 5$
   
   c) $2 : 8 = 1 : 4$

   3. a) 15 : 20 = 3 : 4
   
   $15 \div 20 = \frac{3}{4}$

   b) 18 : 20 = 9 : 10
   
   $18 \div 20 = \frac{9}{10}$

   c) $? : 16 = 3 : 4$
   
   $\frac{2}{16} = \frac{3}{4}$

   d) 20 : 30 = 2 : 3
   
   $\frac{20}{30} = \frac{2}{3}$

   e) $\frac{3}{8} = \frac{9}{24}$

   f) $\frac{2}{3} = \frac{12}{18}$

   g) $\frac{13}{20} = \frac{65}{100}$

   h) $\frac{5}{9} = \frac{40}{72}$

   4. a) 15 : 20 = 3 : 4
   
   $15 \div 20 = \frac{3}{4}$

   b) 18 : 20 = 9 : 10
   
   $18 \div 20 = \frac{9}{10}$

   c) $? : 16 = 3 : 4$
   
   $\frac{2}{16} = \frac{3}{4}$

   d) 20 : 30 = 2 : 3
   
   $\frac{20}{30} = \frac{2}{3}$

   e) $\frac{3}{8} = \frac{9}{24}$

   f) $\frac{2}{3} = \frac{12}{18}$

   g) $\frac{13}{20} = \frac{65}{100}$

   h) $\frac{5}{9} = \frac{40}{72}$

   5. b) $\frac{5}{10}$ are shaded
   
   $\frac{1}{2}$ are shaded
   
   5 is $\frac{1}{2}$ of 10
   
   $5 : 10 = 1 : 2$

   c) $\frac{4}{12}$ are shaded
   
   $\frac{1}{3}$ are shaded
   
   4 is $\frac{1}{3}$ of 12
   
   $4 : 12 = 1 : 3$

   d) $\frac{6}{8}$ are shaded
   
   $\frac{3}{4}$ are shaded
   
   6 is $\frac{3}{4}$ of 8
   
   $6 : 8 = 3 : 4$

   2. b) $6 : 10 = 3 : 5$
   
   c) $2 : 8 = 1 : 4$
d) 20
10. 25% red
   75% not red
11. 42
12. 25

AP Book NS7-74
page 24
1. a) 49%
   b) 61%
   c) 72%
2. Answers will vary –
   teacher to check.
3. 28%
4. Item | Money spent
       |   |   |
       | F | P | $  |
B    | 1/4 | 25% | $125 |
P    | 3/10 | 30% | $150 |
C    | 9/20 | 45% | $225 |
5. 70%
6. Item | S | B | B |
       |   |   |
RP   | $52.00 | $38.96 | $9.80 |
D%   | 10% | 25% | 30% |
DS$  | $5.20 | $9.74 | $2.94 |
SP   | $46.80 | $29.22 | $6.86 |
7. $150 saved
8. chair: 1/4 = 25%
   table: 7/20 = 35%
   sofa: 2/5 = 40%

9. There are now 344
   sturgeon in the lake;
   344/400 = 24.6% of total.

AP Book NS7-75
page 25
1. b  g  c
   a) 8  5  13
   b) 4  7  11
   c) 12 15 27
   d) 11 9 20
2. b  g  c
   a) 5/11 6/11 11
   b) 8/15 7/15 15
3. b: 2 7 15
   g: 1 5 20
   %b: 16 40
   %g: 25 50
4. s  b  g  %b  %g
   a) 20 4/5 1/5 16 4
   b) 30 1/3 2/3 10 20
   c) 28 1/4 3/4 7 21
   d) 26 7/13 6/13 14 12
7. Ron received $20 or $20.
   Ella received $15.
8. Subject | Time
     F   | P   | D   | M   |
     E   | 1/4 | 25% | 0.25 | 15 |
     S   | 1/20 | 5%  | 0.05 | 3  |
     M   | 1/2 | 50% | 0.5  | 30 |
   F   | 1/5 | 20% | 0.2  | 12 |

AP Book NS7-76
page 27
1. b) 2
   c) 2
2. a) 25 boys
   b) 6 boys
3. a) 1 unit = 10 beads
   70 beads total
   b) 1 unit = 4 beads
   32 beads total
   c) 1 unit = 10 beads
   120 beads total
4. Models should include the
   following:
   a) 30 red; 20 green
   b) 9 red; 15 green
5. 60 fish
   Teacher to check model.

AP Book NS7-77
page 28
1. a) 10
   b) 21
   c) 55
2. a) 8
   b) 20
   c) 25
   d) 45
3. a) 3/5
   b) $40
4. 800 students
5. a) 1/2
   b) 7/12
   c) 72 fish
6. 1250 stamps
7. 500 lights

AP Book NS7-78
page 29
1. b) 3/7 + 3/7
   c) 5/11 + 5/11 + 5/11 + 5/11
2. a) 3 x 1/2
   b) 2 x 5/9
   c) 5 x 3/4
3. b) 6/4 = 1 1/2
   c) 8/7 = 1 1/7
   d) 20/11 = 1 9/11
   e) 18/7 = 2 4/7
4. b) 5 x 2/3 = 10/3 = 3 1/3
   c) 3 x 4/5 = 12/5 = 2 2/5
5. b) 6
   c) 2
   d) 2
   e) 3
6. b) 2
   c) 9
   d) 5
   e) b
7. a) 2/2
   b) 6, 6
   c) 4, 4
   d) 15, 15

Answer Keys for AP Book 7.2
INVESTIGATION 1
A. i) \( \frac{8}{4} \div 2 \)
   ii) \( \frac{12}{3} \div 4 \)
   iii) \( \frac{30}{5} \div 6 \)
   iv) \( \frac{60}{6} \div 10 \)
B. No

INVESTIGATION 2
A. i) \( \frac{4}{3} \div \frac{8}{6} = \frac{4}{3} \)
ii) \( \frac{11}{3} \div \frac{22}{6} = \frac{11}{3} \)
B. Yes

AP Book NS7-79
page 31
1. b) \( \frac{1}{15} \)
   c) \( \frac{1}{10} \)
   d) \( \frac{1}{3} \) of \( \frac{1}{3} = \frac{1}{9} \)
   e) \( \frac{1}{6} \) of \( \frac{1}{5} = \frac{1}{30} \)
2. b) \( \frac{1}{3} \times \frac{1}{5} = \frac{1}{15} \)
   c) \( \frac{1}{5} \times \frac{1}{2} = \frac{1}{10} \)
   d) \( \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \)
   e) \( \frac{1}{6} \times \frac{1}{5} = \frac{1}{30} \)
3. b) \( \frac{1}{2} \times \frac{1}{5} = \frac{1}{10} \)
   c) \( \frac{1}{6} \times \frac{1}{2} = \frac{1}{12} \)
   d) \( \frac{1}{5} \times \frac{1}{4} = \frac{1}{20} \)
   e) \( \frac{1}{5} \times \frac{1}{3} = \frac{1}{15} \)
4. \( \frac{1}{a} \times \frac{1}{b} \)
5. a) \( \frac{1}{10} \)
   b) \( \frac{1}{14} \)
   c) \( \frac{1}{18} \)
6. a) \( \frac{6}{28} \) (or \( \frac{3}{14} \))
   b) \( \frac{10}{21} \)
   c) \( \frac{3}{5} \times \frac{2}{5} = \frac{6}{25} \)
   d) \( \frac{3}{5} \times \frac{3}{5} = \frac{9}{25} \)
7. a) \( \frac{a \times c}{b \times d} \)
   b) \( \frac{2}{5} \)
   c) \( \frac{15}{28} \)
   d) \( \frac{6}{40} \) (or \( \frac{3}{20} \))
8. b) \( \frac{3}{10} \)
   c) \( \frac{15}{28} \)
   d) \( \frac{6}{40} \) (or \( \frac{3}{20} \))
9. a\times c
   b\times d
10. a) \( \frac{2}{5} \)
    b) \( \frac{10}{21} \)
    c) \( \frac{4}{15} \)
    d) \( \frac{16}{21} \)
    e) \( \frac{8}{21} \)
11. a) \( \frac{1}{10} \)
    b) \( \frac{1}{15} \)
    c) \( \frac{1}{12} \)
    d) \( \frac{1}{15} \)
    e) \( \frac{1}{15} \)
12. a) Circle: \( \frac{5}{7} \) \( \frac{7}{10} \)
    b) Circle: \( \frac{5}{7} \) \( \frac{2}{10} \) \( \frac{3}{7} \)
    c) \( \frac{15}{16} \) \( \frac{9}{21} \) \( \frac{15}{20} \)

AP Book NS7-80
page 33
1. a) \( 5 \times 3 = 15, 15 \)
    b) \( 4, 4 \)
    c) \( 3 \times 4 = 12, 12 \)
    d) \( 7 \times 4 = 28, 28 \)
2. a) \( 9 \times 5 = 45 \)
    b) \( 8 \times 4 = 32 \)
    c) \( 7 \times 6 = 42 \)
    d) \( 24 \)
    e) \( 36 \)
    f) \( 35 \)
    g) \( 49 \)
    h) \( 72 \)
3. a) \( 9 \times 4 \div 3 = 12 \)
    b) \( 8 \times 5 \div 4 = 10 \)
    c) \( 28 \)
    d) \( 8 \)
    e) \( 12 \)
    f) \( 15 \)
    g) \( 30 \)

AP Book NS7-81
page 35
1. \( \frac{9}{4} \) (or \( \frac{22}{10} \)) cups
2. \( \frac{8}{3} \) (or \( \frac{22}{10} \)) hours
3. 12 people
4. 100 pieces
5. 9 hours
6. \( \frac{2}{15} \) of a pie
7. a) \( \frac{1}{12} \)
8. a) \( \frac{65}{100} \) (or \( \frac{13}{20} \))
    b) 832 cards
    c) 1280 cards
9. 30 years old
10. 9 years old

AP Book NS7-82
page 36
1. 2, right;
   3, right
2. a) 10, 100
   b) 1, left
   c) 0.01
   d) 0.001
   e) 0.0001
   f) 0.5
   g) 0.02
   h) 0.007
3. a) 0.1, 1, 10
   b) 2, left
   c) 3, left
   d) 0.001
   e) 0.0001
   f) 0.0003
   g) 0.025
   h) 0.0139
   i) 0.01
   j) 0.0072
4. a) 0.0005
   b) 0.00032
   c) 0.5
   d) 0.00572
5. less
6. a) 3.6
   b) 0.01, 4.7
   c) 0.001, 0.085
7. a) ii) 0.1
   b) iii) 3
   c) iv) 0.007
   d) 0.0072
   e) 32, 3.2
   f) 4, 300, 3
   g) 5, 90, 0.09
1. a) 1  
   b) 2  
   c) 3  
   d) 6  

2. a) i) 2  
   ii) 2  
   iii) 12  
   iv) 51  
   v) 8.247  
   b) They are the same.  

3. Teacher to check shading.  

4. b) 6  

5. a) 15  
   b) 15  
   c) 15  
   d) 15  

6. The # of zeros in the product = the total # of zeros in the denominators of the multiplied fractions  

7. a) i) 0.2  
   ii) 0.16  
   iii) 0.02  
   iv) 0.001  
   v) 0.000  
   b) The # of decimal places in the product = the total # of decimal places in the multiplied decimals  

8. a) 0.40 (or 0.4)  
   b) 0.63  
   c) 0.12  
   d) 0.120 (or 0.12)  
   e) 0.078  
   f) 0.268  
   g) 0.288  
   h) 0.028  
   i) 0.14  
   j) 0.368  

9. a) 0.004  
   b) 0  
   c) 90  
   d) 0.008  

10. Estimates may vary.  
    Sample answers:  
    a) 2 × 7 = 14  
    b) 3 × 5 = 15  
    c) 20 × 2 = 40  
    d) 10 × 4 = 40  
    e) 40 × 5 = 200  

11. a) 31.2  
    b) 170.88  
    c) 9.345  
    d) 150.9045  

12. a) Correct as is  
    b) Should be 6.97  
    c) Correct as is  
    d) Correct as is  

13. a) 9.72  
    b) 4.964  
    c) 17.55  

14. Teacher to check.  
    Sample answers:  
    a = 0.1, b = 6;  
    a = 1, b = 0.6;  
    a = 2, b = 0.3;  
    a = 3, b = 0.2, etc.  

15. 45.05 m²  

16. 1.75 L  

1. a) 100  
   b) 100000  
   c) 10000  

2. c) 10.  
   d) 100,  
   e) 10.  

3. b) cm, cm  
   c) m, m  
   d) m, m  
   e) dm, dm  
   f) m, m  
   g) m, m  
   h) dm, dm  
   i) km, km  

4. a) 423 pennies, $4.23  
    b) 423 cm, 4.23 m  
    c) The both deal with one unit that is a whole, and a second unit equal to 1/100 of the first.  

5. It is equal to 5.28 m since there are 100 cm in 1 m.  

6. b) 52 m = 52 cm  
    c) 60 dm = 60 mm  
    d) 520 m = 520 mm  

7. b) 4 m + 2/10 m  
    c) 47 dm  
    d) 68 cm  
    e) 873 mm  
    f) 23 mm  

8. a) 2, 3, 5, 7  
    b) 3, 5, 2  
    c) 5, 0, 0, 6  
    d) 8, 0, 4  
    e) 3, 5  
    f) 20, 5, 4  

9. b) 17.52 dm  
    c) 10.79 cm  
    d) 23.459 m  
    e) 58.41 dm  
    f) 121 mm  

10. 2.375 km  

11. The cm are like the pennies since there are 100 of them in 1 whole unit (i.e. m or dollars).
12. Combinations may vary.

Sample answers:

a) \( 5 + 5 + 5 + 3 \) or \( 3 + 3 + 3 + 3 + 3 \)
b) \( 5 + 5 + 3 + 3 + 3 \)
c) \( 5 + 5 \)
d) \( 5 + 5 + 5 + 5 + 3 \)

13. Tree Height (cm)

<table>
<thead>
<tr>
<th>Tree</th>
<th>Height (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>White Birch</td>
<td>2 000</td>
</tr>
<tr>
<td>Lodgepole Pine</td>
<td>3 050</td>
</tr>
<tr>
<td>Western Red Cedar</td>
<td>5 900</td>
</tr>
<tr>
<td>Red Oak</td>
<td>2 400</td>
</tr>
</tbody>
</table>

In order from tallest to shortest:

Western Red Cedar, Lodgepole Pine, Red Oak, White Birch
Measurement – AP Book 7, Part 2: Unit 2

AP Book ME7-22

1.  
   a) $4 \times 5 = 20$  
   b) $20$  
   c) $3$  
   d) $20 \times 3 = 60$  

   $4 \times 4 = 16$  
   $16 \times 3 = 48$

2.  
   a) $l \times w$  
   b) $h$  
   c) $l \times w \times h$  

   d) Volume = length $\times$ width $\times$ height

3.  
   In a volume calculation, the length occurs three times.

4.  
   a) width: 2 blocks  
      length: 2 blocks  
      height: 2 blocks  
      Volume = $8 \text{ blocks}^3$
   
   b) width: 2 m  
      length: 3 m  
      height: 2 m  
      Volume = $12 \text{ m}^3$
   
   c) width: 2 cm  
      length: 4 cm  
      height: 2 cm  
      Volume = $16 \text{ cm}^3$
   
   d) width: 2 mm  
      length: 3 mm  
      height: 5 mm  
      Volume = $30 \text{ mm}^3$

AP Book ME7-23

1.  
   Teacher to check shading of bases is correct.
   a) i) Hexagonal prism  
      ii) Pentagonal prism  
      iii) Triangular prism
   
   b) Rectangles
   c) i) 6  
      ii) 5  
      iii) 3

2.  
   a) Right prisms: A, B, F  
      Skew prisms: D, G  
      Not prisms: C, E
   
   b) $E$: opposite bases aren't connected by parallel lines.
      $C$: bases are not congruent polygons.

3.  
   Teacher to check.

4.  
   a) $1 \times 2$  
   b) $2$

AP Book ME7-24

1.  
   a) $\frac{1}{2}$  
   b) $4$  
   c) $2$

2.  
   a) $\frac{1}{2}$  
   b) $2$

   c) i) $V$ of r.p. = 8  
      ii) $V$ of t.p. = 4
   
   d) i) $V$ of r.p. = 70  
      ii) $V$ of t.p. = 35
   
   e) i) $V$ of r.p. = 40  
      ii) $V$ of t.p. = 20

3.  
   a) i) 1  
      ii) 5  
      iii) 4

   b) i) $1 \times 4 = 4$  
      ii) $5 \times 7 = 35$

   c) The numbers are the same.

AP Book ME7-25

INVESTIGATION 1

A.  
   i) $6 \text{ cm}^2$  
   ii) $12 \text{ cm}^2$
   
   iii) $8 \text{ cm}^2$  
   iv) $12 \text{ cm}^2$

   b) $8 \text{ cm}^3$  
   iii) $16 \text{ cm}^3$
   
   iv) $36 \text{ cm}^3$

C.  
   The numbers are all the same, only the units differ.

D.  
   i) $1 \text{ cm}$  
   ii) $2 \text{ cm}$
   
   iii) $2 \text{ cm}$  
   iv) $3 \text{ cm}$

E.  
   i) $6 \text{ cm}^3$  
   ii) $24 \text{ cm}^3$
   
   iii) $16 \text{ cm}^3$  
   iv) $36 \text{ cm}^3$

F.  
   By the number of layers

G.  
   Volume of a rectangular prism = (area of the base) $\times$ height

INVESTIGATION 2

A.  
   i) $12 \text{ cm}^2$  
   ii) $11 \text{ cm}^2$
   
   iii) $7 \text{ cm}^2$

   b) $12 \text{ cm}^3$

B.  
   i) $12 \text{ cm}^3$
Measurement – AP Book 7, Part 2: Unit 2

11. a) Teacher to check.
   b) \( t = 4 \times 3 = 12 \text{ cm}^2 \)
   c) \( f = 1 \times 3 = 3 \text{ cm}^2 \)

12. a) Teacher to check.
   b) \( 60 \text{ m}^3 \)
   c) \( 24 \text{ m}^3 \)

AP Book ME7-28

page 56

1. a) \( 94 \text{ cm}^2 \)
   b) \( 40 \text{ cm}^2 \)
   c) \( 52 \text{ cm}^2 \)

2. She can multiply 20 by 20 to get the total area of the back, bottom, and left faces, which is equal to the surface area of the front, top, and right faces.

3. Only 3 of the 6 are correct:
   a) Square prism
   b) Triangular prism
   c) Square prism

4. a) Nets 1 and 2
   b) Net 2 only
   c) In a): Net 3 has both bases on the same side. In b): Net 1 has a base that isn’t an equilateral triangle but all rectangular sides are equal; Net 3 has bases oriented in opposite directions.

5. a) Smaller rectangle is missing; teacher to check.
   b) Smallest rectangle is missing; teacher to check.

6. Teacher to check.

7. Teacher to check nets.
   a) Square prism
   b) Rectangular prism
   c) Triangular prism
   d) Square prism

8. b) Teacher to check.
   c) 60 cm
   d) 24 cm

9. a) \( b = 2 \times 3 = 6 \text{ cm}^2 \)
    b) \( f = 1 \times 4 = 4 \text{ cm}^2 \)
    c) \( f = 1 \times 3 = 3 \text{ cm}^2 \)

10. Teacher to check nets.
   a) \( \text{surface area: } 42 \text{ cm}^2 \)
      \( \text{volume: } 18 \text{ cm}^3 \)
   b) \( \text{surface area: } 28 \text{ cm}^2 \)
      \( \text{volume: } 8 \text{ cm}^3 \)

AP Book ME7-27

page 53

1. a) Square prism
   b) Triangular prism
   c) Rectangular prism

2. Teacher to check shading of bases is correct.

3. Teacher to check.

4. Teacher to check.

5. a) Back: 6 cm
    Bottom: 12 cm
    Left: 8 cm

6. Back: 15 cm
    Bottom: 6 cm
    Left: 10 cm

7. a) 1000 mL
    b) 1000 cm
    c) V = 1000 cm
    d) V = 10000 mL

8. a) V = 1000 cm
    b) V = 10000 mL
    c) V = 1 m

9. a) 425 mL
    b) 336 mL
    c) 1125.3 mL

10. a) 1890 cm
    b) 20 cm
    c) 80 cm

BONUS h = (250 + 49) + 4.5

AP Book ME7-26

page 51

1. Teacher to check.

2. a) 1 cm
    b) 3 cm
    c) 5 cm

3. a) Teacher to check.
    b) Front: 3 cm
       Top: 6 cm
       Right side: 2 cm
       Back: 3 cm
       Bottom: 6 cm
       Left side: 2 cm
    c) add everything in b) together: 22 cm

4. Teacher to check.

5. a) Back: 6 cm
    Bottom: 12 cm
    Left: 8 cm

6. a) \( t + b = 15 \times 2 = 30 \text{ cm}^2 \)
    b) \( r + l = 10 \times 2 = 20 \text{ cm}^2 \)
    c) \( f + b = 9 \times 2 = 18 \text{ cm}^2 \)
    d) \( t + b = 12 \times 2 = 24 \text{ cm}^2 \)
    e) \( r + l = 12 \times 2 = 24 \text{ cm}^2 \)

7. a) Yes, since two of the dimensions (length and width) are the same.

8. a) \( b = 2 \times 3 = 6 \text{ cm}^2 \)
    b) \( f = 1 \times 4 = 4 \text{ cm}^2 \)
    c) \( f = 1 \times 3 = 3 \text{ cm}^2 \)

9. a) \( l + b = 10 \times 2 \)
    b) \( r + l = 12 \times 2 \)
    c) \( t + b = 15 \times 2 \)

10. Teacher to check nets.
    a) \( \text{surface area: } 42 \text{ cm}^2 \)
       \( \text{volume: } 18 \text{ cm}^3 \)
    b) \( \text{surface area: } 28 \text{ cm}^2 \)
       \( \text{volume: } 8 \text{ cm}^3 \)

11. a) No, he’s only counted half the faces; it is actually 52 cm².

12. a) \( 12 \text{ cm} \)
    b) \( 336 \text{ cm} \)
    c) \( 125.3 \text{ cm}^3 \)

5. a) \( 1125 \text{ cm}^3 \)
   b) \( 336 \text{ cm}^3 \)

6. Teacher to check.

7. a) Smaller rectangle is correct.
   b) Smallest rectangle is correct.
   c) Smaller rectangle is correct.

8. a) \( \text{Volume} = (\text{Area of base}) \times h \)
    b) \( \text{Volume} = (\text{Area of base}) \times h \)

9. a) \( t = 4 \times 3 = 12 \text{ cm}^2 \)
    b) \( b = 4 \times 3 = 12 \text{ cm}^2 \)

10. Teacher to check nets.
    a) \( \text{surface area: } 42 \text{ cm}^2 \)
       \( \text{volume: } 18 \text{ cm}^3 \)
    b) \( \text{surface area: } 28 \text{ cm}^2 \)
       \( \text{volume: } 8 \text{ cm}^3 \)
9. a) Teacher to check.
   b) Teacher to check.
   c) bottom = 750 cm$^2$
      top = 750 cm$^2$
      front = 600 cm$^2$
      back = 600 cm$^2$
      right side = 500 cm$^2$
      left side = 500 cm$^2$
   d) lid of box = top side:
      25 cm $\times$ 30 cm
   e) 2 950 cm$^2$

10. a) 1 hexagon;
     6 rectangles
   b) Teacher to check.
   c) $520 + [6 \times (10 \times 20)]$
      $= 520 + 1 200$
      $= 1 720$ cm$^2$

11. a) Volume of a right prism = (area of base) $\times$ height
   b) Q9: 15 000 cm$^3$
      Q10: 10 400 cm$^3$

12. a) 2 m
    b) 7 m
    c) 4 m

13. a) No, he’s wrong – he’s only multiplied 2 of the 3 edges, plus all the units must first be made the same (0.5 m = 50 cm).
    b) Volume = 400 000 cm$^3$
       Surface area = 34 000 cm$^2$

14. a) Teacher to check.
    b) Width = 4 cm
    c) Surface area = 94 cm$^2$

15. Be sure to make all the units the same first.
    Volume = 510 000 cm$^3$
    $= 0.51$ m$^3$
    Surface area = 45 800 cm$^2$
    $= 4.58$ m$^2$
1. a) Start at 0, count by 2, stop at 8. Yes, it's good because all the bars are easy to read.

2. a) 2 students
   b) 5 students
   c) 5 students
   d) 10 students

3. a) Yes
   b) Graph A (left)
   c) Graph B (right)
   d) Graph B
   e) Graph A because the proportions are accurate (scale starts at 0).

4. a) 0; 10 000; 50 000; 47 000; 100; 47 300
   b) Graph B
   c) Ms. C

5. a) A; the bars are longer overall
   b) Graph B: it has the proper proportions.

6. It helps when the bars are similar in length (shows detail); can be misleading since it exaggerates the difference.

7. a) A: Minimum Wage in Ontario ($)
    b) B: Minimum Wage in Ontario ($)
    c) Graph A
    d) Graph B
    e) February
    f) April
    g) A: Winter
    h) A: Winter
    B: Summer
    They each had their highest sales in (or leading up to) these seasons.

8. a) Yes; just the order of the colours on the x-axis changes
    b) No; the graph shows frequency of four different categories (nothing to do with changes over time)
    c) A bar graph would be better since it's best for representing frequency/size of different categories.

9. Bar Stem & Leaf
    0 1
    1 0 1
    2 0 1
    3
    4

10. Answers may vary but will likely be:
    a) stem and leaf
    b) both the same
    c) stem and leaf
    d) stem and leaf

[Note: A circle graph would also be a good choice here but isn't covered until later in the unit. Students may remember it from prior years but it shouldn't be expected.]
Probability and Data Management – AP Book 7, Part 2: Unit 3

Answer Keys for AP Book 7.2

page 63

1. a) (French = 4)
   Math; Gym; History; French; Science
   b) ~ 20; since 35 goes into 100 just less than 3 times, you can estimate by multiplying 7 by 3 (21) and picking a number a bit less

2. a) Yes – there are 5 (not 4) As in the list.
   b) Yes: 16 students got an A or B, which is more than half of 23.

AP Book PDM7-10

1. a) 200; 50
   b) The percentage per school is 100% which makes sense because 100% of those surveyed gave an answer.
   2. The columns don’t add to 100% (but to 104%).

AP Book PDM7-11

1. a), b), c)
   Percent Angle
   walk 25% 90°
   bike 100% 360°
   bus 50% 180°
   other 5% 18°
   d) Circle graphs are proportionate – they show the % of the students who prefer rock music, not the actual numbers.
   e) Teacher to check.

Daily Newspaper Habit

Never look at
Delivered to home
Buy occasionally

How Students Spend Money

Savings
Entertainment
Clothes and personal care
Snacks

Favourite Kind of Pie

<table>
<thead>
<tr>
<th></th>
<th>Angle in circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>apple</td>
<td>( \frac{1}{5} \times 360 = 72^\circ )</td>
</tr>
<tr>
<td>blueberry</td>
<td>( \frac{3}{20} \times 360 = 54^\circ )</td>
</tr>
<tr>
<td>cherry</td>
<td>( \frac{1}{10} \times 360 = 36^\circ )</td>
</tr>
<tr>
<td>other</td>
<td>( \frac{11}{20} \times 360 = 198^\circ )</td>
</tr>
</tbody>
</table>

Favourite Indoor Games

<table>
<thead>
<tr>
<th></th>
<th>Fraction</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>board games</td>
<td>( \frac{11}{36} )</td>
<td>110°</td>
</tr>
<tr>
<td>card games</td>
<td>( \frac{1}{36} )</td>
<td>10°</td>
</tr>
<tr>
<td>video games</td>
<td>( \frac{21}{36} )</td>
<td>210°</td>
</tr>
<tr>
<td>other</td>
<td>( \frac{3}{36} )</td>
<td>30°</td>
</tr>
</tbody>
</table>

Favourite Sport

<table>
<thead>
<tr>
<th></th>
<th>Angle</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>hockey</td>
<td>126°</td>
<td>35%</td>
</tr>
<tr>
<td>swimming</td>
<td>108°</td>
<td>30%</td>
</tr>
<tr>
<td>running</td>
<td>72°</td>
<td>20%</td>
</tr>
<tr>
<td>other</td>
<td>54°</td>
<td>15%</td>
</tr>
</tbody>
</table>

AP Book PDM7-12

1. a) 30, 30%
b) 28, 28%
c) 45, 45%
d) 11, 44, 44%
e) 4, 40, 40%
f) 13, 65, 65%

2. a) 162
b) 117
c) 7, 63
d) 7, 70

3. a) 30, 30%
b) 28, 28%
c) 45, 45%
d) 11, 44, 44%
e) 4, 40, 40%
f) 13, 65, 65%

4. a) \( \frac{8}{20} \) = 41.7%
b) \( \frac{9}{20} \) = 20.8%
c) \( \frac{11}{20} \) = 210%
d) \( \frac{12}{20} \) = 210%
e) \( \frac{13}{20} \) = 63.7%

5. a) \( \frac{8}{20} \) = 41.7%
b) \( \frac{9}{20} \) = 20.8%
c) \( \frac{11}{20} \) = 210%
d) \( \frac{12}{20} \) = 210%
e) \( \frac{13}{20} \) = 63.7%

AP Book PDM7-13

INVESTIGATION

A. a) (3 + 3 + 4 + 6 + 6) + 5 = 4.4
b) \( \left( \frac{3 + 3 + 4 + 6 + 6}{5} \right) \times 2 = 10 = 4.4 \)
c) \( \left( \frac{3 + 3 + 4 + 6 + 6}{5} \right) \times 3 = 15 = 4.4 \)
d) \( \left( \frac{2 + 5 + 4 + 1}{4} \right) \times 3 = 3 \)
e) \( \left( \frac{2 + 5 + 4 + 1}{4} \right) \times 3 = 3 \)
B. No, it creates equivalent fractions so the mean doesn’t change.
C. Median changes, mode does not.

AP Book PDM7-14

page 72

1. Circle → Visually displays the relative frequency of results
Stem and Leaf Plot → Makes it easy to see the largest, smallest, and most common data values
Double Bar Graph → Compares two sets of data
Bar Graph → Visually displays the frequency of data.
2. a) Amounts in the chart are in millions of 

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>F/R</td>
<td>60</td>
<td>25</td>
</tr>
<tr>
<td>S</td>
<td>40</td>
<td>25</td>
</tr>
<tr>
<td>A</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>M/S</td>
<td>70</td>
<td>40</td>
</tr>
<tr>
<td>Total</td>
<td>200</td>
<td>100</td>
</tr>
</tbody>
</table>

b) Charity A

f) Circle graph since it gave spending by percent rather than by $ amount.

c) $35
d) $40
e) Answers will vary but likely to B since more money goes toward medical supplies.

3. a) bar
b) double bar
c) circle
d) line
e) stem and leaf

4. a) circle
b) double bar
c) bar
d) stem and leaf
e) line

5. The shopping mall since it is representative rather than biased.

6. a) Miki’s sample is more biased. He only asked the people in his class, who know him so are more likely to vote for him than for someone else.
b) Melanie, since her sample represents the whole school and not a biased sample (students not in Miki’s class will be less likely to vote for him).

7. No, the results aren’t useful because the answers will all belong to the first category.

8. a) B; A
b) Sample answer: “Are you in favour of a school uniform policy?”
1. a) $s \times 4 = t$
   - 1: $4 \times 1 = 4$
   - 2: $4 \times 2 = 8$
   - 3: $4 \times 3 = 12$

   $6 \times s = r$
   - 1: $6 \times 1 = 6$
   - 2: $6 \times 2 = 12$
   - 3: $6 \times 3 = 18$

   $r + 1 = c$
   - 1: $1 + 6 = 7$
   - 2: $2 + 6 = 8$
   - 3: $3 + 6 = 9$

   BONUS $8 \times s = t$
   - 1: $8 \times 1 = 8$
   - 2: $8 \times 2 = 16$
   - 3: $8 \times 3 = 24$

   $r = 4 \times s$; $r = 4 \times s$
   - 1: $1 = 4$
   - 2: $2 = 8$
   - 3: $3 = 12$

   $t = 4 \times s$; $r = 4 \times s$
   - 1: $1 = r$
   - 2: $2 = s$
   - 3: $3 = r$

   $t = 1 \times r$

   2. a) $t = s \times 4$
   b) $t = s \times 5$
   c) $t = s \times 2$
   d) $t = s \times 6$

   3. a) $s \times r$
   - 1: $1 \times 4 = 4$
   - 2: $2 \times 8 = 16$
   - 3: $3 \times 12 = 36$

   4. Wendy will need 28 rectangles.

   5. No, since $6 \times 7 = 42$.

   6. Teacher to check.

   7. Teacher to check.

   8. a) $r + 6 = c$
   - 1: $1 + 6 = 7$
   - 2: $2 + 6 = 8$
   - 3: $3 + 6 = 9$

   b) $r + 9 = c$
   - 1: $1 + 9 = 10$
   - 2: $2 + 9 = 11$
   - 3: $3 + 9 = 12$

   9. a) $r + 7 = c$
   b) $r + 8 = c$
   c) $r + 5 = c$

   10. a) $r + 3 = c$

   11. There would be 13 chairs.

   12. b) Input | Output
   - 4 | 0
   - 5 | 1
   - 6 | 2

   c) Input | Output
   - 4 | 24
   - 7 | 42
   - 8 | 48

   d) Input | Output
   - 18 | 27
   - 19 | 28
   - 20 | 29

   e) Input | Output
   - 28 | 7
   - 16 | 4
   - 40 | 10

   f) Input | Output
   - 4 | 32
   - 6 | 48
   - 9 | 72

   g) Input | Output
   - 26 | 29
   - 11 | 14
   - 46 | 49

   h) Input | Output
   - 15 | 10
   - 19 | 14
   - 23 | 18

   i) Input | Output
   - 4 | 16
   - 7 | 28
   - 12 | 48

   13. a) $I \times 6 = O$
   b) $I + 8 = O$
   c) $I - 3 = O$

   1. a) Multiply by 3; Divide by 3
   b) Add 5; Subtract 5
   c) Multiply by 2; Divide by 2

   2. Multiply becomes divide (and vice versa); addition becomes subtraction (and vice versa)

   3. a) (4,6), (5,7), (6,8)
   b) (10,5), (8,4), (6,3), (4,2)
   c) (5,10), (4,8), (3,6), (2,4)

   2. Teacher to check points marked on line segments.

   a) OP | 1st | 2nd
   - (2,1) | 2 | 1
   - (4,3) | 4 | 3
   - (6,5) | 6 | 5

   b) OP | 1st | 2nd
   - (1,3) | 1 | 3
   - (3,5) | 3 | 5
   - (5,7) | 5 | 7

   c) OP | 1st | 2nd
   - (2,4) | 2 | 4
   - (4,5) | 4 | 5
   - (6,6) | 6 | 6

   or

   Answer Keys for AP Book 7.2
Patterns and Algebra – AP Book 7, Part 2: Unit 4 (continued)

Answer Keys for AP Book 7.2

1. a)  7  b)  17  c)  11

2. a)

<table>
<thead>
<tr>
<th>Term #</th>
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<td>3</td>
<td>5</td>
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<tr>
<td>4</td>
<td>7</td>
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</tbody>
</table>

b)  A: O = (3 × I) – 1  
B: O = I + 2  
C: O = I

3. Teacher to check graph.  
Ordered Pairs:  
(3,1), (4,3), (5,5), (6,7)

4. Teacher to check.

5. Teacher to check.

AP Book PA7-20

page 85

1. a) Teacher to check.  
b)  i) No  
ii) Yes  
iii) Yes  
v) Yes  
vi) No

2. Teacher to check.  
3. Teacher to check.  
4. Q2: lines go up from left to right  
Q3: lines go down from left to right

INVESTIGATION 1

A. ii), iii), and v) are linear  
Gaps  
ii) +3  
iii) +3  
v) –2

B. i), iv), and vi) aren't linear  
Gaps  
i) +3, +2, +6, +3  
iv) +1, +1, +2, +4  
vi) –4, –2, –4, –1

C. The sequence is linear if and only if all of the gaps are equal.

5. a) Sequence B is linear because the gaps are all –3.  
b) Teacher to check.

6. a)  i) +2, +2  
ii) –3, –3  
iii) +2, +5  
iv) i) and iii)  
iv) i) and ii)  
d)  A → ii)  
B → iii)  
C → i)

INVESTIGATION 2

A. i)  Sequence: 1, 3, 5, 7  
Gaps: +2, +2, +2  
ii)  S: 4, 5, 6, 7  
G: +1, +1, +1  
iii)  S: 1, 2, 4, 8  
G: +1, +2, +4  
iv)  S: 2, 7, 4, 9  
G: +5, –3, +5  
v)  S: 2, 7, 5, 2  
G: +5, –2, –3  
vi)  S: 12, 10, 8, 6  
G: –2, –2, –2  
vii)  S: 2, 6, 18, 54  
G: +4, +12, +36  
viii)  S: 96, 48, 24, 12  
G: –48, –24, –12

B. i), ii) and iv)  
C. Teacher to check.  
D. The sequence is linear as long as the rule says to add or subtract a consistent number.

AP Book PA7-21

page 88

1. a)  

<table>
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<th>Output</th>
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<tbody>
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<td>5</td>
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<tr>
<td>3</td>
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</tr>
<tr>
<td>4</td>
<td>11</td>
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Graph B

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<tr>
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<td>5</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
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Graph C

<table>
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<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
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</tbody>
</table>

INVESTIGATION 2

A. i)  Sequence: 1, 3, 5, 7  
Gaps: +2, +2, +2  
i)  S: 4, 5, 6, 7  
G: +1, +1, +1  
ii)  S: 1, 2, 4, 8  
G: +1, +2, +4  
iii)  S: 2, 7, 4, 9  
G: +5, –3, +5  
v)  S: 2, 7, 5, 2  
G: +5, –2, –3  
vi)  S: 12, 10, 8, 6  
G: –2, –2, –2  
vii)  S: 2, 6, 18, 54  
G: +4, +12, +36  
viii)  S: 96, 48, 24, 12  
G: –48, –24, –12

B. i), ii) and iv)  
C. Teacher to check.  
D. The sequence is linear as long as the rule says to add or subtract a consistent number.

5. a)  20 km  
b)  40 km  
c)  Yes, between hours 3 & 4 – the distance doesn’t change so Kathy isn’t moving.  
d)  No: 10 km/h for the first 3 hrs, 5 km/h for the last 2 hrs

6. a)  40 m  
b)  60 m
c) Tom won, by 40 m.

d) 40 m

e) 15 seconds

7. a) i) $8
ⅱ) $12
ⅲ) $10

b) $4

c) Mike’s ($10 versus $10.50 at Vi’s)

AP Book PA7-22

1. n = 2 → 5
ⅱ) n = 3 → 8
ⅲ) n = 4 → 11

2. a) $8
ⅱ) $12
ⅲ) $10

b) $4

c) Mike’s ($10 versus $10.50 at Vi’s)

AP Book PA7-23

1. b) S:
ⅱ) 1, 4, 7, 10, 13
G:
ⅱ) +3
R:
ⅱ) Start at 1, then add 3 each time.

c) S:
ⅱ) 21, 17, 13, 9, 5
G:
ⅱ) –4
R:
ⅱ) Start at 21, then subtract 4 each time.

d) S:
ⅱ) 2, 7, 12, 17, 22
G:
ⅱ) +5
R:
ⅱ) Start at 2, then add 5 each time.

e) S:
ⅱ) 16, 13, 10, 7, 4
G:
ⅱ) –3
R:
ⅱ) Start at 16, then subtract 3 each time.

3. Teacher to check graphs.

a) n 2n + 2

<table>
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<tr>
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<td>1</td>
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<td>8</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
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</table>

(1,4), (2,6), (3,8), (4,10), (5,12)

b) n 3n – 1

<table>
<thead>
<tr>
<th>n</th>
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<td>8</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
</tr>
</tbody>
</table>

(1,2), (2,5), (3,8), (4,11), (5,14)

c) we can easily continue the rule until the 10th term.

AP Book PA7-24

1. b) FN
ⅱ) # Blocks

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<th>FN</th>
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</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

Rule:
ⅱ) Multiply the Figure Number by 2.

c) FN
ⅱ) # Blocks

<table>
<thead>
<tr>
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<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

Rule:
ⅱ) Multiply the Figure Number by 3.

d) FN
ⅱ) # Blocks

<table>
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<th># Blocks</th>
</tr>
</thead>
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<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

Rule:
ⅱ) Multiply the Figure Number by 3.

3. a) TN
ⅱ) n × gap T

<table>
<thead>
<tr>
<th>TN</th>
<th>n × gap T</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
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</tbody>
</table>

Add: 7  Gap: +4

Rule: Multiply by 4, then add 7.

AP Book PA7-25

1. a) 7,11,15
ⅱ) Gap: +4

b) 1, 4, 7
ⅱ) Gap: +3

c) 6, 11, 16
ⅱ) Gap: +5

d) 6, 16, 26
ⅱ) Gap: +10

e) The gap is equal to the number that you multiply by in the rule or formula.

2. a) FN
ⅱ) # Blocks

<table>
<thead>
<tr>
<th>FN</th>
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<tbody>
<tr>
<td>1</td>
<td>2 + 3 = 5</td>
</tr>
<tr>
<td>2</td>
<td>4 + 3 = 7</td>
</tr>
<tr>
<td>3</td>
<td>6 + 3 = 9</td>
</tr>
</tbody>
</table>

Gap: +2

Formula: 2 × FN + 3

c) FN
ⅱ) # Blocks

<table>
<thead>
<tr>
<th>FN</th>
<th># Blocks</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>3 + 1 = 4</td>
</tr>
<tr>
<td>2</td>
<td>6 + 1 = 7</td>
</tr>
<tr>
<td>3</td>
<td>9 + 1 = 10</td>
</tr>
</tbody>
</table>

Gap: +3

Formula: 3 × FN + 1

d) FN
ⅱ) # Blocks

<table>
<thead>
<tr>
<th>FN</th>
<th># Blocks</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>2 × FN</td>
</tr>
<tr>
<td>2</td>
<td>2 × FN + 1</td>
</tr>
<tr>
<td>3</td>
<td>3 × FN + 2</td>
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</tbody>
</table>

Rule: Multiply by 4, then add 7.
Patterns and Algebra – AP Book 7, Part 2: Unit 4 (continued)

1. b) +2
c) +3

d) –2

2. a) +3
b) –4
c) +2
d) –3
e) +5

3. a) 0, 4, 10, 12
   Gaps: +4, +6, +2
b) Term values
c) Vertical axis

d) +4
ii) +2
iii) +3

4. a) i) 7
ii) 5
iii) 2

5. b) i) 2
ii) 5
iii) 7

c) i) C
ii) B
iii) A

6. B, C, A
7. Teacher to check.

AP Book PA7-27

1. a) From left to right: 7, 4
   b) From left to right: 2, 9

2. a) 12
   b) 32
   c) 20
   d) 13

3. a) 25
   b) 33

4. a) 40n
   b) 5n + 1

5. a) i) 7
   ii) 5
   iii) 2

AP Book PA7-28

1. a) 5n – 1, 49
   b) 6n – 2, 58
   c) 6n – 3, 57
   d) 11n – 6, 104
   e) 10n + 1, 101
   f) 2n + 11, 31
   g) 7n – 4, 66
   h) 3n + 41, 71

2. Rule:
   # of letters in nth row
   = n + 1
   K is the 11th letter so there will be 12 K’s.

3 b) i) 4, 9, 14
   ii) 5 × FN – 1
   iii) 5n – 1
   iv) 15(15) – 1 = 74
   c) i) 6, 8, 10
   ii) 2 × FN + 4
   iii) 2n + 4
   iv) 2(15) + 4 = 34

4. a) 3n + 1
   b) i) 3(23) + 1
      = 69 + 1 = 70
   ii) 5 × FN
   iii) 3n
   iv) 4(15) = 60

5. a) 1 2
   2 3
   3 4
   4 5
   5 6
   6 7
   7 8
   8 9
   9 10
   10 11
   11 12
   12 13
   13 14
   14 15
   15 16
   16 17
   17 18
   18 19
   19 20
   20 21
   21 22
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   93 94
   94 95
   95 96
   96 97
   97 98
   98 99

6. Teacher to check.

AP Book PA7-29

1. a) From left to right: 2, 9
   b) From left to right: 3, 10

2. a) 32
   b) 49

3. a) 20
   b) 13

4. a) 25
   b) 33

5. a) 40n
   b) 5n + 1

6. Teacher to check.

7. a) 3n + 1
   b) 3(23) + 1
      = 69 + 1 = 70
   ii) 20th term:
      Add 3 to extend the sequence: 21st term: 64
      22nd term: 67
      23rd term: 70

5. a) 1
   b) 2
   c) 3
   d) 4
   e) 5
Patterns and Algebra – AP Book 7, Part 2: Unit 4 (continued)

11. 

Tank 1 drains at a rate of 40L/min. 

After \(n\) minutes, the water left in Tank 1 is \(540 - 40n\). 

It will empty after 13.5 minutes (solve by setting \(540 - 40n\) to 0). 

Tank 2 doesn’t drain at a consistent rate so we must extend the sequence to see when it will empty: 

500, 490, 470, 440, 400, 350, 290, 220, 140, 50, -50, … 

It will empty after about 9.5 minutes. 

:. Tank 2 will empty first. 

b) 

i) \(5^\text{th}\) triangle: 25 

6\(^\text{th}\) triangle: 36 

ii) the gap follows this sequence: +1, +3, +5, … 

Rule: Start at 1, add 2 each time. 

iii) 7\(^\text{th}\) triangle: 49 

8\(^\text{th}\) triangle: 64 

14. 

a) \(3n + 4\) 

b) \(5n - 4\) 

c) \(7n - 4\) 

d) \(2n + 9\) 

15. 

a) \(4n - 2\) 

\(4(50) - 2 = 198\) 

b) \(8n + 2\) 

\(8(50) + 2 = 402\) 

16. 

a) Teacher to check. 

b) \(n(n+1) = 15(16)\) 

= 240 

c) This will be equal to half the sum of the first 15 even numbers: 

\(240 ÷ 2 = 120\) 

17. 

a) 

\[\begin{array}{ccc}
1 & 1 & 0 \\
2 & 3 & 1 \\
3 & 6 & 3 \\
4 & 10 & 6
\end{array}\] 

b) Continue sequence for shaded triangles: 

1, 3, 6, 10, 15, 21, 28, 36, 45, 55 

:. the 10\(^\text{th}\) figure has 55 shaded triangles 

18. 

a) The graph could only represent the first situation. 

The cost of renting a bike for \(n\) hours increases linearly (that is, the gap is always the same). 

The area of a square with side length \(n\) doesn’t increase linearly.
Geometry – AP Book 7, Part 2: Unit 5

Answer Keys for AP Book 7.2

AP Book G7-13
page 105

1. Teacher to check.
2. 

3. a) Teacher to check.
b) Teacher to check.

4. a) \( VW = EF \)
    \( WX = FG \)
    \( VX = EG \)
b) \( JK = ST \)
    \( KL = TU \)
    \( JL = SU \)

5. a) \( \angle A = \angle D \)
    \( \angle B = \angle E \)
    \( \angle C = \angle F \)
b) \( \angle M = \angle P \)
    \( \angle N = \angle Q \)
    \( \angle O = \angle R \)

6. \( \angle A = \angle D = \angle K \)
    \( \angle B = \angle E = \angle L \)
    \( \angle C = \angle F = \angle M \)

7. a) \( \angle B = \angle E \)
    \( \angle C = \angle F \)
    \( BC = EF \)
    \( AC = DF \)
b) \( \angle R = \angle W \)
    \( \angle S = \angle Y \)
    \( \angle T = \angle X \)
    \( RS = WY \)
    \( ST = YX \)
    \( RT = WX \)
c) \( \angle A = \angle K = \angle C = \angle M \)
    \( \angle B = \angle L \)
    \( AB = KL = BC = LM \)
    \( AC = KM \)
d) \( \angle E = \angle P = \angle G = \angle O \)
    \( \angle F = \angle N \)
    \( EF = PN = FG = NO \)
    \( EG = PO \)

8. No, not necessarily.
   Kali forgot to check that the side length of the two shapes is equal (they might just be similar, not congruent).

9. The angles of a triangle always add to 180° so, if any two corresponding angles are equal, the third has to be equal as well.

10. \( \triangle ABC \cong \triangle RST \)

11. a) i) \( \angle A = \angle E = \angle B = \angle D \)
    \( \angle C = \angle F \)
ii) \( \triangle ABC \cong \triangle EDF \)
iii) Answers may vary – teacher to check.

b) i) All the angles are equal.
ii) Teacher to check – any order is correct.
iii) Teacher to check – any order is correct.

12. \( QR = LM \)
    \( PR = KM \)
    \( \angle Q = \angle L \)
    \( \angle R = \angle M \)

13. \( PR = LM \)
    \( \angle Q = \angle K \)
    \( \angle R = \angle M \)

14. a) Measurements may vary slightly:
    \( AB = 4.8 \text{ mm} \)
    \( AC = 3.4 \text{ mm} \)
    \( BC = 2.0 \text{ mm} \)
    \( DE = 4.6 \text{ mm} \)
    \( EF = 2.0 \text{ mm} \)
    \( DF = 5.0 \text{ mm} \)
    \( HI = 4.6 \text{ mm} \)
    \( IJ = 2.0 \text{ mm} \)
    \( HJ = 5.0 \text{ mm} \)
    \( KL = 5.5 \text{ mm} \)
    \( LM = 3.9 \text{ mm} \)
    \( KM = 2.4 \text{ mm} \)

b) Measurements may vary slightly:
   \( \angle A = 22^\circ \)
   \( \angle B = 38^\circ \)
   \( \angle C = 120^\circ \)
   \( \angle D = 23^\circ \)
   \( \angle E = 91^\circ \)
   \( \angle F = 66^\circ \)
   \( \angle H = 23^\circ \)
   \( \angle I = 91^\circ \)
   \( \angle J = 66^\circ \)
   \( \angle K = 39^\circ \)
   \( \angle L = 23^\circ \)
   \( \angle M = 118^\circ \)
   \( \triangle DEF \cong \triangle HIJ \)

15. None of Tom’s are correct.

INVESTIGATION 1

A. a) Only one unique triangle is possible (though it may be rotated, reflected, etc.)
b) Yes, there is only one way to arrange any combination of side lengths into a triangle.

B. a) There is only one way to complete the triangle (all the triangles drawn will be congruent).
b) Yes, if you know any two side lengths and the angle between those sides, there is only one way to draw the triangle.

C. a) They are congruent.
b) Yes, if you know one side length and the angles on either side of it, only one triangle is possible.

D. a) It is impossible to make more than one unique triangle.
INVESTIGATION 2

A. $BC = DE$, $AC = EF$ and $\angle B = \angle D$

B. No, since neither one has the equal angle between the two equal sides.

C. No

D. No, neither one is a congruence rule. The equal angles must be between the equal sides – otherwise more than one triangle can be drawn according to the specs.

1. a) SAS
   $\triangle ABC \cong \triangle DEF$

b) ASA
   $\triangle RST \cong \triangle ZXY$

c) SSS
   $\triangle MNO \cong \triangle FGH$

2. Teacher to check.

3. Teacher to check.

4. $\angle Q = \angle Y$

5. Need to know if any corresponding sides are equal.

6. a) Teacher to check.
   Possible answers:
   $\angle A \neq \angle D$, $\angle C \neq \angle F$, $BA \neq ED$, $CA \neq FD$

b) As above.

7. a) Teacher to check.
   Sample answer:

b) Teacher to check.
   Sample answer:

8. Yes, $\triangle ABC \cong \triangle GHI$ since corresponding sides and angles are all equal.
   Teacher to check sketch.

9. No, they aren’t always congruent – it depends where the equal sides are in relation to the given angle, e.g:

10. a) Check, from left to right: 2, 3, 6, 7
    b) Circle: 7

BONUS

In order to form a triangle, the sum of the two shorter sides must be greater than the length of the longest side.

AP Book G7-16

page 115

1. a) Left point
    b) Top point
    c) Uppermost (middle) point

2. a) Teacher to check.
    b) Teacher to check.
    c) $OA = OB$

3. a) No
    b) Yes

4. a) 2
    b) 1
    c) 0

5. Teacher to check.

6. Teacher to check.

7. Teacher to check.

8. a) Teacher to check – there are 2 possible locations for point $D$:
    b) $BD = CD$

AP Book G7-17

page 118

1. b) E
    c) D
    d) A
    e) C, F

2. a) SSS, $\angle A = \angle D$
    b) SAA, $BC = EF$
    or SAA, $AB = DE$

3. $A \cong C$ (SAS)

4. a) ASA
    b) SAS
    c) SSS
    d) SSS

5. a) Teacher to check.
    b) Teacher to check.
    c) i) $\triangle ADP \cong \triangle BDP$ by SSS
    ii) $\triangle ADP \cong \triangle BDP$ by SSS
    iii) IMPORTANT
       The sketch should include pre-drawn lines for $AP$ and $BP$,
       as shown here:

    4. a) $AC = BC$
       * radii of equal circles
    b) $AC = AB$
       * radii of the same circle
    c) $AB = AC = BC$
       * radii of equal circles

    5. a) $E$
    b) $C$
    c) $E$
    d) $A$
    e) $B$
    f) $D$
Answer Keys for AP Book 7.2

Geometry – AP Book 7, Part 2: Unit 5 (continued)

AP Book G7-19
page 122
1. Teacher to check.
2. a) Teacher to check.
   b) Teacher to check.
   c) AD is also a median in i), but not in ii).
3. a) Yes
   b) No, only the median through point A is a perpendicular bisector.
4. AB = BC, AD = DC
   \( \triangle ABD \cong \triangle CBD \), SSS
   \( \angle ADB = \angle CDB \)
   \( \angle ADB = \angle CDB \)
   90°
   AC
5. a) Teacher to check.
   b) \( \triangle ABD \cong \triangle CBD \) (SSS)
   c) \( \angle A = \angle C \)
   \( \angle ABD = \angle CBD \)
   \( \angle ADB = \angle CDB \)
   d) \( \angle ADB = \angle CDB \)
6. The following statements are true (the others should be crossed out as false):
   * BM is a median of \( \triangle ABC \)
   * BM is a line of symmetry for \( \triangle ABC \)
   * BM \perp AC
   * BM bisects \( \angle ABC \)
   * BM is a perpendicular bisector of AC
   * \( \triangle ABM \cong \triangle CBM \)
7. DC is an angle bisector. Students can check by measuring to find that BC = CA.

8. \( AC = AB \)
   Teacher to check for median from A to BC, which bisects \( \angle CAB \).
9. Teacher to check.
10. Question 5 shows that \( \triangle ABD \cong \triangle CBD \) by SSS, so we know from this that \( \angle ABD = \angle CBD \).
    Since \( \angle ABD + \angle CBD = \angle ABC \), BD bisects \( \angle ABC \).

AP Book G7-20
page 125
1. a) \( OR = PR, OP = OQ \)
   b) Teacher to check.
2. Teacher to check.
3. Teacher to check.
4. Teacher to check.
5. Teacher to check.
6. a) \( QT = QS \)
   b) \( TU = SU \)
   c) \( \triangle QTU \cong \triangle QSU \)
   d) SSS
7. a) The angles opposite the equal sides are equal (isosceles).
   b) All the angles are equal (equilateral).
8. a) 180° + 3 = 60°
   b) Teacher to check.
   c) \( \angle CAD = 30° \)
   \( \angle CDA = 90° \)
9. a) Use Yen’s method, and then bisect the 60° angle to form two 30° angle.
   b) Teacher to check.

AP Book G7-21
page 127
1. a) Teacher to check.
   b) Teacher to check.

INVESTIGATION
A. Teacher to check.
B. \( \angle D = \angle E \)

AP Book G7-22
page 128
1. The following sides will likely be marked – teacher to check.
   a)
   b)
2. Teacher to check.
3. a) Right angles occur at (and all around) points K and L.
   b) \( 90° + 90° + \angle KFL = 180° + \angle KFL \)
   c) No, the two angles already sum to 180° – this means that \( \angle KFL \) would be 0°, which is impossible.

AP Book G7-23
page 129
1. Teacher to check.
2. \( \angle QOR = 90° \)
   \( \angle POR = 180° \)
3. a) Teacher to check.
   b) SSS

AP Book G7-24
page 131
1. Teacher to check.
2. a) Teacher to check.
   b) \( AD = BD \)
   c) 90°
   d) 90°
   e) \( \triangle ADB \cong \triangle BDC \)

Teacher to check explanation (likely using ITT or SAS).

f) \( AC = BC \)

g) \( AE = BE \)

h) No, any point E along CD will work since \( \triangle AEB \) will always be isosceles.

i) No
INVESTIGATION

AC = BC
A. Isosceles
B. Teacher to check.
C. No, since \( \triangle ABC \) is isosceles, the angle bisector of \( \angle ACB \) must also be the perpendicular bisector of \( AB \).

3. \( BD = 3 \) cm
   Teacher to check explanation.
   
   Sample answer:
   If \( \triangle ABC \) is isosceles and \( \angle A = \angle B = 60^\circ \), it follows that \( \triangle ABC \) is actually an equilateral triangle. This means that \( AB = 6 \) cm. If \( D \) is the midpoint of \( AB \), then \( BD = 3 \) cm.
1. b) –
c) +
e) +
f) +
g) –
2. c) – 5 + 4
d) + 7 – 6
f) + 2 + 4 – 5 + 1
g) – 4 – 7 + 9 + 4
h) + 3 – 2 – 1 + 4
3. b) –
c) –
d) –
e) +
f) –
g) +
h) +
i) +
4. b) – 1
c) 0
d) 0
e) + 2
f) – 4
g) + 6
h) – 4
i) – 8
j) – 4
k) + 4
l) + 8
m) – 5
n) – 3
o) + 5
p) + 2
q) 0
r) – 79
5. b) + 4 – 3 – 2
   = + 4 – 5
c) + 8 – 6 – 4
   = + 8 – 10
d) + 9 + 2 – 6
   = + 11 – 6
BONUS
+ 4 + 2 + 4 + 1 + 2 – 3 – 1
= + 13 – 12
6. a) + 7 + 2 – 6
   = + 9 – 6
   = + 3
b) + 5 + 4 – 7
   = + 9 – 7
   = + 2
d) + 6 + 3 + 2 – 4 – 5
   – 8 – 1
   = + 11 – 18
   = – 7
e) + 5 + 6 + 8 + 1 – 4
   – 3 – 2 – 5 – 4
   = + 20 – 18
   = + 4
7. b) – 2
c) + 4
d) – 4
e) + 7
f) + 2
g) + 3
h) – 8
i) – 8
j) + 8
8. The two +4s should not have been cancelled.

AP Book NS7-88

page 135
1. b) gain of 4
c) loss of 1
d) gain of 9
2. b) (– 2) + (+ 6) + (– 3)
c) 4 + (+ 2) + (– 6)
d) 7 + (– 5) + (– 4)
e) (– 3) + (– 2) + (+ 4)
f) (– 3) + (– 5) + (– 4)
3. b) + 2 + 7
c) – 2 + 7
d) – 2 – 7
e) + a – b
f) + a + b
g) – a + b
h) – a – b
8. b) – 4
c) + 6
d) – 8
e) + 3
f) 0
g) – 10
h) – 7
i) – 12
j) – 2
k) – 5
l) + 5
9. a) True
b) False, sample counter-example:
   – 5 + 7 = + 2
### AP Book NS7-89  
**page 137**

1. Teacher to check number lines.
   - a) – 2
   - b) – 5
   - c) + 4
   - d) – 1
   - e) 0
   - f) 0

2. Teacher to check.

### INVESTIGATION

A. Teacher to check number lines.
   - a) – 8; – 8
   - b) + 6; + 6
   - c) – 10; – 10
   - d) – 4; – 4
   - e) – 3; – 3

B. No

3. a) – 1, – 4, – 7
   - b) – 2, 0, + 2

### AP Book NS7-90  
**page 138**

1. a) 3, left
   - b) 5, right
   - c) 2
   - d) 6
   - e) 3, left
   - f) 3, right
   - g) 2, left
   - h) 2, right
   - i) 1

2. Teacher to check.

### AP Book NS7-91  
**page 140**

1. a) 1°, – 1°
   - b) 4°, – 4°
   - c) 2°, – 2°
   - d) 1°, – 1°
   - e) 3°, – 3°
   - f) 3°, – 3°
   - g) 4°, – 4°

2. They are the negative/opposite of each other:
   - a + b = – (b – a)
   - b) – 3
   - c) Teacher to check.

3. a) – 5, – 5
   - b) – 6, – 6
   - c) – 6, – 6
   - d) – 7, – 7

### AP Book NS7-92  
**page 142**

1. a) 3
   - b) 5
   - c) 2
   - d) 6
   - e) 3
   - f) 9
   - g) 5
   - h) 9
   - i) 9
   - j) – 9
   - k) – 9
   - l) 9

2. b) + 5, + 1
   - c) + 8, + 5
   - d) + 9, + 5
   - e) + 9, + 4
   - f) + 9, + 6

3. (+ a) – b and (– b) – a are the same;
   - b) – 5, + 5
   - c) – 8, + 8
   - d) – 9, + 9
   - e) – 9, + 9
   - f) – 9, + 9

4. b) 7
   - c) 3 – (– 3) = 6
   - d) 4 – (– 8) = 4
   - e) 2 – (– 6) = 8

5. a) 7, – 7
   - b) 7, – 7

6. a) positive
   - b) negative

7. b) + 12
   - c) – 3
   - d) + 5
   - e) – 9
   - f) + 4
5. b) – 3  
c) – 4  
d) – 4  
e) – 1  
f) – 4  
g) 2  
h) – 11  

6. Yes, they are essentially the same question – just the order of the addends has switched.

AP Book NS7-94  
page 144  
1. a) Brooke is right:  
2 000 – (– 300)  
= 2 300 m  
b) 406 – (– 10 911)  
= 11 317 m  

2. – 19°C, – 18°C, 0°C, 3°C, 15°C, 21°C, 24°C  

3. a) 5°C  
b) 0°C  
c) – 10°C  
d) – 25°C  

4. a) 5°C  
b) 10°C  
c) – 15°C  
d) 5°C  

5. Teacher to check that points are properly marked on number lines:  

A  – 2  
B  + 1  
C  + 2  
D  + 3  
E  + 4  
F  – 5  
G  + 9  
H  – 8 or + 8  

6. a) – 2  
b) – 10  

7.  

<table>
<thead>
<tr>
<th></th>
<th>–1</th>
<th>+4</th>
</tr>
</thead>
<tbody>
<tr>
<td>+5</td>
<td>+1</td>
<td>–3</td>
</tr>
<tr>
<td>–2</td>
<td>+3</td>
<td>+2</td>
</tr>
</tbody>
</table>

8. Toronto 25°C  
Montreal 31°C  
Vancouver 26°C  

b) 650°C  
c) Saturn  

10. a) – 5°C  
b) – 7°C  

11. a) 18 + (– 12) = 6°C  
b) 13 – (– 3) = 16°C  

12. a) + 12, if you spun + 6 both times  
b) – 18, if you spun – 9 both times  
c) + 6 – (– 9) = 15  
d) If you spun + 5 and – 5 (in either order)  

13. Monday 10°C  
Tuesday 2°C  
Wednesday 4°C  
Thursday 12°C
Geometry – AP Book 7, Part 2: Unit 7

**AP Book G7-26**
*page 149*

1. a) A (-9,5) B (-10,-6)
   b) C (-5,-8)
   c) D (0,7)
   d) E (7,6)
   e) F (9,0)
   f) G (0,0)
   g) H (-3,5)

2. a) Teacher to check.
   b) y-axis
   c) x-axis

3. a) A (-13.11) B (15.11)
   b) C (15.10) D (-13.10)

4. Teacher to check plot.
   *ABCD* is a rhombus.

5. Teacher to check plots.
   a) Right-angle triangle
   b) Square
   c) Vertical;
      Teacher to check selected points.
      All points on the line *PQ* have an x-coordinate = 3.
   d) No, because the x-coordinate is always < 0 in QII.
   e) Yes, since the x-coordinate = 3.
   f) Teacher to check points.
      Any point below *P* on the line will have a y-coordinate < 2.

8. a) Teacher to check.
   b) Greater than 3
   c) Smaller than 3
   d) To the right of; 5 > 3
   e) To the left of; –2 < 3
   f) On; 3 = 3

9. a) Greater than 3
   b) Smaller than 3
   c) Below; –2 < 3
   d) Above; 4 > 3
   e) On; 3 = 3

10. a) A (6,4) B (-4,4)
    b) C (-4,-6) D (6,-6)
    c) Right of
    d) Left of
    e) Above
    f) Below

**INVESTIGATION**

A. i) (5,-2)
   ii) (-2,-3)
   iii) (3,1)
   iv) (7,0)

B. The x-coordinate
   C. x, 4;
      x – x + 4

D. i) E' (3,-3)
    ii) F' (2,2)

E. Teacher to check plots.
F. (x + 3, y)

G. Teacher to check plots.
   a) P' (3,0)
   b) P' (1,2)
   c) P' (1,2)
   d) x, decreased by 2

7. D, B, A, C

**AP Book G7-28**
*page 154*

1. b) A (2,-1) B (-2,1)
   c) A (1,2) B (-2,-2)

2. Teacher to check plots.
   a) A (-3,2) → A' (3,-1)
   b) A (3,2) → A' (-2,2)
   c) A (2,2) → A' (0,6)

3. Teacher to check plots.
   a) P' (-3,1) → P' (-3,-2)
   b) P (3,0) → P' (3,3)
   c) P (-3,-1) → P' (-3,-4)

4. Teacher to check plots.
   a) A (-5,2) → A' (1,0)
   b) B (-2,2) → B' (4,0)
   c) C (-5,0) → C' (1,2)
b) \(A(-5,3) \rightarrow A'(1,1)\) 
\(B(-2,1) \rightarrow B'(4,-1)\) 
\(C(-5,2) \rightarrow C'(1,0)\)

5. a) 2, up
b) Teacher to check.
c) Teacher to check.
d) 5, 2, up
e) Yes
f) \(A(-1,-1) \rightarrow A'(4,1)\) 
\(B(-3,-1) \rightarrow B'(2,1)\) 
\(C(-4,-3) \rightarrow C'(1,-1)\) 
\(D(-2,-3) \rightarrow D'(3,-1)\)
g) 2, up: 
\((x + 5, y + 2)\).

6. Answers will vary; teacher to check.

7. b) \(+4 + (-3) = +1;\) 1 unit up
c) \((-6) + (+2) = -4;\) 4 units down
d) \((-7) + (-11) = -18;\) 18 units down

8. b) \(+6 + (-4) = +2;\) 2 units right
c) \((-7) + (+2) = -5;\) 5 units left
d) \((-8) + (+12) = +4;\) 4 units right

9. a) \(+3 + (-5) = -2\) 
\(.2 \text{ units down}\)
\(+2 + (+4) = +6\) 
\(.6 \text{ units right}\)
b) \((-4) + (-3) = -7\) 
\(.7 \text{ units down}\)
\(+8 + (-6) = +2\) 
\(.2 \text{ units right}\)
c) \(+2 + (+3) = +5\) 
\(.5 \text{ units up}\)
\((-7) + (+5) = -2\) 
\(.2 \text{ units left}\)

10. a) i) Predictions will vary. The actual coordinates are: 
\(A'(-3,3)\) 
\(B'(-3,5)\) 
\(C'(0,5)\) 
\(D'(0,3)\)

ii) Teacher to check.
iii) \(A^*(0,-2)\) 
\(B^*(0,0)\) 
\(C^*(3,0)\) 
\(D^*(3,-2)\)

Under the two translations, \(ABCD\) moved 3 units down: 
\(+2 + (-5) = -3\) and 1 unit left: 
\((-4) + 3 = -1.\)

b) \(A'(7,4) \rightarrow B'(6,1)\) 
\(C'(4,1) \rightarrow D'(6,4)\)

Under the two translations, \(ABCD\) moved 6 units right: 
\(+4 + (+2) = +6\) and 4 units up: 
\((-3) + (+7) = +4.\)

6. The x-coordinate changes to its opposite (i.e. it changes signs). 
The y-coordinate stays the same.

7. a) \(A(4,3) \rightarrow A'(4,-3)\) 
\(B(4,-2) \rightarrow B'(-4,-2)\) 
\(C(1,1) \rightarrow C'(1,1)\) 
\(D(1,2) \rightarrow D'(-1,2)\)
b) \(A(-1,3) \rightarrow A'(1,3)\) 
\(B(2,1) \rightarrow B'(-2,1)\) 
\(C(-2,2) \rightarrow C'(2,-2)\)
c) \(A(0,3) \rightarrow A'(0,3)\) 
\(B(3,0) \rightarrow B'(-3,0)\) 
\(C(0,-2) \rightarrow C'(0,2)\)

AP Book G7-29
page 157

1. Teacher to check plots.
2. Teacher to check plots.
   a) \(P(2,1) \rightarrow P'(2,1)\) 
   \(Q(1,3) \rightarrow Q'(1,3)\) 
   \(R(-3,2) \rightarrow R'(-3,2)\)
   b) \(P(1,1) \rightarrow P'(1,1)\) 
   \(Q(-3,1) \rightarrow Q'(-3,1)\) 
   \(R(0,3) \rightarrow R'(0,3)\)
   c) \(P(-3,0) \rightarrow P'(-3,0)\) 
   \(Q(2,0) \rightarrow Q'(2,0)\) 
   \(R(0,2) \rightarrow R'(0,2)\)
3. Teacher to check plots.
   a) \(A(-1,1) \rightarrow A'(-1,1)\) 
   \(B(3,1) \rightarrow B'(3,1)\) 
   \(C(1,-3) \rightarrow C'(1,3)\)
   b) \(A(-3,3) \rightarrow A'(-3,-3)\) 
   \(B(2,3) \rightarrow B'(2,-3)\) 
   \(C(2,1) \rightarrow C'(-2,1)\)
   c) \(A(-2,0) \rightarrow A'(-2,0)\) 
   \(B(0,3) \rightarrow B'(0,-3)\) 
   \(C(3,0) \rightarrow C'(3,0)\)

4. Teacher to check plots.
   b) \(P(-3,-2) \rightarrow P'(3,-2)\)
   c) \(P(-4,0) \rightarrow P'(4,0)\)

5. a) \(Q(3,1) \rightarrow Q'(-3,1)\) 
   \(R(1,-3) \rightarrow R'(-1,-3)\)
   b) \(P(-3,1) \rightarrow P'(3,1)\) 
   \(Q(-3,3) \rightarrow Q'(3,3)\) 
   \(R(-2,1) \rightarrow R'(2,-1)\)
   c) \(P(0,3) \rightarrow P'(0,3)\) 
   \(Q(0,-2) \rightarrow Q'(0,2)\) 
   \(R(-3,0) \rightarrow R'(3,0)\)

6. The x-coordinate changes to its opposite (i.e. it changes signs). 
The y-coordinate stays the same.

7. a) \(A(4,3) \rightarrow A'(-4,3)\) 
   \(B(4,-2) \rightarrow B'(-4,-2)\) 
   \(C(1,1) \rightarrow C'(-1,1)\) 
   \(D(1,2) \rightarrow D'(-1,2)\)
   b) \(A(-1,3) \rightarrow A'(1,3)\) 
   \(B(2,1) \rightarrow B'(-2,1)\) 
   \(C(-2,2) \rightarrow C'(2,2)\)
   c) \(A(0,3) \rightarrow A'(0,3)\) 
   \(B(3,0) \rightarrow B'(-3,0)\) 
   \(C(0,-2) \rightarrow C'(0,2)\)

AP Book G7-30
page 159

INVESTIGATION
A. a) Vertical reflection
b) \[
\begin{array}{c|c}
A \text{ and } A' & 6 \\
A \text{ and } M_A & 3 \\
B \text{ and } B' & 8 \\
B \text{ and } M_B & 4 \\
C \text{ and } C' & 4 \\
C \text{ and } M_C & 2 \\
\end{array}
\]

b) \(M_A\) is halfway between \(A\) and \(A'\).
\(M_B\) is halfway between \(B\) and \(B'\).
\(M_C\) is halfway between \(C\) and \(C'\).

2. a) Reflection through the line \(x = 1\)
b) Translation 6 units to the left
c) Reflection through the line \(y = -1\)
d) Yes, Ron is correct.
e) Teacher to check.

3. a) Teacher to check.
b) Teacher to check.
c) Ying is incorrect. 
The triangles each face a different direction so a rotation is also required.

4. a) Teacher to check.
b) Teacher to check.
c) They are the same.

5. a) Teacher to check.
   b) No
   c) Yes
   d) Instead of up or down, directions have to be thought of in terms of the mirror line, e.g. closer or further.
   Moving closer to the mirror line might require you to go either up or down, depending on whether the shape is above or below the line.

6. a) Teacher to check.
   b) Translate 2 units left and 3 units up, then reflect through mirror line t.

AP Book G7-32

page 164

1. Teacher to check.
2. a) $\angle ABO = \angle OB'$
   $\angle ABO = \angle A'B'O$
   $\angle BOB' = \angle AOB'$
   The triangles are congruent.
   b) They are equal.
   c) 90°
   d) 90°
   Teacher to check explanation.
   Sample answer:
   $\angle AOA' = \angle AOB' + \angle A'OB'$
   $\angle BOB' = \angle AOB'$
   and $\angle A'OB' = \angle AOB$ (due to symmetry of $\triangle AOA'$ and $\triangle AOB$)
   $\therefore \angle AOA' = \angle AOB' + \angle A'OB' = \angle AOB' + \angle AOB
   = 90°$
   e) Rotation 90° CW
3. a) I
   b) IV
   c) Teacher to check.
   d) 3, 2, $P'(3,-2)$
4. a) III, II;
   b) 2; 8, $Q'(-2,8)$
   c) $Q'(2,-8)$ is in quadrant IV. $Q'$ has the same horizontal and vertical lengths as $Q$.
5. Yes, they are equal but they are not the same. $Q'$ and $Q''$ are 180° rotations of each other (in either direction).

AP Book G7-33

page 162

1. Teacher to check.
2. 90° CW → 270° CCW
   180° CW → 180° CCW
   270° CW → 90° CCW
3. Teacher to check that drawings are correct.
   a) $P : I$
   $P' : IV$
   b) $P : II$
   $P' : I$
   c) $P : III$
   $P' : II$
   d) $P : IV$
   $P' : III$
4. Teacher to check that drawings are correct.
   a) $P : III$
   $P' : I$
   b) $P : IV$
   $P' : II$
   c) $P : I$
   $P' : II$
   d) $P : II$
   $P' : IV$
5. Yes, they are equal but they are not the same. $Q'$ and $Q''$ are 180° rotations of each other (in either direction).

AP Book G7-33

page 164

1. B: Reflection through the mirror line $y = -1$.
   C: Translation 6 units right and 7 units down.
   D: 90° CW rotation about the origin.
2. Answers will vary depending on assignment of F and G.
   a) F (-2, 1) G (-2, -3)
   b) Translation 5 units left and 2 units down (image is the same size and direction)
   c) Sample answer: Reflection through the x-axis followed by a translation 4 units left (image is the same size but opposite direction)
   d) 4. Teacher to check.
INVESTIGATION 1
A. Teacher to check.
B. Teacher to check.
C. No; translation 3 units left and 5 units down
D. No
Teacher to check explanations.
Sample explanation:
A and B face in different directions so one is not a rotation of the other. They do not face opposite directions so they’re not reflections of each other.
A rotation of A around the origin can’t take (1,1) to (3,0) so it’s also not a rotation around the origin.
Note that a 90° CW rotation around the origin can’t take (1, 1) to (0, 3) so it’s also not a rotation around the origin.

INVESTIGATION 2
A. Teacher to check.
B. Teacher to check.
C. No
D. Reflection through the y-axis.
Students can tell this by noticing that the shapes face in opposite directions to the y-axis.

BONUS
A reflection through the mirror line with equation x = y, i.e. the diagonal line passing through (1,1), (0,0), (-1,-1), etc.
INVESTIGATION

A.

<table>
<thead>
<tr>
<th># of vertices</th>
<th>sum of interior angles</th>
<th>measure of one interior angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>180°</td>
<td>180° ÷ 3 = 60°</td>
</tr>
<tr>
<td>4</td>
<td>360°</td>
<td>360° ÷ 4 = 90°</td>
</tr>
<tr>
<td>5</td>
<td>540°</td>
<td>540° ÷ 5 = 108°</td>
</tr>
<tr>
<td>6</td>
<td>720°</td>
<td>720° ÷ 6 = 120°</td>
</tr>
<tr>
<td>7</td>
<td>900°</td>
<td>900° ÷ 7 = 128.6°</td>
</tr>
<tr>
<td>8</td>
<td>1080°</td>
<td>1080° ÷ 8 = 135°</td>
</tr>
</tbody>
</table>

B. Increases;
Yes, since the pattern increases and a regular octagon has angles of 135°, a regular polygon with more than 8 sides will have angles greater than 135°.

C. a) The number of copies of the polygon around the common vertex
b) Since the polygons can’t overlap or have gaps, the interior angles around the common vertex must divide evenly into 360°.

D.

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<td>128.6°</td>
<td>2.8</td>
</tr>
<tr>
<td>135°</td>
<td>2.5</td>
</tr>
</tbody>
</table>

E. Regular triangles, quadrilaterals and hexagons will tessellate (since each of their interior angles divides evenly into 360°).

F. Teacher to check.

Sample answer:
To create a common vertex, you need at least 3 shapes coming together. As the number of sides in a regular polygon increases, the size of the polygon’s interior angles increase too so any regular polygon with more than 6 sides will produce an angle greater than 360° (120° × 3) when placed around a common vertex. This means they will overlap and therefore won’t tessellate.

G. Regular triangles, quadrilaterals and hexagons will tessellate (since each of their interior angles divides evenly into 360°).
13. a) \( \angle A = \angle D \approx 21^\circ \)
   \( \angle B = \angle E \approx 120^\circ \)
   \( \angle C = \angle F \approx 39^\circ \)
   \( AB : DE = 3 : 4.7 \)
   \( BC : EF = 1.8 : 2.7 \)
   \( CA : FD = 4.4 : 6.6 \)
   All these ratios are equal (\( \approx 0.66 \)).

b) Yes, \( \triangle ABC \) and \( \triangle DEF \) are similar (see a) for evidence).

14. a) \( x = 12, y = 15 \)
   b) \( x = 16, y = 12 \)

15. a) No, two of their angles are different.
   b) No, a pair of the corresponding sides isn’t proportionate.

AP Book G7-37

page 177

1. a) \( OB' : OB = 6 : 1 \)
   b) \( OA' : OA = 3 : 1 \)
   \( OL' : OL = 10 : 3 \)
   \( AM' : AM = 10 : 3 \)
   \( L'M' : LM = 3 : 1 \)

2. \( OK' : OK = 3 : 1 \)
   \( OM' : OM = 4.5 : 1.5 = 9 : 3 = 3 : 1 \)

3. 6, 3, 4, 5

4. Teacher to check.

5. a) ii) Teacher to check for \( P' \):
   \( 6, 15 \)
   b) \( OP' : OP = 3 : 1 \)
   This ratio is the same as the ratios for the horizontal and vertical distances.

6. Teacher to check.

7. Teacher to check.

INVESTIGATION

A. 37°, 53°, 90°, 37°, 53°, 90°
   Students should notice that \( \angle A = \angle A' \), \( \angle B = \angle B' \) and \( \angle C = \angle C' \).
1. a) 6
   [1, 2, 3, 4, 5, 6]
b) 2
   [heads, tails]c) 3
   [win, lose, draw]
2. a) The spinner lands in regions 1, 2, 3 or 4. There are 4 outcomes.
b) The spinner lands in regions 2, 3 or 4. There are 3 outcomes.
3. a) 12, 4, 20
b) 1, 3, 7, 9, 11
c) 11, 12, 20
4. a) 3
b) 4, yes
c) 5, no
5. a) 1; 3
b) 3; 8
6. a) 1; 3
b) 2; 4
7. a) 2: (1,3), (2,2)
b) 2: (1,4), (2,2)
8. 2: (2,3), (3,2)
9. a) 2/6 = 1/3
b) 1/4
c) 3/4
d) 3/8
10. a) 1/6
    b) 2/6 = 1/3
c) 2/6 = 1/3
d) 6/6 = 3/4
e) 3/6
    f) 4/6 = 1/2
11. a) 1, 2, 3, 4, 5, 6
    b) 6
12. a) 2, 4, 6
    b) 3
    c) 3/6 = 1/2
    d) 4/6 = 1/2
13. a) 5, 6
    b) 2
    c) 2/6 = 1/3
14. D A
    C B
15. a) 2/5
    b) 1/5
    c) 1/5
d) 3/5
e) 2/5
    f) 3/5
16. Emma is right.
   There are 6 possible rolls on a die (1, 2, 3, 4, 5, 6) and each one is equally likely, so each roll (including a 5 or a 1) has a probability of 1/6.
17. Teacher to check.
18. Teacher to check.
9. a) | Question | at 1st level | at 2nd level | Total # paths |
<table>
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<td>3</td>
<td>3</td>
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</tbody>
</table>

b) You simply multiply them.

c) It will have 20 × 12 = 240 paths.

AP Book PDM7-20 page 188

1. 1st B B B R R R Y Y Y
   2nd 1 2 3 1 2 3 1 2 3
   a) 3
   b) 3 times each
   c) 9

2. 1st Y Y Y Y G G G G
   2nd 1 2 3 4 1 2 3 4
   a) 4
   b) 4 times each
   c) 8

3. 1st G G Y Y B B
   2nd 1 2 1 2 1 2
   a) 2
   b) 2 times each
   c) 6

4. C H H H T T T
   S 1 2 3 1 2 3
   So the 6 outcomes are:
   (H,1), (H,2), (H,3), (T,1), (T,2), (T, 3)

5. LP RP Value
   Q D 35¢
   Q N 30¢
   D D 20¢
   D N 15¢

6. a) Morning Afternoon
   painting drama
   painting pottery
   painting dance
   music drama
   music pottery
   music dance

   b) Because there are 3 options for the afternoon.

AP Book PDM7-21 page 190

1. a) ii) R, G, G, B, G
   iii) G, G, B, B, G
   b) Yes

2. a) i) R, G, G, B, B, G
   ii) R, G, G, B, B, G
   iii) R, G, G, B, B, G
   b) No

3. Jade's

4. a) Yes
   b) Yes
   c) Yes
   d) Yes
   e) No
   f) No

5. For e), since the coin is not replaced after the first draw.
   For f), since the same student can't be picked as captain twice (the team picking second has fewer choices).

6. a) Bob:

   Nick:

   b) Yes, because the number of outcomes with both a cat and a dog increase.

7. a) (1, A) (2, A) (3, A) (1, B) (2, B) (3, B)
   b) 6
   c) i) 1: (1, A)
      ii) 2: (1, B), (3, B)
   d) i) \( \frac{1}{6} \)
      ii) \( \frac{2}{6} = \frac{1}{3} \)

8. a) (1, A) (2, A) (3, A) (1, B) (2, B) (3, B) (1, C) (2, C) (3, C)
   b) i) \( \frac{1}{9} \)
      ii) \( \frac{2}{9} \)
   c) Answer will vary.

9. a) (L-5, R-10)
   (L-10, R-10)
   (L-5, R-5)
   (L-10, R-5)
   b) \( \frac{1}{4} \)
   c) Answer will vary.
   d) Answer will vary but, yes, it will likely be different.

10. a) Yes, because the number of outcomes with both a cat and a dog increase.
d) 13
e) 34
f) 41
g) 9
h) 11
i) 17
j) 21
k) 16
l) 24

2. a) 19
b) 28
c) 13
d) 14
e) 14
f) 19

3. \(\frac{1}{2}\)

4. a) 6
b) 20
c) 34

5. a) \(\frac{1}{3}\)
b) I would expect to spin red \(\frac{1}{4}\) of the time.

6. a) 6, 11, 23
b) 6, 11, 23

7. Teacher to check.

8. b) 7, 14
c) 26
d) 28
e) 50
f) 36
g) 42
h) 45

9. Teacher to check.

10. a) \(\approx 20\) times
b) \(\approx 25\) times

11. a) \(\frac{1}{8}\)
b) 25

12. a) \(\frac{10}{50} = \frac{1}{5}\)
b) 20%
c) \(\frac{1}{5}\)
d) 200

13. Forest B: the survival rate is lower but the number of surviving birds is higher (6 000 vs 4 000).

14. a) i) \(\frac{200}{1000} = \frac{1}{5}\)
ii) \(\frac{250}{1000} = \frac{1}{4}\)
iii) \(\frac{400}{1000} = \frac{2}{5}\)
b) i) \(\frac{1}{5}\) of 60 = 12
ii) \(\frac{1}{4}\) of 60 = 15
iii) \(\frac{2}{5}\) of 60 = 24

AP Book PDM7-23

1. a) 5
b) No, Daniel is wrong – the game is fair. The fact that so many spins were yellow this time is unlikely, but definitely possible.

2. a)

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b) i) \(\frac{18}{36} = \frac{1}{2}\)
ii) \(\frac{9}{36} = \frac{1}{4}\)
iii) \(\frac{10}{36} = \frac{5}{18}\)
c) 2, and 12
d) 7

6. B is more likely:
\(\frac{1}{2}\) compared to \(\frac{1}{36}\).

b) Circle, as shown:
(4,1), (3,2), (2,3), (1,4)
c) \(\frac{4}{32} = \frac{1}{8}\)
d) \(\frac{1}{8}\) of 40 = 5

3. a) The possible combinations are $5/$10, $5/$50 and $10/$50 but each pair could be pulled out in either order.

b) The chances are unlikely, only 1 in 3.
c) Teacher to check.

4. She wrote the number 1 six times because there are 6 sides on a die so 6 possible rolls for the second die.

5. a) i) \(\frac{3}{36} = \frac{1}{12}\)
ii) \(\frac{5}{36}\)
iii) \(\frac{6}{36} = \frac{1}{6}\)
iv) \(\frac{2}{36} = \frac{1}{18}\)
v) \(\frac{1}{36}\)
1. Reduce the following fractions to lowest terms.
   a) \( \frac{8}{64} \)  
   b) \( \frac{6}{48} \)  
   c) \( \frac{9}{12} \)  
   d) \( \frac{2}{24} \)

2. Decide whether each fraction is a decimal fraction. Write yes or no.
   a) \( \frac{3}{4} \)  
   b) \( \frac{7}{100} \)  
   c) \( \frac{10}{15} \)  
   d) \( \frac{184}{10000} \)

3. Change each fraction to a decimal by first writing it as a decimal fraction.
   a) \( \frac{3}{5} \)  
   b) \( \frac{3}{4} \)  
   c) \( \frac{16}{25} \)  
   BONUS: \( \frac{6}{15} \)

4. Change the fractions to decimals by using long division.
   a) \( \frac{1}{4} = \)  
   b) \( \frac{3}{15} = \)

5. Write the numbers in order from least to greatest: 0.43, 0.43, 0.43, 0.4
   _____ < _____ < _____ < _____

6. Change both numbers to be in the same form. Circle the greater number in each pair.
   a) \( \frac{2}{5} \) and 0.42  
   b) \( \frac{4}{25} \) and 0.15  
   BONUS: \( \frac{3}{8} \) and 0.367
Unit 1: Number Sense

Quiz (Lessons 55–57, 59) — ON

1. a) \(\frac{1}{8}\)
b) \(\frac{1}{8}\)
c) \(\frac{3}{4}\)
d) \(\frac{1}{12}\)

2. a) No
b) Yes
c) No
d) Yes

3. a) \(\frac{3}{5} = \frac{6}{10} = 0.6\)
b) \(\frac{3}{4} = \frac{75}{100} = 0.75\)
c) \(\frac{16}{25} = \frac{64}{100} = 0.64\)

**BONUS**
\(\frac{6}{15} = \frac{2}{5} = \frac{4}{10} = 0.4\)

4. a) 0.25
b) 0.2

5. 0.4 < 0.43 < 0.43 < 0.43

6. a) \(\frac{40}{100} \text{ and } \frac{42}{100}\)
   or 0.4 and 0.42;
   Circle: 0.42
b) \(\frac{16}{100} \text{ and } \frac{15}{100}\)
   or 0.16 and 0.15;
   Circle: \(\frac{4}{25}\)

**BONUS**
\(\frac{375}{1000} \text{ and } \frac{367}{1000}\)
   or 0.375 and 0.367;
   Circle: \(\frac{3}{8}\)
Unit 1: Number Sense

Quiz (Lessons 55–59) — WNCP

1. Reduce the following fractions to lowest terms.

   a) \( \frac{8}{64} \)  
   b) \( \frac{6}{48} \)  
   c) \( \frac{9}{12} \)  
   d) \( \frac{2}{24} \)

2. Write the numbers in order from least to greatest: 0.43  0.43  0.43  0.4

   _____ < _____ < _____ < _____

3. Change both numbers to be in the same form. Circle the greater number in each pair.

   a) \( \frac{2}{5} \) and 0.42  
   b) \( \frac{4}{25} \) and 0.15  
   BONUS: \( \frac{3}{8} \) and 0.367

4. Change the fractions to decimals using long division. If the decimal value of the fraction is repeating, write it using bar notation.

   a) \( \frac{2}{5} = \) _____  
   b) \( \frac{1}{3} = \) _____  
   BONUS: \( \frac{18}{55} = \) _____
Unit 1: Number Sense

Quiz (Lessons 55–59) — WNCP

1. a) \(\frac{1}{8}\)
   b) \(\frac{1}{8}\)
   c) \(\frac{3}{4}\)
   d) \(\frac{1}{12}\)

2. \(0.4 < 0.43 < 0.43 < 0.43\)

3. a) \(\frac{40}{100}\) and \(\frac{42}{100}\)
   or 0.4 and 0.42;
   \(\text{Circle: } 0.42\)
   b) \(\frac{16}{100}\) and \(\frac{15}{100}\)
   or 0.16 and 0.15;
   \(\text{Circle: } \frac{4}{25}\)

BONUS
\(\frac{375}{1000}\) and \(\frac{367}{1000}\)
or 0.375 and 0.367;
\(\text{Circle: } \frac{3}{8}\)

4. a) 0.4
b) 0.3

BONUS
0.327
1. Circle whether the fraction is equivalent to a terminating or repeating decimal.
   a) \( \frac{1}{4} \) Terminating  
      Repeating
   b) \( \frac{1}{12} \) Terminating  
      Repeating
   c) \( \frac{7}{25} \) Terminating  
      Repeating

2. Write the fraction in lowest terms. Then circle whether the fraction is equivalent to a terminating or repeating decimal.
   a) \( \frac{6}{16} = \)  
      Terminating  
      Repeating
   b) \( \frac{45}{100} = \)  
      Terminating  
      Repeating
   c) \( \frac{10}{15} = \)  
      Terminating  
      Repeating

3. Sam thinks that the fraction \( \frac{3}{12} \) is equivalent to a repeating decimal because the denominator cannot be written as a product of only 2s and/or 5s. Is he correct? Explain.

4. Add or subtract the decimals by lining up the decimal places.
   a) \( 0.45 + 0.2 = \)  
   b) \( 0.5 - 0.13 = \)

5. Write the fraction as a repeating decimal.
   a) \( \frac{37}{99} = \)  
   b) \( \frac{5}{9} = \)  
   c) \( \frac{8}{99} = \)  
   d) \( \frac{245}{999} = \)
Unit 1: Number Sense

Quiz (Lessons 60–63) — WNCP

6. Change the fraction to an equivalent fraction with denominator 9 or 99. Then write the answer as a repeating decimal.

   a) \( \frac{14}{33} = \) \[ \quad \]
   b) \( \frac{2}{3} = \) \[ \quad \]
   c) \( \frac{5}{11} = \) \[ \quad \]

   = \[ \quad \] = \[ \quad \] = \[ \quad \]

BONUS: Write \( \frac{7825}{9999} \) as a repeating decimal. \[ \quad \]

ADVANCED: Write \( 0.\overline{47} \) as a fraction by following the steps.

   a) Write 0.4 as a fraction. \[ \quad \]
   b) Write 0.\( \overline{7} \) as a fraction. \[ \quad \]
   c) Write 0.0\( \overline{7} \) as a fraction. \[ \quad \]

   d) \( 0.\overline{47} = 0.4 + 0.0\overline{7} = \) \[ \quad \] + \[ \quad \] = \[ \quad \] + \[ \quad \] = \[ \quad \]
Unit 1: Number Sense

Quiz (Lessons 60–63) — WNCP

1. a) terminating
   b) repeating
   c) terminating

2. a) \( \frac{3}{8} \), terminating
   b) \( \frac{9}{20} \), terminating
   c) \( \frac{2}{3} \), repeating

3. No. The fraction in lowest terms is \( \frac{1}{4} \), which terminates.

4. a) 0.67
   b) 0.42

5. a) 0.37
   b) 0.5
   c) 0.08
   d) 0.245

6. a) \( \frac{42}{99} = 0.42 \)
   b) \( \frac{6}{9} = 0.6 \)
   c) \( \frac{45}{99} = 0.45 \)

BONUS

0.7825

ADVANCED

a) \( \frac{2}{5} \)

b) \( \frac{7}{9} \)

c) \( \frac{7}{90} \)

d) \( \frac{2}{5} + \frac{7}{90} \)
   \[ = \frac{36}{90} + \frac{7}{90} \]
   \[ = \frac{43}{90} \]
1. Circle the unit fractions. Underline the decimal fractions.

   a) \( \frac{23}{100} \)  
   b) \( \frac{1}{8} \)  
   c) \( \frac{1}{4} \)  
   d) \( \frac{7}{100} \)

2. Write the fraction as a decimal.

   a) \( \frac{23}{100} \)  
   b) \( \frac{7}{10} \)  
   c) \( \frac{9}{100} \)  
   d) \( \frac{73}{100} \)

   = _____  
   = _____  
   = _____  
   = _____

3. Write the fraction as a decimal fraction.

   a) \( \frac{3}{10} \)  
   b) \( \frac{9}{20} \)  
   c) \( \frac{3}{4} \)  
   d) \( \frac{31}{50} \)

4. Use long division to write the decimal as a fraction. Keep dividing until the remainder is 0.

   a) \( \frac{3}{4} \)  
   b) \( \frac{5}{8} \)  
   c) \( \frac{7}{16} \)
5. Write the fraction as a decimal. Circle the decimal that is closest to the fraction.

   a) \( \frac{2}{5} = \) 

   \( \frac{2}{5} \) is closest to: 0.2 0.5 0.8 1.1

   b) \( \frac{3}{4} = \) 

   \( \frac{3}{4} \) is closest to: 0.3 0.5 0.7 0.9

6. Use a calculator to write each fraction as a decimal. Round your answer to three decimal places. Circle the fraction that is closest to the decimal.

   \( \frac{7}{8} = \)  

   \( \frac{11}{16} = \)  

   \( \frac{3}{4} = \) 

   0.7 is closest to: \( \frac{7}{8} \) \( \frac{11}{16} \) \( \frac{3}{4} \)

**BONUS:** Use \( \frac{1}{16} = 0.0625 \) to write \( \frac{3}{16} \) as a decimal.
1. unit fractions:
   b), c)
   decimal fractions:
   a), d)

2. a) 0.23
   b) 0.7
   c) 0.09
   d) 0.73

3. a) \( \frac{30}{100} \)
   b) \( \frac{45}{100} \)
   c) \( \frac{75}{100} \)
   d) \( \frac{62}{100} \)

4. a) 0.75
   b) 0.625
   c) 0.4375

5. a) 0.4, 0.5
   b) 0.75, 0.7

6. 0.875, 0.688, 0.75, \( \frac{11}{16} \)

BONUS

\[ 0.0625 \times 3 = 0.1875 \]
Unit 1: Number Sense

Test (Lessons 55–63) — WNCP

1. a) Change the fractions to decimals by first writing them as a decimal fraction.
   
   i) \( \frac{2}{5} \)  
   ii) \( \frac{3}{4} \)  
   iii) \( \frac{11}{20} \)  
   iv) \( \frac{6}{15} \)

   b) Which two questions from part a) have the same answer? Why does that happen?

   c) Choose one of the fractions from part a) and change it to a decimal using long division. Did you get the same answer both times?

2. a) Write the first eight digits after the decimal point of each decimal.
   
   i) \( 0.24 \) = 0. ___ ___ ___ ___ ___ ___ ___ 
   ii) \( 0.5\bar{17} \) = 0. ___ ___ ___ ___ ___ ___ ___ 

   b) Use your answers to part a) to find the first eight digits of \( 0.24 + 0.5\bar{17} \).

   0. ___ ___ ___ ___ ___ ___ ___ 

   c) Use the pattern in the digits of your answer to part b) to write \( 0.24 + 0.5\bar{17} \) in bar notation.

   \( 0.24 + 0.5\bar{17} = \) ________
Unit 1: Number Sense

3. Use long division to change $\frac{17}{44}$ to a decimal. Write your answer in bar notation.

4. Calculate the first four products, then predict the fifth product.

\[
\begin{array}{cccccc}
\text{a)} & 0.01 & \times & 23 & \text{b)} & 0.011 & \times & 23 & \text{c)} & 0.0111 & \times & 23 & \text{d)} & 0.01111 & \times & 23 & \text{e)} & 0.1 & \times & 23
\end{array}
\]

5. a) Write the fractions as decimals. Use bar notation for any repeating decimals.

\[
\begin{align*}
\text{i)} \quad \frac{1}{2} & = \underline{\phantom{0}} \\
\text{ii)} \quad \frac{1}{3} & = \underline{\phantom{0}}
\end{align*}
\]

b) Add the fractions and their equivalent decimals.

\[
\frac{1}{2} + \frac{1}{3} = \underline{\phantom{} + \phantom{}} \quad \text{and} \quad \underline{\phantom{}} + \underline{\phantom{}} = \underline{\phantom{}}
\]

**BONUS:** Verify that your answers to part b) are equivalent.
1. a) i) \( \frac{2}{5} = \frac{4}{10} = 0.4 \)
   
   ii) \( \frac{3}{4} = \frac{75}{100} = 0.75 \)
   
   iii) \( \frac{11}{20} = \frac{55}{100} = 0.55 \)
   
   iv) \( \frac{6}{15} = \frac{2}{5} = \frac{4}{10} = 0.4 \)
   
   b) i) and iv), because they are equivalent fractions
   
   c) Answers will vary, but students should get the same answer as in part a).

2. a) i) 0.24242424
   
   ii) 0.51717171
   
   b) 0.75959595
   
   c) 0.759

3. 0.3863

4. a) 0.23
   
   b) 0.253
   
   c) 0.2553
   
   d) 0.25553
   
   e) 0.25

5. a) i) 0.5
   
   ii) 0.3
   
   b) \( \frac{5}{6} \) and 0.83

BONUS
Students should use long division to verify that \( \frac{5}{6} \) and 0.83 are indeed equivalent.
1. Convert the following decimals and fractions to percents.
   a) \( \frac{4}{5} = \) ______
   b) 0.67 = ______
   c) \( \frac{1}{4} = \) ______
   d) 0.08 = ______

2. Is the fraction closest to 10%, 25%, 50%, 75% or 100%?
   a) \( \frac{4}{10} \)
   b) \( \frac{4}{5} \)
   c) \( \frac{3}{10} \)
   BONUS: \( \frac{2}{9} \)

3. What percent of the figure is shaded?

4. If 20% is 8, what is 80%?

   **BONUS:** If 30% is 24, what is 75%?

5. What is 26% of 44?
1. a) 80%
   b) 67%
   c) 25%
   d) 8%
2. a) 50%
   b) 75%
   c) 25%
   **BONUS**
   25%
3. 60%
4. 32
   **BONUS**
   60
5. 11.44
1. Solve.
   
   a) 8 is 40% of what number?
   
   b) What is 60% of 25?
   
   c) What percent of 30 is 24?

2. In classroom A there are 35 children and 60% are girls.
   
   In classroom B there are 36 children and the ratio of girls to boys is 5 : 4.
   
   Which classroom has more girls?

3. There are \( \frac{4}{5} \) as many green beads as red beads and 6 more red beads than green beads.
   
   Draw a model to find the number of red and green beads.

4. Write a fraction statement for the figure using the word “of.”
   
   a) 

   BONUS: 

   5. a) How many pieces of length \( \frac{1}{5} \) fit into 8? ________
       
       b) How many pieces of length \( \frac{2}{5} \) fit into 8? ________
1. a) 20  
b) 15  
c) 80%  
2. Classroom A has 21 girls, and Classroom B has 20 girls.  
\[ \therefore \text{Classroom A has more.} \]  
3. There are 30 red beads and 24 green beads.  
4. a) \( \frac{1}{4} \) of \( \frac{1}{6} = \frac{1}{24} \)  
   BONUS  
   \( \frac{1}{4} \) of \( \frac{1}{4} = \frac{1}{16} \)  
5. a) 5 \times 8 = 40 pieces  
b) 40 \div 2 = 20 pieces
1. Erica bought $\frac{1}{4}$ of a watermelon. She ate $\frac{2}{3}$ of the watermelon. What fraction of the watermelon did she eat?

2. Calculate.
   a) $5 \times \frac{1}{7}$
   b) $\frac{3}{7} \times \frac{2}{5}$
   c) $6 \div \frac{2}{7}$
   d) $8.4 \times 5.2$

   **BONUS:** $0.75 \times \frac{2}{3}$

3. Estimate the product, then place the decimal point correctly in the answer.
   
   $42.65 \times 351.424 = 149882336$

4. Change each measurement in the chart to the same unit and then order the plants from tallest to shortest.

<table>
<thead>
<tr>
<th>Plant</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Birch tree</td>
<td>9.8 m = ___________</td>
</tr>
<tr>
<td>Caribou moss</td>
<td>98 mm = ___________</td>
</tr>
<tr>
<td>Dandelion</td>
<td>38.3 cm = __________</td>
</tr>
<tr>
<td>Rose</td>
<td>1 m 72 cm = __________</td>
</tr>
<tr>
<td>Sugar maple tree</td>
<td>23 m = ___________</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Plants from Tallest to Shortest</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
</tr>
<tr>
<td>2.</td>
</tr>
<tr>
<td>3.</td>
</tr>
<tr>
<td>4.</td>
</tr>
<tr>
<td>5.</td>
</tr>
</tbody>
</table>
Unit 1: Number Sense

Quiz (Lessons 81–85) — ON

1. \( \frac{1}{6} \)

2. a) \( \frac{5}{7} \)
   b) \( \frac{6}{35} \)
   c) 21
   d) 43.68

   BONUS
   \( \frac{1}{2} \)

3. \( \approx 14,000, 14,988.2336 \)

4. Sample answer:
   980 cm
   9.8 cm
   38.3 cm
   172 cm
   2300 cm
   1. Sugar maple tree
   2. Birch tree
   3. Rose
   4. Dandelion
   5. Caribou moss
1. Calculate.
   a) \(5 \times \frac{1}{7}\)  
   b) \(\frac{3}{7} \times \frac{2}{5}\)  
   c) \(6 \div \frac{2}{7}\)  
   d) \(8.4 \times 5.2\)

   **BONUS:** \(0.75 \times \frac{2}{3}\)

2. Estimate the product, then place the decimal point correctly in the answer.
   \(42.65 \times 351.424 = 149882336\)

3. Change each measurement in the chart to the same unit and then order the plants from tallest to shortest.

<table>
<thead>
<tr>
<th>Plant</th>
<th>Length</th>
<th>Plants from Tallest to Shortest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Birch tree</td>
<td>9.8 m = _______</td>
<td>1.</td>
</tr>
<tr>
<td>Caribou moss</td>
<td>98 mm = _______</td>
<td>2.</td>
</tr>
<tr>
<td>Dandelion</td>
<td>38.3 cm = ______</td>
<td>3.</td>
</tr>
<tr>
<td>Rose</td>
<td>1 m 72 cm = ______</td>
<td>4.</td>
</tr>
<tr>
<td>Sugar maple tree</td>
<td>23 m = ______</td>
<td>5.</td>
</tr>
</tbody>
</table>
Unit 1: Number Sense

Quiz (Lessons 82–85) — WNCP

1. a) \( \frac{5}{7} \)
b) \( \frac{6}{35} \)
c) 21
d) 43.68

**BONUS**

\( \frac{1}{2} \)

2. \( \approx 14 \, 000, 14 \, 988.2336 \)

3. Sample answer:
   - 980 cm
   - 9.8 cm
   - 38.3 cm
   - 172 cm
   - 2300 cm

1. Sugar maple tree
2. Birch tree
3. Rose
4. Dandelion
5. Caribou moss
1. Complete the chart.

<table>
<thead>
<tr>
<th>Picture</th>
<th>Fraction</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Fraction Picture" /></td>
<td>$\frac{100}{100}$</td>
<td>34%</td>
</tr>
<tr>
<td><img src="image2" alt="Fraction Picture" /></td>
<td>$\frac{100}{100}$</td>
<td>7%</td>
</tr>
<tr>
<td><img src="image3" alt="Fraction Picture" /></td>
<td>$\frac{18}{100}$</td>
<td>___%</td>
</tr>
<tr>
<td><img src="image4" alt="Fraction Picture" /></td>
<td>$\frac{100}{100}$</td>
<td>___%</td>
</tr>
</tbody>
</table>

2. In Anna’s school, 30% of the students play soccer, $\frac{3}{5}$ play basketball, and 0.48 play volleyball. Which sport is the most popular? ________________

3. Fill in the missing numbers.

<table>
<thead>
<tr>
<th>Ratio of boys to girls</th>
<th>Fraction of boys</th>
<th>Fraction of girls</th>
<th>Percentage of boys</th>
<th>Percentage of girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 : 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 : 4</td>
<td>$\frac{3}{4}$</td>
<td></td>
<td></td>
<td>50%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4. A lake has about 4 000 fish and 30% of them are trout. As a part of a conservation program, 1 000 more trout are released into the lake.

   a) How many trout are now in the lake? __________

   b) What percent of the fish in the lake are trout? __________

**BONUS:** If 1 000 sturgeon were released into the lake instead of 1 000 trout, what percent of the fish in the lake would be trout?

5. Solve the following problem using the diagram as a model.

One fourth of the fish in a tank are blue. The rest are red and green. There are 8 more green fish than blue fish. There are 32 red fish.

   How many fish are in the tank? ________

6. Calculate.

   a) $4 \times \frac{1}{7}$
   
   b) $\frac{5}{7} \times \frac{2}{3}$

   c) $6 \div \frac{2}{5}$

   d) $8.5 \times 5.3$

**BONUS:** $0.72 \times \frac{2}{3}$

7. Estimate the product, then place the decimal point correctly in the answer.

   $72.65 \times 891.324 = 6 \, 4 \, 7 \, 5 \, 4 \, 6 \, 8 \, 8 \, 6$
8. Change each measurement to the smallest unit used in the chart and then order the animals from shortest to longest.

<table>
<thead>
<tr>
<th>Animal</th>
<th>Length</th>
<th>Animals from Shortest to Longest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mouse</td>
<td>65 mm =</td>
<td>1.</td>
</tr>
<tr>
<td>Elephant</td>
<td>5 m =</td>
<td>2.</td>
</tr>
<tr>
<td>Tiger</td>
<td>20 dm =</td>
<td>3.</td>
</tr>
<tr>
<td>Hawk</td>
<td>50 cm =</td>
<td>4.</td>
</tr>
<tr>
<td>Horse</td>
<td>1.5 m =</td>
<td>5.</td>
</tr>
</tbody>
</table>
1. \( \frac{34}{100} \)
   \( \frac{7}{100} \)
   18\%
   \( \frac{4}{100} \), 4\%

2. Soccer
   \( \frac{30}{100} = 30\% = 0.30 \)

   Basketball
   \( \frac{60}{100} = 60\% = 0.60 \)

   Volleyball
   \( \frac{48}{100} = 48\% = 0.48 \)

   :. basketball is the most popular sport.

3. |
   ---|
   7:3 \[ \frac{7}{10} \; \frac{3}{10} \]
   70\% 30\%
   1:3 \[ \frac{1}{4} \; \frac{3}{4} \]
   25\% 75\%
   1:4 \[ \frac{1}{5} \; \frac{4}{5} \]
   20\% 80\%
   1:1 \[ \frac{1}{2} \; \frac{1}{2} \]
   50\% 50\%

4. a) 2 200
    b) 44\%

    **BONUS**
    24\%

5. 8 + 32 = 40, which must equal the other \( \frac{2}{3} \), or \( \frac{1}{2} \)

   :. there are a total of 80 fish in the tank.

6. a) \( \frac{4}{7} \)
    b) \( \frac{10}{21} \)
    c) 15
    d) 45.05

    **BONUS**
    0.48 or \( \frac{48}{100} = \frac{12}{25} \)

7. 64 754.688 6

8. 1. Mouse (65 mm)
   2. Hawk (500 mm)
   3. Horse (1 500 mm)
   4. Tiger (2 000 mm)
   5. Elephant (5 000 mm)
1. Write the width, length, and height. Then find the volume of the box. Include the units.
   a) width = ________
   b) length = ________
   c) height = ________
   d) volume = ________

2. Match the prism with its name.
   a) hexagonal prism _____
   b) rectangular prism _____
   c) triangular prism _____
   d) pentagonal prism _____

3. Find the volume of the prism. The picture below the prism shows the base of the prism.

   2 cm

   3 cm

   4 cm
4. Find the volume of a prism with the height 20 cm and the base shown.

a) 

\[
\begin{array}{c}
\text{15 cm} \\
\text{8 cm}
\end{array}
\]

b) 

\[
\begin{array}{c}
\text{8 cm} \\
\text{4 cm}
\end{array}
\]

**BONUS:** The area of the base of a prism is 100 cm\(^2\). Its volume is 5 000 cm\(^3\).
What is the height of the prism?
1. a) 2 mm  
b) 3 mm  
c) 5 mm  
d) 30 mm³
2. a) B  
b) D  
c) C  
d) A
3. \[ V = 4 \times 3 + 2 \times 2 \]
   \[ = 12 \text{ cm}³ \]
4. a) \[ V = 15 \times 8 \times 20 \]
    \[ = 120 \times 20 \]
    \[ = 2400 \text{ cm}³ \]
   b) \[ V = (8 + 6) ÷ 2 \times 4 \times 20 \]
    \[ = 7 \times 4 \times 20 \]
    \[ = 560 \text{ cm}³ \]

BONUS
\[ h = 5000 ÷ 100 \]
\[ = 50 \text{ cm} \]
1. Complete the drawing of the net for the prism by drawing the top face on the grid paper.

![Prism Diagram](image)

2. Find the area of each face of the prism in Question 1. Include the units.

   a) left ________ right ________ front ________ back ________

   b) left ________ right ________ front ________ bottom ________

   top ________ total surface area = ________ total surface area = ________
Unit 2: Measurement

Quiz (Lessons 26–28) — ON

3. How many of each type of face would you need to make the prism?

\[
\begin{align*}
\square &= \_\_\_ \\
\pentagon &= \_\_\_
\end{align*}
\]

4. Find the surface area and volume of the prism. Include the units.

\[
\text{surface area} = \_\_\_\_\_\_ \quad \text{volume} = \_\_\_\_\_\_
\]

**BONUS:** A rectangular prism with volume 1 000 cm\(^3\) has width 5 cm and length 4 cm. Find the height of the prism.
Unit 2: Measurement

Quiz (Lessons 26–28) — ON

1. Teacher to check.

2. a) left = 2 cm²
   right = 2 cm²
   front = 3 cm²
   back = 3 cm²
   bottom = 6 cm²
   top = 6 cm²
   total = 22 cm²

   b) left = 5 cm²
   right = 3 cm²
   front = 4 cm²
   bottom = 6 cm²
   top = 6 cm²
   total = 24 cm²

3. 5
   2

4. \( SA = 6 \times 4 \times 2 + 4 \times 2 \times 2 \\
    + 6 \times 2 \times 2 \\
    = 48 + 16 + 24 \\
    = 88 \text{ cm}^2 \\
V = 6 \times 4 \times 2 \\
= 48 \text{ cm}^3 \\

BONUS

   height = 1000 ÷ 5 ÷ 4 \\
   = 50 \text{ cm}
Unit 2: Measurement

Test (Lessons 22–28) — ON

Do not use a calculator for any problems on this test.

1. Sketch a net for this rectangular prism. Mark the dimensions of each face on the net. Then calculate its surface area and volume.

![Rectangular Prism Net](image)

2. The base of a pentagonal prism has an area of $2.4 \text{ m}^2$. The prism has a volume of $4.8 \text{ m}^3$. What is its height?

3. Find the volume and surface area of this prism.

![Prism Dimensions](image)
4. A box has the shape of a right prism with a regular hexagon with sides of 5 cm each for its base. The box is 10 cm high and its base has an area of 130 cm². Find the surface area of the prism.

5. Using the right triangle below as a base, sketch a right triangular prism with a volume of 120 cm³. Label the length of an edge perpendicular to the base.

6. A 1.2 L juice carton in the shape of a rectangular prism has a height of 24 cm. What is the area of its base?

**BONUS:** Sketch a cube with a surface area of 216 cm² and then calculate its volume.
1. \[
\begin{align*}
\text{Surface area} &= 152 \text{ cm}^2 \\
\text{Volume} &= 96 \text{ cm}^3
\end{align*}
\]
2. \(2\) m
3. \[
\begin{align*}
\text{Volume} &= 690 \, 000 \text{ cm}^3 \\
&= 0.69 \text{ m}^3 \\
\text{Surface area} &= 50 \, 300 \text{ cm}^2 \\
&= 5.03 \text{ m}^2
\end{align*}
\]
4. \(6(50) + 2(130) = 560 \text{ cm}^2\)
5. \[
\begin{align*}
\text{Volume} &= 216 \text{ cm}^3
\end{align*}
\]
1. Draw a bar graph for the frequency table.

<table>
<thead>
<tr>
<th>Favourite Colour</th>
<th>Number of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>4</td>
</tr>
<tr>
<td>Blue</td>
<td>6</td>
</tr>
<tr>
<td>Green</td>
<td>3</td>
</tr>
<tr>
<td>Yellow</td>
<td>9</td>
</tr>
</tbody>
</table>

2. Jennifer asked people to vote for their favourite team mascot. Look at the two bar graphs below.

a) Describe the scale on each bar graph.

A. Start at ________, count by ________, stop at ________.

B. Start at ________, count by ________, stop at ________.

b) Which graph makes it easier to tell the difference in the votes? ________

c) Which mascot came in second in the voting? ____________________
3. The table shows car sales by two companies. Complete a double bar graph to show the data.

<table>
<thead>
<tr>
<th>Month</th>
<th>Company A</th>
<th>Company B</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>4 000</td>
<td>3 000</td>
</tr>
<tr>
<td>February</td>
<td>5 800</td>
<td>7 200</td>
</tr>
<tr>
<td>March</td>
<td>2 700</td>
<td>1 400</td>
</tr>
<tr>
<td>April</td>
<td>3 500</td>
<td>2 800</td>
</tr>
</tbody>
</table>

4. The number of cars in a parking lot is shown on the line graph. No cars left before 4:00 pm.

a) How many cars were in the lot at:
   i) 8:00 am? _________
   ii) 9:00 am? _________

b) How many cars arrived between 8:00 am and 9:00 am? _________

c) How many cars left between 4:00 pm and 6:00 pm? _________

**BONUS:** It costs $10 per car to park in the lot. How much money was collected for parking?
Unit 3: Probability and Data Management

Quiz (Lessons 6–8) — ON

1. Teacher to check.

2. a) A: 0, 200, 1,000
   B: 900, 20, 1,000
   b) B
   c) Grizzlies

3. Teacher to check.

4. a) i) 10
    ii) 80
   b) 80 - 10 = 70
   c) 90 - 30 = 60

   BONUS

   90 × $10 = $900
1. Write the fraction as an equivalent fraction over 100 and then as a percent.

a) \( \frac{7}{20} = \quad = \quad % \)  

b) \( \frac{6}{25} = \quad = \quad % \)  

c) \( \frac{13}{50} = \quad = \quad % \)

2. Complete the relative frequency table.

<table>
<thead>
<tr>
<th>Favourite Pastime</th>
<th>Frequency</th>
<th>Fraction</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Watching movies</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Playing video games</td>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Playing sports</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Write the fraction as an equivalent fraction over 360 to find the angle in a circle graph.

a) \( \frac{5}{36} = \quad = \quad \circ \)  

b) \( \frac{3}{20} = \quad = \quad \circ \)  

c) \( \frac{13}{72} = \quad = \quad \circ \)

4. John completed the relative frequency chart. Use a protractor to draw a circle graph.

<table>
<thead>
<tr>
<th>Type of Movie</th>
<th>Frequency</th>
<th>Fraction</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comedy</td>
<td>6</td>
<td>( \frac{6}{20} )</td>
<td>108°</td>
</tr>
<tr>
<td>Drama</td>
<td>11</td>
<td>( \frac{11}{20} )</td>
<td>198°</td>
</tr>
<tr>
<td>Action</td>
<td>3</td>
<td>( \frac{3}{20} )</td>
<td>54°</td>
</tr>
</tbody>
</table>
5. Find the percentage of a circle.

   a) $180^\circ$ 
   \[
   \frac{180}{360} = \frac{1}{2} = \frac{50}{100} = 50\%
   \]

   b) $36^\circ$ 
   \[
   \frac{36}{360} = \frac{1}{10} = \frac{10}{100} = 10\%
   \]

6. Frank measured the angles of the section in the circle graph.

   White: $198^\circ$  Blue: $90^\circ$  Pink: $72^\circ$

   a) Calculate the percentage of pink shirts.

   b) There are 60 shirts altogether. How many are pink?

**BONUS:** In a circle graph, a section with an angle of $36^\circ$ is 10% of the circle graph. What is the angle for 1% of the circle graph?
Unit 3: Probability and Data Management

Quiz (Lessons 9–13) — ON & WNCP

1. a) 35%
   b) 24%
   c) 26%

2. a) \(\frac{12}{25}, 48\%\)
   b) \(\frac{11}{25}, 44\%\)
   c) \(\frac{2}{25}, 8\%\)

3. a) \(\frac{50}{360} = 50^\circ\)
   b) \(\frac{54}{360} = 54^\circ\)
   c) \(\frac{65}{360} = 65^\circ\)

4. Teacher to check.

5. a) \(\frac{180}{360} = \frac{1}{2}\)
   \(= \frac{50}{100} = 50\%\)
   b) \(\frac{36}{360} = \frac{1}{10}\)
   \(= \frac{10}{100} = 10\%\)

6. a) \(\frac{72}{360} = \frac{1}{5}\)
   \(= \frac{20}{100} = 20\%\)
   b) 12

BONUS

3.6°
1. Name the type of graph (line, circle, stem and leaf plot, bar, or double bar) that would be best to:

a) compare two sets of data. __________________________

b) show a trend in data over time. __________________________

c) visually display the frequency of result. __________________________

d) make it easy to see the largest, smallest, and most common data values. __________________________

e) visually display the relative frequency of results. __________________________

2. The bar graph shows the results of an election.

Election Results

Sally thinks it would be better to represent the results using a circle graph. Complete the table and use a protractor to draw the circle graph.

Title: _______________________

<table>
<thead>
<tr>
<th>Candidate</th>
<th>Number of Votes</th>
<th>Relative Frequency</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frank</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Karen</td>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jane</td>
<td>30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3. State whether you would use a census or a sample in the situation.

   a) You and your friends want to choose a restaurant for dinner. __________________________
   b) A department store wants to measure customer satisfaction. __________________________
   c) The health department wants to find the effects of sugary drinks. __________________________
   d) An employer wants to find employee sizes for uniforms. __________________________
   e) The government wants to know the number of people over 65 years of age in Canada. __________________________

4. A principal wants to know how many hours the average student spends playing video games. He has several choices on which students to select for his survey:

   A. ask all the boys
   B. ask only students in Grades 7 and 8
   C. select students randomly from a list

   a) Which method is not biased? _________

   b) Explain a possible bias in each other method.

5. A telephone survey about cell phone usage contacts people on their home landline to ask questions. Explain why this might not be a good design feature of the survey.
Unit 3: Probability and Data Management

Quiz (Lessons 14–16) — ON

1. a) double bar graph  
   b) line graph  
   c) bar graph  
   d) stem and leaf plot  
   e) circle graph

2. 20, \( \frac{20}{100} \cdot 72^\circ \)
   50, \( \frac{50}{100} \cdot 180^\circ \)
   30, \( \frac{30}{100} \cdot 108^\circ \)

   Teacher to check circle graph.

3. a) census  
   b) sample  
   c) sample  
   d) census  
   e) census

4. a) C
   b) Sample answers:  
      Boys might play more video games than girls.  
      Grade 7 & 8 students might play more than students in other grades.

5. Sample answer:  
   Some people who have landlines may not have cell phones.
1. Write the fraction as an equivalent fraction over 100 and then as a percent.

   a) \( \frac{8}{10} = \frac{\phantom{0}}{\phantom{00}} = \phantom{0}\% \)
   
   b) \( \frac{13}{20} = \frac{\phantom{0}}{\phantom{00}} = \phantom{0}\% \)
   
   c) \( \frac{18}{25} = \frac{\phantom{0}}{\phantom{00}} = \phantom{0}\% \)

2. Complete the relative frequency table.

<table>
<thead>
<tr>
<th>Favourite Sport</th>
<th>Frequency</th>
<th>Fraction</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soccer</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Football</td>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hockey</td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Write the fraction as an equivalent fraction over 360 to find the angle in a circle graph.

   a) \( \frac{7}{18} = \frac{\phantom{0}}{\phantom{00}} = \phantom{0}\° \)

   b) \( \frac{11}{36} = \frac{\phantom{0}}{\phantom{00}} = \phantom{0}\° \)

   c) \( \frac{11}{20} = \frac{\phantom{0}}{\phantom{00}} = \phantom{0}\° \)

4. Complete the relative frequency chart. Then use a protractor to draw a circle graph.

<table>
<thead>
<tr>
<th>Type of Car</th>
<th>Frequency</th>
<th>Fraction</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electric</td>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gas</td>
<td>15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hybrid</td>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Title: ______________________
5. Find the percentage of a circle.
   a) 72°
      \[
      \frac{360 \times 72}{100} = \frac{28800}{100} = 288\%
      \]
   b) 90°
      \[
      \frac{360 \times 90}{100} = \frac{32400}{100} = 324\%
      \]

6. In an election for mayor, Tom received 50% of the votes, Jin received 30%, Barbara received 15%, and David received 5%.
   a) Use a calculator to find the angle for each section of a circle graph.

<table>
<thead>
<tr>
<th>Candidate</th>
<th>Percentage</th>
<th>Calculation of Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tom</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Jin</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Barbara</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>David</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

   b) There were 4 500 people that voted in the election. How many votes did Jin receive?

   BONUS: In a circle graph, the section of votes earned by the Independent Party is 1°. If 360 000 votes were cast, how many votes did the Independent Party get?
Unit 3: Probability and Data Management

Test (Lessons 9–13) — WNCP

1. a) 80%
   b) 65%
   c) 72%

2. \(\frac{6}{20}\), 30%
   \(\frac{11}{20}\), 55%
   \(\frac{3}{20}\), 15%

3. a) \(\frac{140}{360}\), 140°
   b) \(\frac{110}{360}\), 110°
   c) \(\frac{198}{360}\), 198°

4. \(\frac{30}{50}\), 216°
   \(\frac{15}{50}\), 108°
   \(\frac{5}{50}\), 36°

5. a) \(\frac{1}{5}\) = \(\frac{20}{100}\) = 20%
   b) \(\frac{1}{4}\) = \(\frac{25}{100}\) = 25%

6. a) \(\frac{50}{100}\) × 360 = 180°
   \(\frac{30}{100}\) × 360 = 108°
   \(\frac{15}{100}\) × 360 = 54°
   \(\frac{5}{100}\) × 360 = 18°
   b) 455 × \(\frac{30}{100}\) = 1,350

BONUS

1,000
1. The bar graph shows the results from a survey that asked people’s favourite type of music.

   a) How many people chose rap? _________

   b) Which type of music was the least favourite?

   ________________________________

   c) How many more people selected pop than country?

   _________

   d) How many people were surveyed in total? _________

2. The bar graphs show the annual salary of employees at a factory.

   A. Employee Salary

   B. Employee Salary

   The employer claims that the employees are paid about the same salary.

   a) Which graph makes it easier to tell the difference in salaries? _________

   b) Which graph should the employer use to support his statement? Explain.

   c) If the scale for the annual earnings starts at 80, counts by 1, and stops at 100, will it make it easier or harder for the employer to support his statement? Explain.
3. Mary’s balance in her bank account at the end of each week is shown in the table on the right.

   a) Draw a line graph to show the data.

   b) At the end of which week did Mary have the highest balance? _________

   c) What is the change in her balance from Week 1 to Week 5? _________

4. The table below shows the results of a survey about favourite drinks. Calculate the relative frequency and percentage for each drink and fill in the table.

<table>
<thead>
<tr>
<th>Drink</th>
<th>Frequency</th>
<th>Fraction</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pop</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Milk</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hot chocolate</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Water</td>
<td>13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td>12</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5. The circle graph shows the results of a vote on where students want to have their prom.

   a) Measure the angle of each section of the circle graph. Then complete the table to find the fraction of the circle and the percentage for each section.

<table>
<thead>
<tr>
<th>Location</th>
<th>Angle</th>
<th>Fraction of Circle</th>
<th>Fraction in Lowest Terms</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>School gym</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dance hall</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   b) There were 120 students surveyed. How many voted for the dance hall?

6. State whether you would use a census or a sample in the situation.

   a) You and your friends want to choose a movie to see at the theatre. _______________

   b) The government wants to find out salaries for calculating taxes. _______________

   c) An advertiser wants to know how many people watch a television show. _______________

7. A survey to find the needs of senior citizens was done by email. Explain why this might not be a good design feature of the survey.

BONUS: 20% of 100 000 people surveyed preferred strawberry ice cream. How many people preferred strawberry ice cream?
1. a) 100
   b) classical
   c) 70
   d) 220
2. a) B
   b) A
   The heights of the bars appear closer together.
   c) Harder.
   The heights of the bars will appear farther apart.
3. a) Teacher to check.
   b) 3
   c) $50
4. a) $\frac{5}{50}, 10\%$
   a) $\frac{12}{50}, 24\%$
   a) $\frac{8}{50}, 16\%$
   a) $\frac{13}{50}, 26\%$
   a) $\frac{12}{50}, 24\%$
5. a) $180^\circ, \frac{180}{360}, \frac{1}{2}, 50\%$
   $144^\circ, \frac{144}{360}, \frac{2}{5}, 40\%$
   $36^\circ, \frac{36}{360}, \frac{1}{10}, 10\%$
   b) 40% of 120 = $\frac{40}{100} \times 120 = 28$
6. a) census
   b) census
   c) sample
7. Sample answer:
   Senior citizens may be less likely to use email.
BONUS
$$\frac{20}{100} \times 100 000 = 20 000$$
1. Write a formula for the number of triangles \( t \) from the number of squares \( s \).

\[
\begin{array}{|c|c|}
\hline
\text{Squares (s)} & \text{Triangles (t)} \\
\hline
1 & 6 \\
2 & 12 \\
3 & 18 \\
4 & 24 \\
\hline
\end{array}
\]

\( t = \text{___________} \)

2. Complete the table. Then write a formula for the number of dots \( d \).

\[
\begin{array}{|c|c|}
\hline
\text{Input} & \text{Output} \\
\hline
6 & 3 \\
7 & 4 \\
8 & 5 \\
9 & 6 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{Input} & \text{Output} \\
\hline
6 & 42 \\
2 & 14 \\
9 & 63 \\
4 & 28 \\
\hline
\end{array}
\]

Output = \text{___________} \]

Output = \text{___________} \]

\[
\begin{array}{|c|c|}
\hline
\text{Row (r)} & \text{Dots (d)} \\
\hline
\hline
\end{array}
\]

\( d = \text{___________} \]

3. Write a rule that will tell you how to make the output numbers from the input numbers.
4. Write a list of ordered pairs for the points shown in the graph and complete the T-table.

![Graph showing points](image)

<table>
<thead>
<tr>
<th>First number</th>
<th>Second number</th>
<th>Ordered pair</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( , )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( , )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( , )</td>
</tr>
</tbody>
</table>

5. Graph the sequence below by first making a list of ordered pairs.

8 6 4 0

( , ) ( , ) ( , ) ( , )

6. The graph shows the distance Lina travelled in a 70 km cycling trip.

a) How far did Lina travel in the first hour? _________

b) How far did Lina travel in 5 hours? _________

c) Did Lina take any breaks during the trip? _________

How long was Lina’s break? _________

**BONUS:** Did Lina cover more distance in the first 3 hours or the last 2 hours of the trip? Explain.
Unit 4: Patterns and Algebra

Quiz (Lessons 16–21) — ON & WNCP

1. a) \( t = s \times 6 \)
b) \( t = s + 3 \)

2. \( d = r + 2 \)

3. a) output = input – 3
b) output = input \times 7

4. (2, 3), (4, 8), (6, 7), (8, 4)

5. (1, 8), (2, 6), (3, 4), (4, 0)
Teacher to check graph.

6. a) 10 km
b) 50 km
c) yes, 1 hour

**BONUS**

She travelled 30 km during the first 3 hours. She travelled 70 – 30 = 40 km in the last 2 hours. So she travelled 10 km farther during the last 2 hours.
1. Evaluate the expression $2n - 1$ for the values $n = 1, 2, 3, 4, \text{ and } 5$.

   a) Write your answers in the T-table and write the ordered pairs.

   
<table>
<thead>
<tr>
<th>$n$</th>
<th>$2n - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

   b) Plot the ordered pairs on the graph.

2. Fill in the chart and write the sequence for the formula. Then find the gaps and write the rule in words.

   Term = $3 \times \text{Term Number} - 2$

   Term number | 1 | 2 | 3 | 4
   ----------- |---|---|---|---
   Term       |   |   |   |   

   Sequence: _____  _____  _____  _____

   Rule: _________________________________________

3. Fill in the chart and write a rule in words for the number of triangles in each figure.

   Rule: _________________________________________
4. Use the gap in the sequence to start writing the formula. Complete the chart and write the formula.

<table>
<thead>
<tr>
<th>Term number (n)</th>
<th>n × Gap</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>22</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>27</td>
</tr>
</tbody>
</table>

Formula: ___________________________

5. Find the gap in the sequence shown by the graph. Then write the formula for the sequence.

Gap: ________

Formula: ___________________________

6. The formula for a sequence is $3n + 10$.

a) Find the gap for the sequence. ________

b) Find the first term for the sequence. ________

BONUS: Find the 100th term for the sequence. ________
Unit 4: Patterns and Algebra

Quiz (Lessons 22–28) — ON & WNCP

7. The graph shows the first three terms in a sequence. Draw a line to join the first three terms. Extend the line to predict the ninth term in the sequence.

ninth term = __________

8. Figure 1       Figure 2             Figure 3                   Figure 4

a) Write a sequence for the number of blocks in each figure.

b) Write an expression for the number of blocks in each figure.

c) Find the number of blocks in the 10th figure.

BONUS: Find the number of blocks in the 1000th figure.
Unit 4: Patterns and Algebra

Quiz (Lessons 22–28) — ON & WNCP

1. a) 1, 3, 5, 7, 9
   b) Teacher to check.

2. 1, 4, 7, 10
   Rule: Start at 1, add 3 each time.

3. 3, 6, 9, 12
   Rule: Multiply the figure number by 3.
   or
   Rule: Start at 3, add 3 each time.

4. 5, 10, 15, 20, 25
   Formula: 5n + 2

5. Gap: 4
   Formula: 4n − 1

6. a) 3
   b) 13

   BONUS
   310

7. Teacher to check graph.
   ninth term = 10

8. a) 1, 4, 7, 10
   b) 3n − 2
   c) 28

   BONUS
   2 998
1. Larissa makes copies of the design below using one square and several triangles of the same size.

Complete the chart and write a formula showing how to find the number of triangles from the number of squares.

<table>
<thead>
<tr>
<th>Squares (s)</th>
<th>Triangles (t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Formula: _________________________________________

2. Mark 3 grid points on the line segment. Then write a list of ordered pairs and complete the T-table.

<table>
<thead>
<tr>
<th>Ordered pair</th>
<th>First number</th>
<th>Second number</th>
</tr>
</thead>
<tbody>
<tr>
<td>(           ,   )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(           ,   )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(           ,   )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Decide which of these sequences is linear by finding the gaps between the terms.

A: 17 15 12 8 3
B: 7 9 11 13 15

_____ is linear because __________________________________________________.
4. The graph shows the cost to rent a bike from two different stores, Bike Pirate and Cycle Star.
   a) Which store charges a flat fee? How much is the flat fee?

   b) How much would it cost to rent a bike for 5 days from Bike Pirate?

   c) How much would it cost to rent a bike for 4 days from Cycle Star?

   d) If you needed a bike for 3 days, which store would you rent from? Explain.

5. Fill in the T-table for the following formula. Then make a list of ordered pairs and plot the points on the graph.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2n + 1 )</th>
<th>Ordered pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
6. a) Write a formula for the number of shaded blocks in each figure.

b) Write a formula for the total number of blocks.

c) In which formula does the number of blocks vary directly with the figure number?

7. For the sequence 4, 6, 8, 10, fill in the table of values for the term numbers and the term values. Draw a graph for your table and extend the graph to find the value of the 6th term.

<table>
<thead>
<tr>
<th>Term number</th>
<th>Term value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

BONUS:

a) Write an expression for the perimeter of each figure.

b) What is the perimeter of Figure 28?
Unit 4: Patterns and Algebra

Test (Lessons 16–28) — ON & WNCP

1. squares Triangles
   1 8
   2 16
   3 24

   Formula: 
   \( t = 8s \) or \( t = 8 \times s \)

2. Any three points from:
   (0, 0) 0 0
   (1, 3) 1 3
   (2, 6) 2 6
   (3, 9) 3 9

3. A
   Gaps: –2, –3, –4, –5

   B
   Gaps: 2, 2, 2, 2
   \( \therefore \) B is linear because the gaps are the same size.

4. a) Bike Pirate, $10
   b) $60
   c) $60
   d) Bike Pirate, because it would cost $40, when Cycle Star would charge more – the Cycle Star line is above 40 at Time = 3 days.

5. \( n \) 2\( n + 1 \) Ordered pair
   1 3 (1, 3)
   2 5 (2, 5)
   3 7 (3, 7)
   4 9 (4, 9)

6. a) 3\( n \)
   b) 3\( n + 3 \)
   c) Formula a), 3\( n \), the number of shaded blocks.

7. Term number Term value
   1 4
   2 6
   3 8
   4 10
   6 14

BONUS
   a) 4\( n + 4 \)
   b) 116
1. $\triangle PQR$ is congruent to $\triangle YXZ$.
   a) Label the vertices of the second triangle.
   b) Mark the equal sides and equal angles in the picture.
   c) Complete the statements of equal sides and angles.
   \[
   \angle____ = \angle____ \quad ____ = ____
   \angle____ = \angle____ \quad ____ = ____
   \angle____ = \angle____ \quad ____ = ____
   \]

2. Which congruence rule tells that the two triangles are congruent?
   Write the vertex letters in the names of the triangles in the correct order.
   Congruence rule: _______________
   $\triangle DEF \cong \triangle ______$

3. Use a compass and a straightedge to construct $\triangle RST$, using line segment $RS$ as the base and with the other side lengths as shown.
1. a–b)

\[ \angle P = \angle Y \]
\[ \angle Q = \angle X \]
\[ \angle R = \angle Z \]
\[ PQ = YX \]
\[ QR = XZ \]
\[ RP = ZY \]

2. Congruency rule: ASA
\[ \triangle DEF \cong \triangle KLJ \]

3. Teacher to check side lengths.
1. \( \triangle ABC \) is an isosceles triangle. \( BM \) is the perpendicular bisector of \( AC \). Name the sides and angles that are equal.

   a) \( AB = \) ______    b) \( AM = \) ______

   c) \( \angle ABM = \) ______   d) \( \angle BAM = \) ______

2. The line segment \( AB \) is 3 cm long. Set your compass so that its diameter is 3 cm. Use your compass to draw line \( OR \) that bisects \( \angle QOP \).

3. a) Construct a perpendicular line to the line segment \( AB \) through \( C \).

   b) Construct a parallel line to the line segment \( AB \) by drawing a perpendicular line through a point in the line you drew in part a).
Unit 5: Geometry

Quiz (Lessons 19–24) — ON & WNCP

4. Construct a perpendicular bisector for the line segment.

5. Construct an angle of exactly 45°.

BONUS: Use the diagram in Question 5 to construct an angle of 22.5°.
Unit 5: Geometry

Quiz (Lessons 19–24) — ON & WNCP

1. a) CB  
   b) CM  
   c) ∠CBM  
   d) ∠BCM

2. Teacher to check.
3. Teacher to check.
4. Teacher to check.
5. Teacher to check.

**BONUS**
Teacher to check.
1. Which two triangles below are congruent? _____ and _____

   Which congruence rule can you use to show it? ________

   ![Triangle A](image1.png)  ![Triangle B](image2.png)  ![Triangle C](image3.png)

2. Use the information provided on the sketch. Which congruence rule would you use to prove that the triangles are congruent?

   a)  ![Diagram](image4.png)
   b)  ![Diagram](image5.png)
   c)  ![Diagram](image6.png)

   Rule: ________  Rule: ________  Rule: ________

3. a) Using a compass and a straightedge, construct the angle bisector of $\angle ABC$.
   b) Mark congruent triangles on your sketch and use them to explain why the line you drew bisects $\angle ABC$. 

   ![Diagram](image7.png)
Unit 5: Geometry

Test (Lessons 13–24) — ON & WNCP

4. Using a compass and a straightedge, construct a line that intersects \( DE \) at a 60° angle.

\[ \begin{array}{c}
D \\
\hline
E \\
\end{array} \]

5. All sides and all angles in an equilateral triangle are equal.
   a) \( \triangle XYZ \) and \( \triangle TUV \) are both equilateral triangles. \( XY = TU \). Are \( \triangle XYZ \) and \( \triangle TUV \) congruent? Start by making a sketch. Explain your answer.

   b) \( \triangle KLM \) and \( \triangle GHJ \) are both equilateral triangles. \( \angle G = \angle K \). Is \( \triangle KLM \cong \triangle GHJ \)? Explain.

**BONUS**: Using only a compass and a straightedge, construct a right isosceles triangle \( XYZ \).
Side \( YZ \) is given below.
1. B and C
   SAS

2. a) ASA
    b) SAS
    c) SAS

3. a) BE = BF (radii)
   The circles centred at B and F have equal radii, so
   FD = ED, and
   BD is the common side, so
   \( \triangle BDF \cong \triangle BDE \)
   by the SSS congruence rule.

4. 

5. a) Yes.
   \( XY = TU \)
   \( XY = YZ = XZ (\triangle XYZ \text{ equilateral}) \)
   \( TU = UV = TV (\triangle TUV \text{ equilateral}) \)
   \( \therefore YZ = UV, XZ = TV, \) and so by SSS,
   \( \triangle XYZ \cong \triangle TUV. \)

b) No.
   In an equilateral triangle all angles are 60°, so any two
   equilateral triangles \( \triangle KLM \) and \( \triangle GHJ \) will have \( \angle G = \angle K. \)
   Any two equilateral triangles with different side lengths will be a counter-example.

BONUS
Teacher to check.
Sample answer:
1. Write the opposite of each integer.
   a) The opposite of $-5$ is ________.
   b) The opposite of $+3$ is ________.
   c) The opposite of $38$ is ________.
   d) The opposite of $-405$ is ________.

2. Rewrite each sum of integers as a sequence of gains and losses, then solve.
   a) $(-5) + (+2) = \phantom{0000}$
      $\phantom{0000}$
   b) $(7) + (-3) = \phantom{0000}$
      $\phantom{0000}$
   c) $(+4) + (-3) + (-2) = \phantom{000}$
      $\phantom{000}$
   d) $(5) + (+6) + (-7) = \phantom{000}$
      $\phantom{000}$

3. Use a number line to add the integers.
   a) $(-2) + (+4) = \phantom{00}$
   b) $(-1) + (-2) = \phantom{00}$
   c) $(+2) + (-3) = \phantom{00}$
   d) $(+4) + (-2) = \phantom{00}$

4. Which two parts of Question 3 have the same answer? ________ and ________

   **BONUS:** Why does that make sense?
Unit 6: Number Sense

Quiz (Lessons 86–89) — ON & WNCP

1. a) +5 or 5
   b) −3
   c) −38
   d) +405 or 405

2. a) = − 5 + 2
    = − 3
   b) = −7 − 3
    = −10
   c) = +4 − 3 − 2
    = −1
   d) = −5 + 6 − 7
    = −6

3. a) (−2) + (+4) = +2

4. a) and d)

   **BONUS**
   
   Because they are adding the same numbers but just in a different order.
Unit 6: Number Sense

Quiz (Lessons 90–94) — ON & WNCP

Name: ______________________

Date: ________________

1. Use a number line to subtract.
   
   a) $(+5) - (+2) = ____$
   
   b) $(-3) - (+2) = ____$

   
   c) $2 - (-3) = ____$
   
   d) $(-3) - (-1) = ____$

2. Write each difference as a sum and then calculate the answer.

   a) $(-2) - (+4) = ____ + ____ = ____$
   
   b) $(+7) - (+4) = ____ + ____ = ____$

   c) $(+5) - (-3) = ____ + ____ = ____$
   
   d) $(-2) - (-4) = ____ + ____ = ____$

3. Use the thermometer model to calculate each expression.

   a) $1 - 3 = ____$
   
   b) $3 - 7 = ____$

   c) $-2 + 6 = ____$
   
   d) $-6 + 5 = ____$

   e) $-3 - 2 = ____$
   
   f) $0 - 6 = ____$

4. Subtract by adding the opposite.

   a) $(-4) - 2 = ____ + ____ = ____$
   
   b) $6 - 8 = ____ + ____ = ____$

   c) $3 - (-4) = ____ + ____ = ____$
   
   d) $(-2) - (-3) = ____ + ____ = ____$

5. How many units apart are the numbers on a number line?

   a) 2 and $-3$ ______
   
   b) $-4$ and 5 ______
   
   c) $-3$ and $-7$ ______
6. Find the missing number.
   a) $2 + ____ = -1$
   b) $-3 + ____ = -4$
   c) $7 + ____ = 5$
   d) $-1 - ____ = -2$
   e) $4 - ____ = 5$
   f) $-3 - ____ = 1$

7. The temperature on Monday is 4°C. The temperature drops by 3 degrees each day for 5 consecutive days. What is the temperature on Saturday?

BONUS: Calculate.

$-1000 + 3000 - (-5000) = \phantom{000}$
1. a) 3  
b) (-5)  
c) 5  
d) (-2)  
2. a) (-2) + (-4) = (-6)  
b) (+7) + (-4) = (+3)  
c) (+5) + (+3) = (+8)  
d) (-2) + (+4) = (+2)  
3. a) -2  
b) -4  
c) 4  
d) -1  
e) -5  
f) -6  
4. a) (-4) + (-2) = (-6)  
b) 6 + (-8) = (-2)  
c) 3 + (+4) = +7  
d) (-2) + (+3) = +1  
5. a) 5  
b) 9  
c) 4  
6. a) (-3)  
b) (-1)  
c) (-2)  
d) 1  
e) (-1)  
f) (-4)  
7. -11  
BONUS  
7 000
1. Draw number lines in the space provided to show how the integers are added, and write the answer.
   a) \((+3) + (-4) = \) ______   
   b) \((-5) + (+3) = \) ______

2. How many units apart are the two whole numbers?
   a) +3 and -3 are ____ units apart.   
   b) -9 and -2 are ____ units apart.

3. Simplify each expression and then add to find the result.
   a) \(+7 + (-1) = \) ____________   
   b) \((-5) - (+2) = \) ____________
   
   \[= \) ____________   
   \[= \) ____________

   c) \((-1) + (+4) = \) ____________   
   d) \((-15) - (-12) = \) ____________
   
   \[= \) ____________   
   \[= \) ____________

   e) \((-2) + (+3) - (-2) = \) ____________   
   f) \(15 - (+9) + (-6) = \) ____________
   
   \[= \) ____________   
   \[= \) ____________

4. On the number line given, use the letter indicated to mark the number that is:
   A. 5 less than 2   
   B. 7 greater than -1   
   C. 5 less than -2   
   D. the same distance from 0 as +8   
   E. opposite of -5   
   F. halfway between -6 and -2

5. Two special six-sided dice have the following markings:
   Die 1: +1, -2, +3, -4, -5, +6   
   Die 2: -1, -2, +3, +4, -5, +6
   a) What is the highest possible total that can be rolled? _____ + _____ = _____
   b) What is the lowest possible total that can be rolled? _____ + _____ = _____
   c) What are two different ways to roll a total of zero? _____ + _____ or _____ + _____
   d) Find two different ways to roll a total of -1. _____ + _____ or _____ + _____
6. Fill in the missing integer that will make the statement true.

   a) \((-5) + _____ = -7\)  
   b) _____ + (-3) = (+3)  
   c) _____ - (-5) = +1  
   d) +2 - _____ = +7

7. The chart shows the average monthly (maximum) temperatures for the city of Ottawa.
   a) Write the six lowest average temperatures in order from least to greatest.
      \[_____ < _____ < _____ < _____ < _____ < _____\]
   b) What is the difference between the average temperatures for June and December? _____ - _____ = _______
   c) What is the difference between the warmest and coldest average temperatures? _____ - _____ = _______
   d) What is the largest drop in temperature from one month to the next? From which month to which month did this occur?
      A drop of _______°C from ________________ to ________________.
   e) What is the largest gain in temperature from one month to the next? From which month to which month did this occur?
      A gain of _______°C from ________________ to ________________.

   

<table>
<thead>
<tr>
<th>Month</th>
<th>Max. (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>-5</td>
</tr>
<tr>
<td>Feb</td>
<td>-4</td>
</tr>
<tr>
<td>Mar</td>
<td>+2</td>
</tr>
<tr>
<td>Apr</td>
<td>+11</td>
</tr>
<tr>
<td>May</td>
<td>+19</td>
</tr>
<tr>
<td>Jun</td>
<td>+24</td>
</tr>
<tr>
<td>Jul</td>
<td>+26</td>
</tr>
<tr>
<td>Aug</td>
<td>+25</td>
</tr>
<tr>
<td>Sep</td>
<td>+20</td>
</tr>
<tr>
<td>Oct</td>
<td>+13</td>
</tr>
<tr>
<td>Nov</td>
<td>+5</td>
</tr>
<tr>
<td>Dec</td>
<td>-2</td>
</tr>
</tbody>
</table>

8. On an autumn day, when the sun sets at 6:00 pm, the temperature is +5°C. The temperature then decreases by 2°C each hour.
   What is the temperature at 9:00 pm?_______

BONUS: Find the following totals.

   \((-1) + (+2) = _____\)  
   \((-1) + (+2) + (-3) + (+4) = _____\)  
   \((-1) + (+2) + (-3) + (+4) + (-5) + (+6) = _____\)

   Continue the pattern in both sides of the equation. Write the next two terms.
   \[\text{______________________________} = _____\]
   \[\text{______________________________} = _____\]

   Predict: \((-1) + (+2) + (-3) + (+4) + (-5) + (+6) + \ldots + (-99) + (+100) = _____\)
Unit 6: Number Sense

Test (Lessons 86–94) — ON & WNCP

1. a) \((+3) + (-4) = -1\)

b) \((-5) + (-3) = -8\)

2. a) 6

b) 7

3. a) \(= + 7 - 1\)

\(= +6\)

b) \(= - 5 - 2\)

\(= -7\)

c) \(= - 1 + 4\)

\(= +3\)

d) \(= - 15 + 12\)

\(= -3\)

e) \(= - 2 + 3 + 2\)

\(= +3\)

f) \(= 15 - 9 - 6\)

\(= 0\)

4. A \(-3\)

B \(+6\)

C \(-7\)

D \(-8\)

E \(+5\)

F \(-4\)

5. a) \((+6) + (+6) = +12\)

b) \((-5) + (-5) = -10\)

c) \((+1) + (-1)\) or \((-4) + (+4)\)

d) any two of \((+1) + (-2)\) or \((-4) + (+3)\) or \((-5) + (+4)\)

6. a) \(-2\)

b) \(+6\)

c) \(-4\)

d) \(-5\)

7. a) \(-5^\circ C, -4^\circ C, -2^\circ C, +2^\circ C, +5^\circ C, +11^\circ C\)

b) \(+24^\circ C - (-2^\circ C)\)

\(= 26^\circ C\)

c) \(+26^\circ C - (-5^\circ C)\)

\(= 31^\circ C\)

d) A drop of 8°C from October to November.

e) A gain of 9°C from March to April.

8. \(-1^\circ C\)

BONUS

1

2

3

\(-1 + 2 - 3 + 4 - 5 + 6 - 7\)

\(+8 = 4\)

\(-1 + 2 - 3 + 4 - 5 + 6 - 7\)

\(+8 - 9 + 10 = 5\)

50
Unit 7: Geometry

Quiz (Lessons 25–26) — ON

1. On the grid shown, plot and label the points.
   
   \[
   \begin{align*}
   A \ &: \ (-3, 2) \\
   B \ &: \ (4, -1) \\
   C \ &: \ (-4, -2) \\
   D \ &: \ (3, 1) \\
   E \ &: \ (-1, 0) \\
   F \ &: \ (0, -4) \\
   G \ &: \ (2, -3) \\
   H \ &: \ (-2, 4) \\
   \end{align*}
   \]

2. Write the quadrant where each point is found.
   
   a) \((17, -6)\) Quadrant ____  
   b) \((-11, 13)\) Quadrant ____  
   c) \((9, 14)\) Quadrant ____  
   d) \((-8, -10)\) Quadrant ____  

3. Write the coordinates of the points.
   
   J (__, __)  K (__, __)  
   L (__, __)  M (__, __)  
   N (__, __)  O (__, __)  
   BONUS: 
   P (__, __)  Q (__, __)
1. 

2. a) IV 
b) II 
c) I 
d) III 

3. J (−10, 20) 
   K (−20, 0) 
   L (−5, −5) 
   M (15, 10) 
   N (0, −15) 
   O (5, −10) 
   BONUS 
   P (−17, −10) 
   Q (20, −10)
Unit 7: Geometry

Quiz (Lessons 25–27) — WNCP

1. a) On the grid shown, plot and label the points.
   
   \[
   \begin{align*}
   A & (\text{–3, 2}) \quad B (4, \text{–1}) \\
   C & (\text{–4, 2}) \quad D (3, 1) \\
   E & (\text{–1, 0}) \quad F (0, \text{–4}) \\
   G & (2, \text{–3}) \quad H (\text{–2, 4})
   \end{align*}
   \]

2. Write the quadrant where each point is found.
   
   a) (17, \text{–6}) Quadrant _____  
   b) (\text{–11, 13}) Quadrant _____  
   c) (9, 14) Quadrant _____  
   d) (\text{–8, 10}) Quadrant _____

3. Write the coordinates of the points.
   
   \[
   \begin{align*}
   A & (\text{___, ___}) \quad B (\text{___, ___}) \\
   C & (\text{___, ___}) \quad D (\text{___, ___}) \\
   E & (\text{___, ___}) \quad F (\text{___, ___}) \\
   G & (\text{___, ___}) \quad H (\text{___, ___})
   \end{align*}
   \]
Unit 7: Geometry

Quiz (Lessons 25–27) — WNCP

4. a) Find the vertical distance between points A and B.
   __________

b) Find the horizontal distance between points C and D.
   __________

5. Find the new coordinates of the point after the translation.
   a) Point A (3, 5) is moved 7 units to the right.  __________
   b) Point B (−1, −4) is moved 2 units to the left.  __________
   c) Point C (1, −5) is moved 3 units up.  __________
   d) Point B (−4, 2) is moved 5 units down.  __________

BONUS: Find the coordinates of A (1, 3) after it has been moved 100 units to the right and 300 units down.
1. Teacher to check.

2. 
   a) IV
   b) II
   c) I
   d) III

3. 
   A (1, −3)
   B (2, 0)
   C (0, 3)
   D (0, −1)
   E (−3, 4)
   F (−4, 0)
   G (−4, −2)
   H (0, 0)

4. 
   a) 4
   b) 5

5. 
   a) (10, 5)
   b) (−3, −4)
   c) (1, −2)
   d) (−4, −3)

BONUS
   (101, −297)
1. How many units right or left and how many units up or down did the dot slide from A to B?

   a)
   
   b)

   _____ units ________

   _____ units ________

2. Translate \(\triangle ABC\) 2 units right and 4 units down.

3. a) Reflect \(\triangle ABC\) in the x-axis.
   Label the new triangle \(\triangle DEF\).

   b) Reflect \(\triangle ABC\) in the y-axis.
   Label the new triangle \(\triangle KLM\).
Unit 7: Geometry

Quiz (Lessons 28–34) — WNCP

4.  a) Rotate $\triangle ABC$ 90° clockwise around the origin. Label the new triangle $\triangle AEF$.

   b) Rotate $\triangle ABC$ 90° counter-clockwise around the origin. Label the new triangle $\triangle ALM$.

5. Draw 4 more of each shape to show how the shape can be used to tessellate the grid.

   a) \hspace{1cm} \hspace{1cm} b)

ADVANCED: Draw the mirror line when $\triangle ABC$ is reflected in the line to get $\triangle DEF$.

BONUS: The point $B (2, -3)$ is a reflection of a point $A$ in the $x$-axis.

What are the coordinates of the point $A$? _____________
Unit 7: Geometry

Quiz (Lessons 28–34) — WNCP

1. a) 5 units right
   4 units down
   
   b) 6 units left
   3 units down

2. Teacher to check.
3. Teacher to check.
4. Teacher to check.
5. Teacher to check.

ADVANCED

Teacher to check.

BONUS

(2, 3)
1. Draw 4 more of each shape to show how the shape can be used to tessellate the grid.
   a) 
   b) 

2. Find the interior angle of a regular hexagon by following these steps.
   a) Divide the hexagon into triangles formed by diagonals from point $A$.
   b) How many triangles can be drawn in this way? ______
   c) Find the sum of the interior angles of one triangle. ______°
   d) Find the sum of the interior angles of all the triangles. ______°
   e) Divide to find the measure of $\angle A$. ______° ÷ ______ = ______°

3. a) Name a triangle congruent to $\triangle ABC$. $\triangle ______$
   b) Name a triangle similar to $\triangle ABC$. $\triangle ______$
   c) Explain why the other triangle is neither congruent nor similar to $\triangle ABC$.
   
   ____________________________________________________________
   ____________________________________________________________
   ____________________________________________________________
4. Perform a dilatation using O as the centre. Use scale factor 2.

5. Identify the transformation between the shapes as a rotation, reflection, translation, or dilatation.

   a)        b)  
   _______________________  ______________________

   c)        d)  
   ________________________  ____________________

BONUS: Name the two transformations that were performed.
1. Teacher to check.

2. a) Teacher to check.
   b) $4$
   c) $180^\circ$
   d) $720^\circ$
   e) $720^\circ + 6 = 120^\circ$

3. a) $\triangle IHG$
   b) $\triangle LJK$
   c) $\triangle EDF$ is not the same size and shape as $\triangle ABC$ so it is not congruent to $\triangle ABC$. It is not the same shape so it is not similar to $\triangle ABC$.

4. Teacher to check.

5. a) reflection
   b) rotation
   c) dilatation
   d) translation

**BONUS**

dilatation, reflection
Unit 7: Geometry
Test (Lessons 25–26, 34–38) — ON

1. Draw the mirror line for each pair of images.
   a) ![Image a]
   b) ![Image b]

2. a) Create a tessellation that includes at least seven more “I” shapes.
   b) Label two of your shapes B and C. Describe the transformations required to move shape A to shapes B and C.

   **BONUS:** Find a single transformation of a different type that takes shape A to one of the shapes B or C.

3. These triangles are similar.
   Find $x$ and $y$.

4. Label the vertices of the images ($A'$, $B'$, ...).
   Find the scale factor of the dilatation.
   \[
   \frac{A'B'}{AB} = \quad = \\
   \text{Scale factor} =
   \]
5. Perform a dilatation of ORQP with scale factor = 3 using O as the centre.

Label the image $O' R' Q' P'$.

Find the coordinates of the points.

$O (____, ____)$  $O' (____, ____)$

$P (____, ____)$  $P' (____, ____)$

$Q (____, ____)$  $Q' (____, ____)$

$R (____, ____)$  $R' (____, ____)$

6. a) Find the sum of the angles in a hexagon by breaking the regular hexagon at right into triangles. Show your work.

b) Calculate the measure of each interior angle within a regular hexagon.

c) Find the measure of angle $V$.

d) Explain why a regular hexagon tessellates.
1. a) 

b) 

2. a) Answers may vary. Teacher to check.
Sample answer:

b) Answers will vary. Teacher to check.
In the sample above:
A to B: Translate 4 units down.
A to C: Translate 4 units down and 4 units right.

**BONUS**
A to B: Reflect A in the bottom side.

3. \( x = 15, \ y = 18 \)

4. \[
\begin{align*}
\frac{AB'}{AB} &= \frac{2}{8} = \frac{1}{4} \\
\text{Scale factor} &= 0.25
\end{align*}
\]

5. 

6. a) Answers may vary. Teacher to check.
Sample answer 1: 

The sum of the angles in a hexagon is \( 4 \times 180^\circ = 720^\circ \).

Sample answer 2:
Break the regular hexagon into 6 equilateral triangles.

The angles of equilateral triangles are all \( 60^\circ \), so the sum of the angles in the hexagon is \( 6 \times 2 \times 60^\circ = 720^\circ \).

b) \( 720^\circ \div 6 = 120^\circ \)

c) \( 360^\circ - (120^\circ + 120^\circ) = 120^\circ \)

d) The interior angle of a hexagon perfectly matches angle V, so a third hexagon will fit perfectly above the first two.
Unit 7: Geometry
Test (Lessons 25–34) — WNCP

1. a) On the grid shown, plot and label the points.
   a) \( A (-3, 2) \)
   b) \( A', \) the reflection of \( A \) in the \( x \)-axis
   c) \( B (4, -1) \)
   d) \( B', \) the reflection of \( B \) in the \( y \)-axis

   b) Write the coordinates of points \( A' (\_, \_) \) and \( B' (\_, \_) \).

2. Write the quadrant where each point is found.
   a) \((17, -6)\) Quadrant __
   b) \((-11, 13)\) Quadrant __
   c) \((9, 14)\) Quadrant __
   d) \((-8, -10)\) Quadrant __

3. Find the horizontal and the vertical distance between the points. Hint: Use a sketch.

<table>
<thead>
<tr>
<th>Point</th>
<th>Image</th>
<th>Horizontal distance</th>
<th>Vertical distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A (-2, -3) )</td>
<td>( A' (-2, 4) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( B (-3, -1) )</td>
<td>( B' (-8, -1) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C (5, -4) )</td>
<td>( C' (5, 1) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. a) Plot each pair of points.
   \( A (-2, -3) \quad A' (2, 4) \)
   \( B (-3, -1) \quad B' (-1, -4) \)

   b) For each pair of points, write the translation needed to take the original point to the image point.

   \( A \to A' \)
   \( B \to B' \)
Unit 7: Geometry
Test (Lessons 25–34) — WNCP

5. Rotate the figure 90° clockwise around the origin by first rotating the vertices. Find the coordinates of the vertices of both figures.

- \( J(\_,\_) \quad J'(\_,\_) \)
- \( K(\_,\_) \quad K'(\_,\_) \)
- \( L(\_,\_) \quad L'(\_,\_) \)
- \( M(\_,\_) \quad M'(\_,\_) \)

6. Describe a single transformation that takes shape A onto the other shapes in the design. Specify the mirror line or the centre of rotation or the amount and the direction of translation.
   a) Shape B ____________________________________
   b) Shape C ____________________________________
   c) Shape D ____________________________________
   d) Shape E ____________________________________
   BONUS: Shape F _________________________________

7. Describe a sequence of two transformations that will map the quadrilateral \( WXYZ \) onto its image, \( W'X'Y'Z' \).

   BONUS:
   Write the coordinates of the two points that are:
   a) located 4 units away, horizontally, from the point \((-2, 2)\). (\_,\_) and (\_,\_)
   b) located 7 units away, vertically, from the point \((-3, 1)\). (\_,\_) and (\_,\_)
1. a) 

   \[ A' (-3, -2) \]

   \[ B' (-4, -1) \]

   b) 

   2. a) IV  
        b) II  
        c) I  
        d) III  

3. | Horizontal Distance | Vertical Distance |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

4. a) 

   \[ A \rightarrow A': 4 \text{ units right,} \]
   \[ 7 \text{ units up} \]

   \[ B \rightarrow B': 2 \text{ units right,} \]
   \[ 3 \text{ units down} \]

   b) 

   5. \( J (-10, 20) \quad J' (20, 10) \)
     \( K (-20, 0) \quad K' (0, 20) \)
     \( L (-5, -5) \quad L' (-5, 5) \)
     \( M (-10, 5) \quad M' (5, 10) \)

6. a) Reflection in the horizontal line through point (0, 4)  
    b) Reflection through the y-axis  
    c) Translation 4 units left and 4 units down  
    d) Rotation 180° clockwise (or counter-clockwise) around the point (0, 4)  

   **BONUS**  
   Rotation 180° clockwise (or counter-clockwise) around the point (2, 6)  

7. Sample answer:  
   A reflection in the y-axis, followed by a translation 2 units right and 1 unit up  

   **BONUS**  
   a) \((-6, 2)\) and \((2, 2)\)  
   b) \((-3, -6)\) and \((-3, 8)\)
1. a) Create a tree diagram to show all the possible outcomes from tossing a coin then rolling a four-sided die labelled 1, 2, 3, 4.

b) How many possible outcomes are there? _____

c) What is the probability of tossing heads, then rolling an even number? _____

2. For the two spinners shown at right:

   a) How many possible outcomes are there? _____

   b) What is the probability of spinning “2 D”? _____

3. There are two bags of coins. Each bag contains one nickel and one dime. You are going to pull out one coin from each bag.

   a) List all possible outcomes in the table.

      +-----+-----+
      | Bag1 | Bag2 |
      +-----+-----+
      | N    | D    |
      +-----+-----+

   b) What is the probability of pulling out a total of 15¢? _____

4. Sketch a spinner that you would expect to spin red 50% of the time, blue 25% of the time, and white 25% of the time.

   a) How many times out of 28 would you expect to spin red? _____

   b) How many times out of 28 would you expect to spin white? _____

   c) How many times out of 28 would you expect to spin a colour that is not blue? _____
5. Suppose that you have two spinners. The first spinner has the numbers 1 and 2. The second spinner has the numbers 1, 2, and 3.

   a) Create an organized list of all the possible outcomes of spinning both spinners.

   b) What is the probability that the sum of the numbers will be …

      i) 3? 
      b) 5? 
      c) 6?

**BONUS:** Two spinners each have the numbers from 1 to 100. What is the probability that the total of the numbers when you spin both spinners will be 2?
1. a) H T
   1 2 3 4 1 2 3 4
b) 8
c) \( \frac{2}{8} \)

2. a) 12
b) \( \frac{1}{12} \)

3. a) 5, 5, 10, 10
    5, 10, 5, 10
b) \( \frac{2}{4} \)

4. a) 14
b) 7
c) 21

5. a) Spin A: 1, 1, 1, 2, 2, 2
    Spin B: 1, 2, 3, 1, 2, 3
b) i) \( \frac{2}{6} \)
    ii) \( \frac{1}{6} \)
    iii) 0

BONUS
\( \frac{1}{10000} \)
1. a) Create a tree diagram to show all of the possible outcomes from first rolling a four-sided die labelled 1, 2, 3, 4, then flipping a coin.

b) How many possible outcomes are there? ______

c) What is the probability of rolling a 2 and flipping tails? ______

d) What is the probability of rolling an odd number and flipping heads? ______

2. For the two spinners shown at right:

a) Show all the combinations you could spin on the two spinners.

b) How many ways can you spin an odd number on the first spinner and a Y on the second? ______

c) What is the probability of spinning the combination 2X? ______

2. In your right pocket, you have a $5 bill and a $10 bill. In your left pocket, you have a $5 bill and a $20 bill. You reach into each pocket and pull out one bill.

a) List all possible outcomes in the table to the right.

<table>
<thead>
<tr>
<th>Right Pocket</th>
<th>Left Pocket</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5 bill</td>
<td>$5 bill</td>
</tr>
<tr>
<td>$10 bill</td>
<td>$5 bill</td>
</tr>
<tr>
<td>$5 bill</td>
<td>$20 bill</td>
</tr>
</tbody>
</table>

b) What is the probability of pulling out a total of $10? ______

c) What is the probability of pulling out a total of at least $25? ______

4. A scientist observes a group of 24 killer whales and finds that only 6 of them are male.

a) What is the experimental probability of a whale in this group being male? ______ female? ______

b) What percentage of the group is female? ______

c) Based on this small group, how many females does the scientist expect to see in a community of 300 whales? ______
Unit 8: Probability and Data Management

Test (Lessons 17–23) — ON

5. a) Sketch a spinner on which you would expect
to spin green 75% of the time.

   b) How many times out of 60 spins would you expect to spin green? ______

   c) How many times out of 24 spins would you expect to spin something other than green? ______

6. For each of the following scenarios, write “yes” if the two events are independent and
   “no” if they are not. If the events are not independent, describe how to change the
   situation to make the events independent.

   a) Flip a coin and roll a die. ______

   b) Pull two playing cards from a deck, one after the other (without replacing the first card). ______

7. Give an example of a real-life situation where probability would be expressed in this form.

   a) 80% _____________________________________________________________

   b) 0.435 ____________________________________________________________

   c) \( \frac{1}{3} \) ________________________________________________________

8. Suppose that you roll 2 four-sided dice, each labelled 1, 2, 3, 4.

   a) Create an organized list of all of the possible outcomes.

   b) What is the probability of rolling:

      i) a total of 4? ______       ii) a total of 3? ______

   c) If you roll the dice 64 times, how many times would you expect to roll:

      i) a total of 4? ______       ii) a total of 3? ______

   **BONUS:** Which of the possible totals is most likely to occur? ______
1. a) Heads
   Tails
   Heads
   Tails
   Heads
   Tails

   b) No – would be independent if the 1st card was replaced before drawing 2nd card
7. Answers may vary. Teacher to check.
   Sample answers:
   a) The probability of precipitation tomorrow is 80%.
   b) My batting average is 0.435.
   c) The chance of rolling a number greater than 4 on a die is 1/3.

2. a) (1,X), (1,Y), (1,Z),
   (2,X), (2,Y), (2,Z),
   (3,X), (3,Y), (3,Z)

   b) 2
   c) 1/6
   d) 2/6 or 1/4

3. a) | R | 5 5 10 10 |
    L | 5 20 5 20

   b) 1/4
   c) 2/4 or 1/2

4. a) male 1/4, female 3/4
   b) 75%
   c) 225

5. a) Answers may vary, but three quarters of spinner should be green.
    Sample answer:

   b) 45
   c) 6

6. a) Yes
1. For each spinner, write the probability of the given events.
   a) P(Green) = _____
   b) P(Blue) = _____

2. Match the net for a die to the correct statement.
   a) The probability of rolling a 4 is \(\approx 0.33\).
   b) The probability of rolling an odd number is \(\frac{2}{3}\).
   c) The probability of rolling a 2 is 50%.
   d) The probability of rolling a 1 is the same as the probability of rolling a 6.

3. Express the probability of spinning each result as a fraction, a decimal and a percent.
   a) P(S)  _____  _____  _____
   b) P(U)  _____  _____  _____

4. a) Create a tree diagram to show all of the possible outcomes from first rolling a four-sided die labelled 1, 2, 3, 4, then flipping a coin.

   b) How many possible outcomes are there? ______
   c) What is the probability of rolling a 2 and flipping tails? ______
   d) What is the probability of rolling an odd number and flipping heads? ______

5. In your right pocket, you have a $5 bill and a $10 bill. In your left pocket, you have a $5 bill and a $20 bill. You reach into each pocket and pull out one bill.
   a) List all possible outcomes in the table to the right.
   b) What is the probability of pulling out a total of $10? ______
   c) What is the probability of pulling out a total of at least $25? ______
6. For the two spinners shown at right:
   a) Show all the combinations you could spin on the two spinners.
      \[ \begin{array}{ccc}
      3 & 1 & 2 \\
      Y & Z & X
      \end{array} \]
   b) How many ways can you spin an odd number on the first spinner and a Y on the second? _____
   c) What is the probability of spinning the combination 2X? _____

7. For each of the following scenarios, write “yes” if the two events are independent and “no” if they are not. If the events are not independent, describe how to change the situation to make the events independent.
   a) Flip a coin and roll a die. ______
   b) Pull two playing cards from a deck, one after the other (without replacing the first card). ______

8. Suppose that you roll 2 four-sided dice, each labelled 1, 2, 3, 4.
   a) Create an organized list of all of the possible outcomes.
   b) What is the probability of rolling:
      i) a total of 4? _____
      ii) a total of 3? _____
   c) If you roll the dice 64 times, how many times would you expect to roll:
      i) a total of 4? _____
      ii) a total of 3? _____
   **BONUS:** Which of the possible totals is most likely to occur? _____
1. a) $P(\text{Green}) = \frac{1}{2}$  
    $P(\text{Blue}) = \frac{1}{4}$  
    b) $P(\text{Blue}) = \frac{1}{4}$  
    $P(\text{Yellow}) = \frac{3}{8}$

2. C B A D

3. a) $\frac{1}{5}$, 0.2, 20%  
    b) $\frac{1}{5}$, 0.4, 40%

4. 
   Heads
   \[ \begin{array}{c}
   1 \\
   2 \\
   3 \\
   4 \\
   \end{array} \]
   Tails

   i) 8  
   b) $\frac{1}{8}$  
   c) $\frac{2}{8}$ or $\frac{1}{4}$

5. a) 
   \[
   \begin{array}{c|c|c|c}
   & R & 5 & 10 \\
   \hline
   L & 5 & 20 & 5 \\
   \hline
   \end{array}
   \]
   b) $\frac{1}{4}$  
   c) $\frac{2}{4}$ or $\frac{1}{2}$

6. a) (1,X), (1,Y), (1,Z), (2,X), (2,Y), (2,Z), (3,X), (3,Y), (3,Z)  
   b) 2  
   c) $\frac{1}{9}$

7. a) Yes

BONUS

5
## Contents

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number Sense and Numeration</td>
<td>3</td>
</tr>
<tr>
<td>Measurement</td>
<td>6</td>
</tr>
<tr>
<td>Geometry and Spatial Sense</td>
<td>8</td>
</tr>
<tr>
<td>Patterning and Algebra</td>
<td>10</td>
</tr>
<tr>
<td>Data Management and Probability</td>
<td>12</td>
</tr>
</tbody>
</table>
Notes

To ensure that the curriculum is fully covered, use the worksheets with the lessons plans in the Teacher’s Guide.

Underlined lesson numbers indicate relevant preparatory exercises.

OCUP: Ontario Curriculum Unit Planner

JUMP Math workbook units are represented by:

- **NS**: Number Sense
- **PA**: Patterns and Algebra
- **ME**: Measurement
- **G**: Geometry
- **PDM**: Probability and Data Management
Number Sense and Numeration

Overall Expectations
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>OCUP Code</th>
<th>Overall Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>7m8</td>
<td>represent, compare, and order numbers, including integers;</td>
</tr>
<tr>
<td>7m9</td>
<td>demonstrate an understanding of addition and subtraction of fractions and integers, and apply a variety of computational strategies to solve problems involving whole numbers and decimal numbers;</td>
</tr>
<tr>
<td>7m10</td>
<td>demonstrate an understanding of proportional relationships using percent, ratio, and rate.</td>
</tr>
</tbody>
</table>

Quantity Relationships
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>-----------</td>
<td>---------------------</td>
</tr>
<tr>
<td>7m11</td>
<td>represent, compare, and order decimals to hundredths and fractions, using a variety of tools;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>7m12</td>
<td>generate multiples and factors, using a variety of tools and strategies;</td>
</tr>
<tr>
<td>7m13</td>
<td>identify and compare integers found in real-life contexts;</td>
</tr>
<tr>
<td>7m14</td>
<td>represent and order integers, using a variety of tools;</td>
</tr>
<tr>
<td>7m15</td>
<td>select and justify the most appropriate representation of a quantity (i.e., fraction, decimal, percent) for a given context;</td>
</tr>
<tr>
<td>7m16</td>
<td>represent perfect squares and square roots, using a variety of tools;</td>
</tr>
<tr>
<td>7m17</td>
<td>explain the relationship between exponential notation and the measurement of area and volume.</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Operational Sense
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code Specific Expectation</td>
<td>Part Unit Lesson</td>
</tr>
<tr>
<td>7m18 divide whole numbers by simple fractions and by decimal numbers to hundredths, using concrete materials;</td>
<td>1 5:NS 52, 53</td>
</tr>
<tr>
<td></td>
<td>1 6:ME 2</td>
</tr>
<tr>
<td></td>
<td>2 1:NS 79–81</td>
</tr>
<tr>
<td>7m19 use a variety of mental strategies to solve problems involving the addition and subtraction of fractions and decimals;</td>
<td>1 5:NS 45–48, 49, 50, 54</td>
</tr>
<tr>
<td></td>
<td>1 6:ME 2</td>
</tr>
<tr>
<td></td>
<td>1 2:PA 15</td>
</tr>
<tr>
<td></td>
<td>2 1:NS 78, 80–83</td>
</tr>
<tr>
<td>7m20 solve problems involving the multiplication and division of decimal numbers to thousandths by one-digit whole numbers, using a variety of tools and strategies;</td>
<td>1 5:NS 45–48, 49–54</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>7m21 solve multi-step problems arising from real-life contexts and involving whole numbers and decimals, using a variety of tools and strategies;</td>
<td>1 5:NS 40, 41, 54</td>
</tr>
<tr>
<td></td>
<td>2 1:NS 82, 83</td>
</tr>
<tr>
<td>7m22 use estimation when solving problems involving operations with whole numbers, decimals, and percents, to help judge the reasonableness of a solution;</td>
<td>1 5:NS 39, 42, 43, 44, 54</td>
</tr>
<tr>
<td></td>
<td>2 1:NS 67, 82, 83</td>
</tr>
<tr>
<td>7m23 evaluate expressions that involve whole numbers and decimals, including expressions that contain brackets, using order of operations;</td>
<td>1 1:NS 2</td>
</tr>
<tr>
<td></td>
<td>1 6:ME 8–10</td>
</tr>
<tr>
<td>7m24 add and subtract fractions with simple like and unlike denominators, using a variety of tools and algorithms;</td>
<td>1 3:NS 27–31</td>
</tr>
<tr>
<td></td>
<td>2 1:NS 56, 78</td>
</tr>
<tr>
<td>7m25 demonstrate, using concrete materials, the relationship between the repeated addition of fractions and the multiplication of that fraction by a whole number;</td>
<td>1 3:NS 22</td>
</tr>
<tr>
<td>7m26 add and subtract integers, using a variety of tools.</td>
<td>2 6:NS 90–94</td>
</tr>
</tbody>
</table>
# Proportional Relationships

By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OCUP Code</strong></td>
<td><strong>Specific Expectation</strong></td>
</tr>
<tr>
<td>7m27</td>
<td>determine, through investigation, the relationships among fractions, decimals, percents, and ratios;</td>
</tr>
<tr>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>7m28</td>
<td>solve problems that involve determining whole number percents, using a variety of tools;</td>
</tr>
<tr>
<td>7m29</td>
<td>demonstrate an understanding of rate as a comparison, or ratio, of two measurements with different units;</td>
</tr>
<tr>
<td>7m30</td>
<td>solve problems involving the calculation of unit rates.</td>
</tr>
</tbody>
</table>
# Measurement

## Overall Expectations

By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>OCUP Code</th>
<th>Overall Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>7m31</td>
<td>report on research into real-life applications of area measurements;</td>
</tr>
<tr>
<td>7m32</td>
<td>determine the relationships among units and measurable attributes, including the area of a trapezoid and the volume of a right prism.</td>
</tr>
</tbody>
</table>

## Attributes, Units and Measurement Sense

By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>-----------</td>
<td>----------------------</td>
</tr>
<tr>
<td>7m33</td>
<td>research and report on real-life applications of area measurements.</td>
</tr>
</tbody>
</table>

## Measurement Relationships

By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>-----------</td>
<td>----------------------</td>
</tr>
<tr>
<td>7m34</td>
<td>sketch different polygonal prisms that share the same volume;</td>
</tr>
<tr>
<td>7m35</td>
<td>solve problems that require conversion between metric units of measure;</td>
</tr>
<tr>
<td>7m36</td>
<td>solve problems that require conversion between metric units of area (i.e., square centimetres, square metres);</td>
</tr>
<tr>
<td>7m37</td>
<td>determine, through investigation using a variety of tools and strategies, the relationship for calculating the area of a trapezoid, and generalize to develop the formula [i.e., Area = (sum of lengths of parallel sides x height) ÷ 2];</td>
</tr>
<tr>
<td>7m38</td>
<td>solve problems involving the estimation and calculation of the area of a trapezoid;</td>
</tr>
<tr>
<td>7m39</td>
<td>estimate and calculate the area of composite two-dimensional shapes by decomposing into shapes with known area relationships;</td>
</tr>
</tbody>
</table>
### Measurement Relationships (continued)

By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OCUP Code</strong></td>
<td><strong>Specific Expectation</strong></td>
</tr>
<tr>
<td>7m40</td>
<td>determine, through investigation using a variety of tools and strategies, the relationship between the height, the area of the base, and the volume of right prisms with simple polygonal bases and generalize to develop the formula (i.e., Volume = area of base x height);</td>
</tr>
<tr>
<td>7m41</td>
<td>determine, through investigation using a variety of tools, the surface area of right prisms;</td>
</tr>
<tr>
<td>7m42</td>
<td>solve problems that involve the surface area and volume of right prisms and that require conversion between metric measures of capacity and volume (i.e., millilitres and cubic centimetres).</td>
</tr>
</tbody>
</table>
Geometry and Spatial Sense

Overall Expectations
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>OCUP Code</th>
<th>Overall Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>7m43</td>
<td>construct related lines, and classify triangles, quadrilaterals, and prisms;</td>
</tr>
<tr>
<td>7m44</td>
<td>develop an understanding of similarity, and distinguish similarity and congruence;</td>
</tr>
<tr>
<td>7m45</td>
<td>describe location in the four quadrants of a coordinate system, dilate two-dimensional shapes, and apply transformations to create and analyze designs.</td>
</tr>
</tbody>
</table>

Geometric Properties
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
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</tr>
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<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>-----------</td>
<td>----------------------</td>
</tr>
<tr>
<td>7m46</td>
<td>construct related lines (i.e., parallel; perpendicular; intersecting at 30°, 45°, and 60°), using angle properties and a variety of tools and strategies;</td>
</tr>
<tr>
<td>7m46</td>
<td>construct related lines (i.e., parallel; perpendicular; intersecting at 30°, 45°, and 60°), using angle properties and a variety of tools and strategies;</td>
</tr>
<tr>
<td>7m46</td>
<td>construct related lines (i.e., parallel; perpendicular; intersecting at 30°, 45°, and 60°), using angle properties and a variety of tools and strategies;</td>
</tr>
<tr>
<td>7m47</td>
<td>sort and classify triangles and quadrilaterals by geometric properties related to symmetry, angles, and sides, through investigation using a variety of tools and strategies;</td>
</tr>
<tr>
<td>7m47</td>
<td>sort and classify triangles and quadrilaterals by geometric properties related to symmetry, angles, and sides, through investigation using a variety of tools and strategies;</td>
</tr>
<tr>
<td>7m48</td>
<td>construct angle bisectors and perpendicular bisectors, using a variety of tools and strategies, and represent equal angles and equal lengths using mathematical notation;</td>
</tr>
<tr>
<td>7m48</td>
<td>construct angle bisectors and perpendicular bisectors, using a variety of tools and strategies, and represent equal angles and equal lengths using mathematical notation;</td>
</tr>
<tr>
<td>7m49</td>
<td>investigate, using concrete materials, the angles between the faces of a prism, and identify right prisms.</td>
</tr>
</tbody>
</table>

Geometric Relationships
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>-----------</td>
<td>----------------------</td>
</tr>
<tr>
<td>7m50</td>
<td>identify, through investigation, the minimum side and angle information (i.e., side-side-side; side-angle-side; angle-side-angle) needed to describe a unique triangle;</td>
</tr>
<tr>
<td>7m51</td>
<td>determine, through investigation using a variety of tools, relationships among area, perimeter, corresponding side lengths, and corresponding angles of congruent shapes;</td>
</tr>
</tbody>
</table>
### Geometric Relationships (continued)
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>7m52</td>
<td>demonstrate an understanding that enlarging or reducing two-dimensional shapes creates similar shapes;</td>
</tr>
<tr>
<td>7m53</td>
<td>distinguish between and compare similar shapes and congruent shapes, using a variety of tools and strategies.</td>
</tr>
</tbody>
</table>

### Location and Movement
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>7m54</td>
<td>plot points using all four quadrants of the Cartesian coordinate plane;</td>
</tr>
<tr>
<td>7m55</td>
<td>identify, perform, and describe dilatations (i.e., enlargements and reductions), through investigation using a variety of tools;</td>
</tr>
<tr>
<td>7m56</td>
<td>create and analyse designs involving translations, reflections, dilatations, and/or simple rotations of two-dimensional shapes, using a variety of tools and strategies;</td>
</tr>
<tr>
<td>7m57</td>
<td>determine, through investigation using a variety of tools, polygons or combinations of polygons that tile a plane, and describe the transformation(s) involved.</td>
</tr>
</tbody>
</table>
Patterning and Algebra

Overall Expectations
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>OCUP Code</th>
<th>Overall Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>7m58</td>
<td>represent linear growing patterns (where the terms are whole numbers) using concrete materials, graphs, and algebraic expressions;</td>
</tr>
<tr>
<td>7m59</td>
<td>model real-life linear relationships graphically and algebraically, and solve simple algebraic equations using a variety of strategies, including inspection and guess and check.</td>
</tr>
</tbody>
</table>

Patterns and Relationships
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code Specific Expectation</td>
<td>Part</td>
</tr>
<tr>
<td>7m60 represent linear growing patterns, using a variety of tools and strategies;</td>
<td>2</td>
</tr>
<tr>
<td>7m61 make predictions about linear growing patterns, through investigation with concrete materials;</td>
<td>2</td>
</tr>
<tr>
<td>7m62 develop and represent the general term of a linear growing pattern, using algebraic expressions involving one operation;</td>
<td>2</td>
</tr>
<tr>
<td>7m63 compare pattern rules that generate a pattern by adding or subtracting a constant, or multiplying or dividing by a constant, to get the next term with pattern rules that use the term number to describe the general term.</td>
<td>2</td>
</tr>
</tbody>
</table>
Variables, Expressions, and Equations

By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OCUP Code</strong></td>
<td><strong>Specific Expectation</strong></td>
</tr>
<tr>
<td>7m64</td>
<td>model real-life relationships involving constant rates where the initial condition starts at 0, through investigation using tables of values and graphs;</td>
</tr>
<tr>
<td>7m65</td>
<td>model real-life relationships involving constant rates, using algebraic equations with variables to represent the changing quantities in the relationship;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>7m66</td>
<td>translate phrases describing simple mathematical relationships into algebraic expressions, using concrete materials;</td>
</tr>
<tr>
<td>7m67</td>
<td>evaluate algebraic expressions by substituting natural numbers for the variables;</td>
</tr>
<tr>
<td>7m68</td>
<td>make connections between evaluating algebraic expressions and determining the term in a pattern using the general term;</td>
</tr>
<tr>
<td>7m69</td>
<td>solve linear equations of the form ( ax = c ) or ( c = ax ) and ( ax + b = c ) or variations such as ( b + ax = c ) and ( c = bx + a ) (where ( a, b, ) and ( c ) are natural numbers) by modelling with concrete materials, by inspection, or by guess and check, with and without the aid of a calculator.</td>
</tr>
</tbody>
</table>
Data Management and Probability

**Overall Expectations**
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>OCUP Code</th>
<th>Overall Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>7m70</td>
<td>collect and organize categorical, discrete, or continuous primary data and secondary data and display the data using charts and graphs, including relative frequency tables and circle graphs;</td>
</tr>
<tr>
<td>7m71</td>
<td>make and evaluate convincing arguments, based on the analysis of data;</td>
</tr>
<tr>
<td>7m72</td>
<td>compare experimental probabilities with the theoretical probability of an outcome involving two independent events.</td>
</tr>
</tbody>
</table>

**Collection and Organization of Data**
By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>7m73</td>
<td>collect data by conducting a survey or an experiment to do with themselves, their environment, issues in their school or community, or content from another subject and record observations or measurements;</td>
</tr>
<tr>
<td>7m74</td>
<td>collect and organize categorical, discrete, or continuous primary data and secondary data and display the data in charts, tables, and graphs (including relative frequency tables and circle graphs) that have appropriate titles, labels, and scales that suit the range and distribution of the data, using a variety of tools;</td>
</tr>
<tr>
<td>7m75</td>
<td>select an appropriate type of graph to represent a set of data, graph the data using technology, and justify the choice of graph (i.e., from types of graphs already studied);</td>
</tr>
<tr>
<td>7m76</td>
<td>distinguish between a census and a sample from a population;</td>
</tr>
<tr>
<td>7m77</td>
<td>identify bias in data collection methods.</td>
</tr>
</tbody>
</table>
## Data Relationships

By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OCUP Code</strong></td>
<td><strong>Specific Expectation</strong></td>
</tr>
<tr>
<td>7m78</td>
<td>read, interpret, and draw conclusions from primary data and from secondary data presented in charts, tables, and graphs (including relative frequency tables and circle graphs);</td>
</tr>
<tr>
<td>7m79</td>
<td>identify, through investigation, graphs that present data in misleading ways;</td>
</tr>
<tr>
<td>7m80</td>
<td>determine, through investigation, the effect on a measure of central tendency (i.e., mean, median, and mode) of adding or removing a value or values;</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>7m81</td>
<td>identify and describe trends, based on the distribution of the data presented in tables and graphs, using informal language;</td>
</tr>
<tr>
<td>7m82</td>
<td>make inferences and convincing arguments that are based on the analysis of charts, tables, and graphs.</td>
</tr>
</tbody>
</table>

## Probability

By the end of Grade 7, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OCUP Code</strong></td>
<td><strong>Specific Expectation</strong></td>
</tr>
<tr>
<td>7m83</td>
<td>research and report on real-world applications of probabilities expressed in fraction, decimal, and percent form;</td>
</tr>
<tr>
<td>7m84</td>
<td>make predictions about a population when given a probability;</td>
</tr>
<tr>
<td>7m85</td>
<td>represent in a variety of ways all the possible outcomes of a probability experiment involving two independent events (i.e., one event does not affect the other event), and determine the theoretical probability of a specific outcome involving two independent events;</td>
</tr>
<tr>
<td>7m86</td>
<td>perform a simple probability experiment involving two independent events, and compare the experimental probability with the theoretical probability of a specific outcome.</td>
</tr>
</tbody>
</table>
## Contents

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>3</td>
</tr>
<tr>
<td>Patterns and Relations</td>
<td>8</td>
</tr>
<tr>
<td>Shape and Space</td>
<td>12</td>
</tr>
<tr>
<td>Statistics and Probability</td>
<td>16</td>
</tr>
</tbody>
</table>
Notes

To ensure that the curriculum is fully covered, use the worksheets with the lessons plans in the Teacher’s Guide.

Underlined lesson numbers indicate relevant preparatory exercises.

WNCP Abbreviations:

[C] Communication
[CN] Connections
[ME] Mental Mathematics and Estimation
[PS] Problem Solving
[R] Reasoning
[T] Technology
[V] Visualization

JUMP Math workbook units are represented by:

NS Number Sense
PA Patterns and Algebra
ME Measurement
G Geometry
PDM Probability and Data Management
## Number

### General Outcome
Develop number sense.

### Specific Outcomes
It is expected that students will:

1. **WNCP CURRICULUM**  
   **Specific Outcome**  
   Determine and explain why a number is divisible by 2, 3, 4, 5, 6, 8, 9 or 10, and why a number cannot be divided by 0. [C, R]

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determine if a given number is divisible by 2, 3, 4, 5, 6, 8, 9 or 10 and explain why.</td>
<td>1</td>
<td>3:NS</td>
<td>13–17</td>
</tr>
<tr>
<td>Sort a given set of numbers based upon their divisibility using organizers, such as Venn and Carroll diagrams.</td>
<td>1</td>
<td>3:NS</td>
<td>14–16</td>
</tr>
<tr>
<td>Determine the factors of a given number using the divisibility rules.</td>
<td>1</td>
<td>3:NS</td>
<td>17</td>
</tr>
<tr>
<td>Explain, using an example, why numbers cannot be divided by 0.</td>
<td>1</td>
<td>1:NS</td>
<td>5</td>
</tr>
</tbody>
</table>

2. **WNCP CURRICULUM**  
   **Specific Outcome**  
   Demonstrate an understanding of the addition, subtraction, multiplication and division of decimals (for more than 1-digit divisors or 2-digit multipliers, the use of technology is expected) to solve problems. [ME, PS, T]

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solve a given problem involving the addition of two or more decimal numbers.</td>
<td>1</td>
<td>5:NS</td>
<td>32–39, 40, 54</td>
</tr>
<tr>
<td>Solve a given problem involving the subtraction of decimal numbers.</td>
<td>1</td>
<td>5:NS</td>
<td>41</td>
</tr>
<tr>
<td>Solve a given problem involving the multiplication of decimal numbers.</td>
<td>1</td>
<td>6:ME</td>
<td>2, 7, 15, 16, 20, 21</td>
</tr>
<tr>
<td>2</td>
<td>1:NS</td>
<td>82, 83</td>
<td></td>
</tr>
<tr>
<td>Solve a given problem involving the multiplication or division of decimal numbers with 2-digit multipliers or 1-digit divisors (whole numbers or decimals) without the use of technology.</td>
<td>1</td>
<td>5:NS</td>
<td>45–54</td>
</tr>
<tr>
<td>1</td>
<td>6:ME</td>
<td>2, 7, 15, 16, 20, 21</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1:NS</td>
<td>82, 83</td>
<td></td>
</tr>
</tbody>
</table>
2. **Achievement Indicators**

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5:NS</td>
<td>42, 43</td>
</tr>
<tr>
<td>2</td>
<td>1:NS</td>
<td>82, 83</td>
</tr>
<tr>
<td>3</td>
<td>5:NS</td>
<td>42, 43, 44</td>
</tr>
<tr>
<td>2</td>
<td>1:NS</td>
<td>83</td>
</tr>
<tr>
<td>1</td>
<td>5:NS</td>
<td>42, 43</td>
</tr>
<tr>
<td>2</td>
<td>1:NS</td>
<td>43, 44</td>
</tr>
</tbody>
</table>

Place the decimal in a given problem involving the multiplication or division of decimal numbers with more than a 2-digit multiplier or 1-digit divisor (whole number or decimal), with the use of technology.

Place the decimal in a problem using front-end estimation, e.g., for $4.5 + 0.73 + 256.458$, think $4 + 256$, so the sum is greater than 260.

Place the decimal in a sum or difference using front-end estimation, e.g., for $12.33 \times 2.4$, think $12 \times 2$, so the product is greater than $24$.

Place the decimal in a quotient using front-end estimation, e.g., for $51.50 \div 2.1$, think $50 \div 2$, so the quotient is approximately 25 m.

Check the reasonableness of solutions using estimation.

Solve the reasonableness of solutions using estimation.

Solve a given problem that involves finding a percent.

Solve a given problem involving the multiplication or division of decimal numbers with more than a 2-digit multiplier or 1-digit divisor (whole number or decimal), with the use of technology.

3. **WNCP CURRICULUM**

**Specific Outcome**

Solve problems involving percents from 1% to 100%. [C, CN, PS, R, T]

**Achievement Indicators**

Express a given percent as a decimal or fraction.

Solve a given problem that involves finding a percent.

Determine the answer to a given percent problem where the answer requires rounding and explain why an approximate answer is needed, e.g., total cost including taxes.

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1:NS</td>
<td>64–69</td>
</tr>
<tr>
<td>2</td>
<td>1:NS</td>
<td>70–77</td>
</tr>
<tr>
<td>2</td>
<td>1:NS</td>
<td>70, 71, 74</td>
</tr>
</tbody>
</table>

4. **WNCP CURRICULUM**

**Specific Outcome**

Demonstrate an understanding of the relationship between positive repeating decimals and positive fractions, and positive terminating decimals and positive fractions. [C, CN, R, T]

**Achievement Indicators**

Predict the decimal representation of a given fraction using patterns, e.g., $\frac{1}{11} = 0.0\overline{9}$, $\frac{2}{11} = 0.1\overline{8}$, $\frac{3}{11} = ?$ ...
(continued)

<table>
<thead>
<tr>
<th>4. <strong>Achievement Indicators</strong></th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Match a given set of fractions to their decimal representations.</td>
<td>2</td>
<td>1:NS</td>
<td>55, 57, 58</td>
</tr>
<tr>
<td>Sort a given set of fractions as repeating or terminating decimals.</td>
<td>2</td>
<td>1:NS</td>
<td>57–59, 60</td>
</tr>
<tr>
<td>Express a given fraction as a terminating or repeating decimal.</td>
<td>2</td>
<td>1:NS</td>
<td>60–62</td>
</tr>
<tr>
<td>Express a given repeating decimal as a fraction.</td>
<td>2</td>
<td>1:NS</td>
<td>62, 63</td>
</tr>
<tr>
<td>Express a given terminating decimal as a fraction.</td>
<td>2</td>
<td>1:NS</td>
<td>55, 56</td>
</tr>
<tr>
<td>Provide an example where the decimal representation of a fraction is an approximation of its exact value.</td>
<td>2</td>
<td>1:NS</td>
<td>58, 60</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>5. <strong>WNCP CURRICULUM</strong></th>
<th><strong>JUMP MATH LESSONS</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Specific Outcome</strong></td>
<td></td>
</tr>
<tr>
<td>Demonstrate an understanding of adding and subtracting positive fractions and mixed numbers, with like and unlike denominators, concretely, pictorially and symbolically (limited to positive sums and differences). [C, CN, ME, PS, R, V]</td>
<td></td>
</tr>
<tr>
<td><strong>Achievement Indicators</strong></td>
<td>Part</td>
</tr>
<tr>
<td>Model addition and subtraction of a given positive fraction or a given mixed number using concrete representations, and record symbolically.</td>
<td>1</td>
</tr>
<tr>
<td>Determine the sum of two given positive fractions or mixed numbers with like denominators.</td>
<td>1</td>
</tr>
<tr>
<td>Determine the difference of two given positive fractions or mixed numbers with like denominators.</td>
<td>1</td>
</tr>
<tr>
<td>Determine a common denominator for a given set of positive fractions or mixed numbers.</td>
<td>1</td>
</tr>
<tr>
<td>Determine the sum of two given positive fractions or mixed numbers with unlike denominators.</td>
<td>1</td>
</tr>
<tr>
<td>Determine the difference of two given positive fractions or mixed numbers with unlike denominators.</td>
<td>1</td>
</tr>
<tr>
<td>Simplify a given positive fraction or mixed number by identifying the common factor between the numerator and denominator.</td>
<td>1</td>
</tr>
<tr>
<td>Simplify the solution to a given problem involving the sum or difference of two positive fractions or mixed numbers.</td>
<td>1</td>
</tr>
<tr>
<td>Solve a given problem involving the addition or subtraction of positive fractions or mixed numbers and determine if the solution is reasonable.</td>
<td>1</td>
</tr>
</tbody>
</table>
### 6. WNCP CURRICULUM

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Achievement Indicators</strong></td>
<td>Part</td>
</tr>
<tr>
<td>Demonstrate an understanding of addition and subtraction of integers, concretely, pictorially and symbolically. [C, CN, PS, R, V]</td>
<td>2</td>
</tr>
<tr>
<td>Explain, using concrete materials such as integer tiles and diagrams, that the sum of opposite integers is zero.</td>
<td>2</td>
</tr>
<tr>
<td>Illustrate, using a number line, the results of adding or subtracting negative and positive integers, e.g., a move in one direction followed by an equivalent move in the opposite direction results in no net change in position.</td>
<td>2</td>
</tr>
<tr>
<td>Add two given integers using concrete materials or pictorial representations and record the process symbolically.</td>
<td>2</td>
</tr>
<tr>
<td>Subtract two given integers using concrete materials or pictorial representations and record the process symbolically.</td>
<td>2</td>
</tr>
<tr>
<td>Solve a given problem involving the addition and subtraction of integers.</td>
<td>2</td>
</tr>
</tbody>
</table>

### 7. WNCP CURRICULUM

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Achievement Indicators</strong></td>
<td>Part</td>
</tr>
<tr>
<td>Compare and order positive fractions, positive decimals (to thousandths) and whole numbers by using:</td>
<td>1</td>
</tr>
<tr>
<td>• benchmarks</td>
<td>2</td>
</tr>
<tr>
<td>• place value</td>
<td>1</td>
</tr>
<tr>
<td>• equivalent fractions and/or decimals.</td>
<td>1</td>
</tr>
<tr>
<td>Order the numbers of a given set that includes positive fractions, positive decimals and/or whole numbers in ascending or descending order, and verify the result using a variety of strategies.</td>
<td>1</td>
</tr>
<tr>
<td>Identify a number that would be between two given numbers in an ordered sequence or on a number line.</td>
<td>1</td>
</tr>
<tr>
<td>Identify incorrectly placed numbers in an ordered sequence or on a number line.</td>
<td>1</td>
</tr>
<tr>
<td>Position fractions with like and unlike denominators from a given set on a number line and explain strategies used to determine order.</td>
<td>1</td>
</tr>
</tbody>
</table>
7. **Achievement Indicators**

<table>
<thead>
<tr>
<th>Order the numbers of a given set by placing them on a number line that contains benchmarks, such as 0 and 1 or 0 and 5.</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5:NS</td>
<td>37</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Position a given set of positive fractions, including mixed numbers and improper fractions, on a number line and explain strategies used to determine position.</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3:NS</td>
<td>19–21</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1:NS</td>
<td>67, 68</td>
<td></td>
</tr>
</tbody>
</table>
Patterns and Relations

General Outcomes
• Patterns: Use patterns to describe the world and solve problems.
• Variables and Equations: Represent algebraic expressions in multiple ways.

Patterns
It is expected that students will:

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demonstrate an understanding of oral and written patterns and their equivalent linear relations. [C, CN, R]</td>
<td></td>
</tr>
<tr>
<td><strong>Achievement Indicators</strong></td>
<td><strong>Part</strong></td>
</tr>
<tr>
<td>Formulate a linear relation to represent the relationship in a given oral or written pattern.</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Provide a context for a given linear relation that represents a pattern.</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Represent a pattern in the environment using a linear relation.</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Create a table of values from a linear relation, graph the table of values, and analyze the graph to draw conclusions and solve problems. [C, CN, R, V]</td>
<td></td>
</tr>
<tr>
<td><strong>Achievement Indicators</strong></td>
<td><strong>Part</strong></td>
</tr>
<tr>
<td>Create a table of values for a given linear relation by substituting values for the variable.</td>
<td>2</td>
</tr>
<tr>
<td>Create a table of values using a linear relation and graph the table of values (limited to discrete elements).</td>
<td>2</td>
</tr>
<tr>
<td>Sketch the graph from a table of values created for a given linear relation and describe the patterns found in the graph to draw conclusions, e.g., graph the relationship between ( n ) and ( 2n + 3 ).</td>
<td>2</td>
</tr>
</tbody>
</table>
Variables and Equations

It is expected that students will:

3. **WNCP CURRICULUM**

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demonstrate an understanding of preservation of equality by:</td>
<td></td>
</tr>
<tr>
<td>• modelling preservation of equality, concretely, pictorially and symbolically</td>
<td></td>
</tr>
<tr>
<td>• applying preservation of equality to solve equations. [C, CN, PS, R, V]</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model the preservation of equality for each of the four operations using concrete</td>
<td>1</td>
<td>2:PA</td>
<td>5–7,</td>
</tr>
<tr>
<td>materials or using pictorial representations, explain the process orally and record</td>
<td></td>
<td></td>
<td>8–11</td>
</tr>
<tr>
<td>it symbolically.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solve a given problem by applying preservation of equality.</td>
<td>1</td>
<td>2:PA</td>
<td>5–7,</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>8–11</td>
</tr>
</tbody>
</table>

4. **WNCP CURRICULUM**

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explain the difference between an expression and an equation. [C, CN]</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identify and provide an example of a constant term, a numerical coefficient and a</td>
<td>1</td>
<td>2:PA</td>
<td>12</td>
</tr>
<tr>
<td>variable in an expression and an equation.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Explain what a variable is and how it is used in a given expression.</td>
<td>1</td>
<td>2:PA</td>
<td>12</td>
</tr>
<tr>
<td>Provide an example of an expression and an equation, and explain how they are</td>
<td>1</td>
<td>2:PA</td>
<td>12</td>
</tr>
<tr>
<td>similar and different.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5. **WNCP CURRICULUM**

**Specific Outcome**
Evaluate an expression given the value of the variable(s). [CN, R]

**Achievement Indicators**
Substitute a value for an unknown in a given expression and evaluate the expression.

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2:PA</td>
<td>5</td>
</tr>
</tbody>
</table>

6. **WNCP CURRICULUM**

**Specific Outcome**
Model and solve problems that can be represented by one-step linear equations of the form \( x + a = b \), concretely, pictorially and symbolically, where \( a \) and \( b \) are integers. [CN, PS, R, V]

**Achievement Indicators**
- Represent a given problem with a linear equation and solve the equation using concrete models, e.g., counters, integer tiles.
- Draw a visual representation of the steps required to solve a given linear equation.
- Solve a given problem using a linear equation.
- Verify the solution to a given linear equation using concrete materials and diagrams.
- Substitute a possible solution for the variable in a given linear equation into the original linear equation to verify the equality.

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2:PA</td>
<td>6, 7</td>
</tr>
<tr>
<td>1</td>
<td>2:PA</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>2:PA</td>
<td>9, 14</td>
</tr>
<tr>
<td>1</td>
<td>2:PA</td>
<td>7, 10</td>
</tr>
<tr>
<td>1</td>
<td>2:PA</td>
<td>8, 9</td>
</tr>
</tbody>
</table>

7. **WNCP CURRICULUM**

**Specific Outcome**
Model and solve problems that can be represented by linear equations of the form:
- \( ax + b = c \)
- \( ax = b \)
- \( ax = b, a \neq 0 \)

concretely, pictorially and symbolically, where \( a, b \) and \( c \) are whole numbers. [CN, PS, R, V]

**Achievement Indicators**
Model a given problem with a linear equation and solve the equation using concrete models, e.g., counters, integer tiles.

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2:PA</td>
<td>6, 7</td>
</tr>
</tbody>
</table>
(continued)

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Draw a visual representation of the steps used to solve a given linear equation.</td>
<td>1</td>
<td>2:PA</td>
<td>7, 10, 13</td>
</tr>
<tr>
<td>Solve a given problem using a linear equation and record the process.</td>
<td>1</td>
<td>2:PA</td>
<td>9, 13, 14</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4:PA</td>
<td>28</td>
</tr>
<tr>
<td>Verify the solution to a given linear equation using concrete materials and diagrams.</td>
<td>1</td>
<td>2:PA</td>
<td>7, 10, 13</td>
</tr>
<tr>
<td>Substitute a possible solution for the variable in a given linear equation into the original linear equation to verify the equality.</td>
<td>1</td>
<td>2:PA</td>
<td>8, 9</td>
</tr>
</tbody>
</table>
Shape and Space

General Outcomes
• Measurement: Use direct or indirect measurement to solve problems.
• 3-D Objects and 2-D Shapes: Describe the characteristics of 3-D objects and 2-D shapes, and analyze the relationships among them.
• Transformations: Describe and analyze position and motion of objects and shapes.

Measurement
It is expected that students will:

1. WNCP CURRICULUM

Specific Outcome
Demonstrate an understanding of circles by:
• describing the relationships among radius, diameter and circumference of circles
• relating circumference to pi
• determining the sum of the central angles
• constructing circles with a given radius or diameter
• solving problems involving the radii, diameters and circumferences of circles. [C, CN, R, V]

Achievement Indicators

<table>
<thead>
<tr>
<th>Illustrate and explain that the diameter is twice the radius in a given circle.</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>6:ME</td>
<td>18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Illustrate and explain that the circumference is approximately three times the diameter in a given circle.</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>6:ME</td>
<td>19</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Explain that, for all circles, pi is the ratio of the circumference to the diameter ( \frac{C}{d} ), and its value is approximately 3.14.</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>6:ME</td>
<td>19</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Explain, using an illustration, that the sum of the central angles of a circle is 360°.</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>6:ME</td>
<td>18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Draw a circle with a given radius or diameter with and without a compass.</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>6:ME</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5:G</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Solve a given contextual problem involving circles.</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>6:ME</td>
<td>1, 3, 21</td>
</tr>
</tbody>
</table>
WNCP CURRICULUM

Specific Outcome
Develop and apply a formula for determining the area of:
• triangles
• parallelograms
• circles. [CN, PS, R, V]

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Illustrate and explain how the area of a rectangle can be used to determine the area of a triangle.</td>
<td>1</td>
<td>6:ME</td>
<td>5</td>
</tr>
<tr>
<td>Generalize a rule to create a formula for determining the area of triangles.</td>
<td>1</td>
<td>6:ME</td>
<td>5, 6</td>
</tr>
<tr>
<td>Illustrate and explain how the area of a rectangle can be used to determine the area of a parallelogram.</td>
<td>1</td>
<td>6:ME</td>
<td>4</td>
</tr>
<tr>
<td>Generalize a rule to create a formula for determining the area of parallelograms.</td>
<td>1</td>
<td>6:ME</td>
<td>4</td>
</tr>
<tr>
<td>Illustrate and explain how to estimate the area of a circle without the use of a formula.</td>
<td>1</td>
<td>6:ME</td>
<td>20</td>
</tr>
<tr>
<td>Apply a formula for determining the area of a given circle.</td>
<td>1</td>
<td>6:ME</td>
<td>1, 3, 20, 21</td>
</tr>
<tr>
<td>Solve a given problem involving the area of triangles, parallelograms and/or circles.</td>
<td>1</td>
<td>6:ME</td>
<td>1, 3, 4, 7, 21</td>
</tr>
</tbody>
</table>

### 3-D Objects and 2-D Shapes

It is expected that students will:

<table>
<thead>
<tr>
<th>WNCP CURRICULUM</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific Outcome</td>
<td></td>
</tr>
<tr>
<td>Perform geometric constructions, including:</td>
<td></td>
</tr>
<tr>
<td>• perpendicular line segments</td>
<td></td>
</tr>
<tr>
<td>• parallel line segments</td>
<td></td>
</tr>
<tr>
<td>• perpendicular bisectors</td>
<td></td>
</tr>
<tr>
<td>• angle bisectors. [CN, R, V]</td>
<td></td>
</tr>
<tr>
<td>Achievement Indicators</td>
<td>Part</td>
</tr>
<tr>
<td>Describe examples of parallel line segments, perpendicular line segments, perpendicular bisectors and angle bisectors in the environment.</td>
<td>1</td>
</tr>
</tbody>
</table>
3. **Achievement Indicators**

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4:G</td>
<td>1–3, 4</td>
</tr>
<tr>
<td>2</td>
<td>5:G</td>
<td>17, 18</td>
</tr>
<tr>
<td>1</td>
<td>4:G</td>
<td>1–3, 4</td>
</tr>
<tr>
<td>2</td>
<td>5:G</td>
<td>13–17, 18, 19, 21</td>
</tr>
<tr>
<td>1</td>
<td>4:G</td>
<td>1–3, 6</td>
</tr>
<tr>
<td>2</td>
<td>5:G</td>
<td>13–16, 22</td>
</tr>
<tr>
<td>1</td>
<td>4:G</td>
<td>1–3, 9, 11</td>
</tr>
<tr>
<td>2</td>
<td>5:G</td>
<td>13–17, 18–20</td>
</tr>
<tr>
<td>1</td>
<td>4:G</td>
<td>1–3, 5, 11</td>
</tr>
<tr>
<td>2</td>
<td>5:G</td>
<td>13–17, 18, 19, 23, 24</td>
</tr>
</tbody>
</table>

**Transformations**

It is expected that students will:

4. **WNCP CURRICULUM**

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identify and plot points in the four quadrants of a Cartesian plane using integral ordered pairs. [ C, CN, V ]</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Label the axes of a four quadrant Cartesian plane and identify the origin.</td>
<td>2</td>
<td>7:G</td>
<td>25</td>
</tr>
<tr>
<td>Identify the location of a given point in any quadrant of a Cartesian plane using an integral ordered pair.</td>
<td>2</td>
<td>7:G</td>
<td>25, 26</td>
</tr>
<tr>
<td>Plot the point corresponding to a given integral ordered pair on a Cartesian plane with units of 1, 2, 5 or 10 on its axes.</td>
<td>2</td>
<td>7:G</td>
<td>26</td>
</tr>
<tr>
<td>Draw shapes and designs, using given integral ordered pairs, in a Cartesian plane.</td>
<td>2</td>
<td>7:G</td>
<td>26</td>
</tr>
<tr>
<td>Create shapes and designs, and identify the points used to produce the shapes and designs in any quadrant of a Cartesian plane.</td>
<td>2</td>
<td>7:G</td>
<td>26, 34</td>
</tr>
<tr>
<td>Specific Outcome</td>
<td>JUMP MATH LESSONS</td>
<td></td>
<td></td>
</tr>
<tr>
<td>---------------------------------------------------------------------------------</td>
<td>-------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Perform and describe transformations (translations, rotations or reflections) of a 2-D shape in all four quadrants of a Cartesian plane (limited to integral number vertices). [CN, PS, T, V]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NOTE: It is intended that the original shape and its image have vertices with integral coordinates.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identify the coordinates of the vertices of a given 2-D shape on a Cartesian plane.</td>
<td>2</td>
<td>7:G</td>
<td>27–29, 32</td>
</tr>
<tr>
<td>Describe the horizontal and vertical movement required to move from a given point to another point on a Cartesian plane.</td>
<td>2</td>
<td>7:G</td>
<td>27, 28, 33</td>
</tr>
<tr>
<td>Describe the positional change of the vertices of a given 2-D shape to the corresponding vertices of its image as a result of a transformation or successive transformations on a Cartesian plane.</td>
<td>2</td>
<td>7:G</td>
<td>27–30, 31, 32</td>
</tr>
<tr>
<td>Determine the distance between points along horizontal and vertical lines in a Cartesian plane.</td>
<td>2</td>
<td>7:G</td>
<td>27, 30, 32</td>
</tr>
<tr>
<td>Perform a transformation or consecutive transformations on a given 2-D shape and identify coordinates of the vertices of the image.</td>
<td>2</td>
<td>7:G</td>
<td>28–30, 31, 32, 33</td>
</tr>
<tr>
<td>Describe the positional change of the vertices of a 2-D shape to the corresponding vertices of its image as a result of a transformation or a combination of successive transformations.</td>
<td>2</td>
<td>7:G</td>
<td>28–30, 31, 32, 33</td>
</tr>
<tr>
<td>Describe the image resulting from the transformation of a given 2-D shape on a Cartesian plane by identifying the coordinates of the vertices of the image.</td>
<td>2</td>
<td>7:G</td>
<td>28–30, 32, 33</td>
</tr>
</tbody>
</table>
### Statistics and Probability

#### General Outcomes
- Data Analysis: Collect, display and analyze data to solve problems.
- Chance and Uncertainty: Use experimental or theoretical probabilities to represent and solve problems involving uncertainty.

#### Data Analysis
It is expected that students will:

1. **WNCP CURRICULUM**
   **Specific Outcome**
   Demonstrate an understanding of central tendency and range by:
   - determining the measures of central tendency (mean, median, mode) and range
   - determining the most appropriate measures of central tendency to report findings. [C, PS, R, T]

   **Achievement Indicators**
   - Determine mean, median and mode for a given set of data, and explain why these values may be the same or different.
   - Determine the range of given sets of data.
   - Provide a context in which the mean, median or mode is the most appropriate measure of central tendency to use when reporting findings.
   - Solve a given problem involving the measures of central tendency.

   **JUMP MATH LESSONS**
<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7:PDM</td>
<td>1–5</td>
</tr>
<tr>
<td>1</td>
<td>7:PDM</td>
<td>1, 3, 4</td>
</tr>
<tr>
<td>1</td>
<td>7:PDM</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>7:PDM</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3:PDM</td>
<td>13</td>
</tr>
</tbody>
</table>

2. **WNCP CURRICULUM**
   **Specific Outcome**
   Determine the effect on the mean, median and mode when an outlier is included in a data set. [C, CN, PS, R]

   **Achievement Indicators**
   - Analyze a given set of data to identify any outliers.
   - Explain the effect of outliers on the measures of central tendency for a given data set.
   - Identify outliers in a given set of data and justify whether or not they are to be included in the reporting of the measures of central tendency.
   - Provide examples of situations in which outliers would and would not be used in reporting the measures of central tendency.

   **JUMP MATH LESSONS**
<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7:PDM</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>7:PDM</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>7:PDM</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>7:PDM</td>
<td>5</td>
</tr>
</tbody>
</table>
3. **WNCP CURRICULUM**

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Construct, label and interpret circle graphs to solve problems. [C, CN, PS, R, T, V]</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identify common attributes of circle graphs, such as: • title, label or legend • the sum of the central angles is 360° • the data is reported as a percent of the total and the sum of the percents is equal to 100%.</td>
<td>2</td>
<td>3:PDM</td>
<td>10–12</td>
</tr>
<tr>
<td>Create and label a circle graph, with and without technology, to display a given set of data.</td>
<td>2</td>
<td>3:PDM</td>
<td>9, 10, 11</td>
</tr>
<tr>
<td>Find and compare circle graphs in a variety of print and electronic media, such as newspapers, magazines and the Internet.</td>
<td>2</td>
<td>3:PDM</td>
<td>10, 14</td>
</tr>
<tr>
<td>Translate percentages displayed in a circle graph into quantities to solve a given problem.</td>
<td>2</td>
<td>3:PDM</td>
<td>12</td>
</tr>
<tr>
<td>Interpret a given circle graph to answer questions.</td>
<td>2</td>
<td>3:PDM</td>
<td>12, 13</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8:PDM</td>
<td>22</td>
</tr>
</tbody>
</table>

**Chance and Uncertainty**

It is expected that students will:

4. **WNCP CURRICULUM**

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Express probabilities as ratios, fractions and percents. [C, CN, R, V, T]</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determine the probability of a given outcome occurring for a given probability experiment, and express it as a ratio, fraction and percent.</td>
<td>2</td>
<td>8:PDM</td>
<td>17, 18, 22</td>
</tr>
<tr>
<td>Provide an example of an event with a probability of 0 or 0% (impossible) and an event with a probability of 1 or 100% (certain).</td>
<td>2</td>
<td>8:PDM</td>
<td>18</td>
</tr>
</tbody>
</table>
5. **WNCP CURRICULUM**

**Specific Outcome**

Identify the sample space (where the combined sample space has 36 or fewer elements) for a probability experiment involving two independent events. [C, ME, PS]

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Provide an example of two independent events, such as:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• spinning a four section spinner and an eight-sided die</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• tossing a coin and rolling a twelve-sided die</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• tossing two coins</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• rolling two dice</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>and explain why they are independent.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Identify the sample space (all possible outcomes) for each of two independent events using a tree diagram, table or another graphic organizer.</td>
<td>2</td>
<td>8:PDM</td>
<td>19, 20, 23</td>
</tr>
</tbody>
</table>

6. **WNCP CURRICULUM**

**Specific Outcome**

Conduct a probability experiment to compare the theoretical probability (determined using a tree diagram, table or another graphic organizer) and experimental probability of two independent events. [C, PS, R, T]

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determine the theoretical probability of a given outcome involving two independent events.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conduct a probability experiment for an outcome involving two independent events, with and without technology, to compare the experimental probability to the theoretical probability.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solve a given probability problem involving two independent events.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8:PDM</td>
<td>21, 23</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8:PDM</td>
<td>21–23</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8:PDM</td>
<td>21–23</td>
</tr>
</tbody>
</table>