# Teacher Resources: Grade 8

# JUMP Math 8

## Contents

### A Introduction
- Features of the Teacher Resources for Grade 8
- Mental Math
- Contents of Assessment & Practice Books 8.1 and 8.2

### Part 1 Lesson Plans and Blackline Masters

<table>
<thead>
<tr>
<th>Letter</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>Unit 1: Number Sense</td>
</tr>
<tr>
<td>C</td>
<td>Unit 2: Patterns and Algebra</td>
</tr>
<tr>
<td>D</td>
<td>Unit 3: Number Sense</td>
</tr>
<tr>
<td>E</td>
<td>Unit 4: Probability and Data Management</td>
</tr>
<tr>
<td>F</td>
<td>Unit 5: Geometry</td>
</tr>
<tr>
<td>G</td>
<td>Unit 6: Measurement</td>
</tr>
<tr>
<td>H</td>
<td>Unit 7: Number Sense</td>
</tr>
<tr>
<td>I</td>
<td>Unit 8: Geometry</td>
</tr>
<tr>
<td>J</td>
<td>Generic Blackline Masters</td>
</tr>
<tr>
<td>K</td>
<td>Answer Keys for Assessment &amp; Practice Book 8.1</td>
</tr>
<tr>
<td>L</td>
<td>Unit Quizzes and Tests</td>
</tr>
</tbody>
</table>

### Part 2 Lesson Plans and Blackline Masters

<table>
<thead>
<tr>
<th>Letter</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>Unit 1: Number Sense</td>
</tr>
<tr>
<td>N</td>
<td>Unit 2: Probability and Data Management</td>
</tr>
<tr>
<td>O</td>
<td>Unit 3: Geometry</td>
</tr>
<tr>
<td>P</td>
<td>Unit 4: Patterns and Algebra</td>
</tr>
<tr>
<td>Q</td>
<td>Unit 5: Number Sense</td>
</tr>
<tr>
<td>R</td>
<td>Unit 6: Measurement</td>
</tr>
<tr>
<td>S</td>
<td>Unit 7: Geometry</td>
</tr>
<tr>
<td>T</td>
<td>Unit 8: Probability and Data Management</td>
</tr>
<tr>
<td>U</td>
<td>Generic Blackline Masters</td>
</tr>
<tr>
<td>V</td>
<td>Answer Keys for Assessment &amp; Practice Book 8.2</td>
</tr>
<tr>
<td>W</td>
<td>Unit Quizzes and Tests</td>
</tr>
</tbody>
</table>

### Extra Resources

<table>
<thead>
<tr>
<th>Letter</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>Grade 8 Ontario Curriculum Correlation</td>
</tr>
<tr>
<td>Y</td>
<td>Grade 8 WNCP Curriculum Correlation</td>
</tr>
</tbody>
</table>
Unit 1  Number Sense

In this unit, students will convert between repeating or terminating decimals and their fraction equivalents. Students will also convert between fractions, percents and decimals, will use proportions to solve percent problems, and will expand their knowledge of percents to decimal percentages and to percentages greater than 100%. Students will also solve problems involving ratios and rates.

**BLM Three Types of Percent Problems** (p M-108) is a summary BLM for the material in lessons NS8-90, NS8-91, NS8-92, NS8-96, and NS8-97. This BLM summarizes how to solve the different types of percent problems, with examples for each.

**Meeting Your Curriculum**

WNCP students should cover all topics in this unit. For Ontario students, a basic understanding of repeating decimals from lesson NS8-76 is essential, but the lessons NS8-78 to NS8-81 which deepen this understanding is optional, as the material will be covered in Grade 9. The lesson NS8-100 on three-term ratios is also optional for Ontario students.
Review long division with whole numbers and decimals. Remind students that when doing long division $a \div b$, they should be careful not to do $b \div a$. It doesn’t matter which is bigger, $a$ or $b$. What matters is which number is being divided by which number.

**ASK:** How would you write the notation for long division to do $75 \div 3$? Is it $3|75$ or $75|3$? (it’s $3|75$) How would you write the notation for $2 \div 5$? Which number is the 2 like—the 3 or the 75? (the 75) How do you know? (because it comes first; it is the number being divided into) **NOTE:** Some students may answer the 3, because it is the smaller number. Emphasize that it doesn’t matter which number is bigger—the notation only tells you which number to divide into which. We are used to dividing a smaller number into a larger number but we can also divide a larger number into a smaller number. Since 2 replaces 75 and 5 replaces 3, we can use our familiarity with the simpler case to get the correct way of writing $2 \div 5$: $5|2$.

Remind students that 2 is the same as 2.0 or 2.00 or 2.000. Remind students that to divide a decimal by a whole number, simply line up the decimal point above the division sign and divide as though the decimal is a whole number. Have students use decimal long division to find:

- **a)** $5|2.0$
- **b)** $8|2.00$
- **c)** $25|2.00$
- **d)** $16|2.000$

**Relate fractions to division.** Write on the board the fraction $12/3$. **ASK:** What number does this represent? (4) How do you know? (because $12 \div 3 = 4$) Demonstrate why this works as follows. **ASK:** 12 is the numerator—what does the numerator of a fraction tell us? (the number of parts we are considering) What does the denominator tell us? (the number of parts in one whole) **ASK:** The fraction is $12/3$. How many parts are in one whole? (3) Then draw on the board:

```
one whole
```

```
**ASK:** How many parts are we considering altogether? **PROMPT:** What is the numerator of the fraction? (12) Explain that we have to keep adding parts until we have 12 parts altogether. We end up with 4 wholes:

<table>
<thead>
<tr>
<th>one whole</th>
<th>two wholes</th>
<th>three wholes</th>
<th>four wholes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

By asking how much do we have if we have 12/3, we are asking how many wholes do we have if we have 12 pieces and each whole has 3 pieces. We are dividing a set of 12 pieces into groups of size 3 and asking how many groups we have. This answer is 12 ÷ 3.

Now write on the board the fraction 5/3. **ASK:** How many wholes is that? Is it more than one whole? (yes, one whole is 3 pieces and we have 5) Is it more than two wholes? (no, two wholes is 6 pieces and we have only 5) Draw on the board:

<table>
<thead>
<tr>
<th>one whole</th>
<th>two wholes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The second picture is only 2/3 of a whole, so we have 1 2/3 wholes. Explain that we are dividing 5 pieces into groups of size 3 (one whole is a group of size 3) and asking how many groups (wholes) we have. The answer is 5 ÷ 3.

Have students write these fractions as division:

a) \( \frac{1}{4} \)  
b) \( \frac{2}{5} \)  
c) \( \frac{4}{5} \)  
d) \( \frac{3}{10} \)  
e) \( \frac{4}{25} \)  
f) \( \frac{11}{20} \)

Then have students convert the fractions into decimals in two ways:

a) Convert the fraction to a decimal fraction and then to a decimal.  
b) Use \(a/b = a \div b\) and do the long division.

Tell students to ensure that they get the same answer both ways; if they don’t, they know to look for a mistake.

**Divide to find decimal equivalents for equivalent fractions.** Have students divide to find:

\[
\frac{3}{4} = 3 \div 4 = \quad \frac{6}{8} = 6 \div 8 = \quad \frac{9}{12} = 9 \div 12 =
\]

**ASK:** What do you notice about your answers? (they are all the same: 0.75) Why did that happen? (because all the fractions are equivalent)

Ask students to use a calculator to determine whether these fractions are equivalent: 17/200 and 41/500. (no, 17 ÷ 200 = 0.085 and 41 ÷ 500 = 0.082)

Have students redo the problem by finding equivalent fractions for each that have the same denominator. (17/200 = 85/1000 and 41/500 = 82/1000)

Point out that the calculator just did the work of finding the numerator when the denominator is 1000.
Extending patterns in decimals. Have students divide to find 1/40, 2/40, 3/40, 4/40, 5/40. **ANSWERS:** (0.025, 0.05, 0.075, 0.1, 0.125) **ASK:** What kind of pattern do you see? **PROMPT:** Write the decimals to three decimal places. **ASK:** What is the rule for obtaining the next term in the pattern? (add 0.025) Write on the board:

\[
\begin{align*}
6 & = 0.15, \\
7 & = 0.175, \\
8 & = 0.2, \\
9 & = 0.225; \\
10 & = 0.25.
\end{align*}
\]

Have students continue the pattern in the decimals. Then have students reduce to lowest terms the fractions that are not already in lowest terms (6/40 = 3/20, 8/40 = 1/5, 10/40 = 1/4) and create for all fractions an equivalent decimal fraction and then an equivalent decimal:

\[
\begin{align*}
6 & = \frac{15}{100} = 0.15, \\
7 & = \frac{175}{1000} = 0.175, \\
8 & = \frac{2}{10} = 0.2, \\
9 & = \frac{225}{1000} = 0.225; \\
10 & = \frac{25}{100} = 0.25.
\end{align*}
\]

Do students get the same answer by turning the fraction into a decimal as they did by extending the pattern? (yes)
Goal

Students will use division to represent fractions as repeating or terminating decimals. Students will compare and order decimals, including repeating and terminating decimals.

PRIOR KNOWLEDGE REQUIRED

Can compare and order terminating
decimals to thousandths
Can use long division to write fractions
with terminating decimal
equivalents as decimals

MATERIALS

2-cm grid paper

Compare the number of digits in the quotient to the number of subtractions in the long division. Remind students that to write 4/5 as a decimal, they can divide 4 ÷ 5 and use long division. Go through this as a class to get 0.8. **ASK:** How did we know when to stop? (when the “remainder” is 0) Have students do 3 ÷ 16 to write 3/16 as a decimal (0.1875). Provide grid paper to help students align the numbers correctly. **ASK:** How long did it take to get a remainder of 0 this time—how many subtractions in the long division did you have to do? (4 subtractions)

Have students do several long divisions, and then copy and complete this chart:

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Number of decimal places in the quotient</th>
<th>Number of place values in the quotient</th>
<th>Number of subtractions before getting 0 remainder</th>
</tr>
</thead>
<tbody>
<tr>
<td>4/5</td>
<td>0.8</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3/16</td>
<td>0.1875</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>15/8</td>
<td>1.125</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>7/20</td>
<td>___</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3/8</td>
<td>___</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>98/5</td>
<td>___</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7/10</td>
<td>___</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19/25</td>
<td>___</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**ASK:** What do you notice? Which two columns are always equal? Which two columns are only usually equal? Emphasize that it is the total number of place values in the quotient that tells us the number of subtractions we will need to do before getting a 0 remainder. This is because every time you add a digit, you do another subtraction. When you get 0 as a remainder and there are no more digits from the dividend to bring down, you can stop adding digits to the quotient.

Now have students divide 5 into 15.23. Ask students how many subtractions they did (3 or 4), and how many non-zero digits are in the quotient (3). Tell students that if a digit in the quotient is 0, they can skip that subtraction. Show this on the board:

\[
\begin{array}{c}
\text{5} \\
\hline
15.23 \\
\text{15} \\
\text{023} \\
\text{0} \\
\text{23} \\
\text{20} \\
\text{30} \\
\text{30} \\
\hline
\text{0}
\end{array}
\quad \text{becomes} \quad
\begin{array}{c}
\text{5} \\
\hline
15.230 \\
\text{15} \\
\text{023} \\
\text{023} \\
\text{20} \\
\text{30} \\
\text{30} \\
\hline
\text{0}
\end{array}
\]

Summarize by emphasizing that the number of non-zero place values in the quotient is equal to the number of subtractions you need to do in the long division, because you have to do a subtraction for every non-zero digit in the quotient.

**PROCESS EXPECTATION**

Looking for a pattern

**Introduce repeating decimals.** Write $\frac{1}{3}$ on the board and **ASK:** What long division can we do to write $\frac{1}{3}$ as a decimal? ($1 \div 3$) Have students do this and tell them to stop after they get a remainder of 0. After letting them work for a few minutes, **ASK:** Who is finished? Did anyone get a remainder of 0 yet? Do you think you will ever get a remainder of 0? (no) How can you tell? What pattern do you see in the long division? (we always get $3 \times 3 = 9$ and then $10 - 9 = 1$ as the remainder, so the pattern in the remainders is a very simple repeating pattern: $1 1 1 1\ldots$; this never becomes 0) **ASK:** If the remainder never equals 0, how many non-zero digits will the quotient have? (infinitely many digits—the decimal continues forever)

Tell students that some decimals never stop—they keep going for ever and ever. Decimals that do stop are called *terminating* decimals. If your students are familiar with the area and circumference of circles, **ASK:** Where have you seen other decimals that don’t terminate? In any case, tell students that the ratio between the distance around a circle and the distance across a circle, called $\pi$, is another decimal that doesn’t stop. Write on the board:

\[
\pi = 3.1415926\ldots
\]

Tell students that this is just the first few digits—the decimal goes on forever.
ASK: Can you guess what the next digit is? After students guess, tell them it is 5, and write the next digit for them:

\[ \pi = 3.14159265... \]

Repeat having students guess the next digit and then writing what it actually is:

3.141592653
3.1415926535
3.14159265358
3.141592653589
3.1415926535897

If students have a scientific calculator, have them press the "pi" button. What number comes up?

\[ \pi = 3.1415926535897932384626433832795... \] (NOTE: Different calculators may show a different number of digits, here and below. Just use whatever you see on your calculator display.) Now write on the board:

\[ \frac{1}{3} = 0.333333333333... \]

ASK: Can you guess what the next digit is? (3) Repeat several times. (the next digit is always 3) Tell students that because there is a pattern, the next digit is easy to find. ASK: What type of pattern do you see in the digits? (a repeating pattern) Explain that because the digits form a repeating pattern, the next digit is easy to predict, and the decimal is called a repeating decimal.

Some decimals, like \( \pi = 3.1415926... \), are not terminating and not repeating, but most decimals students will deal with this year will be either terminating or repeating. In fact, mathematicians have shown that any decimal that they can get from a fraction will be either terminating or repeating.

Finding repeating decimals. Have students find more decimals by long division and determine whether they are terminating or repeating:

\[ \begin{array}{ccccccc}
\text{a) } & \frac{5}{6} & \quad & \text{b) } & \frac{7}{12} & \quad & \text{c) } & \frac{5}{8} & \quad & \text{d) } & \frac{4}{15} & \quad & \text{e) } & \frac{5}{11} & \quad & \text{f) } & \frac{56}{125} \\
\text{ANSWERS:} & 0.833333... \text{ repeating} & \quad & 0.5833333... \text{ repeating} & \quad & 0.626 \text{ terminating} & \quad & 0.2666666... \text{ repeating} & \quad & 0.45454545... \text{ repeating} & \quad & 0.448 \text{ terminating} \\
\end{array} \]

ASK: How are these repeating decimals different from \( \frac{1}{3} = 0.333333... \)\? Notice that the digit for parts a), b), and d) repeat, but not right away. In part e), there is not a single repeating digit but a repeating pattern (4, 5, then repeat), so this decimal is still called a repeating decimal.

Calculators can be misleading. Have students calculate \( \frac{3}{7} \) using long division until they either get 0 as a remainder or they can prove that they won’t because they find a pattern that goes on forever. They should get 0.428571428571... The repeating pattern is 4, 2, 8, 5, 7, 1, then repeat.
Discuss the answers. **ASK:** Is 3 ever a digit in 3/7? (no) Tell students that your calculator says it is. Have students calculate 3 \(\div\) 7 on their calculator, then show them what you get on yours: 0.42857142857142857142857143.

**ASK:** Why does my calculator tell me there is a digit 3 when I know there isn’t one? (because the calculator cannot show an infinite number of digits, so it rounds the answer; the last digit displayed is actually 2, but the next digit is 8, so the calculator rounds the 2 up to get 3)

**Introduce bar notation.** Tell students that mathematicians have invented a way to write repeating decimals exactly, without rounding and without having to write digits forever (because that would take a long time!). Mathematicians have decided to put a bar over the digits that repeat. Write on the board:

\[
0.428571428571428571428571428571428571\ldots \text{ becomes } 0.\overline{428571}.
\]

\[
0.83333333 \text{ becomes } 0.8\overline{3}.
\]

Have students write these repeating decimals using bar notation:

a) 0.5833333…  b) 0.26666666…  c) 0.45454545…

Then have students write the first eight digits of these repeating decimals:

a) 0.32\overline{1}  b) 0.32\overline{1}  c) 0.32\overline{1}  d) 0.43\overline{2}

**ANSWERS:**

a) 0.32132132  b) 0.32121212  c) 0.32111111  d) 0.43213213

Tell students that we only put a bar over numbers after the decimal point. So, for example, 0.321032103210… would be written 0.3210, not 0.321.

Have students write these decimals using bar notation:

a) 351.351351351…  b) 42.62626262…  c) 538.538383838…

**ANSWERS:**

a) 351.\overline{351}  b) 42.\overline{62}  c) 538.\overline{538}

Write 0.3\overline{23} on the board, clearly placing the bar over the 2 only. **ASK:** Does this notation make sense? (no, this would say to repeat the 2 forever, but then you can never add a 3 after) Tell students that the bar always goes over the last digits that you write. How many digits it covers depends on which digits repeat.

**Review how to compare terminating decimals.** Remind students how to compare terminating decimals: look for the largest place value where they have different digits. Have students compare:

a) 0.3746  0.3728  b) 0.4589  0.4723  c) 0.35  0.32141

**ANSWERS:**

a) 0.3746 \(>\) 0.3728  b) 0.4589 \(<\) 0.4723  c) 0.35 \(>\) 0.32141

Use part c) to emphasize that the number of digits after the decimal point tells you nothing about how large a number is; 0.35 has fewer digits after the decimal point than 0.32141, but it is larger. This is in contrast to the number of digits before the decimal point, which tells you a lot about how
large a number is (e.g., 3412.5 is larger than 99.999 because, even though
the digits are smaller, there are more digits before the decimal point).

**Introduce how to compare repeating decimals.** Show students how to
compare decimals by writing their first few digits after the decimal point at
least until they differ (you may need to add zeroes to terminating decimals).
See the teaching box on Workbook page 5. Give students grid paper and
have them do more problems like Questions 8 and 9 on Workbook page 5.

**EXTRA PRACTICE for Workbook Question 8:**

a) 0.358  0.358  b) 0.254  0.25  c) 0.754  0.75

d) 0.25  0.45  e) 0.25  0.23  f) 5.123  5.132

g) 0.382  0.382  h) 0.312  0.312  i) 0.318  0.318
**Goal**

Students will use decimal equivalents to compare fractions and to determine fractions that are close to given decimals.

**Prior Knowledge Required**

- Can compare fractions by finding a common denominator
- Can convert fractions to decimals (repeating or terminating) by using long division
- Can use subtraction to find the difference between two numbers
- Can subtract terminating decimals
- Can compare and order repeating decimals
- Can round decimal numbers

**Note:** Many concepts in this lesson assume familiarity with repeating decimals. Students who are not familiar with repeating decimals can still do most of this lesson. To convert fractions to decimals, use a calculator and round the answers to 1, 2, or 3 decimal places, as required. Students can thus compare, order, and subtract terminating decimals to answer all but one question (5 b) on Workbook page 4.

**Using decimals to compare fractions.** Now that students know how to convert fractions to decimals, and to compare decimals, have students compare fractions by first converting the fractions to equivalent decimals and then comparing the decimals. Allow students to use grid paper to help align the place values. See Workbook page 4 Questions 1, 3, and 7.

**Compare fractions and decimals two ways.** Have students compare $\frac{4}{9}$ and 0.435, first by writing both as fractions with a common denominator ($\frac{4}{9} = \frac{4000}{9000}$ and $\frac{435}{1000} = \frac{3895}{9000}$, so $\frac{4}{9}$ is larger) and then by writing both as decimals ($\frac{4}{9} = 0.\overline{4} = 0.4444\ldots > 0.435$ because the largest place value where they differ is hundredths and 4 hundredths is more than 3 hundredths). **ASK:** Did you get the same answer both ways? (yes) Which way did you like better? Have students do more examples both ways, as in Workbook Question 4, but keep in mind that students have not yet learned to convert repeating decimals to fractions, so include only terminating decimals in decimal format for the comparison.

**Which one is closer?** Have students convert these fractions to decimals: 3/11 and 3/13. **Answers:** 0.27 and 0.230769. **ASK:** Which one is closer to $\frac{1}{4} = 0.25$? To guide students, write on the board:

- $0.27272727\ldots$
- $0.25000000\ldots$
- $0.230769230769\ldots$

**Curriculum Expectations**

Ontario: 7m11, 7m27, 8m1, 8m3, 8m6, 8m7, 8m17

WNCP: 7N4, 7N7, 8N3, [R, ME, C, CN]

**Vocabulary**

rounding
fraction
decimal
equivalent
ASK: To know how far one number is from another, what do we have to do? (subtract them) PROMPT: What operation will we use? Then tell students that we don’t yet know how to subtract repeating decimals. How can we do this problem without subtracting repeating decimals? PROMPT: What do we know how to subtract? (we know how to subtract terminating decimals)

ASK: How can we change a repeating decimal into a terminating decimal and still keep the answer the same? (if we round the repeating decimals to a few decimal places after they start being different, and then subtract, we should still get the right answer as to which difference is smaller) Point out that when we compared repeating decimals, we didn’t have to compare all the place values, only the early place values. It’s the same here: we only need to compare the differences in the early place values, and rounding the repeating decimals will allow us to do that.

Have students round each decimal to 5 decimal places and then find the differences:

\[
\begin{array}{cc}
0.27273 & 0.25000 \\
-0.25000 & -0.23077 \\
0.02273 & 0.01923
\end{array}
\]

So the difference between 1/4 and 3/13 is smaller than the difference between 3/11 and 1/4. This means that 3/13 is closer to 1/4 than 3/11 is. Point out that 1/4 = 3/12, and write down this sequence of fractions:

\[
\frac{3}{1} \quad \frac{3}{2} \quad \frac{3}{3} \quad \frac{3}{4} \quad \frac{3}{5} \quad \frac{3}{6} \quad \frac{3}{7} \quad \frac{3}{8} \quad \frac{3}{9} \quad \frac{3}{10} \quad \frac{3}{11} \quad \frac{3}{12} \quad \frac{3}{13}
\]

It is easy to see the decimal equivalents for the first few terms in this sequence—they are all terminating decimals:

\[
3 \quad 1.5 \quad 1 \quad 0.75 \quad 0.6 \quad 0.5
\]

ASK: Do these fractions appear to be getting closer together or further apart? (closer together) Explain that the further we go in the sequence, the closer together the numbers are. ASK: Does this agree with the answer we got? (yes, we said that 3/13 is closer to 3/12 than 3/11 is) Discuss the differences between the two methods. Which method do students like better, and why?

Have students do Workbook page 4 Questions 2 and 5.
**Goal**

Students will determine whether a fraction’s decimal equivalent will terminate or repeat by looking at the denominator of the fraction in reduced form.

**Prior Knowledge Required**

Can convert terminating decimals to decimal fractions
Can convert decimal fractions to terminating decimals

**Introduce the lesson topic.** Tell students that they will now investigate how to determine, by looking at a fraction, whether its decimal equivalent will terminate or repeat. (You will work through the Investigation on Workbook page 5 over the course of the lesson.)

**Review writing terminating decimals as decimal fractions.** Have students write each of these terminating decimals as a decimal fraction:

a) 0.3  
b) 0.34  
c) 0.342  
d) 0.3425  
e) 0.32457  
f) 0.324579

ANSWERS:

a) \( \frac{3}{10} \)  
b) \( \frac{34}{100} \)  
c) \( \frac{324}{1000} \)  
d) \( \frac{3425}{10000} \)  
e) \( \frac{324457}{1000000} \)  
f) \( \frac{342579}{1000000} \)

Have students do Parts A and B of the Investigation on Workbook page 7.

Tell students that a terminating decimal can be very long, for example: 0.123 456 789 876 543 421

Have students write that terminating decimal as a decimal fraction.

ANSWER: 123 456 789 876 543 421/1 000 000 000 000 000 000

**Ask:** How did you know what to write in the denominator? (the number of zeroes is the number of decimal places in the decimal) Tell students that any terminating decimal can be written as a decimal fraction. For example, consider a decimal with two decimal places, such as 3.12. Since there are two decimal places, we can read the number of hundredths: there are 312 hundredths, so 3.12 = 312/100. Similarly, 0.045 has 3 decimal places, so we can read the number of thousandths: there are 45 thousandths, so 0.045 = 45/1000.

Tell students that terminating decimals can also look very much like the start of a repeating decimal. For example, 0.323232323232 is a terminating decimal. Have students write that decimal as a decimal fraction.

ANSWER: 323 232 323 232/1 000 000 000 000
Review writing decimal fractions as terminating decimals. ASK: Can any decimal fraction be written as a terminating decimal? (yes) Ask students to articulate the reason, then explain that the smallest place value in the decimal will be determined by the denominator of the decimal fraction. For example, if the decimal fraction has denominator 10,000, then the smallest place value in the decimal will be ten thousandths, which means the decimal has only 4 place values.

Have students write these decimal fractions as terminating decimals:

a) \( \frac{36}{100} \)  
b) \( \frac{5}{1000} \)  
c) \( \frac{2341}{1000} \)  
d) \( \frac{23456}{100000} \)

ANSWERS: a) 0.36   b) 0.005   c) 2.341   d) 0.23456

Can you always use a calculator to decide whether a fraction has a terminating decimal equivalent? Have students calculate these quotients using a calculator:

\[
\frac{1}{25} = 0.04 \quad \frac{1}{27} = 0.037037037037037037037037 \quad \frac{1}{29} = 0.034482758620689655172413793103448
\]

ASK: Which of the fractions looks like it has a decimal equivalent that terminates? (1/25 = 0.04 does) Can you tell for sure from the calculator display for all three fractions?

Tell students that your calculator says that \( \frac{1}{27} \) = 0.037037037037037037037037. This could either be a very long terminating decimal or it could be the repeating decimal 0.037. Also, tell students that your calculator says that \( \frac{1}{29} \) = 0.034482758620689655172413793103448. This also could be a terminating decimal or it could be the repeating decimal 0.0344827586206896551724137931. SAY: Notice how close to the end of my calculator display we see the digits start to repeat. Some calculators may not even show enough digits for you to see that they repeat at all! In fact, 1/29 is a repeating decimal with a very long (28 digits!) block that repeats.

Decide whether a fraction has a terminating or repeating decimal equivalent. Have students do Workbook page 5 Investigation Part C. Tell students that if, after dividing a numerator by a denominator, the digits do not go to the end of the calculator display—the most digits that the calculator can show—then students can be sure that the equivalent decimal terminates, since the calculator would have shown more digits if there were any more to show. If the digits do go to the end of the calculator display, students won’t be able to tell for sure whether the equivalent decimal repeats or not. Tell students to guess by looking at the digits whether there is a repeating pattern or not.

After students complete the exercise, remind them that a fraction will have a terminating decimal equivalent if it has an equivalent fraction that is a decimal fraction. Then have students look at the fractions they said were repeating. Have them try to find an equivalent decimal fraction and to think about why they can’t. Explain that to prove that it’s not possible to do something, mathematicians often start by trying to do it! Sometimes only
when they start trying to do something can they see why it won’t work. After allowing students to work for several minutes, have students work in pairs to articulate a reason why 7/15 has no equivalent decimal fraction. Then have students get into groups of four (two pairs) and agree on a reason. Allow each group of four to articulate their reason then summarize the results as follows: The fraction 7/15 is in lowest terms, so to make any equivalent fraction we have to multiply the numerator and denominator by the same number. Since we want to find an equivalent decimal fraction, we’re looking for a fraction that has denominator a power of 10. Suggest trying each possibility in turn. Write on the board:

\[
\begin{align*}
\frac{7}{15} \times \frac{10}{10} &= \frac{70}{150} \\
\frac{7}{15} \times \frac{100}{100} &= \frac{700}{1500} \\
\frac{7}{15} \times \frac{1000}{1000} &= \frac{7000}{15000} \\
\frac{7}{15} \times \frac{10000}{10000} &= \frac{70000}{150000}
\end{align*}
\]

\[\text{ASK:} \text{ Does 15 divide evenly into any power of 10? Students can try this by long division and see that there will always be a remainder of 10.}\]

Then have students do Workbook page 5 Investigation Part D. After students finish, tell them that any power of 10 can always be written as a product of 2s and 5s. Now tell students that 15 = 3 × 5. \text{ASK:} Can 15 × something be equal to a product of 2s and 5s? (no) Why not? (because of the 3) Tell students that 20 = 4 × 5, but 20 does divide evenly into 100, a power of 10. What makes the 3 different from the 4? (4 can be written as \(2 \times 2\), so 20 is still a product of 2s and 5s, but 3 cannot be written as a product of 2s and 5s) Tell students that 3 is prime, and if a number divides evenly into a power of 10, the only prime numbers that divide into it are 2 and 5. \text{ASK:} Can 15 be written as a product of 2s and 5s? (no, because the prime number 3 divides into it)

Now look again at the fractions from Investigation Part C. They are all in lowest terms. \text{ASK:} Which fractions have denominators that are products of only 2s and/or 5s? (5/8 because 8 = \(2 \times 2 \times 2\) and 13/2000 because 2000 = \(2 \times 2 \times 2 \times 5 \times 5 \times 5\)—students can find the answers by continually dividing by 2 and then by 5, or they can use prime factorizations if they are familiar with them). Are these the same as the fractions you said were terminating? (yes) Is 3/17 terminating? (no) How do you know? (because 17 is not a product of 2s and 5s)

Now have students complete the Investigation on Workbook page 5.

\text{EXTRA PRACTICE:}

1. Write out all the fifteenths from 1/15 to 14/15, and then write them in lowest terms. Decide from the denominators which fractions will terminate as decimals, then check on a calculator.

2. Write these fractions in lowest terms and then decide if their decimal equivalents will terminate or not.

\[
\begin{align*}
a) \frac{7}{25} & \quad b) \frac{13}{20} & \quad c) \frac{9}{150} & \quad d) \frac{5}{7} & \quad e) \frac{6}{21} & \quad f) \frac{4}{110}
\end{align*}
\]

Check your answers on a calculator.

\text{PROCESS EXPECTATION} 
Organizing data

\text{PROCESS EXPECTATION} 
Looking for a pattern

\text{PROCESS EXPECTATION} 
Investigating and making conjectures
Does a calculator display the exact value for a fraction? Tell students that calculators never show a decimal as repeating—you will never see the bar above repeating digits on a calculator, so calculators always make decimals look terminating. This means that when you calculate a fraction that has a repeating decimal equivalent on a calculator, it will not give you the exact answer, only an approximation. Have students do the following exercise:

A calculator shows $\frac{50}{97} = 0.5154639175257732$. Is this an exact value or an approximation? How do you know? (it is an approximation because 97 is not a product of 2s and 5s, so its decimal equivalent is repeating, not terminating)

Create fractions with a repeating or terminating decimal equivalent.
Challenge students to create a fraction with a terminating decimal equivalent that
a) has a 1-digit denominator,
b) has a 2-digit denominator,
c) has a 3-digit denominator,
d) has a 4-digit denominator.

SAMPLE ANSWERS: a) $\frac{1}{4}$  b) $\frac{11}{100}$  c) $\frac{111}{1000}$  d) $\frac{1111}{1000}$

Repeat for fractions with a repeating decimal equivalent.

SAMPLE ANSWERS: $\frac{1}{3}$, $\frac{1}{33}$, $\frac{1}{333}$, $\frac{1}{3333}$. Have students explain how they chose the denominators.

Extension

Why is the decimal representation of every fraction either repeating or terminating? Why can’t it be like pi, which doesn’t terminate or repeat? Think of what actually happens in the long division algorithm. Take for example $\frac{2}{7} = 2 \div 7$.

```
2.857142
7 | 2.000000
 14
 60  
 56
 40
 35
 50
 49
 10
 7
 30
 28
 20
```
At each step of the algorithm, we are dividing 7 into ten times whatever remainder we get. We start with remainder 2 because $2 \div 7 = 0$ Remainder 2, and divide 7 into 20 to get 2 Remainder 6. Our next step is to divide 7 into 60 and determine the remainder. As soon as we get a remainder that we’ve already had, then the division algorithm becomes exactly the same as it was from the first time we saw that remainder. If we get 0 as a remainder, the algorithm stops and the decimal terminates, but if we never get 0, then we have to continue forever. But there are only 6 possible remainders when we divide by 7 (not including 0), so if we do the algorithm forever, and each time the remainder is either 1, 2, 3, 4, 5, or 6, we are eventually going to repeat a remainder. Once we do that, the decimal starts repeating. This reasoning works for any number. For example, to find $1/29 = 1 \div 29$, we are always dividing by 29, so our remainder will always be less than 29. It might take a while to find a repeat remainder, but if we go on forever, we eventually will.

No matter what we divide by, if we keep dividing forever (because we never find a 0 remainder), we will eventually get the same remainder twice. The decimal will repeat from the point at which that remainder occurred the first time, because we are now doing the exact same divisions that we did to get all the digits in the decimal from the first time that remainder showed up!

**NOTE:** Some numbers, like $\pi$, do not come from fractions, and hence cannot be calculated by using long division of one whole number by another. The decimal for $\pi$ neither terminates nor repeats.
Review adding and subtracting terminating decimals to thousandths. Remind students that it is important to line up the place values, to ensure they are always adding the same place values together. An easy way to do this is to always line up the decimal point. The decimal point is always between the ones and the tenths, so if the decimal points are lined up, every place value will be lined up. Give students grid paper to help them line up the decimal points and have students solve:

\[
\begin{align*}
a) \quad & 3.451 + 2.764 \\
b) \quad & 8.45 + 3.582 \\
c) \quad & 13.45 + 1.345 \\
d) \quad & 43.218 + 7.469 \\
BONUS \quad & 8.9452 + 3.479 \\
& 9.415 + 23.74 + 12.678
\end{align*}
\]

**ANSWERS:**
0.72, 0.76, 0.764, 0.7644, 0.76444

**Introduction adding repeating decimals with no regrouping.** Write on the board: 0.32 + 0.4. **ASK:** How is this different from the other problems we have done so far? (this one asks us to add repeating decimals) Have students solve these problems:

\[
\begin{align*}
a) \quad & 0.32 \\
b) \quad & 0.32 \\
c) \quad & 0.32 \\
d) \quad & 0.32 \\
e) \quad & 0.32 \\
+ 0.4 & \quad + 0.44 & \quad + 0.444 & \quad + 0.4444 & \quad + 0.44444
\end{align*}
\]

**ANSWERS:** 0.72, 0.76, 0.764, 0.7644, 0.76444

Then have students continue the pattern in the questions and the answers. What are the next two sums?

Ask students to predict 0.32 + 0.4. **ANSWER:** 0.764. Explain that the sum will have 4 in every place value from thousandths on because in those place values we are always adding 0 (from 0.3200000...) and 4 (from 0.7644444...).
Have students practise adding a repeating decimal with a terminating decimal by lining up the first few decimal places. Point out that this is how we compared repeating decimals too—we pretended that the decimal didn’t go on forever and that was enough to know which was greater. Similarly, adding the first few decimal places might be enough to see the repeating pattern in the sum. Suggest that students write at least eight decimal places so that they can see the repeating pattern in the sum. **EXAMPLES:**

a) $0.42 + 0.5$  

b) $0.45 + 0.3$  

c) $0.3\bar{2} + 0.47\bar{5}$  

**Adding repeating decimals with repeating parts of the same length.**

Then demonstrate how to add two repeating decimals with repeating parts both the same length, by lining up the decimal places. Point out that students should start adding from the left since they cannot start adding from the right—there is no right-most digit. Emphasize that they can do this as long as no regrouping is required. **EXAMPLE:** $0.5\bar{2}3 + 0.1\bar{6}$

```
0.52323232…
+ 0.16161616…
0.68484848…
```

So $0.5\bar{2}3 + 0.1\bar{6} = 0.6\bar{8}4$

Have students practise. **EXAMPLES:**

a) $0.27 + 0.3\bar{1}$  

b) $4.1\bar{3} + 0.3\bar{1}$  

c) $0.4\bar{1} + 0.2\bar{3}$  

**Adding repeating decimals with repeating parts of different lengths.**

Now write on the board: $0.26 + 0.3\bar{1}\bar{2}$. **ASK:** How is this different from what we have done so far? **PROMPT:** Look at the length of the repeating parts. Have we ever looked at a problem like this before? Do you think it will be very different? Tell students that they solve it in the same way as before, but that sometimes the repeating part of the sum might be quite a bit longer than the repeating part of the addends, so they might have to keep track of more decimal places. Have students practise, and suggest that they write out the core in the answer twice, so that they are sure it repeats. **EXAMPLES:**

a) $0.3\bar{4} + 0.1\bar{3}2$  

b) $0.2\bar{1}4 + 0.1\bar{3}2$  

c) $0.3\bar{1}5\bar{5} + 0.2\bar{1}$  

**Bonus**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>d) $0.2\bar{4}1 + 0.3\bar{1}5\bar{5}$</td>
<td>e) $0.2\bar{3} + 0.3\bar{1}5\bar{2}5$</td>
</tr>
<tr>
<td>f) $0.2\bar{3}1 + 0.3\bar{1}5\bar{2}5$</td>
<td></td>
</tr>
</tbody>
</table>

When students finish, **ASK:** How was adding two decimals with different lengths of repeating blocks different from adding two decimals with the same length of repeating blocks? (the repeating part of the sum was a lot longer than the repeating parts of the addends) Did this surprise you?

**Subtracting repeating decimals with no regrouping.** Tell students that subtracting repeating decimals is just like adding them: line the first few decimal places up and subtract from left to right until you see a pattern in the difference. **EXAMPLE:** $0.8\bar{7}4 - 0.3\bar{4}$
Emphasize that they can subtract from left to right because there is no regrouping required. Have students practise. As above, students should write at least two cores in the answer, to be sure it repeats.

**EXAMPLES:**

a) \(0.874 - 0.234\)  
b) \(0.874 - 0.870\)  
c) \(0.874 - 0.34\)  
d) \(0.874 - 0.034\)

### The pattern in ninths.

Show students the beginning of the pattern:

\[
\begin{align*}
\frac{1}{9} &= 0.1 \\
\frac{2}{9} &= 0.2 \\
\frac{3}{9} &= 0.3
\end{align*}
\]

Have students extend the pattern to express the rest of the ninths in decimal form. Then have students do Workbook page 8 Questions 2 and 3. When students finish, discuss the results: When we don’t need to regroup, adding repeating decimals is easy, but when we need to regroup, we can’t start at the left! So we have to add the decimals to more decimal places each time and hope we can see how the pattern continues. Sometimes it is easier to add the fractions.

### Adding and subtracting with regrouping when the decimal being subtracted is terminating.

Show students how to subtract \(0.\overline{4} - 0.27\). First, subtract up to hundredths, because that is where 0.27 ends:

\[
\begin{align*}
0.44 & \quad \underline{- \ 0.27} \\
0.17 & \quad \underline{\hspace{1cm}}
\end{align*}
\]

Now we can subtract the rest:

\[
\begin{align*}
0.44444444\ldots & \quad \underline{- \ 0.27} \\
0.17444444\ldots & \quad \underline{\hspace{1cm}}
\end{align*}
\]

so \(0.\overline{4} - 0.27 = 0.17\overline{4}\)

Have students practise:

a) \(0.\overline{7} - 0.18\)  
b) \(0.\overline{572} - 0.39\)  
c) \(0.\overline{3} - 0.485\)

**ANSWERS:**

a) \(0.59\overline{7}\)  
b) \(0.18\overline{27}\)  
c) \(0.151\overline{36}\)

Encourage students to check their answers on a calculator.
Bonus: Remind students that, to subtract terminating decimals, if the larger decimal has fewer decimal places, they can keep adding 0s after its smallest place value. Then have students do these questions:

a) \(0.35 - 0.2\)  

b) \(0.35 - 0.22\)  
c) \(0.35 - 0.222\)  
d) \(0.35 - 0.2222\)  
e) \(0.35 - 0.22222\)  
f) \(0.35 - 0.222222\)  
g) \(0.35 - 0.2222222\)  
h) \(0.35 - 0.2\)

**ANSWERS:** a) 0.15  
b) 0.13  
c) 0.128  
d) 0.1278  
e) 0.12778  
f) 0.127778  
g) 0.1277778  
h) 0.12

Before students do Workbook page 6 Question 4, display a list of the common fraction-decimal conversions that they will need to do the question:

\[
\frac{1}{4} = 0.25 \quad \frac{1}{2} = 0.5 \quad \frac{1}{3} = \frac{3}{9} = 0.3 \quad \frac{4}{9} = 0.4
\]

**Extensions**

1. Repeating decimals that both have repeating blocks of length 3 can add to a decimal with repeating block of length 1. For example, \(0.345 + 0.432 = 0.777 = 0.7\). Challenge students to find an example of two such decimals that add to 0.8. (There are many, e.g., \(0.345 + 0.543\).)

Challenge students to find:

- two decimals with repeating blocks of length 2 that add to a decimal with a repeating block of length 1 (SAMPLE ANSWER: \(0.41 + 0.36 = 0.7\))
- three decimals with repeating blocks of length 2 that add to a decimal with a repeating block of length 1 (SAMPLE ANSWER: \(0.41 + 0.13 + 0.23 = 0.7\))
- two decimals with repeating blocks of length 4 that add to a decimal with a repeating block of length 2 (SAMPLE ANSWER: \(0.4123 + 0.3755 = 0.78\))

Then **ASK:** Can you find two decimals with repeating blocks of length 3 that add to a decimal with a repeating block of length 2? (The answer, perhaps surprisingly, is no. It is worth allowing students to struggle with this first before telling them that mathematicians have shown that this is not possible).

2. **Adding two repeating decimals always results in a decimal that either repeats or terminates.** We can think of a terminating decimal as a repeating decimal that repeats 0s. For example, \(0.25 = 0.250000\ldots\)

Investigate: If you know the length of the repeating blocks of each addend, what can you say about the length of the repeating block of the sum? Is there a maximum number it can be?
Have students fill in a chart like the one below. Students can add their own decimals to the chart. (Point out that the repeating pattern in some of the sums will have a very long core, so students will have to write out a lot of digits to be sure that the decimal repeats.)

<table>
<thead>
<tr>
<th>First addend</th>
<th>Length of repeating block</th>
<th>Second addend</th>
<th>Length of repeating block</th>
<th>Sum</th>
<th>Length of repeating block</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>2</td>
<td>0.354</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.34 = 0.3400…</td>
<td>1</td>
<td>0.52</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.342</td>
<td>3</td>
<td>0.251</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.45</td>
<td>2</td>
<td>0.54</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.45</td>
<td>2</td>
<td>0.13212</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The answer is that the length of the repeating block of the sum will be at most the lowest common multiple (LCM) of that for the two addends. The digits after the decimal point form a repeating pattern. If one pattern has core length 2 and another has core length 3, what happens when we add them? Consider the decimals. 0.712 and 0.354. Start at the place value where both are repeating:

\[0. \quad 7 \quad 1 \quad 2 \quad 1 \quad 2 \quad 1 \quad 2 \quad 1 \quad 2 \quad \ldots \]
\[+ \quad 0. \quad 3 \quad 5 \quad 4 \quad 3 \quad 5 \quad 4 \quad 3 \quad 5 \quad \ldots \]

Starting at the hundredths, the repeating patterns are 1, 2, repeat and 5, 4, 3, repeat. The first pattern will start over 2 places later, 4 places later, 6 places later, and so on (all the multiples of 2). The second pattern will start over 3 places later, 6 places later, and so on (all the multiples of 3). Simply find the smallest multiple of both to see when we can be sure they will both start over together. Then the sum will repeat from here on too (although it might start repeating earlier as well).
Write these fractions on the board:

\[
\frac{1}{11} \quad \frac{2}{11} \quad \frac{3}{11} \quad \frac{4}{11}
\]

Have students use long division to write these fractions as decimals, as in Workbook page 7 Question 1 a). **ASK:** What do you notice about your answers? Is there a pattern? Can you predict what 5/11 will be as a decimal? **PROMPT:** Look at the first two digits after the decimal point: 09, 18, 27, 36. How should we continue this pattern? (add 9) So the next term is 45, which means we expect that 5/11 = 0.45. **ASK:** Does this make sense? Let’s compare both to 1/2 to check. Is 5/11 more or less than 1/2? (less) How much less—a lot or a little? (just a little) How about 0.45—is it more or less than 1/2? **PROMPT:** What is 1/2 as a decimal? (0.5) Is 0.45 more or less than 0.5? (less) How much less—a lot or a little? (just a little) They are both a little less than one half, so our prediction makes sense. Then have students extend the pattern to find 6/11, 7/11, 8/11, 9/11, 10/11, and 11/11, as in Workbook page 7 Question 1 b).

Using products to write all the elevenths as decimals. Write down:

\[
\frac{1}{11} = 0.09 = 0.090909090909\ldots
\]

Remind students that 5/11 = 1/11 + 1/11 + 1/11 + 1/11 + 1/11 = 5 × 1/11. So 5/11 = 5 × 1/11. But 1/11 = 0.09, so 5/11 = 5 × 0.09. Have students calculate each product:

\[
0.09 \times 5 \quad 0.0909 \times 5 \quad 0.090909 \times 5 \quad 0.09090909 \times 5
\]

Then have students predict 0.09 × 5. **ANSWER:** 0.45 Express admiration: tell students that you knew they could multiply whole numbers by decimals with lots of decimal places, but you didn’t know they could multiply whole numbers by decimals with infinitely many decimal places!

Then verify that this result is the same as their prediction for 5/11. It is, because we predicted that the digits that repeat would be 9 × 5 = 45, and this is what we found.

Have students finish writing all the elevenths, up to 11/11, as repeating decimals.
Discovering that \(0.\overline{9} = 1\). Use the pattern in ninths from Workbook page 6 Question 2 to write \(9/9\) as a repeating decimal. Then compare the decimal representations of \(9/9\) and \(11/11\). ASK: Why are these the same? (\(9/9\) and \(11/11\) are both 1) ASK: Does this mean that \(1 = 0.9999...\)? (yes—if you write 9s forever after the decimal point, this number is equal to 1) SAY: Most students don’t learn that \(1 = 0.\overline{9}\) until they are in high school but now you’ve proven it twice! Once by finding the pattern for elevenths to find \(11/11\) and once by finding the pattern for ninths to find \(9/9\).

Writing fractions with denominator 99 as repeating decimals. Have students do Workbook page 7 Question 2. Then explain that just as \(17/99 = 17 \times 1/99\), any fraction with denominator 99 can be written as the product of the numerator and 1/99. Have students predict the decimal representation of 13/99 by using multiplication: \(13/99 = 13 \times 0.\overline{01}\), so successively multiply 0.01 \(\times\) 13, 0.0101 \(\times\) 13, 0.010101 \(\times\) 13, and look for a pattern in the answers. ANSWER: The products are 0.13, 0.1313, and 0.131313, so predict 13/99 = 0.\(\overline{13}\). Then have students check by long division: \(99 \overline{\sqrt{13}}\).

Repeat for 8/99 and 58/99. When students finish, emphasize that it is very easy to write fractions with denominator 9 or 99 as decimals: write the numerator with a repeating bar over it, but ensure that the repeating part has the same number of digits as the denominator (so \(8/99 = 0.0\overline{8}\), not 0.\(\overline{8}\)).

Then ask students to write many fractions with denominator 99 as repeating decimals and to check their answer with a calculator instead of by long division. EXAMPLES: 5/99, 52/99, 47/99, 7/99.

PROCESS EXPECTATION Connecting

Have students calculate the decimal for 1/11 by first writing it as an equivalent fraction with denominator 99. Does this method produce the same results as dividing directly, \(1 \div 11\)? (1/11 = 9/99 = 0.\(\overline{09}\). It fits!)

PROCESS EXPECTATION Connecting

Making connections between numbers that are close together. Have students compare how to find the equivalent decimal for fractions with denominator 99 and fractions with denominator 100. For example, compare the decimal representations for 35/99 and 35/100. They are 0.\(\overline{35}\) = 0.35353535... and 0.35. ASK: Are these decimals close to each other? (yes) Why does this make sense? (because the fractions are close to each other)

PROCESS EXPECTATION Representing

Writing fractions as decimals using equivalent fractions with denominator 9, 10, 99, or 100. Have students find an equivalent fraction with denominator 99, and then write the fraction as a decimal, as in Workbook page 7 Question 4. Students should check their answers with a calculator. EXAMPLES:

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>2/33</td>
<td>0.06</td>
</tr>
<tr>
<td>7/66</td>
<td>0.1092</td>
</tr>
</tbody>
</table>

Have students find an equivalent fraction with denominator 9, 10, 99, or 100, and write the equivalent decimals, again checking with a calculator.

EXAMPLES:

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>12/18</td>
<td>0.6666</td>
</tr>
<tr>
<td>9/50</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Number Sense 8-80
Writing repeating decimals that begin repeating right away as fractions.

Now start with the repeating decimal and write it as a fraction. **ASK:** What is \(0.\overline{01}\) as a fraction? Tell students that the answer is right in their workbook, on page 7. (1/99) **ASK:** What is \(0.\overline{02}\) as a fraction? (2/99)

Write on the board:

\[
0.\overline{01} + 0.\overline{01} = 0.0101010101\ldots + 0.0101010101\ldots = 0.0202020202\ldots = 0.\overline{02}
\]

So \(0.\overline{02} = \frac{1}{99} + \frac{1}{99} = \frac{2}{99}\). **ASK:** What is \(0.\overline{03}\) as a fraction? (1/99 + 1/99 + 1/99 = 3/99) What is \(0.\overline{12}\) as a fraction? (12 \times 1/99 = 12/99)

Write on the board: \(0.\overline{1} = 1/9\) and \(0.\overline{01} = 1/99\). Have students use these to convert decimals to fractions, as on Workbook page 9 Question 5.

After students finish Workbook Question 6, point out that any repeating decimal that begins repeating right away can be written as a fraction with a denominator of the form 9, 99, 999, etc.

**PROCESS ASSESSMENT**

8m1, 8m6, [T, R, CN]

Workbook Question 6

Tell students that you divided two numbers, \(a \div b\), on your calculator and it displayed the answer as:

\[
0.63636363636363636363636363636364
\]

Have students write two different fractions \(a/b\) which you could have been calculating, one of which has a terminating decimal equivalent and the other a repeating decimal equivalent. Hint: Use your answer to Workbook page 7 Question 1.

**ANSWERS:**

<table>
<thead>
<tr>
<th>Terminating decimal</th>
<th>Repeating decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>63636363636363636363636363636364</td>
<td>7</td>
</tr>
<tr>
<td>1000000000000000000000000000000000000000000000000000000000000000</td>
<td>11</td>
</tr>
</tbody>
</table>
NS8-81 Writing Repeating Decimals as Fractions (Advanced)

Goal
Students will write any repeating decimal as a fraction.

Prior Knowledge Required
Can use long division to find the decimal equivalent of a fraction
Can multiply terminating decimals
Can convert decimals that begin repeating right away to fractions
Can convert fractions with denominator 9, 99, or 999 to decimals
Can multiply and divide terminating decimals by 10
Can add terminating and repeating decimals
Can add fractions with like and unlike denominators
Understands that multiplication by a whole number is repeated addition
Understands that \( a \div (b \times c) = a \div b \div c \)

Multiplying and dividing repeating decimals by 10, 100, and 1000.
Review multiplying and dividing terminating decimals by 10, 100, and 1000. Then have students calculate the products in Questions a)–e) and then look for a pattern to find the product in f):

a) \( 3.54 \times 100 \)
b) \( 3.544 \times 100 \)
c) \( 3.5444 \times 100 \)
d) \( 3.54444 \times 100 \)
e) \( 3.544444 \times 100 \)
f) \( 3.54 \times 100 \)

Then point out that you can use the same method to multiply and divide repeating decimals by 10, 100, and 1000 as you use to multiply and divide terminating decimals by 10, 100, and 1000—simply shift the decimal point 1, 2, or 3 places to the right or left. Demonstrate with two Examples:

\[
3.\overline{54} \times 100 = 3.5444444\ldots \times 100 \\
= 354.44444\ldots \\
= 354.\overline{4}
\]

\[
3.\overline{54} \div 100 = 3.5444444\ldots \div 100 \\
= 0.035444444 \\
= 0.0354\overline{4}
\]

Have students do Workbook page 8 Question 2.

Review multiplying fractions by 10. Write on the board:

\[
10 \times \frac{3}{71} = \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71} + \frac{3}{71}
\]

\[
= \frac{3 + 3 + 3 + 3 + 3 + 3 + 3 + 3}{71}
\]

\[
= \frac{10 \times 3}{71}
\]
So to multiply a fraction by 10, multiply its numerator by 10. Have students multiply these fractions by 10:

a) \( \frac{3}{99} \)  
b) \( \frac{4}{9} \)  
c) \( \frac{7}{999} \)  
d) \( \frac{82}{999} \)

**ANSWERS:**

a) \( \frac{30}{99} \)  
b) \( \frac{40}{9} \)  
c) \( \frac{70}{999} \)  
d) \( \frac{820}{999} \)

**Making connections between equivalent numbers.** Have students find the equivalent decimals for the fractions above, and the equivalent decimals for the fractions found when multiplying by 10. For **EXAMPLE:** \( \frac{3}{99} = 0.03 \) and \( \frac{30}{99} = 0.30 \). Have students multiply the decimal for \( \frac{3}{99} \) by 10: Do they get the decimal for \( \frac{30}{99} \)? (yes)

Then have students add the decimal representations for \( \frac{2}{99} \) and \( \frac{30}{99} \). \( (0.02 + 0.30 = 0.32) \) **ASK:** What fraction should this be equivalent to? \( \frac{32}{99} \) Is it? (yes)

**Review dividing fractions by 10.** Write on the board:

\[
\frac{1}{3} \div 10 = \left( \frac{1 \div 3}{3} \right) \div 10
\]

\[
= \frac{1 \div 3}{10}
\]

\[
= \left( \frac{1}{3} \times 10 \right)
\]

\[
= \frac{1}{30}
\]

Point out that we just multiplied the denominator by 10 to divide the fraction by 10. Have students divide each of these fractions by 10, by multiplying the denominator by 10.

a) \( \frac{4}{9} \)  
b) \( \frac{56}{99} \)  
c) \( \frac{17}{999} \)  
d) \( \frac{8}{99} \)

**ANSWERS:**

a) \( \frac{4}{90} \)  
b) \( \frac{56}{990} \)  
c) \( \frac{17}{9990} \)  
d) \( \frac{8}{990} \)

**Dividing fractions by 100 and 1000.** **ASK:** If to divide a fraction by 10 you multiply its denominator by 10, how can you divide a fraction by 100? (multiply its denominator by 100) How would you divide a fraction by 1000? (multiply its denominator by 1000) Have students practise.

a) \( \frac{13}{99} \div 10 \)  
b) \( \frac{4}{9} \div 100 \)  
c) \( \frac{8}{3} \div 1000 \)

**ANSWERS:**

a) \( \frac{13}{990} \)  
b) \( \frac{4}{900} \)  
c) \( \frac{8}{3000} \)

**Finding decimal equivalents for fractions with denominator 9, 90, 900, or 9000.** Remind students that \( \frac{4}{9} = 0.4 \). **ASK:** What decimal is equivalent to \( \frac{4}{90} \)? **PROMPT:** We divided the fraction by 10, so let’s divide its equivalent decimal by 10.
0.4 \div 10 = 0.44444... \div 10 = 0.044444... = 0.04

So \(\frac{4}{90} = 0.04\). Ask students to write these fractions as decimals:

a) \(\frac{4}{900}\)  
b) \(\frac{4}{9000}\)  
c) \(\frac{7}{90}\)  
d) \(\frac{7}{9000}\)  
e) \(\frac{8}{9000000}\)  

**Bonus**

Finding decimal equivalents for fractions with denominator a power of 10 times 99 or 999.** Tell students that now that they know how to find decimal equivalents for fractions with denominator 9, 99, or 999, and can divide fractions by 10, 100, and 1000, they can find decimal equivalents for lots of different fractions. Have students try writing these fractions as decimals (do a) and b) together as a class):

a) \(\frac{18}{990}\)  
b) \(\frac{29}{33300}\)  
c) \(\frac{95}{1110}\)  
d) \(\frac{35}{9900}\)  
e) \(\frac{21}{33000}\)

Students should check their answers on a calculator.

Now write on the board: 83/90. Tell students that you want to find \(\frac{83}{90}\) as a decimal. **ASK:** What is different about this problem? (you have to divide an improper fraction by 10 to get it) Remind students that an improper fraction can be easily changed to a decimal by first changing it to a mixed number, and then changing the fractional part to a decimal. Write on the board:

\[
\frac{83}{90} = \frac{83}{9} \div 10 = \frac{9.2}{9} \div 10 = 9.2 \div 10 \quad \text{(since} \quad \frac{2}{9} = 0.\overline{2}) \\
= 0.9\overline{2}
\]

Now have students use this method to write these fractions as decimals:

a) \(\frac{718}{990}\)  
b) \(\frac{895}{1110}\)  
c) \(\frac{837}{3330}\)

Students should check their answer with a calculator.

**ANSWERS:**

a) \(\frac{718}{99} \div 10 = 7.25\overline{99}\)  
b) \(\frac{895}{111} \div 10 = 8.7\overline{111}\)  
c) \(\frac{837}{333} \div 10 = 2.\overline{4}171\overline{333}\)

Writing any repeating decimal as a fraction. Review writing repeating decimals that begin repeating right away as fractions. Have students find the equivalent fractions for these repeating decimals:

a) 0.47  
b) 0.30\overline{2}  
c) 0.\overline{7}  
d) 0.08\overline{2}

**ANSWERS:**

a) \(\frac{47}{99}\)  
b) \(\frac{302}{999}\)  
c) \(\frac{7}{9}\)  
d) \(\frac{82}{999}\)
As always, students should check the answers on a calculator by dividing the numerator by the denominator.

Now have students find equivalent fractions for:

a) $0.047$  
   b) $0.0302$  
   c) $0.07$  
   d) $0.0082$

**ANSWERS:**

a) $\frac{47}{990}$  
   b) $\frac{302}{9990}$  
   c) $\frac{7}{90}$  
   d) $\frac{82}{9990}$

Show students how to find the equivalent fraction for $0.247$. First, write on the board: $0.247 = 0.2 + 0.047$. **SAY:** We know how to find the fractions for both of these decimals and we know how to add fractions, so we know how to do this problem:

$$0.247 = 0.2 + 0.047 = \frac{1}{5} + \frac{47}{990}$$

$$= \frac{99 \times 2}{990} + \frac{47}{990}$$

$$= \frac{198}{990} + \frac{47}{990}$$

$$= \frac{245}{990}$$

Have students find $245 \div 990$ on their calculators to check this answer.

Then have students use this method to find an equivalent fraction (not necessarily reduced) for each of these repeating decimals:

a) $0.347$  
   b) $0.4302$  
   c) $0.27$  
   d) $0.5082$  
   e) $0.2547$  
   f) $0.16302$  
   g) $0.757$  
   h) $0.13082$

**ANSWERS:**

a) $\frac{344}{990}$  
   b) $\frac{4298}{9990}$  
   c) $\frac{25}{90}$  
   d) $\frac{5077}{9990}$  
   e) $\frac{2522}{9900}$  
   f) $\frac{16286}{99900}$  
   g) $\frac{682}{900}$  
   h) $\frac{13069}{99900}$

Have students check their answers using a calculator.

**Extension**

Write always, sometimes, or never in the blanks. Justify your answer.

a) Adding two terminating decimals will _____________ result in a terminating decimal.

b) Adding two repeating decimals will _____________ result in a repeating decimal.

c) Adding a terminating decimal and a repeating decimal will _____________ result in a repeating decimal.
d) Adding two repeating decimals will ______________ result in a decimal that goes on forever but does not repeat.

ANSWERS:

a) always—a decimal that does not terminate does not have a smallest place value—it goes on forever!—and so cannot be the sum of two terminating decimals, which each have smallest place values. For example, adding 2 hundredths + 35 ten thousandths will result in a smallest place value of ten thousandths (0.02 + 0.0035 = 0.0235).

b) sometimes—for example, \(\frac{2}{7} + \frac{1}{7} = \frac{3}{7}\) results in a repeating decimal, but \(\frac{2}{3} + \frac{1}{3} = 1\) results in a terminating decimal.

c) always—after the point where the terminating decimal stops, the sum will be the same as the repeating decimal, which means that it also goes on forever and becomes a repeating decimal.

d) never—adding two fractions always results in a fraction! (See Extension 2 from NS7-61 for a more complete explanation of why adding two repeating decimals always results in a repeating or terminating decimal.)
Introduce the word percent. Ask students what the word per means in these sentences:

- Rita can type 60 words per minute.
- Anna scores 3 goals per game.
- John makes $10 per hour.
- The car travels at a speed of up to 140 kilometres per hour.

Then write percent on the board. Point out that percent is made up of two words: per and cent. ASK: Has anyone seen the word cent before? What does it mean? Does anyone know a French word that is spelled the same way? What does that word mean? (cent is French for 100) Explain that percent means “for every 100,” “out of 100,” or “out of every 100.”

For example, a score of 84% on a test would mean that you got 84 out of every 100 marks or points. Another example: if a survey reports that 72% of people read the newspaper every day, that means 72 out of every 100 people surveyed read the newspaper daily.

Have students rephrase the percents in the following statements using the phrases “for every 100” or “out of every 100.”

a) 52% of students in the school are girls (For every 100 students, 52 are girls OR 52 out of every 100 students in the school are girls.)

b) 40% of tickets sold were on sale (For every 100 tickets sold, 40 were on sale OR 40 out of every 100 tickets were on sale.)

c) Alejandra scored 95% on the test (For every 100 possible points, Alejandra scored 95 points on the test OR Alejandra got 95 out of every 100 points on the test.)

d) About 60% of your body weight is water (For every 100 kg of body weight, about 60 kg is water OR 60 kg out of every 100 kg of body weight is water.)

A percent is a ratio that compares a number to 100. Explain to students that a percent is a part-to-whole ratio that compares a number to 100. For example, $45\% = 45 : 100$. Remind students that a ratio refers to “for every.”
If there are 74 girls for every 100 people at a party, 74% of people at the party are girls. However, if there were 74 girls for every 100 boys, we could not say that 74% of the people are girls. We could say that the number of girls is 74% of the number of boys. But since girls are not part of the group of boys, we cannot say that 74% of the boys are girls. Have students complete Workbook page 11.

Have students rewrite each sentence using percents.

a) Sally got 83 marks for every 100 possible marks on a test.
   ANSWERS: Sally got 83% on a test.

b) The ratio of boys to students in a class is 65 to 100.
   ANSWERS: 65% of the students in a class are boys.

c) 36 out of every 100 people surveyed said they would vote for Annette.
   ANSWERS: 36% of people surveyed said they would vote for Annette.

Have students rewrite each sentence in terms of a ratio comparing a number to 100.

a) 30% of people in a city are visible minorities.
   ANSWERS: For every 100 people in a city, 30 are visible minorities.

b) John got 80% on his math test.
   ANSWERS: For every 100 possible marks on a test, John got 80 marks.

c) Jacob paid the friendly waiter a 25% tip.
   ANSWERS: For every $100 Jacob paid for the food, he gave $25 to the friendly waiter as a tip.

**PROCESS ASSESSMENT**

Have students rewrite each sentence using percents.

8m6, [CN]

a) Sally got 83 marks for every 100 possible marks on a test.
   ANSWERS: Sally got 83% on a test.

b) The ratio of boys to students in a class is 65 to 100.
   ANSWERS: 65% of the students in a class are boys.

c) 36 out of every 100 people surveyed said they would vote for Annette.
   ANSWERS: 36% of people surveyed said they would vote for Annette.

**PROCESS EXPECTATION**

Reflecting on what made a problem easy or hard

**PROCESS EXPECTATION**

Representing

**PROCESS ASSESSMENT**

Then have students write each percent as a fraction:

8m6, [V, CN]

Workbook Question 7

a) 6%   b) 19%   c) 8%   d) 54%   e) 79%   f) 97%
Adding and subtracting percents. First, review writing percents as fractions with denominator 100. Then review adding and subtracting fractions with denominator 100. Then combine the two steps to add and subtract percents. EXAMPLES:

Write the percents as fractions.

a) 34%  
   b) 18%  
   c) 94%

\[
\begin{align*}
\text{ANSWERS: } & \frac{34}{100} \quad \frac{18}{100} \quad \frac{94}{100} \\
\end{align*}
\]

Add the fractions.

a) \[\frac{2}{100} + \frac{7}{100}\]  
   b) \[\frac{5}{100} + \frac{18}{100}\]  
   c) \[\frac{23}{100} + \frac{61}{100}\]

\[
\begin{align*}
\text{Bonus } & \frac{8}{100} + \frac{7}{100} + \frac{14}{100} + \frac{18}{100} \\
\end{align*}
\]

\[
\begin{align*}
\text{ANSWERS: } & \frac{9}{100} \quad \frac{23}{100} \quad \frac{84}{100} \quad \text{Bonus } \frac{47}{100} \\
\end{align*}
\]

Change the percents to fractions with denominator 100 to help you add the percents.

a) 8% + 7%  
   b) 13% + 14%  
   c) 20% + 16%

\[
\begin{align*}
\text{Bonus } & 9% + 30% + 43% \\
\end{align*}
\]

\[
\begin{align*}
\text{ANSWERS: } & 15% \quad 27% \quad 36% \quad \text{Bonus } 82% \\
\end{align*}
\]

ASK: How do you add percents? (add the numbers—you can think of the percent sign as a short form for a denominator of 100, and the percents being added are like the numerators of fractions with denominator 100).

The total percent is always 100%. Tell students that one student wrote a test and got 85% of the questions right. ASK: What percent did the student get wrong? (15%) How do you know? (because the total percent is 100%, so I subtracted 100% – 85%) Have students do Workbook page 12.
Question 4. Emphasize that students should total the percents given and then subtract that total from 100% to find the missing percent.

Money and percents. Remind students that percent means “out of every 100.” Ask: How many pennies are in a dollar? (100) Ask: What percent of a dollar is a penny? (1%) What percent of a dollar is a dime? (10%, because a dime is worth 10 pennies)

Writing percents as decimal hundredths. Review writing percents as fractions with denominator 100. Then review writing fractions with denominator 100 as decimals. Combine the two steps to write percents as decimals. Examples:

| a) | 35% = \frac{35}{100} = 0.35 | b) | 8% = \frac{8}{100} = 0.08 |
| c) | 94% = \frac{94}{100} = 0.94 |

Writing decimal hundredths as percents. Review writing decimal hundredths as fractions with denominator 100. Then review writing fractions with denominator 100 as percents. Combine the two steps to write decimal hundredths as percents. Examples:

| a) | 0.41 | b) | 0.83 | c) | 0.05 | d) | 0.07 |
| ANSWERS: | a) \frac{41}{100} = 41% | b) \frac{83}{100} = 83% |
| c) | \frac{5}{100} = 5% | d) | \frac{7}{100} = 7% |

Writing decimal tenths as percents. Have students write various decimal tenths as percents by first changing the decimal to a fraction with denominator 100. Examples:

| a) | 0.2 = \frac{20}{100} = 20% | b) | 0.3 | c) | 0.9 | d) | 0.7 | e) | 0.5 |

Writing decimals as percents by moving the decimal point two places to the right. Have students look at their answers to writing decimal hundredths and tenths as percents. Ask: How can you get the percent from the decimal? (multiply by 100, or move the decimal point two places to the right) Have students write decimal tenths or hundredths as percents directly, by moving the decimal point two places to the right. Examples:

| a) | 0.43 | b) | 0.9 | c) | 0.07 | d) | 0.74 | e) | 0.2 |

What percent is shaded? Explain to students that they can find a percent of a figure just as they can find a fraction of a figure. Ask students to decide first what fraction and then what percent of each figure is shaded:

| a) | ![Image](image1.png) | b) | ![Image](image2.png) | c) | ![Image](image3.png) |
**ANSWERS:**

a) \( \frac{4}{10} = \frac{40}{100} = 40\% \) shaded
b) \( \frac{1}{4} = \frac{25}{100} = 25\% \) shaded
c) \( \frac{7}{20} = \frac{35}{100} = 35\% \) shaded

**Bonus**

![Diagram of shaded regions](image)

**Answer:** Divide the region into smaller regions.

The areas of the three shaded regions are, respectively, 2, 4, and 5. Since the area of the whole region is 20 square units, the fraction shaded is \( \frac{11}{20} = \frac{55}{100} = 55\% \).

**Extension**

Sara did a survey and found that 30\% of the girls in her school like action movies and 80\% of the boys like action movies. She sees that the total percent is 110\%, not 100\%, and thinks she made a mistake. Explain to her why her reasoning is wrong.

**Sample Answer:** The 30\% and the 80\% are not percents of the same thing, so we wouldn’t expect them to add to 100\%.

**Rounding decimals to the nearest whole number percent.** Review rounding decimals to the nearest hundredth. Then write the hundredth as a whole number percent.

**Word Problems Practice:**

a) Jane got 16 out of 25 marks on a test. What was her percentage grade?

b) Ron sold 17 of his 20 books at a yard sale. What percentage of his books did he sell?

c) Nomi collects sports cards. 20\% of her collection is baseball cards and 35\% of her collection is hockey cards. What percentage of her collection is neither hockey cards nor baseball cards?

**Answers:** a) 64\% b) 85\% c) 45\%
Changing reduced fractions to percents. Write the fraction 3/5 on the board and have a volunteer find an equivalent fraction with denominator 100. ASK: If 3 out of every 5 students at a school are girls, how many out of every 100 students are girls? (60) What percent of the students are girls? (60%). Write on the board:

\[ \frac{3}{5} = \frac{60}{100} = 60\% . \]

Then have volunteers find the equivalent fraction with denominator 100 and the equivalent percent for more fractions with denominator 5.

EXAMPLES: \( \frac{4}{5} \) and \( \frac{1}{5} \)

Repeat for more simple fractions, that is, fractions with denominators that divide evenly into 100. EXAMPLES:

a) \( \frac{2}{5} \)  

b) \( \frac{7}{10} \)  

c) \( \frac{9}{20} \)  

d) \( \frac{37}{50} \)  

e) \( \frac{18}{25} \)

ANSWERS: a) 40\%  

b) 70\%  

c) 45\%  

d) 74\%  

e) 72\%

Have students use the equivalent percents to put the above fractions in order from least to greatest.

ANSWERS: 2/5 < 9/20 < 7/10 < 18/25 < 37/50

Fractions that need reducing before the denominator divides evenly into 100. Write the fraction 9/15 on the board. Tell students that you want to find an equivalent fraction with denominator 100. ASK: How is this fraction different from previous fractions you have changed to percents? (The denominator does not divide evenly into 100.) Is there any way to find an equivalent fraction whose denominator does divide evenly into 100?
(Reduce the fraction by dividing both the numerator and the denominator by 3. Now the denominator is a factor of 100.) Write on the board:

\[
\frac{9}{15} = \frac{3}{5} = \frac{60}{100} = 60\%
\]

Summarize the steps for finding the equivalent percent of a fraction.

1. Reduce the fraction so that the denominator is a factor of 100.
2. Find an equivalent fraction with denominator 100.
3. Write the fraction with denominator 100 as a percent.

Have students write various fractions as percents:

a) \(\frac{3}{12}\)  
b) \(\frac{6}{30}\)  
c) \(\frac{24}{30}\)  
d) \(\frac{3}{75}\)  
e) \(\frac{6}{15}\)  
f) \(\frac{36}{48}\)  
g) \(\frac{60}{75}\)

ANSWERS: a) 25%  b) 20%  c) 80%  d) 4%  e) 40%  f) 75%  g) 80%

Changing fractions to percents using division. Have students change the fractions above to percents, but this time by using division to obtain the decimal. EXAMPLE: \(\frac{3}{12} = 3 \div 12 = 0.25 = 25\%\).

Changing percents to fractions in lowest terms. First write the percent as a fraction with denominator 100, and then reduce to lowest terms.

Which percent is a fraction closest to? Show students a double number line with fractions above and percents below:

<table>
<thead>
<tr>
<th>0%</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
<th>50%</th>
<th>60%</th>
<th>70%</th>
<th>80%</th>
<th>90%</th>
<th>100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>100</td>
</tr>
</tbody>
</table>

Have students copy the number line onto 1 cm grid paper (using a ruler) with 2 mm representing 1 percent (so that 2 cm represent 10 percent) and mark on it the fractions from Workbook page 12 Questions 6. Students should first convert each fraction to a percent. The easiest way to do this is to convert each fraction to a fraction with denominator 100. You could also encourage your students to check their answers using long division. If any students struggle with motor skills, have the double number line pre-drawn for them. Students can then use the number line to complete Question 6.

Not all fractions can be written as a whole number percent. Tell students that you want to change \(\frac{3}{7}\) to a percent. But 7 does not divide evenly into 100 and \(\frac{3}{7}\) is already reduced! You can’t reduce the fraction to make the denominator divide evenly into 100. So you have to use division to change the fraction to a decimal: \(\frac{3}{7} = 3 \div 7 = 0.428571 \approx 0.43 = 43\%

Have students write these fractions to the nearest whole number percent by following these steps:

Step 1: division \((a/b = a \div b)\);  
Step 2: round the resulting decimal to the nearest hundredths;  
Step 3: Write the decimal hundredth as a whole number percent.
Estimating fractions as percents. Write the fraction 9/40 on the board.
Tell students that you want to estimate what percent this is close to. **SAY:**
Let’s first think about other fractions that are easy to turn into percents.
Which fractions can you think of that are easy to turn into percents? (1/2,
1/4, 1/5, 1/10, 1/20, 1/25, 1/50, and any fraction with these denominators)
Is 9/40 close to one of these fractions? **PROMPT:** What fraction with
denominator 40 is equivalent to 1/4? Is 9/40 close to that fraction?
(10/40 = 1/4, and this is close to 9/40 because 10 is close to 9) What
percent is 10/40? (25%) Should 9/40 be a little more or a little less than
25%? (a little less) How do you know? (because 9 is a little less than 10)
Have students find 9 ÷ 40 on a calculator or using long division.
(9/40 = 0.225 ≈ 0.23 = 23%) **ASK:** Is this a little less than 25%? (yes)
Explain that this tells us that our estimate was good. Have students practise
estimating more fractions as percents:

a) \( \frac{23}{30} \)  

b) \( \frac{43}{70} \)  

c) \( \frac{22}{35} \)  

d) \( \frac{43}{80} \)

**ANSWERS:**

a) a little less than \( \frac{24}{30} = \frac{8}{10} = 80\% \)  

b) a little more than \( \frac{42}{70} = \frac{6}{10} = 60\% \)  

c) a little more than \( \frac{21}{35} = \frac{3}{5} = 60\% \)  

d) a little less than \( \frac{44}{80} = \frac{11}{20} = 55\% \)

Have students divide to change each fraction to a decimal (round to
two decimal places) and then to a percent. Was their estimate close to
the answer?

**ANSWERS:**

a) \( 23 \div 30 \approx 0.77 = 77\% \)  

b) \( 43 \div 70 \approx 0.61 = 61\% \)  

c) \( 22 \div 35 \approx 0.63 = 63\% \)  

d) \( 43 \div 80 \approx 0.54 = 54\% \)

Yes, the estimate was close in all cases.

Then tell students that so far, they have only had to change the numerator
to find an equivalent fraction with denominator 2, 4, 5, 10, 20, 25, 50, or
100. Sometimes it is convenient to start by changing the denominator. For
example, write on the board the fraction 45/99. **ASK:** What denominator
that is easy to work with is close to 99? (100) What makes it easy to
work with denominator 100? (a fraction with denominator 100 can be
easily changed to a percent because percents are out of 100) Is the
fraction 45/99 close to 45/100? (yes) How do you know? (they are both
45 parts, and the parts are almost the same size) Which fraction is
bigger? (45/99 because ninety-ninths are slightly bigger than hundredths)
Have students calculate 45/99 on a calculator, round the answer to
the nearest hundredth, and then convert it to a whole-number percent.
(45/99 = 0.4545454545… \( \approx 0.45 = 45\% \) ) Notice that although 45/99 is
slightly more than 45%, it is 45% to the nearest whole number percent—
it is closer to 45% than to 46%. Now have students practise estimating
with these **EXAMPLES:**

a) \( \frac{20}{49} \)  

b) \( \frac{17}{24} \)  

c) \( \frac{17}{52} \)  

d) \( \frac{12}{19} \)
ANSWERS:

a) a little more than $\frac{20}{50} = 40\%$

b) a little more than $\frac{17}{25} = 68\%$

c) a little less than $\frac{17}{50} = 34\%$

d) a little more than $\frac{12}{20} = 60\%$

Have students use a calculator to change each fraction to a decimal (rounded to two decimal places) and then to a percent. Was their estimate close to the answer?

ANSWERS:

a) $\frac{20}{49} \approx 0.41 = 41\%$

b) $\frac{17}{24} \approx 0.71 = 71\%$

c) $\frac{17}{52} \approx 0.33 = 33\%$

d) $\frac{12}{19} \approx 0.63 = 63\%$

Yes, the estimate was close in all cases.

Extensions

1. **ASK:** How many degrees are in a circle? (360) If I rotate an object 90° counter-clockwise, what percent of a complete turn has the object made? (90/360 = 1/4 = 25/100 = 25%)

Repeat for various degrees: 180°, 18°, 126°, 270°, 72°, 216°.

2. Investigate: Which fractions can be written as whole number percents and which need to be rounded to whole number percents.

Try these examples plus two more of your own.

a) $\frac{7}{25}$  
b) $\frac{18}{30}$  
c) $\frac{7}{9}$  
d) $\frac{21}{28}$  
e) $\frac{21}{35}$  
f) $\frac{8}{14}$

Articulate how you can tell without writing the fraction as a decimal whether it is equivalent to a whole number percent or not. **ANSWER:** If the fraction can be reduced so that the denominator divides evenly into 100, then the fraction is equivalent to a whole number percent.
**NS8-86** Visual Representations of Percents  
**NS8-87** Comparing Fractions, Decimals, and Percents

**Curriculum Expectations**  
Ontario: 6m15, 6m27, 7m15, 8m3, 8m5, 8m6, 8m7, 8m13, 8m14, 8m18, 8m28  
WNCP: 7N3, 7N7, 8N3, 8N5, [CN, C, R, V]

---

**Goals**  
Students will develop visual representations of percents and will compare fractions, percents, and decimals.

**Prior Knowledge Required**  
Can compare and order fractions with like and unlike denominators  
Can compare and order percents  
Can compare and order decimals  
Can convert between fractions, percents, and decimals

**Materials**  
a metre stick

**Visual representations of percents.** Draw a hundreds block on the board and have students write what part of the block is shaded in three different ways: a fraction, a decimal, and a percent. **Example:**

\[
\begin{array}{c}
\includegraphics{hundreds_block.png} \\
\frac{39}{100}, \ 0.39, \ 39\% \\
\end{array}
\]

Now draw the shapes below on the board and have students write which fraction is shaded. To change the fraction to a percent, students should find an equivalent fraction with denominator 100 to change to a decimal and then a percent.

**Answers:**  
a) \(\frac{7}{10} = \frac{70}{100} = 0.70 = 70\%\)  
b) \(\frac{1}{5} = \frac{20}{100} = 0.20 = 20\%\)  
c) \(\frac{9}{25} = \frac{36}{100} = 0.36 = 36\%\)  
d) \(\frac{14}{20} = \frac{70}{100} = 0.70 = 70\%\)  
e) \(\frac{11}{20} = \frac{55}{100} = 0.55 = 55\%\)
Estimating percents of line segments. Draw a line on the board and ask a volunteer to mark 50% on the line:

Use a metre stick to check the volunteer’s estimate. Then invite students to draw 5 lines of different lengths independently on a blank sheet of paper and to mark a different percent on each one: for example, 20% on the first line, 75% on the second line, and so on. To check their estimates, students can use regular rulers or “percent” rulers made from elastics. (A percent ruler is a wide elastic band on which you make 11 evenly spaced markings labelled with percents: 0, 10, 20, … 100. To use this ruler, stretch the elastic so that the 0 and 100 markings line up with the ends of a line segment. You could make 3 or 4 such rulers and have students share them.)

Bonus: Draw two lines such that 20% of the first line is longer than 50% of the second line.

Now draw a line segment and identify what percent of a line the segment represents. Invite a volunteer to extend the line segment to its full length, that is, to show 100%. EXAMPLES:

50% ANSWER: __________________________  __________________________

40% ANSWER: __________________________  __________________________

Notice that 50% = 1/2, so the given line is 1 of 2 equal parts—simply draw another equal part. Also, 40% = 4/10 = 2/5, so the given line is 2 of 5 equal parts—divide the given line into two equal parts and draw three more identical parts.

Draw on the board a line one metre long, and have students estimate the percent of various marks on the number line (to the nearest 10%). Then, using the metre stick, draw another line of the same length divided into 10 equal parts above or below the first line, so that students can check their estimates.

Comparing simple fractions, decimal hundredths, and whole number percents. Review comparing fractions with the same denominator. Then review changing simple fractions, decimal hundredths, and whole number percents to fractions with denominator 100. Then have students do Workbook page 14 Question 2.

Then have students redo three or four parts of Question 2 (they can choose the parts) by changing all the fractions and percents to decimals. ASK: Did you get the same answer both ways?

After students do Workbook page 14 Question 3, have students redo the question by changing all the numbers to decimal fractions with the same denominator. ASK: Did you get the same answer both ways? Ensure that students understand that if they didn’t get the same answer both ways, they should check their answer with another student or with you.
Tell students that Sally wrote five math tests this year. Each test had a different number of questions, all worth 1 mark each. Her marks were:

<table>
<thead>
<tr>
<th>Test 1</th>
<th>Test 2</th>
<th>Test 3</th>
<th>Test 4</th>
<th>Test 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>6</td>
<td>18</td>
<td>24</td>
<td>17</td>
</tr>
<tr>
<td>25</td>
<td>10</td>
<td>24</td>
<td>30</td>
<td>20</td>
</tr>
</tbody>
</table>

Have students answer these questions:

a) Change each of Sally’s grades to percent and decimal form. Copy and complete this chart:

<table>
<thead>
<tr>
<th>Test Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fraction</td>
<td>17/25</td>
<td>6/10</td>
<td>18/24</td>
<td>24/30</td>
<td>17/20</td>
</tr>
<tr>
<td>Percent</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Decimal</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Decide which form of Sally’s grades (fraction, percent, or decimal) you would use to answer Sally’s questions below. Justify your choice.

i) Sally wants to know if her grades are improving.

ii) To study for a cumulative test, Sally will study only the questions she got wrong. How many questions does she have to study?

iii) Sally wants to know her average (mean) test score.

c) Answer each question in part b.

**ANSWERS:**

a) | Percent | Decimal |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>68%</td>
<td>0.68</td>
</tr>
<tr>
<td>60%</td>
<td>0.6</td>
</tr>
<tr>
<td>75%</td>
<td>0.75</td>
</tr>
<tr>
<td>80%</td>
<td>0.8</td>
</tr>
<tr>
<td>85%</td>
<td>0.85</td>
</tr>
</tbody>
</table>

b) i) Percents and decimals are definitely easier to compare than fractions with unlike denominators, and percents are easiest because they are whole numbers.

ii) The fractions tell you how many questions Sally got wrong: subtract the numerator (number of questions she got right) from the denominator (total number of questions).

iii) Percents and decimals are definitely easier to add than fractions with unlike denominators, and percents are probably easiest because they are whole numbers.
c)  

i) Her grades tend to be improving—the only exception is from the first to second test, but all the rest are improving and are better than the first test.

ii) \[ 8 + 4 + 6 + 6 + 3 = 27 \]

iii) \[ \text{mean score} = \frac{68 + 60 + 75 + 80 + 85}{5} = \frac{368}{5} = 0.736 \approx 74\% \]

**Word Problems Practice:**

a) If 13 out of 20 students in a class like skiing, what percent of the students like skiing?

b) Audrey got 16/20 on a math test, 19/25 on a science test, and 78% on a history test. On which test did she do best?

c) In Bilal’s city, 54% of the population are visible minorities. In Bilal’s class, 18 of 30 students are visible minorities. Does Bilal’s city or Bilal’s class have a greater percentage of visible minorities?

**ANSWERS:**

a) 65%

b) Audrey got 80% in math, 76% in science, and 78% in history, so she did best on her math test.

c) In Bilal’s class, \[ \frac{18}{30} = \frac{6}{10} = 60\% \] of students are visible minorities. This is greater than the city-wide average of 54%.
Finding one tenth of a number using base ten materials. Tell students that you will use one thousands block (the big cube) to represent one whole. Remind students that a thousands block is called a thousands block because it is made up of 1,000 little cubes, but it doesn’t always have to represent 1,000. **ASK:** If the thousands block represents one whole, what does a hundreds block represent? (one tenth) A hundreds block has one tenth the number of cubes of a thousands block, so if the thousands block is a whole, the hundreds block is one tenth of a whole, or $\frac{1}{10}$. **ASK:** If the thousands block represents a whole, what does a tens block represent? (one hundredth) What does a ones block represent? (one thousandth) These relationships are summarized in the teaching box on Workbook page 17.

Ask students to identify the fraction and the decimal each model represents:

**ANSWERS:**

a) $0.1, \frac{1}{10}$  

b) $0.01, \frac{1}{100}$  

c) $0.03, \frac{3}{100}$  

d) $0.005, \frac{5}{1000}$  

e) $0.033, \frac{33}{1000}$  

f) $0.136, \frac{136}{1000}$

Then tell students that you want to make a model of the number 1.6, again using the thousands block as one whole. **ASK:** What do I need to make the
model? (1 thousands block, 6 hundreds blocks) How can I show 1/10 of 1.6? (one tenth of a thousands block is a hundreds block, and one tenth of a hundreds block is a tens block, so I need 1 hundreds block and 6 tens blocks to make 1/10 of 1.6) What number is 1/10 of 1.6? (0.16, since this is what the base ten materials show) Find 1/10 of more numbers together, then have students do so independently using base ten materials.

**EXAMPLES:**

a) 0.1 

b) 0.01 

c) 0.1 

d) 7 

e) 2.3 

f) 0.41 

g) 5.01

**ANSWERS:**

a) 0.1 

b) 0.001 

c) 0.01 

d) 0.7 

e) 0.23 

f) 0.041 

g) 0.501

**Finding one tenth of a number by moving the decimal point one place left.** Remind students that when they find one tenth of a number, each place value becomes worth one tenth as much: if there were 2 ones, there are now 2 tenths; if there were 3 tenths, there are now 3 hundredths. **ASK:** How can you move the decimal point to make each place value worth one tenth as much? (move the decimal point one place left) **Demonstrate this with 1/10 of 2.3 = 0.23.** **ASK:** Does this remind you of a rule for dividing by something? (yes, to divide by 10, move the decimal point one place left) Emphasize that to find 1/10 of anything, you divide it into 10 equal parts; to find 1/10 of a number, you divide the number by 10.

**Finding 10% of a number by moving the decimal point one place left.** Have students convert 1/10 to a decimal (1/10 = 0.1) and to a percent (1/10 = 10/100 = 10%). Emphasize that by finding 1 tenth of a number, they are finding 10% of it. Ask your students to find 10% of each number by just moving the decimal point.

a) 40 

b) 4 

c) 7.3 

d) 500 

e) 408 

f) 3.07 

g) 432.5609

**Finding multiples of 10% of a number.** Show this number line:

```
<table>
<thead>
<tr>
<th>0%</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
<th>50%</th>
<th>60%</th>
<th>70%</th>
<th>80%</th>
<th>90%</th>
<th>100%</th>
</tr>
</thead>
</table>
```

Have a volunteer fill in the missing numbers on the number line. (because 10% of 30 is 3, skip count by 3s to find the remaining percents) Then ask students to look at the completed number line and identify 10% of 30, 40% of 30, 90% of 30, and 70% of 30.

Repeat the exercise for a number line from 0 to 21. (Since 10% of 21 is 2.1, skip count by 2.1. It might be easier to skip count by 21 and divide by 10 as you go: 21 (the first mark is 2.1), 42 (the next mark is 4.2), 63 (the next mark is 6.3) and so on.) **ASK:** If you know 10% of a number, how can you find 30% of that number? (multiply 10% of the number by 3) Tell students that you would like to find 70% of 12. **ASK:** What is 10% of 12? (1.2) If I know that 10% of 12 is 1.2, how can I find 70% of 12? (multiply 1.2 × 7)

Have students use this method to find:

a) 60% of 15 

b) 40% of 40 

c) 60% of 4 

d) 20% of 1.5 

e) 90% of 8.2 

f) 70% of 4.3 

g) 80% of 5.5
Visualizing percents. Have students calculate:

a) 40% of 33  b) 30% of 42  c) 60% of 85  d) 90% of 21

**ANSWERS:** a) 13.2  b) 12.6  c) 51  d) 18.9

Have students check their answers using **BLM Percent Strips**. Students should cut out the percent strip from the bottom of the sheet and line it up with the other strips. Here is the answer for part a) above:

So 40% of 33 is a little more than 13, as calculated.

**Finding 5% and 15% of a number using 10% of the number.** **ASK:** If I know 10% of 42 is 4.2, how can I find 5% of 42? (5% is half of 10%, so if 10% is 4.2, then 5% is 2.1) Have students find 5% of the following numbers by first finding 10% then dividing by 2. (Students should use long division on a separate piece of paper.)

<table>
<thead>
<tr>
<th>Number</th>
<th>5% Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 80</td>
<td>4</td>
</tr>
<tr>
<td>b) 16</td>
<td>0.8</td>
</tr>
<tr>
<td>c) 72</td>
<td>3.6</td>
</tr>
<tr>
<td>d) 50</td>
<td>2.5</td>
</tr>
<tr>
<td>e) 3.2</td>
<td>0.16</td>
</tr>
<tr>
<td>f) 2.34</td>
<td>0.117</td>
</tr>
</tbody>
</table>

**ANSWERS:** a) 4  b) 0.8  c) 3.6  d) 2.5  e) 0.16  f) 0.117

**SAY:** I know that 10% of 42 is 4.2, and 5% of 42 is 2.1. What is 15% of 42? (15% of a number is 10% of the number plus 5% of the number, so 15% of 42 is 4.2 + 2.1 = 6.3. Have students verify this answer using **BLM Percent Strips** by lining up the percent strip with the “42” strip. **ASK:** Does 10% look like it lines up with 4.2? (yes) Does 15% look like it lines up with 6.3? (yes)

Then have students calculate 15% of each number below by finding 10% and 5% and then adding:

<table>
<thead>
<tr>
<th>Number</th>
<th>15% Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 60</td>
<td>9</td>
</tr>
<tr>
<td>b) 240</td>
<td>36</td>
</tr>
<tr>
<td>c) 12</td>
<td>1.8</td>
</tr>
<tr>
<td>d) 7.2</td>
<td>1.05</td>
</tr>
<tr>
<td>e) 3.80</td>
<td>0.57</td>
</tr>
<tr>
<td>f) 6.10</td>
<td>0.915</td>
</tr>
</tbody>
</table>

**PROCESS EXPECTATION**

Real World

**PROCESS EXPECTATION**

Mental math and estimation

**PROCESS EXPECTATION**

Reflecting on the reasonableness of an answer

Tell students that when going out to eat at a restaurant, people are expected to leave a tip. Depending on how good the service is, the tip could be anywhere from 10% to 20% of the bill, before taxes. Assuming you want to leave a 15% tip, have students calculate, in their heads, what tip they should leave for various total costs of a meal. Remind students that 15% = 10% + 5%.

<table>
<thead>
<tr>
<th>Meal Cost</th>
<th>Tip Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $30</td>
<td>b) $31.50</td>
</tr>
<tr>
<td>b) $22</td>
<td>c) $33.30</td>
</tr>
<tr>
<td>c) $18</td>
<td>d) $34.50</td>
</tr>
<tr>
<td>d) $35</td>
<td>e) $36.50</td>
</tr>
<tr>
<td>e) $47</td>
<td></td>
</tr>
</tbody>
</table>

**ANSWERS:** a) $4.50  b) $3.30  c) $2.70  d) $5.25  e) $7.05

As a check on their calculations, have students order the meal costs from smallest to greatest and the tips they calculated from smallest to greatest—these should be in the same order! (c, b, a, d, e)
Relate the different percents to each other. Remind students that 5% is half of 10%, and that 15% is the sum of 10% and 5%. Ask: If I know 4% of a number is 32, how can I find 2%? (find half of 32, so 2% is 16) What is 1%? (1% is half of 2% = 16, so 1% is 8) What is 100%? (800, since 100% of a number is just the number) Ask: Could I have gotten that directly from knowing that 4% of the number is 32? Prompt: What can I multiply 4% by to get 100%? (25) So calculate 32 × 25 = 800 to get 100% of the number.

Ask: If 30% of a number is 27, what is the number? To guide students, suggest that they first find 10% of the number, then find the number. (10% is 9, so the number is 90)

Easy percents to find. Tell students that since percents can be changed to fractions, and some fractions of numbers are easy to find, some percents are easy to find too. Ask: What is 20% as a fraction? (1/5) How can you find 20% of a number easily, without using 10% of the number? (divide the number by 5) Have students calculate 20% of the numbers from BLM Percent Strips (33, 42, 85, and 21) by dividing the number by 5, and then have them check their answer using the BLM. Repeat for 25% of the numbers (25% = 1/4, so divide the number by 4) Have students find 75% of the same numbers using 25% of each number (which they already found). Students can check their answer on BLM Percent Strips.

Finding 1% of a number. Have students convert 1% to a decimal and to a fraction. (1% = 0.01 = 1/100) Ask: Finding 1% of a number is the same as dividing the number by what? (100) How can you move the decimal point to find 1% of a number? (move it two places left) Have students find 1% of various numbers. Examples:

a) 27  b) 3.2  c) 773  d) 12.3  e) 68

Answers: a) 0.27  b) 0.032  c) 7.73  d) 0.123  e) 0.68

Finding any whole number percent of a number. Say: I know that 1% of 400 is 4. Ask: What is 2% of 400? (2 × 4 = 8) What is 3% of 400? (3 × 4 = 12) What is 17% of 400? (17 × 4 = 68) Explain that to find any percent of a number, you can find 1% of the number and then multiply. Have students use this method to calculate:

a) 64% of 33  b) 37% of 42  c) 94% of 85  d) 83% of 21

Students can check their answers visually using BLM Percent Strips.

Tell students that the Harmonized Sales Tax (HST) used in Ontario is 13%. Have students determine the amount of HST on something that costs:

a) $33  b) $42  c) $85  d) $21

Alternatively, use the tax rate in your region instead of the Ontario HST. Ask: How can you use the percent strips to check your answer?

Note: Have students keep their percent strips to refer to next class.
Review determining percents of numbers. In the last lesson, students learned to find the percents of numbers by using division to find 1% and then multiplying. For example, to find 64% of 33, they found 1% of 33 by calculating $33 \div 100$. Then they multiplied 1% of 33 by 64 to find 64% of 33 since 64% of a number is 64 times greater than 1% of the number. So they calculated: $64 \times 33 \div 100$.

Connect this method to finding a fraction of a whole number. We know that $64\% = \frac{64}{100}$, so 64% of 33 is $\frac{64}{100}$ of 33. But this is just $64 \times 33 \div 100$, as above. Have students use this method to calculate 53% of 12 (53% of 12 is 50% of 12, which is 6, which is lower than the actual answer because 50% is less than 53%).

Using easy percents to check if an answer is reasonable. Tell students that you want to know if the answer is reasonable. Have students take out their percent strips from last class. ASK: Can I check using the percent strips? (no) Why not? (because we don’t have a percent strip for 12) ASK: What is another way to check if our answer is reasonable? PROMPT: Is there a percent of 12 that is close to 53% and easy to calculate? (yes, 50%) Will this estimate be lower or higher than the actual answer? (50% of 12 is 6, which is lower than the actual answer because 50% is less than 53%) Is the answer we found just a little higher than 6? (yes) Conclude that the answer is reasonable. Have students calculate the following percents and then use an easy percent to determine if their answers are reasonable.

a) 76% of 24 (should be slightly more than 3/4 of 24, which is $6 \times 3 = 18$)
b) 19% of 25 (should be slightly less than 1/5 of 25, which is $25 \div 5 = 5$)
c) 48% of 76 (should be slightly less than 1/2 of 76, which is 38)
d) 11% of 32 (should be slightly more than 1/10 of 32, which is 3.2)
**PROCESS EXPECTATION**

- Reflecting on other ways to solve a problem

**Commutativity of percents.** Have students predict which is greater: 20% of 60 or 60% of 20. Tell students that the first is a smaller percentage of a larger number, so you’re not sure which is bigger. Then have students calculate both. (both are 12) **SAY:** These are both the same. In hindsight, is there a reason we should have been able to predict this? **PROMPT:** How does 20% of 60 compare to 20% of 20? (it is 3 times greater) How does 60% of 20 compare to 20% of 20? (it is also 3 times greater) So they are both equal.

**PROCESS EXPECTATION**

- Looking for a pattern

Have students calculate and compare:

<table>
<thead>
<tr>
<th>a) 30% of 50 and 50% of 30</th>
<th>b) 40% of 20 and 20% of 40</th>
</tr>
</thead>
<tbody>
<tr>
<td>c) 70% of 90 and 90% of 70</td>
<td>d) 80% of 60 and 60% of 80</td>
</tr>
<tr>
<td>e) 50% of 40 and 40% of 50</td>
<td>f) 36% of 24 and 24% of 36</td>
</tr>
<tr>
<td>g) 17% of 35 and 35% of 17</td>
<td>h) 29% of 78 and 78% of 29</td>
</tr>
<tr>
<td>i) 48% of 52 and 52% of 48</td>
<td></td>
</tr>
</tbody>
</table>

What pattern do students see? Challenge them to figure out why this pattern holds. **PROMPT:** What calculations do you need to do to get 24% of 36? (24 \times 36 \div 100) How about 36% of 24? (36 \times 24 \div 100) What rule can you use to explain why these calculations will get the same answer? (multiplication is commutative—e.g. 36 \times 24 = 24 \times 36)

**NOTE:** In the following discussion and questions, we presume the sales tax is 13%. If the sales tax in your region is different, please adjust the questions accordingly. For example, we calculate 10% and 30% in preparation for calculating 10% and 3%. If your sales tax is 12%, you could change this to 10% and 20% in preparation for calculating 10% and 2%.

**PROCESS EXPECTATION**

- Selecting tools and strategies, Mental math and estimation

**Estimating percents of whole numbers and decimals.** Tell students that you want to estimate 30% of 48. Discuss different strategies.

1. 10% of 48 is about 5, so 30% is about 15.
2. 30% of 48 = 48% of 30, which is about 50% of 30, or 15.
3. 30% of 48 is about half of 30% of 100, which is 30, so 30% of 48 is about 15.

Have students estimate:

<table>
<thead>
<tr>
<th>a) 1% of 732</th>
<th>b) 3% of 732</th>
<th>c) 3% of 7.32</th>
</tr>
</thead>
</table>

**ANSWERS:** a) 7  b) 21  c) 0.21

**ASK:** What is a good estimate for 10% of 732? (73 or 70—73 is a more accurate estimate but 70 is easier to work with) What about 13% of 732? (73 + 21 = 94 or 70 + 21 = 91) And 13% of 7.32? (0.91 or 0.9)

Remind students that 1% of a dollar is a cent. **ASK:** What is 1% of $12? (12 cents) Have students estimate:

<table>
<thead>
<tr>
<th>a) 13% of $12.49</th>
<th>b) 13% of $8.51</th>
<th>c) 13% of $9.00</th>
<th>d) 13% of $25.99</th>
</tr>
</thead>
</table>

**ANSWERS:**

a) 1% is about 12¢, so 13% is about
\[
13 \times 12¢ = 10 \times 12 + 3 \times 12 = 120 + 36 = 156,
\]
so 13% of $12.49 is about $1.60.

b) 13% of 851¢ is about \(85¢ + 25¢ = 110¢\). So 13% of $8.51 is about $1.10.

c) 13% of 900¢ is \(90¢ + 27¢\), or about 120¢, so 13% of $9.00 is about $1.20.

d) 13% of 2599¢ is about 13% of 2600¢, or about \(260¢ + 75¢ = 335¢\), so 13% of $25.99 is about $3.35.

Point out that rounding $25.99 to $26 will not affect the answer, since it will only be off by 13% of 1 cent.

**PROCESS EXPECTATION**

Reflecting on the reasonableness of an answer

**ASK:** Which two answers are close together? (13% of 851 is close to 13% of 900) Why does this make sense? (because 851 is close to 900) Which answer is about twice as much as another answer? (13% of $25.99 is about twice as much as 13% of $12.49) Why does this make sense? ($25.99 is about $26 and $12.49 is close to $13, and 26 is twice as much as 13, so 13% of the first quantity will be about twice as much as 13% of the second)

Have students calculate the exact sales tax on the amounts above. **ASK:** My calculator tells me that 13% of $12.49 is 1.6237—what is the sales tax on a book that costs $12.49? (the sales tax is $1.62) Why do we need to round to two decimal places? (because the lowest coin we have is a penny, worth one hundredth of a dollar, we can’t pay .37 of a cent.)

Have students use a calculator to add a sales tax of 13% to the following prices. They should round the total price to two decimal places.

a) original price: $27.85
b) original price: $26.44
c) original price: $119.99
d) original price: $74.00

**ANSWERS:** a) $3.62  b) $3.42  c) $15.60  d) $9.62

**ASK:** Which question did you not need to round for? Why not? (d, because 13% of 74 has only 2 decimal places)

Solve the following problem together as a class: A book costs $18.49. The salesperson tells you that the total price is $22.37. If books are taxed at 13% of the cost of the book, is the total price reasonable?

**ASK:** What whole number price that is easy to work with is close to $18.49? (19 or 20) Tell students that you find 20 easier to work with, even though it is not as close to 18.49 as 19 is. Have students calculate 13% of $20. (10% is 2, 1% is 0.2, 3% is 0.6, and 13% is 10% + 3% = $2.60) **ASK:** The tax on $20 would be $2.60. Will the tax on $18.49 be more or less than $2.60? (less) Why? (because 13% of 18.49 is less than 13% of 20) To estimate the total cost of the book, students can add the tax on a $20-dollar book to $18.49
and know that the cost will be a little less than the answer: $18.49 + \$2.60 is about $18.50 + \$2.50 = \$21.00, so the total price of the book should be a little less than \$21. But the salesperson told you that the price is more than \$22! This is highly unreasonable. Have students use a calculator to determine the actual exact cost of the book. (13% of \$18.49 is \$2.40 so the total cost is \$20.89)

Either the salesperson made an honest mistake, or she is pocketing the extra money. **ASK:** If the salesperson is pocketing the extra money, how much did she pocket? ($22.37 − \$20.89 = \$1.48)

Have students determine, without using a calculator, which total prices (after taxes) are reasonable if the tax is 13%:

- a) original price: \$15.00  
  price after taxes: \$16.95
- b) original price: \$23.00  
  price after taxes: \$26.99
- c) original price: \$21.49  
  price after taxes: \$24.28
- d) original price: \$3.24  
  price after taxes: \$3.66
- e) original price: \$5.83  
  price after taxes: \$7.29
- f) original price: \$83.29  
  price after taxes: \$961.48

**ANSWERS:**

- a) 10% is \$1.50 and 3% is \$0.45, so 13% is \$1.95. Yes, \$16.95 is reasonable.
- b) 10% is \$2.30 and 3% is \$0.69, so 13% is about \$3.00, but the price quoted is nearly \$4 more than the original price. No, this is not reasonable.
- c) 10% is about \$2.10 and 3% is about \$0.60, so the total price should be about \$21.50 + \$2.70 or about \$24.20, so yes, this is reasonable.
- d) 10% is about \$0.32, and 3% is about \$0.09, so the total price should be about \$0.40 cents more than original, as it is, so the price given is reasonable.
- e) 10% is about \$0.60 and 3% is about \$0.18 so the total price should be less than \$0.80 more than the original price, but it is over a dollar more, so the price given is not reasonable.
- f) 10% is about \$83 and 3% is about \$25, so the total price should be about \$110 more than the original price, but it is about \$130 more, so the price given is not reasonable.

Have students use a calculator to check their estimates. Were they correct about which prices after taxes are reasonable?
ACTIVITY

The Honest Cashier

In this whole-class activity, five students will act as cashiers and the rest will be shoppers in a bookstore. One cashier is honest and calculates the total price of the books correctly all the time. The remaining cashiers calculate the total price incorrectly at different rates. This activity will give students lots of practice calculating prices after taxes (in other words, calculating percents). The five cashiers will not get this practice, so the game should be repeated with different cashiers.

Invite five volunteers to be the cashiers. The cashiers behave as follows:

Cashier 1 calculates the total price correctly all the time.

Cashier 2 adds $1.00 to the total price every time.

Cashier 3 calculates the total price correctly half the time, and adds $1.00 to the price the other half of the time.

Cashier 4 calculates the total price correctly one quarter of the time, and adds $1.00 to the price the other three quarters of the time.

Cashier 5 calculates the total price correctly three quarters of the time, and adds $1.00 to the price the other one quarter of the time.

Each cashier receives a copy of BLM Price Chart. The BLM lists 50 prices before and after taxes (the tax rate is 13%). The total prices have been calculated according to the roles listed above. On each cashier’s copy, the teacher circles the column of prices that the cashier should use. The cashiers can determine which cashier they are (1, 2, 3, 4, or 5) by looking at the list. Take the cashiers aside to discuss what makes this easy to determine.

Then have the cashiers stand around the room and invite the shoppers to make purchases. You can attach a different price to 50 different books, or just write the different prices on slips of paper. Shoppers should follow these rules:

1. If the cashier cheats the student, the student lines up at a different cashier.

2. If the cashier is honest, the student lines up again at the same cashier.

3. Once you enter a line, you cannot leave it even if you see other people ahead of you were cheated.

As students make purchases, they check the total price the cashier charged them to see if it is correct, and line up again (with the same or a different book), the line up for the honest cashier will get longer and the line ups for dishonest cashiers will get shorter. The goal is to have all the shoppers lined up at the honest cashier.

VARIATION: Students try to figure out which cashier is playing which role. They confer with each other about which cashiers cheated them to help them decide. All students must agree before they are allowed to make their guess.
Extensions

1. Sara says that to find 10% of a number, she divides the number by 10, so to find 5% of a number, she divides the number by 5. Is she right? Explain. (No—5% of a number is 5/100 or 1/20 of the number, so to find 5%, or 1/20, of the number, she should divide it by 20.)

2. Revisit this problem: A book costs $18.49, and the salesperson tells you that the total price is $22.37. If books are taxed at 13% of the cost, is the total price reasonable?

Tell students that sometimes it is easy to do such problems by using fractions as benchmarks. Since taxes are 13%, it is useful to use benchmarks that are near 13%. For example:

- 1/5 is 20%
- 1/6 is a little less than 17%
- 1/7 is a little more than 14%
- 1/8 is halfway between 12% and 13%.

To identify which benchmark is best to use in a particular case, think about what the number you are working with is easy to divide by. In this case, the cost of the book is about $18, and 18 is easily divided by 6. So using a benchmark of 1/6 would work well in this case. We know that 1/6 of $18 is $3, so if the tax rate is 17%, we expect to be taxed about $3. But we were taxed at 13% and the tax was almost $4, so right away we can see that the total price is unreasonable.

Have students redo problems a) – f) above (determining which prices are reasonable), using this new strategy.

ANSWERS:

a) $15.00 divided by 8 is almost $2, so yes, this is reasonable.

b) $23 divided by 8 is almost $3, but the price quoted is $4 more than the original price, so no, this is not reasonable.

c) $21 ÷ 7 is $3, so if the tax rate was 14%, we would expect about $3 in taxes. In fact, we were charged about $2.70 in taxes, which is reasonable since the tax rate is 13%.

d) $3.20 ÷ 8 = $0.40, so the total price should be about $0.40 more than the original, which it is, so the given price is reasonable.

e) 1/6 of $6 is $1, so if the original price was $6 and we were taxed at 17% we should expect $1 in taxes. In fact, we were charged more for a lower original price and a lower tax rate, so the given price is not reasonable.

f) The taxes we pay should be less than 832 ÷ 7, which in turn is less than 840 ÷ 7 = $120, so $120 should be more than the taxes we pay, but in fact we are paying more than that, so no the given price is not reasonable.
NS8-90  Writing Equivalent Statements for Proportions

Pages 18–19

CURRICULUM EXPECTATIONS
Ontario: 7m28, 8m1, 8m6, 8m27
WNCP: 7N3, 8N3, [R, CN]

VOCABULARY
proportion
equivalent ratio
percent

Goals
Students will write equivalent statements for proportions by keeping track of the part and the whole.

PRIOR KNOWLEDGE REQUIRED
Can write equivalent ratios
Can name a ratio from a picture

MATERIALS
BLM Three Types of Percent Problems (p M-108)

PROCESS EXPECTATION  Review equivalent ratios using pictures. The following picture shows that 6 : 9 = 2 : 3.

6 of the 9 circles are shaded.
2/3 of the 9 circles are shaded.
So 6 is 2/3 of 9.

Have students do Workbook Questions 1, 2, and 3. Have students write the ratio as part : whole.

PROCESS EXPECTATION  Writing ratios with missing parts. Have students write each number in the correct place in the proportion, but replace the missing number with a question mark.

a) 3 is 1/2 of what number? (1 : 2 = 3 : ?)
b) 4 is 1/3 of what number? (1 : 3 = 4 : ?)
c) 6 is 2/5 of what number? (2 : 5 = 6 : ?)
d) What number is 3/4 of 20? (3 : 4 = ? : 20)
e) What number is 4/5 of 20? (4 : 5 = ? : 20)
f) What number is 2/7 of 21? (2 : 7 = ? : 21)

Then have students write their answers in fraction form as well.

For example, the answer for part a) is $\frac{1}{2} = \frac{3}{?}$.

PROCESS EXPECTATION  Write on the board: 12 is how many fifths of 30? Underline “how many fifths” and point out that this is the same as “?/5.” The denominator tells you that the size of the parts is a fifth, and the numerator, the unknown, tells you the number of fifths. So “12 is how many fifths of 30” is another way of saying “12 is ?/5 of 30.” This is now easy to change to an equivalent ratio using the method of Workbook page 18 Question 3. (? : 5 = 12 : 30) Have students write equivalent ratios for these questions:
a) 8 is how many thirds of 12?
b) 21 is how many quarters of 28?
c) 18 is how many tenths of 30?

Again, have students write the fraction form as well.

**Writing percent statements in terms of ratios.** Remind students that asking how many hundredths is like asking for \(?/100\). **ASK:** What is another name for a fraction with denominator 100? **PROMPT:** What do we use to compare numbers to 100? (a percent) Since students can write fraction statements as equivalent ratios, and a percent is just a fraction with denominator 100, students can now write percent statements as equivalent ratios. Have students write proportions for these questions:

a) 19 is how many hundredths of 20?
b) 13 is how many hundredths of 50?
c) 36 is how many hundredths of 60?

Have students use their answers above to write proportions for these questions (without solving them):

a) 19 is what percent of 20?
b) 13 is what percent of 50?
c) 36 is what percent of 60?

Have students write the proportion in terms of fractions as well. For example, \(19/20 = ?/100\).

Have students write these questions as a proportion, in both ratio and fraction form:

a) What is 15% of 40? (? : 40 = 15 : 100, or \(?/40 = 15/100\))
b) What is 32% of 50? (? : 50 = 32 : 100, or \(?/50 = 32/100\))
c) What is 75% of 48? (? : 48 = 75 : 100, or \(?/48 = 75/100\))

Repeat for these problems:

a) 24 is 80% of what number?
b) 62 is 25% of what number?
c) 12 is 30% of what number?

And then combine all three types of questions above. (These are summarized on **BLM Three Types of Percent Problems**). Have students do Workbook page 19 Questions 5 and 6.

**Word Problems Practice:**
Have students use their answer to each problem below to obtain the answer to the next problem. Discuss the similarities and differences between each problem and the next.

a) 12 is how many fifths of 30?
b) How many fifths of 30 is 12?
c) 12 is what percent of 30?
d) What percent of 30 is 12?
e) A shirt costs $30, and $12 was taken off. What percent was taken off?
ANSWERS: a) 2  b) 2  c) 40  d) 40  e) 40

Have students solve these word problems.

1. A shirt costs $25. After taxes, it cost $30. What percent of the original price are the taxes?

2. A shirt costs $40. After taxes, it cost $46. At what rate was the shirt taxed?

3. A shirt costs $40. It was on sale for $28. What percent was taken off?

**Bonus**  A shirt costs $20. It was on sale at 15% off. A 15% tax was then added. What was the final price?

**ANSWER:** The sale price is $17. After a 15% tax, the price becomes $19.55.

**Extensions**

1. Include non-whole numbers as the value of a percent. **EXAMPLES:**
   a) What percent of 30 is 16.5?
   b) What percent of 18 is 2.7?
   c) What percent of 14 is 2.8?

2. Give word problems involving non-whole numbers as the value of a percent.
   a) A book that costs $18 came to $20.70 after taxes.
      i) How much were the taxes?
      ii) What percent is the tax?
   b) The regular price of a book is $18. The sale price is $12.60.
      i) How much was taken off the regular price?
      ii) What percent was taken off the regular price?
Review solving proportions. See Workbook page 20 Questions 1–3.

Using proportions to solve percent problems. Have students rewrite each question as a proportion in fraction form, and then solve the proportion:

- a) What percent of 60 is 12?
- b) What is 24% of 20?
- c) 6 is what percent of 20?
- d) 6 is 30% of what number?

Continue with proportions that require reducing one of the ratios before it can be solved.

- a) What percent of 80 is 16?
- b) What is 60% of 15?
- c) 9 is what percent of 30?
- d) 6 is 40% of what number?

When students do Workbook page 21 Question 8, note that they should be comparing the proportions in Question 7 to parts a–e of Question 4. (The proportions in Question 4 parts a–e can be solved directly without reducing either ratio; the proportions in Question 7 cannot be.)
Number Sense 8-92  Solving Percent Problems — Advanced
Pages 22–24

CURRICULUM EXPECTATIONS
Ontario: 8m1, 8m3, 8m5, 8m7, 8m17, 8m27, 8m28
WNCP: 8N3, [CN, R, ME, C]

VOCABULARY
cross-multiply

Goals
Students will cross-multiply to solve percent problems that involve proportions.

PRIOR KNOWLEDGE REQUIRED
Can convert a fraction a/b to a decimal by dividing \( a \div b \)
Can find equivalent fractions by multiplying the numerator and denominator by the same number
Can write a proportion to solve percent problems
Can write an equivalent multiplication statement for a given division statement

Review writing a fraction as a division statement. Remind students that we can calculate the value of a fraction like 3/4 by dividing 3 ÷ 4. For a quick reminder of why this is true, SAY: To find 1/4 of something, I would divide it into 4 equal groups, so to find 1/4 of something, divide it by 4. You can think of 1/4 as 1/4 of 1, so that is 1 ÷ 4. But 3/4 is 3 times as much as 1/4, so 3/4 is 3 × 1 ÷ 4 = 3 ÷ 4. Have students calculate:

a) 3/5  b) 5/8  c) 7/20  d) 3/10  e) 8/25

Writing fraction statements as equivalent multiplication statements.
Remind students that a division statement can be written as a multiplication statement. For example, 12 ÷ 3 = 4 can be rewritten as 12 = 3 × 4. Write on the board the answers to the previous questions:

a) 3 ÷ 5 = 0.6  b) 5 ÷ 8 = 0.625  c) 7 ÷ 20 = 0.35  d) 3 ÷ 10 = 0.3  e) 8 ÷ 25 = 0.32

Have students change the division statements to multiplication statements.

ANSWERS: a) 3 = 5 × 0.6  b) 5 = 8 × 0.625  c) 7 = 0.35 × 20  d) 3 = 10 × 0.3  e) 8 = 25 × 0.32

Have students change these fraction statements to division statements and then to multiplication statements:

a) \( \frac{7}{8} = 0.875 \)  b) \( \frac{1}{4} = 0.25 \)  c) \( \frac{17}{20} = 0.85 \)  d) \( \frac{4}{5} = 0.8 \)

To guide students, write on the board:

\[
\frac{\text{_____}}{\text{_____}} = \text{_____} \\
\text{so} \quad \text{_____} = \text{_____} \times \text{_____}
\]

ANSWERS:

a) \( 7 \div 8 = 0.875 \)  b) \( 1 \div 4 = 0.25 \)  c) \( 17 \div 20 = 0.85 \)  d) \( 4 \div 5 = 0.8 \)

so \( 7 = 8 \times 0.875 \)  so \( 1 = 4 \times 0.25 \)  so \( 17 = 20 \times 0.85 \)  so \( 4 = 5 \times 0.8 \)
Now have students look at their answers above. **ASK:** If we know the value of a fraction as a decimal, how can we use that to write a multiplication statement? Write $\frac{7}{4} = \big(\big)$ on the board. **ASK:** Where does the numerator—the top number—of the fraction go? In which blank? (the first one) What number goes in the second blank, the denominator or the value? (It doesn’t matter, because multiplication commutes) Explain that when you know the decimal value of a fraction, the numerator of the fraction can be written as the product of the denominator and the decimal value.

Have students write the numerators of these fractions as products:

a) $\frac{7}{4} = \big(\big)$  
**ANSWERS:** a) $7 = 4 \times 1.75$ (or $7 = 1.75 \times 4$)  
b) $\frac{9}{20} = \big(\big)$  
**ANSWERS:** b) $9 = 20 \times 0.45$
c) $\frac{7}{35} = \big(\big)$  
**ANSWERS:** c) $7 = 35 \times 0.2$  
d) $\frac{9}{25} = \big(\big)$  
**ANSWERS:** d) $9 = 25 \times 0.36$

Now have students calculate the value of each fraction and then write a multiplication statement:

a) $\frac{2}{5}$  
**ANSWERS:** a) $2 = 5 \times 0.4$  
b) $\frac{9}{10}$  
**ANSWERS:** b) $9 = 10 \times 0.9$  
c) $\frac{21}{25}$  
**ANSWERS:** c) $21 = 25 \times 0.84$  
d) $\frac{19}{20}$  
**ANSWERS:** d) $19 = 20 \times 0.95$

**Writing fraction statements that involve variables as a product.** Write on the board: $\frac{10}{x} = 2$. **SAY:** I don’t know what number $x$ is, but I know that whatever it is, 2 times $x$ is equal to 10. Write on the board: $2x = 10$.

Have students change each of the following equations to an equation that involves multiplication instead of division.

a) $\frac{24}{x} = 2$  
**ANSWERS:** a) $24 = 2x$  
b) $\frac{24}{x} = 3$  
**ANSWERS:** b) $24 = 3x$  
c) $\frac{24}{x} = 4$  
**ANSWERS:** c) $24 = 4x$  
d) $\frac{x}{3} = 5$  
**ANSWERS:** d) $x = 15$  
e) $\frac{x}{6} = 5$  
**ANSWERS:** e) $x = 30$  
f) $\frac{x}{7} = 8$  
**ANSWERS:** f) $x = 56$
g) $\frac{8}{2} = x$  
**ANSWERS:** g) $x = 4$  
h) $\frac{15}{3} = x$  
**ANSWERS:** h) $x = 5$  
i) $\frac{18}{2} = x$  
**ANSWERS:** i) $x = 9$  
j) $\frac{18}{3} = x$  
**ANSWERS:** j) $x = 6$  
k) $\frac{33}{3} = x$  
**ANSWERS:** k) $x = 11$  
l) $\frac{x}{2} = 15$

**Solving equations that involve fractions.** Have students solve the equations they found above.

**ANSWERS:** a) $x = 12$  
b) $x = 8$  
c) $x = 6$  
d) $x = 15$  
e) $x = 30$
f) $x = 56$  
g) $x = 4$  
h) $x = 5$  
i) $x = 9$  
j) $x = 6$  
k) $x = 11$
l) $x = 30$

Have students rewrite the following equations so that they involve multiplication, and then solve for $x$. Encourage students to check their answers by substitution.

a) $\frac{20}{x} = 5$  
**ANSWERS:** a) $20 = 5x$  
b) $\frac{x}{6} = 7$  
**ANSWERS:** b) $x = 42$  
c) $\frac{26}{x} = x$  
**ANSWERS:** c) $26 = x^2$  
d) $\frac{60}{x} = 15$  
**ANSWERS:** d) $60 = 15x$  
e) $\frac{x}{7} = 9$
**ANSWERS:** e) $x = 63$  
f) $\frac{48}{8} = x$
**ANSWERS:** f) $x = 6$
ANSWERS:

a) $20 = 5x$ so $x = \frac{20}{5} = 4$

b) $x = 6 \times 7 = 42$

c) $2x = 26$, so $x = \frac{26}{2} = 13$

d) $60 = 15x$, so $x = \frac{60}{15} = 4$

e) $x = 7 \times 9 = 63$

f) $48 = 8x$ so $x = \frac{48}{8} = 6$

Point out that c) and f) do not even need to be rewritten, as they can be done directly. For example, c) says directly that $26 \div 2 = x$, so we don’t even need to write first that $26 = 2x$.

Changing an equation of division statements to an equation of multiplication statements. Write on the board: $3 \div 5 = 12 \div 20$. Have students verify this equation by doing long division. Tell students that you find it easier to work with multiplication than with division. SAY: I would like to be able to verify this equality by using multiplication instead of division, and I saw a trick that lets me change the equation so I can do that. Work through the steps below:

\[
\begin{align*}
3 \div 5 &= 12 \div 20 \\
5 \times 3 \div 5 &= 12 \div 20 \times 5 \\
3 &= 12 \div 20 \times 5 \\
20 \times 3 &= 12 \times 20 \div 20 \times 5 \\
20 \times 3 &= 12 \times 5
\end{align*}
\]

Point out that we have now created an equation of multiplication statements instead of division statements. Have students verify the equation by using multiplication. ASK: Was it easier to use multiplication or division to verify the equation? (the multiplication)

Have students work through the steps above to change these equations to equations that involve multiplication statements:

a) $2 \div 3 = 8 \div 12$

b) $2 \div 5 = 6 \div 15$

c) $5 \div 9 = 10 \div 18$

d) $3 \div 8 = 9 \div 24$

ANSWERS:

a) $2 \times 12 = 3 \times 8$

b) $2 \times 15 = 5 \times 6$

c) $5 \times 18 = 9 \times 10$

d) $3 \times 24 = 8 \times 9$

Cross-multiplying to verify equivalent fractions. Point out that the fractions $3/5$ and $12/20$ are equivalent fractions. ASK: How do I know?
PROMPT: What number can we multiply both the numerator and denominator by in $3/5$ to get $12/20$? (multiply 3 by 4 to get 12 and 5 by 4 to get 20) Write on the board:

\[
\frac{3 \times 4}{5 \times 4} = \frac{12}{20}
\]

Remind students that we can write the fractions as division statements. Write on the board: $3 \div 5 = 12 \div 20$. ASK: How can we change this to an equation with multiplication instead? (we did it above—it was $20 \times 3 = 12 \times 5$) Have students change the following equivalent fractions to equivalent division statements and then to equivalent multiplication statements.

a) $\frac{3}{4} = \frac{15}{20}$

b) $\frac{3}{5} = \frac{9}{15}$

c) $\frac{7}{9} = \frac{21}{27}$

d) $\frac{3}{8} = \frac{15}{40}$
Sample Solution for a):

3 ÷ 4 = 15 ÷ 20 since 3/4 = 15/20
4 × 3 ÷ 4 = 4 × 15 ÷ 20 multiply both sides by 4
3 = 4 × 15 ÷ 20 rewrite the left side
20 × 3 = 4 × 15 ÷ 20 multiply both sides by 20
20 × 3 = 4 × 15 rewrite the right side

Have students look at their answers to the four questions above. Ask:

How can you find which numbers to multiply together from the fractions?

Prompt: Do you multiply both numerators together? (no) What do you multiply together? (the numerator of one with the denominator of the other)

Go through each one, point to the answer, and verify that this is indeed what students did for each question—join the numerator of each fraction with the denominator of the other to emphasize this point. Tell students that because the products from equivalent fractions can be found by drawing an X, we call this process cross-multiplying. See the box near the bottom of Workbook page 22.

Have students verify that the fractions in each pair above are in fact equivalent by verifying that the products they found are equal.

Answers:

a) 2 × 12 = 3 × 8  b) 2 × 15 = 5 × 6  c) 5 × 18 = 9 × 10
24 = 24  30 = 30  90 = 90

d) 3 × 24 = 8 × 9  e) 3 × 20 = 4 × 15  f) 3 × 15 = 5 × 9
72 = 72  60 = 60  45 = 45

g) 7 × 27 = 9 × 21  h) 3 × 40 = 8 × 15
189 = 189  120 = 120

Have students verify that each pair of fractions below is equivalent by using cross-multiplication.

Answers:

a) 3 = 6  b) 4 = 12  c) 3 = 15  Bonus  d) 19 = 133
7 14 5 15 8 40

Cross-multiply to identify equivalent fractions. Have students decide whether the fractions in each pair are equivalent by multiplying the numerator of each fraction with the denominator of the other and checking if the two products are equal or not.

Answers: a) not equivalent  b) equivalent  c) not equivalent
d) not equivalent  e) not equivalent  f) equivalent  g) equivalent

Bonus not equivalent
Cross-multiply to solve equations. Show students how to cross-multiply to write an equation when there is a variable in one of the fractions. For example, \( \frac{6}{x} = \frac{3}{5} \) becomes \( 6 \times 5 = 3x \), which becomes \( 30 = 3x \). This says that 30 and 3x are the same number, so we can write \( 3x = 30 \). Tell students that if they find an equation with the variable on the right side, they can always rewrite the equation so that the variable is on the left side. Have students rewrite these equations so that the variable is on the left side:

a) \( 20 = 4x \)  

**ANSWERS:** a) \( 4x = 20 \)  

b) \( 27 = 2 + 5x \)  

c) \( 8 = 3x - 4 \)  

**ANSWERS:** a) \( 4x = 20 \)  

b) \( 2 + 5x = 27 \)  

c) \( 3x - 4 = 8 \)  

Have students write equations from these proportions. (Students should rewrite the equation so that the variable is on the left side, if necessary.)

a) \( \frac{5}{x} = \frac{10}{8} \)  

b) \( \frac{x}{9} = \frac{2}{6} \)  

c) \( \frac{15}{6} = \frac{x}{2} \)  

d) \( \frac{2}{3} = \frac{8}{x} \)  

**ANSWERS:** a) \( 10x = 5 \times 8 = 40 \)  

b) \( 6x = 18 \)  

c) \( 6x = 30 \)  

d) \( 2x = 24 \)  

Have students solve the equations they found. Encourage students to check their answers by substituting the answer they found for \( x \) into the original equation.

**ANSWERS:** a) \( x = 4 \)  

b) \( x = 3 \)  

c) \( x = 5 \)  

d) \( x = 12 \)  

Sample check for a): \( 5/4 = 10/8 \) is true because \( 5 \times 2 = 10 \) and \( 4 \times 2 = 8 \).

Have students cross-multiply to solve for \( x \), then check their answers by substituting their answer for \( x \) to see if the fractions are equivalent.

a) \( \frac{3}{5} = \frac{x}{15} \)  

b) \( \frac{x}{8} = \frac{5}{2} \)  

c) \( \frac{12}{20} = \frac{3}{x} \)  

d) \( \frac{18}{8} = \frac{9}{x} \)  

**ANSWERS:** a) \( 5x = 45 \), so \( x = 9 \)  

b) \( 2x = 40 \), so \( x = 20 \)  

c) \( 12x = 60 \), so \( x = 5 \)  

d) \( 18x = 72 \), so \( x = 4 \)  

Sample check: a) \( 3/5 = 9/15 \) is true because multiplying each of the numerator and denominator of 3/5 by 3 results in 9/15 (\( 3 \times 3 = 9 \) and \( 5 \times 3 = 15 \)).

**Bonus**

a) \( \frac{3x}{8} = \frac{15}{20} \)  

b) \( \frac{12}{45} = \frac{32}{5x} \)  

c) \( \frac{9}{x^2} = \frac{1}{4} \)  

d) \( \frac{1}{45} = \frac{20}{x^2} \)  

e) \( \frac{x}{20} = \frac{5}{x} \)  

f) \( \frac{2}{x} = \frac{18}{x} \)  

g) \( \frac{6}{7x} = \frac{3x}{14} \)  

h) \( \frac{5}{3x} = \frac{2x}{270} \)  

**ANSWERS:** a) \( 2 \)  

b) \( 24 \)  

c) \( 6 \)  

d) \( 30 \)  

e) \( 10 \)  

f) \( 6 \)  

g) \( 2 \)  

h) \( 15 \)  

Cross-multiplying when the answers are decimal numbers. Tell students to again cross-multiply to solve for \( x \), but this time their answers will be decimal numbers. This means that it is not equivalent fractions they are comparing, but equivalent ratios. Remind students that we can write ratios in fraction form even when both terms are not whole numbers. Tell students to round their answers to one decimal place.

a) \( \frac{x}{3} = \frac{5}{6} \)  

b) \( \frac{10}{3} = \frac{3}{x} \)  

c) \( \frac{7}{5} = \frac{x}{6} \)  

d) \( \frac{7}{x} = \frac{3}{5} \)  

e) \( \frac{9}{5} = \frac{2}{x} \)  

f) \( \frac{6}{11} = \frac{5}{x} \)  

**PROCESS EXPECTATION**

Representing

Number Sense 8-92  

M-61
If students are familiar with repeating decimals, they can write the exact answer to these questions.

**Bonus** Solve for \( x \) and write each answer to two decimal places.
Use a calculator.

\[
\begin{align*}
&\text{a) } \frac{2}{3x} = \frac{5}{7} \quad \text{b) } \frac{1}{7} = \frac{5}{x^2} \\
&\text{c) } \frac{5}{x} = \frac{x}{9} \quad \text{d) } \frac{3}{2x} = \frac{x}{10}
\end{align*}
\]

**ANSWERS:**
\[
\begin{align*}
&\text{a) } 15x = 14 \text{ so } x = \frac{14}{15} = 0.93 \\
&\text{b) } x^2 = 35 \text{ so } x = \sqrt{35} \approx 5.92 \\
&\text{c) } x^2 = 45 \text{ so } x = \sqrt{45} \approx 6.71 \\
&\text{d) } 2x^2 = 30 \text{ so } x^2 = 15. \text{ Then } x = \sqrt{15} \approx 3.87
\end{align*}
\]

**Solving percent problems using proportions.** Review writing a proportion to solve a percent problem and then demonstrate how using cross-multiplication makes the problem easy. Use the teaching box on Workbook page 23 to demonstrate. Have students practice solving percent problems using cross-multiplication by doing Workbook Questions 8–19.

**Extensions**

1. Solve for \( x \).

\[
\begin{align*}
&\text{a) } \frac{3}{x+5} = \frac{1}{4} \\
&\text{b) } \frac{3x+4}{20} = \frac{4}{5}
\end{align*}
\]

**ANSWERS:**
\[
\begin{align*}
&\text{a) } 3 \times 4 = x + 5, \text{ so } 12 = x + 5. \text{ Rewrite the equation so that the variable is on the left: } x + 5 = 12, \text{ so } x = 7 \\
&\text{b) } 5(3x + 4) = 4 \times 20 \quad \text{OR} \quad 5(3x + 4) = 4 \times 20 \\
&\quad 15x + 20 = 8 \quad \text{OR} \quad 5(3x + 4) = 80 \\
&\quad 15x + 20 - 20 = 80 - 20 \quad \text{OR} \quad 5(3x + 4) \div 5 = 80 \div 5 \\
&\quad 15x = 60 \quad \text{OR} \quad 3x + 4 = 16 \\
&\quad x = 4 \quad \text{OR} \quad 3x + 4 - 4 = 16 - 4 \\
&\quad \quad \quad \quad \quad \quad \text{OR} \quad 3x = 12 \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \text{OR} \quad x = 4
\end{align*}
\]

2. Have students investigate this question: If fractions \( \frac{a}{b} \) and \( \frac{c}{d} \) are equivalent, what fraction is \( \frac{a}{c} \) equivalent to?

Write on the board:
\[
\frac{3}{5} = \frac{6}{10} \text{ so } 3 \times 10 = 5 \times 6
\]

\[
\frac{?}{5} = \frac{?}{10} \text{ so } 3 \times 10 = 6 \times 5
\]

Have students decide what fractions we can use to get \( 3 \times 10 = 6 \times 5 \) the same way we used the fractions \( \frac{3}{5} \) and \( \frac{6}{10} \) to get \( 3 \times 10 = 5 \times 6 \).
Suggest that students look for where each part of each fraction goes to in the equation and compare how the equations are different. **PROMPT:** Which numbers switched positions, and which numbers stayed in the same position?

**ANSWER:** 3 and 10 are in the same position, but 5 and 6 get switched. So the corresponding fractions are \( \frac{3}{6} = \frac{5}{10} \).

Then have students do more questions of this sort, that is, make a new pair of equivalent fractions by cross-multiplying and then using the commutative property of multiplication for one of the products (not both!) to make a new pair of equivalent fractions.

\[
a) \quad \frac{2}{3} = \frac{6}{9} \quad \text{so} \quad ____ = ____ \\
b) \quad \frac{1}{4} = \frac{5}{20} \quad \text{so} \quad ____ = ____ \\
c) \quad \frac{3}{5} = \frac{9}{15} \quad \text{so} \quad ____ = ____ \\
d) \quad \frac{a}{b} = \frac{c}{d} \quad \text{so} \quad ____ = ____
\]

**ANSWERS:**

\[
a) \quad \frac{2}{6} = \frac{3}{9} \\
b) \quad \frac{1}{5} = \frac{4}{20} \\
c) \quad \frac{3}{9} = \frac{5}{15} \\
d) \quad \frac{a}{c} = \frac{b}{d}
\]

Emphasize to students that to find the second pair of equivalent fractions they can read the numbers from the first pair across, from left to right.

Tell students that you know someone who changed the fractions in part b to 1/20 = 5/4. **ASK:** How can you tell immediately that this is wrong? (1/20 is less than 1, but 5/4 is more than 1)

3. This extension emphasizes the importance of checking the answer by substitution.

a) Can \( \frac{x}{3} = \frac{x}{5} \) be solved? Explain.  
   b) Can \( \frac{3}{x} = \frac{5}{x} \) be solved? Explain.

**ANSWERS:**

a) Cross-multiply to obtain 5x = 3x. Then 2x = 0, so x = 0. Checking in the original equation, this works.

b) Cross-multiplying gives the same equation, so again x = 0. But checking in the original equation results in 3/0 = 5/0. These fractions do not make sense! This equation cannot be solved.

4. In this lesson, we derived the rule for cross-multiplication by using the fact that a fraction can be rewritten as a division statement. For example, consider the equivalent fractions \( \frac{3}{4} = \frac{6}{8} \). To derive the cross-multiplication rule \( 3 \times 8 = 4 \times 6 \), we started with \( 3 \div 4 = 6 \div 8 \).

Tell students that we can get the same cross-multiplication rule by starting instead with the definition of equivalent fractions. If two fractions are equivalent, we can multiply the numerator and denominator of one fraction by the same number (not necessarily a whole number) to get the numerator and denominator of the other.
\[
\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8}
\]
This says that \(6 \div 3\) and \(8 \div 4\) are both 2. In general, the numerator of the second fraction divided by the numerator of the first fraction is equal to the denominator of the second fraction divided by the denominator of the first fraction:
\[
6 \div 3 = 8 \div 4
\]
Have students use these equivalent division statements to find equivalent multiplication statements, using steps similar to those used in the lesson. **ASK:** Do you get the same equality of products?

**ANSWER:**
\[
6 \div 3 = 8 \div 4
\]
\[
3 \times 6 \div 3 = 3 \times 8 \div 4 \quad \text{multiply both sides by 3}
\]
\[
6 = 3 \times 8 \div 4 \quad \text{rewrite the left side}
\]
\[
6 \times 4 = 3 \times 8 \div 4 \times 4 \quad \text{multiply both sides by 4}
\]
\[
6 \times 4 = 3 \times 8 \quad \text{rewrite the right side}
\]
This is the same equality of products as before—we still multiply the numerator of one by the denominator of the other.
Goals
Students will represent decimal percents between 0% and 1% visually, on a grid, and will convert between decimal percents, decimals, and fractions.

PRIOR KNOWLEDGE REQUIRED
Can convert between decimals, fractions, and whole-number percents
Can visualize whole-number percents on a hundreds grid

MATERIALS
BLM Thousandths Grid (p M-109)
BLM Thousandths Grid Practice (pp M-110–M-111)

Decimal percents. Write on the board:
\[
\frac{5}{200}
\]
Tell students that you want to write this as a percent. **ASK**: Is this more or less than 2%? **PROMPT**: Write 2% as a fraction with denominator 200. (2% = \(\frac{2}{100} = \frac{4}{200}\) is less than \(\frac{5}{200}\)) **ASK**: Is \(\frac{5}{200}\) more or less than 3%? (3% = \(\frac{6}{200}\) is more than \(\frac{5}{200}\)) Tell students that \(\frac{5}{200}\) is between 2% and 3%. In fact, it is halfway between the two percents. **ASK**: What number is halfway between 2 and 3? (2 1/2 or 2.5) Tell students that \(\frac{5}{200}\) is 2 1/2% or 2.5%. Write on the board:
\[
\frac{5}{200} = \frac{2.5}{100}
\]
Remind students that a percent is just a ratio that compares a number to 100. Since 5 : 200 is an equivalent ratio to 2.5 : 100, \(\frac{5}{200} = 2.5\%\).

**PROCESS EXPECTATION**
Looking for a pattern

**Writing fractions as percents.** Show students the following pattern:
\[
\begin{align*}
100\% & = 1 \\
10\% & = \frac{1}{10} \\
1\% & = \frac{1}{100} \\
\frac{1}{1000} & = \frac{1}{1000}
\end{align*}
\]
Ask students to decide what number goes in the blank and to explain their answer in their notebook. **ANSWER**: 0.1 because every number is being divided by 10.
Show students how to write a fraction with denominator 1000 as a percent: write the fraction as a decimal and then multiply by 100. **EXAMPLE:**

\[
\frac{17}{1000} = 0.017 = 1.7\%.
\]

Ask students to write these fractions as percents:

\[
\begin{align*}
a) & \quad \frac{3}{1000} \\
b) & \quad \frac{23}{1000} \\
c) & \quad \frac{7}{1000} \\
d) & \quad \frac{47}{1000} \\
e) & \quad \frac{347}{1000} \\
f) & \quad \frac{804}{1000}
\end{align*}
\]

**ANSWERS:** a) 0.3% b) 2.3% c) 0.7% d) 4.7% e) 34.7% f) 80.4%

**Review writing decimal tenths and hundredths as percents.** Write decimals as fractions with denominator 100 and then as percents.

**EXAMPLES:**

\[
\begin{align*}
a) & \quad 0.73 \\
b) & \quad 0.4 \\
c) & \quad 0.42 \\
d) & \quad 0.8 \\
e) & \quad 0.15
\end{align*}
\]

**ANSWERS:**

\[
\begin{align*}
a) & \quad \frac{73}{100} = 73\% \\
b) & \quad 40\% \\
c) & \quad 42\% \\
d) & \quad 80\% \\
e) & \quad 15\%
\end{align*}
\]

Emphasize that students are just multiplying the decimal by 100 and writing the result as a percent. We can do this because a percent is one hundredth, so to write a decimal number as a percent, we write how many hundredths are in the decimal. To find how many hundredths are in the decimal, multiply the decimal by 100.

**Writing decimal thousandths as decimal percents.** Write decimals as ratios of a number to 100, and then as decimal percents. **EXAMPLES:**

\[
\begin{align*}
a) & \quad 0.003 \\
b) & \quad 0.037 \\
c) & \quad 0.437 \\
d) & \quad 0.307 \\
e) & \quad 0.512 \\
f) & \quad 0.904
\end{align*}
\]

**ANSWERS:**

\[
\begin{align*}
a) & \quad 3 : 1000 = 0.3 : 100 = 0.3\% \\
b) & \quad 3.7\% \\
c) & \quad 43.7\% \\
d) & \quad 30.7\% \\
e) & \quad 51.2\% \\
f) & \quad 90.4\%
\end{align*}
\]

Emphasize again that students are just multiplying the decimal number by 100 to write the decimal as a percent. Again, this makes sense because the number of percents in a decimal is the number of hundredths in the decimal. Have students write these decimals as percents:

\[
\begin{align*}
a) & \quad 0.401 \\
b) & \quad 0.5 \\
c) & \quad 0.007 \\
d) & \quad 0.04 \\
e) & \quad 0.081 \\
f) & \quad 0.492
\end{align*}
\]

**ANSWERS:**

\[
\begin{align*}
a) & \quad 40.1\% \\
b) & \quad 50\% \\
c) & \quad 0.7\% \\
d) & \quad 4\% \\
e) & \quad 8.1\% \\
f) & \quad 49.2\%
\end{align*}
\]

**Writing decimal percents as decimals.** **SAY:** To write a decimal as a percent, we have to multiply the decimal by 100. **ASK:** If we start with the percent, how can we write it as a decimal? **PROMPT:** How can we get back where we started after we multiply by 100? What operation undoes multiplication? (division, so divide by 100 to write a percent as a decimal) Have students write these percents as decimals:

\[
\begin{align*}
a) & \quad 38.4\% \\
b) & \quad 0.9\% \\
c) & \quad 6.5\% \\
d) & \quad 10.5\% \\
e) & \quad 0.05\% \\
f) & \quad 0.36\%
\end{align*}
\]
ANSWERS:
a) 0.384  
b) 0.009  
c) 0.065  
d) 0.105  
e) 0.0005  
f) 0.0036

Point out that by writing a percent as a decimal, we are changing units from percents to ones. The new units are one hundred times larger, so we need 100 times fewer units. So, for example, 38.4% = 38.4 hundredths = 0.384 ones.

Percents of a thousandths grid. Photocopy BLM Thousandths Grid onto an overhead transparency. Project the BLM onto the board and shade 34.5% of the grid. ASK: How many thousandths are shaded? (345 thousandths) Write on the board: 0.345 of the grid is shaded. Have students write the shaded amount as a percent of the grid. Then shade various other amounts (EXAMPLES: 1%, 3%, 3.8%, 15.8% 40.9%, 62.7%) and have students write each shaded amount as a percent.

Writing fractional percents as decimals. Write on the board: 21%. ASK: How would you write this percent as a fraction? (21/100) Repeat for 22% (22/100). Now write on the board: $21\frac{1}{3}$%. ASK: Can we just write this number with denominator 100 to write it as a fraction? (no) Why not? (because it is not a whole number and both the numerator and denominator of a fraction must be whole numbers) ASK: What is $21\frac{1}{3}$ as an improper fraction? (64/3) Remind students that to write a percent as a decimal, we divide the number of percents by 100. We can do the same to write a percent as a fraction—divide the number of percents by 100. ASK: What is $64/3 \div 100$? (64/300) Have students write this fraction in lowest terms. (16/75) Have students write the following fractional percents as fractions in lowest terms.

a) $37\frac{1}{2}$%  
b) $5\frac{3}{8}$%  
c) $16\frac{2}{3}$%  
d) $83\frac{1}{3}$%

ANSWERS:
a) $37\frac{1}{2}$% = $75/2$% = $75/200$ = $3/8$
b) $5\frac{3}{8}$% = $43/8$% = $43/800$
c) $16\frac{2}{3}$% = $50/3$% = $50/300$ = $1/6$
d) $83\frac{1}{3}$% = $250/3$% = $250/300$ = $5/6$

Finally, have students write various fractional and decimal percents in reduced fraction form. See Workbook page 25 Question 5.
Review adding and subtracting percents. Remind students that a percent is just a hundredth, so if they know how to add and subtract hundredths, then they know how to add and subtract percents. Have students add and subtract the hundredths, fractions, and percents below.

a) \(18 \text{ hundredths} + 42 \text{ hundredths} = \) hundredths
\[
\frac{18}{100} + \frac{42}{100} = \frac{60}{100} = 60\%
\]

b) \(43 \text{ hundredths} - 13 \text{ hundredths} = \) hundredths
\[
\frac{43}{100} - \frac{13}{100} = \frac{30}{100} = 30\%
\]

c) \(47 \text{ hundredths} + 29 \text{ hundredths} = \) hundredths
\[
\frac{47}{100} + \frac{29}{100} = \frac{76}{100} = 76\%
\]

d) \(12 \text{ hundredths} + 18 \text{ hundredths} = \) hundredths
\[
\frac{12}{100} + \frac{18}{100} = \frac{30}{100} = 30\%
\]

e) \(45 \text{ hundredths} - 36 \text{ hundredths} = \) hundredths
\[
\frac{45}{100} - \frac{36}{100} = \frac{9}{100} = 9\%
\]

ANSWERS:

a) 60 hundredths, 60/100, 60%  
b) 30 hundredths, 30/100, 30%  
c) 76 hundredths, 76/100, 76%  
d) 30 hundredths, 30/100, 30%  
e) 9 hundredths, 9/100, 9%
Have students solve these problems:

a) $30\% + 40\% = \underline{\phantom{0000}}$

b) $32\% + 27\% = \underline{\phantom{0000}}$

c) $46\% + 38\% = \underline{\phantom{0000}}$

d) $20\% + \underline{\phantom{0000}} = 60\%$

e) $24\% + \underline{\phantom{0000}} = 37\%$

f) $38\% + \underline{\phantom{0000}} = 72\%$

g) $50\% - 10\% = \underline{\phantom{0000}}$

h) $78\% - 24\% = \underline{\phantom{0000}}$

i) $62\% - 45\% = \underline{\phantom{0000}}$

j) $43\% - \underline{\phantom{0000}} = 18\%$

k) $34\% - 12\% + 56\% = \underline{\phantom{0000}}$

ANSWERS: a) 70%   b) 59%   c) 84%   d) 40%   e) 13%

f) 34%   g) 40%   h) 54%   i) 17%   j) 25%   k) 78%

Introduce percents greater than 100%. Remind students that fractions can represent more than one whole. Have students change these fractions to percents:

a) $\frac{3}{5}$   b) $\frac{4}{5}$   c) $\frac{3}{5} + \frac{4}{5} = \frac{7}{5}$

ANSWERS: a) 60%   b) 80%   c) 60% + 80% = 140%

Point out that, just as fractions can represent a number more than 1, so can percents. Tell students that some of them may already have seen percents greater than 100% if they answered correctly a bonus question and all the other questions on a math test.

Converting between fractions, decimals, and percents. Remind students that percents can be written in decimal form by dividing the percent by 100. So, $140\% = 1.4$. Have students write the following fractions as percents, add the percents, and then write their answer in decimal form.

a) $\frac{3}{10} + \frac{8}{10}$   b) $\frac{5}{8} + \frac{7}{8}$

Bonus Write the fractions as percents to the nearest tenth, add the percents, and change the answer to a decimal: $\frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{4}{7} + \frac{5}{7} + \frac{6}{7}$

ANSWERS: a) 30% + 80% = 110% = 1.1   b) 62.5% + 87.5% = 150% = 1.5

Bonus 14.3% + 28.6% + 42.9% + 57.1% + 71.4% + 85.7% = 300% = 3

Have students add the fractions and change the answers to decimals. Do they get the same answers? Which way was easier, and why?

ANSWERS: a) $\frac{11}{10} = 1.1$   b) $\frac{12}{8} = 1.5$   Bonus $\frac{21}{7} = 3$ (Yes, all answers are the same. Adding the fractions first is easier because you only have to divide once at the end, instead of dividing for each addend.)

Visualizing percents. Draw a line one metre long and tell students that you have drawn 50% of a line. ASK: How would I draw the whole line, that is, 100%? (double it) How do you know? (because 100 is twice as much as 50) How would I draw 150%? (make the line three times as long) How do you know? (because 150 is three times as much as 50) Demonstrate drawing both lines. Then have students draw a line 3 cm long in their notebook.
Tell students that the line is 30%. Have students draw 120%. The have students draw the following lines:

a) A line 4 cm long that is 75%. What is 150%? (8 cm)
b) A line 2 cm long that is 50%. What is 200%? (8 cm)
c) A line 5 cm long that is 60%. What is 150%? (12.5 cm)

**Hint:** What is 30%?

**Convert between ratios, fractions, percents, and decimals.** Remind students that a percent is a ratio that compares a number to 100. Then have students do Questions 8–14 on Workbook pp. 26–27.

**Determining percents mentally.** Tell students that you want to know what 300% of 12 is. **ASK:** What is 100% of 12? (12) How can I calculate 300% from 100%? (multiply by 3) So what is 300% of 12? (36) How can I calculate 110% of 12? **PROMPT:** Split 110% as a sum of two smaller and easy-to-compute percents (110% = 100% + 10%). **ASK:** What is 10% of 12? (1.2) What is 100% + 10% of 12? (12 + 1.2 = 13.2) Have students calculate these percents:

a) 10% of 15 
   b) 10% of 13 
   c) 10% of 18

30% of 15  
200% of 13  
5% of 18

130% of 15  
210% of 13  
105% of 18

d) 230% of 5  
e) 140% of 16  
f) 305% of 4  
g) 820% of 3

**ANSWERS:**

a) 1.5, 4.5, 19.5  
b) 1.3, 26, 27.3  
c) 1.8, 0.9, 18.9

d) 11.5  
e) 22.4  
f) 12.2  
g) 24.6

**Have students check their answers on a calculator by changing the percents to decimals.** **SAMPLE ANSWER:** f) 3.05 × 4 = 12.2

**Finding 100% mentally.** Have students calculate these percents:

a) If 40% is 60,  
b) If 5% is 8,  
c) If 150% is 300

then 10% is _____  
than 10% is _____  
then 50% is _____

and 100% is _____  
and 100% is _____

d) If 200% is 80, what is 100%?  
e) If 120% is 6, what is 100%?  
f) If 450% is 9, what is 100%?  
g) If 160% is 80, what is 100%

**ANSWERS:**

a) 15, 150  
b) 16, 160  
c) 100, 200  
d) 40

e) 5  
f) 2  
g) 50

**Estimating percents.** **ASK:** About what percent of 35 is 68? **PROMPT:** Is 68 more or less than 35? (more) Will 68 be more or less than 100% of 35? (more) What is 200% of 35? (70) Is 68 close to 70? (yes) Explain that 68 is close to 200% of 35, so 195% is a good estimate.

Show students how to visualize this on a number line:

0%  100%  200%

0  35  70
**ASK:** What is 20% of 35? **PROMPT:** What is 1/5 of the way from 0% to 100%? (7 is 1/5 of 35, so 7 is 20% of 35) Have students fill in the number line by counting by 7s. Then **ASK:** Where is 68 on the number line? Does 195% look like a good estimate? (yes) Have students find the exact percent to determine how good the estimate is. (68 ÷ 35 ≈ 1.943 = 194.3%) **ASK:** What percent of 5 is 8? Draw a number line to help students visualize the numbers:

![Number Line]

Since 8 is 3/5 of the way from 100% to 200%, 8 is 160% of 5. Tell students that visualizing the numbers in their heads is a good way to estimate what percent one number is of another. Have students estimate the following percents of numbers, then check by calculating the actual percents (round to one decimal place):

a) What percent of 3 is 16?  
b) What percent of 3.5 is 10.3?  
c) What percent of 8 is 15?  
d) What percent of 54 is 100?

**ANSWERS:** The actual percents are: a) 533.3%, b) 294.3%, c) 187.5%, and d) 185.2%. If any of their estimates were not very close to the actual percent, can students explain where they might have made a mistake in their estimation, or what they could have done differently to get a better estimate?

**A context for percents greater than 100%**. Tell students that you are comparing the population growth of three cities over the last 50 years.

<table>
<thead>
<tr>
<th>City</th>
<th>Population 50 years ago</th>
<th>Population now</th>
</tr>
</thead>
<tbody>
<tr>
<td>City A</td>
<td>45 000</td>
<td>50 000</td>
</tr>
<tr>
<td>City B</td>
<td>2 000</td>
<td>11 000</td>
</tr>
<tr>
<td>City C</td>
<td>1 600 000</td>
<td>4 800 000</td>
</tr>
</tbody>
</table>

**ASK:** Are these populations exact? How do you know? (no, they are not exact, because the chances that they would all be an exact thousand are very slim) Explain to students that populations change very rapidly, so it wouldn’t even make sense to list them as exact numbers, even if we could. A number that was accurate 5 minutes ago probably wouldn’t be accurate right now.

**ASK:** How much did each population increase by? (A: 5 000, B: 9 000, C: 3 200 000) **ASK:** Which city’s population increased the most? (City C) Tell students that you want to compare how much each city’s population increased by. **ASK:** How can we do that when each population was different to start? What can we use? (percents) Tell students that you will check what percent of the original population each city’s population increased by. Write on the board: City A’s population increased by 5 000 people. **ASK:** What percent of 45 000 is 5 000? (5 000 is 1/9 of 45 000, so 5 000 is about 11.1% of 45 000) Write the result on the board underneath the previous sentence:
City A’s population increased by 5 000 people.  
City A’s population increased by 11.1%.

Now write on the board:

City B’s population increased by 9 000 people.  
City B’s population increased by ______%.

Tell students that you want to know what percent of 2 000 the 9 000 is.

**ASK:** How is this problem different from the first problem? (9 000 is more than 2 000) Tell students that 9 000 is more than 100% of 2 000. **ASK:** What is 100% of 2 000? (2 000) What is 200% of 2 000? **PROMPT:** 200% is double 100%. What is the double of 2 000? (4 000) What is 300% of 2 000? (6 000) What is 400% of 2 000? (8 000) What is 500% of 2 000? (10 000) Tell students that 9 000 is halfway between 400% and 500% of 2 000. What percent of 2 000 is 9 000? (450%) Write 450 in the blank shown above.

Now write on the board:

City C’s population increased by 3 200 000 people.  
City C’s population increased by ______%.

**ASK:** What percent of 1 600 000 is 3 200 000? Is it more than 100%? (yes) How do you know? (because 3 200 000 is more than 1 600 000) Write on the board:

\[
\begin{array}{c}
??? \% \\
- 100 \% \\
- 1 600 000 \\
\hline
1 600 000
\end{array}
\]

Tell students that the increase was actually two times the original population. What percent of the original population is that? To guide students write on the board:

\[
3 200 000 = 1 600 000 + 1 600 000 \\
= 100\% + 100\% \\
= ______\%
\]

**ANSWER:** 200%

Add a row to the chart above:

<table>
<thead>
<tr>
<th></th>
<th>City A</th>
<th>City B</th>
<th>City C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population 50 years ago</td>
<td>45 000</td>
<td>2 000</td>
<td>1 600 000</td>
</tr>
<tr>
<td>Population now</td>
<td>50 000</td>
<td>11 000</td>
<td>4 800 000</td>
</tr>
<tr>
<td>Percent increase</td>
<td>11.1%</td>
<td>450%</td>
<td>200%</td>
</tr>
</tbody>
</table>

**ASK:** Which city’s population increased by the greatest percentage? (City B) Tell students that even though City C increased by a large amount, and is growing quite rapidly, we would say that City B is growing faster. City C’s population increased by a larger number, but City B’s population increased by a greater percentage of its original population.
Recognizing the part and the whole. To solve questions involving fractions, ratios, and percents, students need to be able to recognize the part and the whole (and to express the ratio of the part to the whole as a fraction). Use simple word problems to illustrate and work through different cases. EXAMPLES:

a) There are 5 boys among 9 children. (In this case, you are given the whole and one of the parts.) **ASK:** What fraction of the children are boys? (5/9) What fraction of the children are girls? (first, subtract the number of boys from the number of children to find the number of girls: 9 − 5 = 4. Therefore, 4/9 of the children are girls.)

b) There are 5 boys and 6 girls. (In this case, you are given two parts). **ASK:** What is the fraction of boys and girls? (first find out how many children there are (this is the whole): 5 + 6 = 11, so 5/11 of the children are boys and 6/11 are girls)

c) The ratio of boys to girls is 3 : 4. **ASK:** What is the fraction of boys and girls? (For every 3 boys there are 4 girls, so 7 children in total (i.e., add the parts of the ratio: 3 + 4 = 7). So 3/7 of the children are boys and 4/7 are girls)

Finding indirect information. Ask students how many boys (b) are in their class today, how many girls (g), and how many children (c) altogether:

\[ b = g = c = \]

**ASK:** Did you count everyone one by one or was there an easier way once you found the number of boys and girls? Tell students that sometimes they are not given all the information that they need, but can find the information indirectly. Ask students to fill in the numbers of boys, girls, and children given various pieces of information:

a) 7 girls and 8 boys \[ b = g = c = \]

b) 6 girls in a class of 20 \[ b = g = c = \]

c) 12 boys in a class of 30 \[ b = g = c = \]

d) 17 girls in a class of 28 \[ b = g = c = \]
Then have students determine the numbers of boys, girls, and children, the fraction of girls, and the fraction of boys in these classes:

a) There are 6 boys and 5 girls.
b) There are 14 boys in a class of 23.
c) There are 15 girls in a class of 26.

Now have students write the fraction and percent of girls and boys in these classes.

a) There are 3 boys and 7 girls in a class.
b) There are 7 boys and 20 children in a class.
c) There are 8 girls and 25 children in a class.
d) The ratio of boys to girls is 1:3.
e) The ratio of girls to boys is 2:3.
f) The ratio of boys to girls is 12:13.
g) The ratio of boys to girls is 13:12.
h) The ratio of girls to boys is 13:12.

ANSWERS:

a) 7/10 = 70% girls; 3/10 = 30% boys
b) 13/20 = 65% girls; 7/20 = 35% boys
c) 8/25 = 32% girls; 17/25 = 68% boys
d) 3/4 = 75% girls; 1/4 = 25% boys
e) 2/5 = 40% girls; 3/5 = 60% boys
f) 13/25 = 52% girls; 12/25 = 48% boys
g) 12/25 = 48% girls; 13/25 = 52% boys
h) 13/25 = 52% girls; 12/25 = 48% boys

ASK: Which two of the last three questions have the same answer? (parts f) and h) have the same answer) Can you think of a question that has the same answer as part g)? (The ratio of girls to boys is 12:13.)

Determining the number of girls and boys given the total and a fraction or ratio. Write this problem on the board: There are 28 children in a class and 3/7 are girls. How many girls and how many boys are in the class?

Have students write the fraction as a part-to-whole ratio: There are 3 girls for every 7 . (children) Show students how to model the situation. Represent girls with shaded circles and boys with unshaded circles:

3 of the 7 students are girls.

We are told that there are 28 students, so we keep drawing 7 students (3 girls and 4 boys) until we have drawn 28 students:

ASK: How many girls are in the class? (12) How many boys are in the class? (16) Have students determine the number of girls and boys in more classes:
a) There are 30 children in a class and 3/5 are girls.
b) There are 36 children in a class and 4/9 are girls.
c) There are 21 children in a class and 4/7 are boys.

Then go back to the original problem (3/7 are girls in a class of 28) and show students how they can solve it without drawing a model. To find the number of girls, they must solve this proportion:

\[
\frac{3}{7} = \frac{?}{28}
\]

To solve, notice that 7 \times 4 = 28, so multiply the numerator by 4 as well:

\[
\frac{3 \times 4}{7 \times 4} = \frac{?}{28}, \text{ so } ? = 12
\]

ASK: How does the model show that there are 3 \times 4 girls? (the shaded circles are in 4 rows of 3) How does the model show that there are 7 \times 4 students? (the circles are in 4 rows of 7) Have students redo the other three questions (a – c) by solving proportions instead of drawing the model.

Now write this problem on the board: There are 36 children in a class. The ratio of boys to girls is 4 : 5. How many boys and how many girls are in the class?

Demonstrate using a model to solve this one. There are 4 boys for every 5 girls, so draw 4 unshaded circles for every 5 shaded circles. Continue until there are 36 circles in total:

The model shows that there are 16 girls and 20 boys. Now demonstrate using proportions to solve the problem, and make the connection between the diagram and solving the proportion, as above. Then have students do the problems below, both by using a model and by solving a proportion.

a) There are 18 children in a class. The ratio of boys to girls is 7:2.
b) There are 18 children in a class. The ratio of girls to boys is 2:7.
c) There are 18 children in a class. The ratio of boys to girls is 2:7.

Stop to discuss which 2 of the 3 are the same (parts a and b) and how to formulate another question the same as part c.

d) There are 30 children in the class and 60% are girls.
e) There are 45 children in the class and 40% are girls.

**Extension**

You can estimate what percent one number is of another by changing one or both of the numbers slightly. For example, to estimate what percent 5 is of 11, change the 11 to 10, because 11 is close to 10 and 10 is easy to find percents of. Since 5 is 50% of 10, 5 is close to 50% of 11.
The chart below shows the lengths (in feet) of different types of whales at birth (when they are called calves) and as adults. Approximately what percent of the adult length is the calf’s length? Did you need to know how long a foot is to answer this question?

<table>
<thead>
<tr>
<th>Type of Whale</th>
<th>Killer</th>
<th>Humpback</th>
<th>Narwhal</th>
<th>Fin Backed</th>
<th>Sei</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calf Length (feet)</td>
<td>7</td>
<td>15</td>
<td>5</td>
<td>20</td>
<td>14</td>
</tr>
<tr>
<td>Adult Length (feet)</td>
<td>32</td>
<td>48</td>
<td>15</td>
<td>78</td>
<td>50</td>
</tr>
</tbody>
</table>

**SAMPLE SOLUTIONS:** 7 is 25% of 28 and 20% of 35; 32 is closer to 35 than to 28, so estimate that 7 is about 22% of 32 OR 8 is 25% of 32, so 7 is slightly less, say 23% of 32.

Students should check their estimates with a calculator. **EXAMPLE:**

\[
7 \div 32 = 0.21875 \approx 0.22 = 22%.
\]
Drawing linear models. Note that if we are given that there are 2/3 as many boys as girls, as in the teaching box on Workbook page 27, then we can draw two bars, one of which is 2/3 the length of the other. The easiest way to do that is to draw 2 equal length bars end to end for boys and 3 of the same bars, end to end, for girls. For solving word problems, it is convenient to have both bars divided into unit bars of the same length.

EXTRA PRACTICE:
Draw two bars so that:

a) Bar A is 1/4 of Bar B
b) Bar A is 3/5 of Bar B

Then have students draw bars to represent girls and boys if:

c) The bar for boys is 3/4 of the bar for girls
d) The bar for girls is 4/5 of the bar for boys
e) The bar for boys if 2/5 as long as the bar for girls
f) The bar for girls is 5/9 as long as the bar for boys

Solving problems when given the total number of boys and girls.
Find the number of students in one unit of the bars by dividing: number of students ÷ number of units in the bars.

EXTRA PRACTICE:
Find the number of boys and girls by drawing a model.

a) There are 30 students on a bus. There are 1/2 as many girls as boys.
b) There are 32 students on a bus. There are 3/5 as many boys as girls.
c) There are 35 students on a bus. There are 2/3 as many girls as boys.

ANSWERS:

a) G [ ]
B [ ]

There are 3 units and 30 students, so 30 ÷ 3 = 10 students in each unit. So there are 10 girls and 20 boys.
b) B 
G
There are 8 units and 32 students, so $32 \div 8 = 4$ students in each unit. There are 12 boys and 20 girls.

c) B 
G
There are 5 units and 35 students, so $35 \div 5 = 7$ students in each unit. There are 14 boys and 21 girls.

Solving problems when given the difference between the number of boys and girls. In this case, we have a different way to determine the number that each unit in the bars represent. For example, suppose we are given that there are $\frac{4}{7}$ as many boys as girls:

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Tell students that we don’t know how many students there are in total, but we know that there are 6 more girls than boys. How can we tell how many students each unit on the bars represent? **PROMPT:** There are 6 extra girls. How many extra units does the bar for girls have? (3) So 3 units represent 6 students. How many students does 1 unit represent? ($6 \div 3 = 2$) Since we know how many students each unit represents, and how many units represent boys and girls, we can calculate how many boys and girls there are. There are $4 \times 2 = 8$ boys and $7 \times 2 = 14$ girls.

Have students model the following situations to determine the number of boys and girls:

a) There are $\frac{4}{5}$ as many boys as girls. There are 3 more girls than boys.

b) There are $\frac{3}{5}$ as many boys as girls. There are 10 more girls than boys.

c) There are $\frac{3}{7}$ as many girls as boys. There are 12 more boys than girls.

**ANSWERS:**

a) 12 boys and 15 girls   b) 15 boys and 25 girls   c) 9 girls and 21 boys

Students should verify that their answers satisfy the given information.

**EXAMPLE:** 12 is $\frac{4}{5}$ of 15 and there are indeed 3 more girls than boys.

**PROCESS EXPECTATION**

- Modelling

**PROCESS EXPECTATION**

- Organizing data

Solve the same problems by using a chart. Show students how they could solve the same type of problem using a chart instead. **EXAMPLES:**

a) There are $\frac{4}{7}$ as many boys as girls. There are 6 more girls than boys. Continue adding 4 boys and 7 girls until there are 6 more girls than boys.

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
<th>How many more girls than boys</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>6</td>
</tr>
</tbody>
</table>

There are 8 boys and 14 girls.
b) There are 3/5 as many girls as boys. There are 40 students altogether. Continue adding 5 boys and 3 girls until there are 40 students altogether.

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
<th>How many boys and girls in total</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>16</td>
</tr>
<tr>
<td>15</td>
<td>9</td>
<td>24</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>32</td>
</tr>
<tr>
<td>25</td>
<td>15</td>
<td>40</td>
</tr>
</tbody>
</table>

There are 25 boys and 15 girls.

PROCESS EXPECTATION

Reflecting on other ways to solve a problem

Discuss which method (draw a linear model or use a chart) students like better.

Finding the whole from the part. Write on the board: 2/3 of a number is 100. What is the number? **ASK**: Is 100 the part or the whole? (the part) What is the whole? (the number that we don’t know) Tell students that this is a part-to-whole ratio, and write on the board:

\[
\frac{2}{3} \times \frac{100}{?} = \frac{\text{part}}{\text{whole}}
\]

Have students solve the proportion. (? = 150) Then have students do Workbook page 28 Question 1.

Now write on the board: 2/3 of the beads in a box are red. 100 beads are red. How many beads are in the box? (This is the same problem as before written a little differently. Use the following sentences and prompts to show this.) Then write:

2/3 of the number of beads in the box is the number of red beads in the box.

**ASK**: What is the number of red beads in the box? (100) Now, underneath the previous sentence, write:

2/3 of the number of beads in the box is 100.

Now underline part of the sentence: 2/3 of the number of beads in the box is 100. Tell students this underlined part is what we want to know, so now underneath the previous sentence, write:

2/3 of what number is 100?

This is exactly the problem we solved earlier.
Have students do Workbook page 31 Question 2 by recognizing that the problems are just like those in Question 1 and rewriting them accordingly.

Problems with three groups instead of two. Write the following problem on the board:

2/5 of the people (boys, girls, adults) at the park are boys.
There are 3 more girls than boys.
There are 7 adults.
How many people are at the park?

Do this problem together both ways—using a model and using a chart.
Start with the chart. It should have a column for the number of each type of person (boys, girls, adults), as well as the total number of people. **SAY:** We know that there are 2 boys for every 5 people, do we can add 2 boys for every 5 people in our chart. Begin filling the chart in:

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
<th>Adults</th>
<th>People</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td>15</td>
</tr>
</tbody>
</table>

**ASK:** If there are 2 boys, how many girls are there? (5) How do you know? (there are 3 more girls than boys) How many adults are there? (7) How do you know? (it says so right in the question) Fill in the first row completely.

**ASK:** Is this possible? Can there be 2 boys, 5 girls, 7 adults, and 5 people altogether? (no, there would be 14 people altogether) Have students continue in this way until they find a combination that works. Students should add rows to the chart as needed.

When students finish, challenge them to find a model to help solve the problem. One possible model could be to draw a bar divided into fifths and then add the given information to it:

```
    3    7
  boys girls adults
```

Have students use this model, or their own, to solve the problem. When students finish, ask them again which way they like better—using a chart or using a model. A model in this case makes the problem quite easy. From the picture, it is clear that 1/5 of the total number of people is 10, so there are 50 people altogether.
PROCESS EXPECTATION

Selecting tools and strategies

Have students decide which method to use—modelling or a chart—to do these problems. Then do the problems both ways. Was their prediction about which way would be easier correct?

a) There are 5 adults at a park.
   There are 7 more girls than boys at a park.
   3/7 of the people at the park are boys.
   How many people are at the park?

b) 80% of the people at a park are children.
   There are 24 more children than adults at the park.
   How many people are at the park?

c) There are 18 boys at a park.
   3/5 of the people (boys, girls, and adults) are girls.
   There are 6 fewer adults than boys.
   How many people are at the park?

ANSWERS: a) 84   b) 40   c) 75

Note that all of these problems are quite easy using modeling, and more difficult using a chart.

Other contexts. Draw a model and then solve these problems:


b) In a parking lot, 3/4 of the vehicles are cars, 1/5 are trucks, and the rest are buses. There are 4 buses. How many vehicles are in the parking lot? (80)

c) Katie has a rock collection. She found 2/5 of her rocks in British Columbia, 1/3 in Alberta, and the rest in Ontario. She found 8 more rocks in Alberta than she did in Ontario.
   i) What fraction of her rocks did she find in Ontario? (4/15)
   ii) What fraction of the total number of rocks do the 8 rocks represent?
       Hint: 8 is the difference between the number found in Alberta and the number found in Ontario. (1/15)
   iii) How many rocks does Katie have altogether? (15 × 8 = 120)

d) At a school, 3/7 of the people are boys, 2/5 are girls, and the rest are adults. There are 6 more students of one gender than the other.
   i) Are there more boys or girls? (3/7 = 15/35 and 2/5 = 14/35, so there are more boys)
   ii) What fraction of the total number of people do the 6 extra students of one gender represent? (3/7 − 2/5 = 15/35 − 14/35 = 1/35)
   iii) How many people are there in the school altogether? (6 × 35 = 210)
In Jacob’s coin collection:

- 1/5 are from China,
- 1/4 are from Europe,
- some are from Canada,
- and the rest are from Mexico.

There are 12 more coins from Europe than from Mexico.
There are 48 more coins from Canada than from China.

How many coins does Jacob have altogether?

**ANSWER:** 360

Have students solve the problems below.

**a)** Two-thirds of Helen’s age is half of Dale’s age. Dale is 10 years older than Helen.

How old is Helen? (30)

**b)** Ron’s age is two-thirds of Mark’s age. Mark’s age is three-fifths of Sara’s age.

Sara is 9 years older than Ron. How old is Mark? (9)

**Extensions**

1. Carene emptied her piggy bank that only holds pennies. The contents weigh about 1 kg 300 g in total. If each penny weighs \( \frac{2\frac{1}{3}}{\text{g}} \), about how much money was in the piggy bank? Round your answer to the nearest half dollar.

**ANSWER:** \( 1300 \div 2\frac{1}{3} = 1300 \div \frac{7}{3} = 1300 \times 3 \div 7 \approx 557 \) pennies = $5.57, so there is about $5.50. **NOTE:** To divide by a mixed number, it is essential that students change the mixed number to an improper fraction. We cannot write \( 1300 \div (2\frac{1}{3}) = 1300 \div (2 + \frac{1}{3}) = 1300 \div 2 + 1300 \div \frac{1}{3} \), because the distributive law does not apply here. If you try to use the distribute law, you should see right away that doing so gives an answer that can’t be correct: \( 1300 \div 2 \) is larger than what you started with, \( 1300 \div (2\frac{1}{3}) \), because 2 is less than 2 1/3 and dividing by a smaller number gives a larger result.

2. a) \( \frac{3}{5} \) of \( \frac{4}{5} \) of a number is 60. What is the number? (Solve in two steps: If \( \frac{3}{5} \) of something is 60, then the something is 100, so \( \frac{4}{5} \) of a number is 100. This means the number is 125.)

b) \( \frac{2}{3} \) of \( \frac{3}{4} \) of a number is 36. What is the number? (72)

c) \( \frac{2}{3} \) of \( \frac{3}{4} \) of \( \frac{4}{5} \) of \( \frac{5}{6} \) of \( \frac{6}{7} \) of a number is 18. What is the number? (Solve in five steps: If \( \frac{2}{3} \) of (\( \frac{3}{4} \) of \( \frac{4}{5} \) of \( \frac{5}{6} \) of \( \frac{6}{7} \) of a number) is 18, then the part in brackets is 27. So \( \frac{3}{4} \) of \( \frac{4}{5} \) of \( \frac{5}{6} \) of \( \frac{6}{7} \) of a number is 27. Continue in this way.)
Finding 100% from other percents. Review finding 100% mentally when given other simple percents. EXAMPLES:

a) 80% is 16. What is 100%?  
b) 150% is 90. What is 100%?  
c) 300% is 15. What is 100%?  
d) 250% is 600. What is 100%?

Then show students how to calculate 100% when given any percent. Write on the board: 112% of a number is 17. What is 100%?  

ASK: How is this problem harder than the previous problems? (You can’t easily find 10%, 50%, or 100% of the percent given.) 

ASK: If I know 112%, how can I find 1%? (divide by 112). Write on the board: 1% is 17 \( \div 112 \approx 0.152 \).

So what is 100%? (1% \( \times \) 100 is about 15.2) Have students use a calculator to determine 100%, to the nearest tenth, if:

a) 93% is 21  
b) 158% is 65  
c) 212% is 98  
d) 53% is 8.2

ANSWERS: a) 22.6  
b) 41.1  
c) 46.2  
d) 15.5

Percents of percents. Remind students that we can take a fraction of a fraction.  

ASK: What is 2/5 of 3/5? (2/5 \( \times \) 3/5 = 6/25) Write on the board:

What is 2/5 of 3/5?  
What is 40% of 60%?

ASK: How are these two questions the same? How are they different? (2/5 is another way of writing 40% and 3/5 is another way of writing 60%—the questions mean the same thing, so will have the same answer, but the numbers are written as percents instead of fractions in the second one) 

ASK: The answer to the first question is 6/25. What is the answer to the second question, that is, the answer as a percent? (6/25 = 24%, so the answer to the second question is 24%)

Now remind students that percents and fractions can be written as decimals. Write on the board, underneath the first two questions:

What is 0.4 of 0.6?
Have students convert the answer, 24%, to a decimal. (0.24)

Remind students that “of” can be replaced with “×”, just as we did for fractions, so 0.4 of 0.6 is 0.4 × 0.6 = 0.24. Have students find percents of percents by changing the percents to decimals, multiplying, and then changing the decimals back to percents:

a) 30% of 50%  

b) 25% of 60%  

c) 35% of 42%  

d) 80% of 150%

**ANSWERS:**  
a) $0.3 \times 0.5 = 0.15 = 15\%$  
b) $0.25 \times 0.6 = 0.15 = 15\%$  
c) $0.35 \times 0.42 = 0.147 = 14.7\%$  
d) $0.8 \times 1.5 = 1.2 = 120\%$

**Contexts for percent problems.** Tell students that you pay 13% sales tax on a shirt that costs $32.00. **ASK:** What percent of the original price (the price before taxes) is the price after taxes? To guide students, write on the board:

\[
\text{price after taxes} = \text{price before taxes} + \text{sales tax} = \text{100\% of price before taxes + 13\% of price before taxes} = \text{113\% of price before taxes.}
\]

Have students calculate the price of the shirt after taxes in two ways:

a) Find 13\% of $32.00 and add the result to $32.00.  
b) Find 113\% of $32.00.

**ASK:** Did you get the same answer both ways?

Have students find the price after taxes in two ways.

a) The original price is $18.00 and the taxes are 15\%.  
b) The original price is $50.00 and the taxes are 6\%.

**ANSWERS:**  
a) $18.00 \times 15\% = 18.00 \times 0.15 = 2.70$, so the price after taxes is $20.70  
AND $18.00 \times 115\% = 18.00 \times 1.15 = 20.70$  
b) $50.00 \times 6\% = 50.00 \times 0.06 = 3$, so the price after taxes is $53  
AND $50.00 \times 106\% = 50.00 \times 1.06 = 53$

**ASK:** Which way do you like better? Why?

**EXTRA PRACTICE:**  
A T-shirt costs $16.00 and is on sale for 30\% off.

a) What percent of the original price is the sale price? (70\%)  
b) Calculate the sale price in two ways:

i) Calculate 30\% of the original price, and subtract the amount from the original price. (30\% of $16 = 4.80$, so the sale price is $16 - 4.80 = 11.20$)

ii) Calculate 70\% of the original price. (70\% of $16 = 11.20$)

Are the two answers the same? (yes)
c) A 15% tax is added to the sale price. What is the final price of the T-shirt? ($11.20 \times 115% = 11.20 \times 1.15 = 12.88$)

Tell students that a book had a 14% tax on it, and the sales clerk charged you, after taxes, $35.33. **ASK:** What was the price before taxes? Write on the board:

The price after taxes is $_______.

The price after taxes is _______% of the price before taxes.

So $_______. is _______% of the price before taxes.

Fill in the first two blanks together. Explain that in the last sentence students should substitute the value for the price after taxes from the first sentence. Have students copy and complete the last sentence in their notebooks. (So $35.33 is 114% of the price before taxes.) Write on the board:

$35.33 is 114% of what number?

Review solving this percent problem using cross-multiplication. Start with the proportion $35.33/x = 114/100$. This means $3533 = 114x$, and $x = 3533 ÷ 114 = 30.99$. The original price is therefore $30.99.

Have students solve these problems:

a) price after taxes = $80.46 taxes = 8% price before taxes = ? ($74.50)

b) price after taxes = $56.71 taxes = 7% price before taxes = ? ($53.00)

c) You are charged $22.59 for a DVD. The tax is 13%. What was the price before taxes? ($19.99)

d) You are charged $370.99 for a bike. The tax is 6%. What was the price before taxes? ($349.99)

Tell students that in some provinces, people pay both a provincial sales tax and a federal sales tax when they make certain purchases. Assume the provincial sales tax is 6% of the price and the federal sales tax is 7% of the price. When you buy a computer monitor that costs $100, you pay $6 in provincial taxes and $7 in federal taxes, for a total price of $113. **ASK:** What percent of the original price are you paying in taxes? (13 is 13% of 100, so I paid 13% in taxes) Have students determine how much they would pay in provincial and federal taxes, and then the total amount of taxes, for these original prices:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>$60</td>
<td>$150</td>
<td>$260</td>
</tr>
<tr>
<td>provincial: $3.60</td>
<td>provincial: $9.00</td>
<td>provincial: $15.60</td>
<td></td>
</tr>
<tr>
<td>federal: $4.20</td>
<td>federal: $10.50</td>
<td>federal: $18.20</td>
<td></td>
</tr>
<tr>
<td>total: $7.80</td>
<td>total: $19.50</td>
<td>total: $33.80</td>
<td></td>
</tr>
</tbody>
</table>

**ANSWERS:**
**ASK:** Did you calculate the provincial tax first or the federal tax? Have students calculate the other tax first. **ASK:** Do you get the same answer? (yes) Discuss how students calculated the second tax. 

**ASK:** Did you use your answer to the provincial tax to get the federal tax? Emphasize that they can just add 1% of the price to get the federal tax from the provincial tax instead of multiplying 1% × 7 to get the federal tax. If students calculated the federal tax first, they could subtract 1% of the price to get the provincial tax. 

**ASK:** Which do you find easier, adding or subtracting? Tell students that because you find adding easier than subtracting, you find it easier to calculate the 6% tax first. Since you will get the same answer both ways, you might as well pick the easier one to do. 

**ASK:** What percent of the original price are the total taxes? (13%) Point out that 6% + 7% = 13% and 7% + 6% = 13%. It doesn’t matter which one you add first, the total is still 13%.

Tell students that last year, a monthly bus pass cost $80. This year, the price increased by 10%. 

**ASK:** What is the new price? ($80 × 110% = $88) If the price increases by 10% again next year, what will the new price be? ($88 × 110% = $96.80) 

**SAY:** The price increased by 10% each year. Did it increase by 20% in total? 

**PROMPT:** What would a 20%-increase from the original price be? ($80 × 120% = $96) The actual price is slightly more than 20% of the price from two years ago. Are any students surprised by this? If the price increases by 10% twice, it seems like it should increase by 20% in total because 10% + 10% = 20%. But this is not what happens. You could explain it like this:

This year, the new price is $80 + 10% of $80 = $80 + $8. Now to get the new price next year, we need to find 10% of $88. This is more than 10% of $80.

\[
\begin{align*}
\text{Total change in price} &= 10\% \text{ of } 80 + 10\% \text{ of } 88 \\
\text{But } 20\% \text{ of } 80 &= 10\% \text{ of } 80 + 10\% \text{ of } 80 \\
\end{align*}
\]

To find the new price, we don’t add just 10% of $80 the first year and then 10% of $80 the second year—we add 10% of $80 the first year and 10% of $88 the second year. Because we are adding 10% of a larger number the second year, we are adding more than 10% of the original price from last year.

Tell students that the number of students at a school increased by 10% one year and 20% the next. 

**ASK:** By what percent did the number of students increase over two years if

a) there were originally 100 students at the school.

b) there were originally 50 students at the school.

c) there were originally 400 students at the school.

d) there were originally \(x\) students at the school.

**SOLUTION:**

a) There are 110 students after one year and then 120% × 110 = 132 students after two years. This is 32% more than the original 100 students.
b) There are 55 students after one year and then \(120\% \times 55 = 66\) students after two years. By what percent is 66 more than 50? This is like asking, 16 is what percent of 50?

Solve the proportion \(\frac{16}{50} = \frac{?}{100}\).

So 66 is 32\% more than the original 50 students.

c) There are 440 students after one year and then \(120\% \times 440 = 528\) students after two years. 528 is what percent more than 400? That is asking: 128 is what percent of 400?

Solve the proportion: \(\frac{128}{400} = \frac{?}{100}\).

So 128 is 32\% more than the original 400 students.

d) There are \(x\) students originally, so \((110\% \times x)\) students after one year, and then \((120\% \times 110\% \times x)\) students after two years. This is \(1.2 \times 1.1 \times x = 1.32x\) students after two years. This is 132\% of \(x\), so the number of students increased by 32\% after two years.

Have students explain why an increase of 10\% one year and an increase of 20\% the next year should be more than an increase of 30\% from the first year.

Word Problems Practice:

1. Cheryl buys a watch for $60 and pays $69 after taxes. She buys a bike for $500 and pays $545 after taxes.

   a) Which is taxed at a higher rate, watches or bikes? Explain how you know.

   b) Why might a province decide to tax some items at a higher rate than others?

   **ANSWERS:** a) $9 is 15\% of $60 and $45 is 9\% of $500, so watches are taxed at a higher rate than bikes OR $9 is more than 10\% of $60, but $45 is less than 10\% of $500, so watches are taxed at a higher rate than bikes.

   b) A province might want to encourage or discourage certain purchases. For example, most governments do not tax vegetables but do tax candy, because vegetables are healthier and consuming them reduces long-term health care costs. Bikes might be encouraged for environmental reasons, whereas watches are purely luxury items.

2. Tegan photocopied a poem, then gave the original to a friend. She accidentally reduced the size to 80\% of the original. How should she photocopy the copy to bring it back to the original size?

   **ANSWER:** She needs to find \(\frac{100\%}{80\%} \times 80\% = 100\%\).

   So \(\frac{100\%}{80\%} = 10 \div 8 = 1.25 = 125\%\).
Tegan needs to set photocopier to enlarge the copy by 125% to bring it back to the original size.

3. To clean a baby’s clogged nose, you can use a 0.9% salt solution (99.1% water). How much water and how much salt do you need to produce 25 mL of the salt solution?

**ANSWER:** To find how much salt, solve the proportion: \( \frac{x}{25} = \frac{0.9}{100} \)

So \( x = 0.225 \), which means we need 0.225 mL of salt and \( 25 - 0.225 = 24.775 \) mL of water.

**Bonus** A hair colouring kit contains a 50-mL bottle of 6% hydrogen peroxide solution (94% water). To staunch a bleeding wound, you can use a 3% hydrogen peroxide solution (97% water). If you use 10 mL of 6% solution, how much water do you need to add to get a 3% solution?

**ANSWER:** 6% of 10 mL = 0.6 mL of hydrogen peroxide. So 10 mL of the solution contains 0.6 mL of hydrogen peroxide, and we want this 0.6 mL to be 3% of the solution, so 0.6 mL is 3% of what number? Solve the proportion:

\( \frac{0.6}{x} = \frac{3}{100} \)

So \( x = 20 \) mL. This means the total amount of solution must be 20 mL, but we only have 10 mL so far, so we need to add 10 mL of water.
Tell students that a team won 2 games, lost 1 game, and tied 4 games.

**ASK:** How many games did the team play? (7) What fraction of games did the team win? (2/7) Based on this fraction, does it look like the team is good or not very good? What is the ratio of games won to games lost? (2 to 1) Is this still a fraction? (no) Why not? (the games won are not a part of the games lost) What kind of ratio is this: part to whole or part to part? (part to part) Does the ratio of games won to games lost make the team look good or not very good? (good, because they won twice as many games as they lost) Which ratio do you think gives a truer picture of the team? (Neither ratio tells the whole story: the part-to-whole ratio, 2 : 7, makes the team look very bad; the part-to-part ratio, 2 : 1, makes the team look fantastic. The reality is that the team tied more than half of its games—it’s about average.)

Tell students that the way to tell the whole story is to say how many games the team has won, how many it has lost, and how many it has tied. We could write a ratio including all three parts. Write on the board: wins : losses : ties = 2 : 1 : 4. This is called a **3-term ratio** because it is a ratio with three terms. Just as the ratio 2 : 1 means there are two wins for every one loss, the ratio 2 : 1 : 4 means there are 2 wins and 1 loss for every 4 ties.

**Writing 2-term ratios from 3-term ratios.** Tell students that at a camp, people are divided into groups with 3 adults, 8 girls, and 7 boys. The ratio of adults : girls : boys is 3 : 8 : 7 because for every 3 adults, there are 8 girls and 7 boys. Show this by using a model (A for adults, G for girls, and B for boys).

```
A A A G G G G G B B B B B
```

Explain that we know that the people at the camp can be divided into groups of 3 adults, 8 girls and 7 boys, but we don’t know how many such groups they can be divided into. **ASK:** How can you tell from the model what the ratio of adults to girls is? (there are 3 adults for every 8 girls) What is the ratio of girls to boys? (8 : 7) How do you know? (because for every 8 girls there are 7 boys) Have students finish writing these ratios:

- a) adults : girls = ______ : ______
- b) girls : boys = ______ : ______
- c) adults : boys = ______ : ______
- d) boys : girls = ______ : ______
- e) boys : adults = ______ : ______
- f) girls : adults = ______ : ______
Tell students that you can get three of the ratios from the other three. **ASK:** If the ratio of girls to boys is 8 to 7, what is the ratio of boys to girls? (7 to 8—just switch the numbers) Explain that when you have a 3-term ratio, you can rewrite it as three 2-term ratios. Have students write the three ratios if the ratio of adults : girls : boys is 5 : 18 : 20. **ANSWER:** adults to girls = 5 : 18, adults to boys = 5 : 20, and girls to boys = 18 : 20. Have students write the ratios in lowest terms. **ANSWERS:** adults to girls = 5 : 18, adults to boys = 1 : 4, and girls to boys = 9 : 10.

**Why fraction notation cannot be used for 3-term ratios.** Tell students that the ratio of girls to boys to adults on a school trip is 6 : 7 : 2. This means that for every 6 girls, there are 7 boys and 2 adults. **ASK:** For every 6 girls, how many boys are there? (7) Write on the board: girls : boys = 6 : 7. **ASK:** For every 6 girls, how many adults are there? (2) Write: girls : adults = 6 : 2. For every 7 boys, how many adults are there? (2) Write on the board: boys : adults = 7 : 2.

Remind students that 2-term ratios can be written using fraction notation, even when there is no fraction involved. If the ratio of girls to boys is 6 : 7, we can write girls : boys = 6 : 7 as girls/boys = 6/7. This doesn’t mean that the girls are 6/7 of any set, but that the number of girls is 6/7 of the number of boys. However, a fraction can only have two parts—a numerator and a denominator. If there are three terms in a ratio, we can’t write the number of girls as a “fraction” of the number of boys and adults. The expression 6/7/2 doesn’t make sense as a fraction; there is no fraction called “six sevenths halves.” We can, however, split the 3-term ratio into three 2-term ratios and make three fractions. The number of girls is 6/7 the number of boys and 6/2 of the number of adults. The number of boys is 7/2 the number of adults.

**The other notation for 3-term ratios.** Remind students that the 2-term ratio 6 : 7 can also be written as 6 to 7 or 6/7. Although 6/7 cannot be used for 3-term ratios, the form 6 to 7 can be. We can write the ratio 6 : 7 : 2 as 6 to 7 to 2. Have students write the following ratios using both forms that make sense for 3-term ratios.

a) 2 adults for every 5 boys and 8 girls  
b) 3 adults for every 9 girls and 10 boys

**ANSWERS:**  
a) adults to boys to girls = 2 to 5 to 8 = 2 : 5 : 8  
b) adults to girls to boys = 3 to 9 to 10 = 3 : 9 : 10

**Equivalent 3-term ratios.** Tell students that there are 3 red marbles for every 5 blue marbles and 6 green marbles in a jar. Write on the board: red : blue : green = 3 : 5 : 6. Have students determine the ratio, in lowest terms, of the following:

a) red to blue marbles  
c) blue to green marbles  
e) blue to all marbles  

b) red to green marbles  
d) red to all marbles  
f) green to all marbles
ANSWERS: a) 3 to 5 b) 3 to 6 = 1 to 2 c) 5 to 6 d) 3 to 14 e) 5 to 14 f) 6 to 14 = 3 to 7

ASK: If there are 20 blue marbles, how many green marbles are there? To answer this question, draw a model for the marbles on the board, with R for red, B for blue, and G for green. (If possible draw the model using the corresponding colours.)

```
R R R B B B B G G G G G G
R R R B B B B G G G G G G
R R R B B B B G G G G G G
R R R B B B B G G G G G G
```

Explain that you need to keep drawing groups of 3 red, 5 blue, and 6 green marbles until you have 20 blue marbles. ASK: How many groups should I draw? (4, because $4 \times 5 = 20$)

```
R R R B B B B G G G G G G
R R R B B B B G G G G G G
R R R B B B B G G G G G G
```

ASK: How many green marbles are there? (24) Explain that there are 4 groups, and 6 green marbles in each group, so there are 24 green marbles if there are 20 blue marbles.

PROCESS EXPECTATION Modelling

Remind students how they can obtain the same result by using a proportion instead of a model:

\[
\frac{\text{blue}}{\text{green}} = \frac{5}{6} = \frac{20}{x}
\]

Emphasize that to solve the problem using the proportion, students follow exactly the same steps they did to solve the problem using the model. How many groups do we need to draw? Write on the board:

5 blue marbles in each group \( \times \) number of groups = 20 blue marbles

So solve $5 \times ? = 20$. Then $? = 4$. This is exactly how we would start solving the proportion. We would ask, What do we multiply the first numerator by to get the second numerator? Then we would multiply the first denominator by that number to get the second denominator. This is also exactly what we do when we use the model. The number we multiply by is the number of groups:

6 green marbles in each group \( \times \) 4 groups = 24 green marbles altogether

Have students answer these questions using 2-term proportions:

a) If there are 42 green marbles, how many red marbles are there? (21)

b) If there are 30 blue marbles, how many green marbles are there? (36)

c) John says there are 20 green marbles. Can he be right? Explain? (no, because each group has 6 marbles in it and 20 is not a multiple of 6)
EXTRA PRACTICE:
1. Fill in the blanks using the ratios in the equation $3 : 5 : 6 = 12 : 20 : 24$
   a) $3 : 5 = \underline{ } : \underline{ }$
   b) $3 : 6 = \underline{ } : \underline{ }$
   c) $5 : 6 = \underline{ } : \underline{ }$
   d) $6 : 5 = \underline{ } : \underline{ }$
   e) $5 : 3 = \underline{ } : \underline{ }$
   f) $6 : 3 = \underline{ } : \underline{ }$

   ANSWERS: a) $12 : 20$
   b) $12 : 24$
   c) $20 : 24$
   d) $24 : 20$
   e) $20 : 12$
   f) $24 : 12$

2. In the question above, which three ratios can be obtained from the other three?

   ANSWERS: The ratios in parts d, e, and f can be obtained from the ratios in parts c, a, and b, respectively.

Solving proportions. Write on the board the following 3-term proportion:

   \[ 2 : x : 4 = y : 9 : 12 \]

Have students write three 2-term proportions from it.

a) $2 : x = \underline{ } : \underline{ }$
   b) $2 : 4 = \underline{ } : \underline{ }$
   c) $x : 4 = \underline{ } : \underline{ }$

   ANSWERS: a) $y : 9$
   b) $y : 12$
   c) $9 : 12$

Point to part a and ASK: Can this one be solved? (no) Why not? (there are two values that we don’t know) Repeat for b and c (yes, they can both be solved because there is only one value that we don’t know)

Have students solve b) and c).

   ANSWERS: b) $2 : 4 = 6 : 12$
   c) $3 : 4 = 9 : 12$

Conclude that $x = 3$ and $y = 6$, so the 3-term proportion is $2 : 3 : 4 = 6 : 9 : 12$. Point out that we didn’t need to solve the proportion with two unknowns; it was enough to solve the two proportions that each have only one unknown.

EXTRA PRACTICE:
1. Solve the 3-term proportions.
   a) $4 : 5 : 6 = 28 : x : y$
   b) $4 : 7 : x = 20 : y : 40$
   c) $8 : 10 : x = y : 30 : 27$

   ANSWERS: a) $x = 35, y = 42$
   b) $x = 8, y = 35$
   c) $x = 9, y = 24$

2. The ratio of adults to boys to girls at a camp is $5 : 18 : 20$.
   a) Write the following ratios in lowest terms:
      i) adults to boys
      ii) adults to girls
      iii) boys to girls
   b) If there are 30 adults, how many girls are there? (120)
c) If there are 36 boys, how many adults are there? (10)

d) If there are 60 girls, how many boys are there? (54)

e) If there are 20 adults, how many boys are there? (72)

f) The camp counsellor holds a general meeting and counts 106 boys and 119 girls. The counsellor knows there are 30 adults. How many boys and girls should the counsellor look for? (Solve $5 : 18 : 20 = 30 : x : y$ to get $x = 108$ and $y = 120$, so the counsellor should look for 2 boys and 1 girl.)

**Extension**

a) Recall that the angles in a triangle always add to 180°. Find the angles in a triangle if the ratio of angle measures is:

- A. 1 : 2 : 3
- B. 2 : 3 : 4
- C. 2 : 3 : 5
- D. 1 : 4 : 5
- E. 1 : 5 : 6
- F. 2 : 2 : 5

b) Which triangles from a) are:

- i) acute
- ii) right
- iii) obtuse
- iv) isosceles

**ANSWERS:**

a) A. 30-60-90  B. 40-60-80  C. 36-54-90  
D. 18-72-90  E. 15-75-90  F. 40-40-100

b) i) B is acute  ii) A, C, D, and E are right  iii) F is obtuse  iv) F is isosceles
Decimals in ratios. Tell students that ratios, even part-to-whole ratios, are different from fractions in one important way: The two terms do not have to be whole numbers! Have students solve various problems. **EXAMPLES:**

1. If 10 bus tickets cost $31.50, how much will 30 bus tickets cost?
2. If 9 bus tickets cost $28.50, how many bus tickets can I buy for $57?
3. On a map, 1 cm represents 6 km. If two cities are 6 cm apart on the map, how far apart are they in real life?
4. On a map, 2.5 cm represents 1.6 km.
   i) Two cities are 25 cm apart on the map. How far apart are they in real life?
   ii) Two cities are 8 km apart in real life. How far apart are they on the map?
5. Mike and Sandra practise running long distances by keeping the same pace for the entire distance.
   i) Mike runs 2.5 laps in the first 6 minutes. How many laps can he run in 18 minutes?
   ii) Sandra runs 2.5 laps in the first 4.4 minutes. How many laps can she run in 44 minutes?

**Introduce rate.** Explain that a rate is like a ratio, but the two terms in a rate have different units. Salaries or allowances are examples of rates. For example, $20 for every 3 hours of work is a rate. We can write this as:

\[
\frac{20}{3 \text{ hours}} \quad \text{or} \quad \frac{20}{3 \text{ hours}}
\]

Another example would be an allowance of $25 every 2 weeks. Have students write this as a rate in two ways.
**Per means for every or for each.** Explain that we sometimes say “per” instead of “for every” or “for each,” but they mean the same thing. For example, when talking about speeds, we usually talk about kilometres per hour. **EXAMPLE:** She drove 80 km per hour (instead of 80 km for every hour).

**Unit rates.** Tell students that Lisa can type 78 words in 3 minutes and John can type 96 words in 4 minutes. Have students write the two ratios. Explain that these are both examples of rates because the units are different for both terms. **ASK:** What are the different units? (words and minutes) Tell students that you want to know who types faster. **ASK:** What makes it hard to tell? (John needed 4 minutes to type his words but Lisa needed only 3 minutes, so even though John typed more words, he might still be slower.) **PROMPT:** I think John types faster because he typed more words. Do you think I’m right? (no because John had more time than Lisa, 4 minutes instead of 3 minutes) **ASK:** How can we find out who types faster? Allow students time to brainstorm or discuss with a partner. Then, if no one suggests it, emphasize that we can find out how many words John and Lisa can each type in 1 minute. By making the second term in both rates the same (in this case 1), it is easier to compare them; it is like making the denominators the same to compare fractions. Have students do this for each rate:

- Lisa: \( \frac{78 \text{ words}}{3 \text{ minutes}} = \frac{\text{words}}{1 \text{ minute}} \)
- John: \( \frac{96 \text{ words}}{4 \text{ minutes}} = \frac{\text{words}}{1 \text{ minute}} \)

**ASK:** Who types faster? (Lisa does because she types 26 words per minute and John types only 24 words per minute)

Explain that when we write an equivalent rate with second term 1, the rate is called a **unit rate**. Unit rates are useful for comparing different rates because when the second term is 1, we just have to compare the first term. Have students calculate unit rates for the following trips Jane made on her bike:

a) 35 km in 2 hours  b) 45 km in 3 hours  
c) 55 km in 4 hours  d) 63 km in 5 hours

**ANSWERS:** a) 17.5 km/h  b) 15 km/h  c) 13.75 km/h  d) 12.6 km/h

**PROCESS EXPECTATION**  
Connecting

**PROCESS EXPECTATION**  
Reflecting on the reasonableness of an answer

**Notation for unit rates.** When writing a unit rate, we often leave out the 1 in the second term. **EXAMPLES:** 80 km/h instead of 80 km/1h, 26 words/minute instead of 26 words/1 minute.

**Unit ratios.** We can talk about unit ratios as well as unit rates. The only difference between a unit ratio and a unit rate is that a unit ratio doesn’t have units, so the ratio can be thought of as a single number. For example, the ratio 7:2 is equivalent to 3.5:1, so you can think of 7 : 2 as the number 3.5. Have students write these ratios first as a unit ratio and then as a number:
(a) $5 : 4 \ (5 \div 4 : 4 \div 4 = 1.25 : 1 \text{ or just } 1.25)\\(b) \ 7 : 5 \ (1.4 : 1 \text{ or just } 1.4)\\(c) \ 6 : 8 \ (0.75 : 1 \text{ or just } 0.75)\\Have \ students \ order \ these \ ratios, \ from \ smallest \ to \ largest.\\(6 : 8 < 5 : 4 < 7 : 5)\\

NOTE: When talking about pi as the ratio of a circle’s circumference to its diameter (ME8-6), it will be important for students to understand that we can think of a ratio as a number by writing it as a unit ratio.

Have students write each ratio as a fraction and then as a percent.

(a) $\frac{3}{8}$ $\text{ANSWERS:} \quad \frac{3}{8} = 37.5\% \\
(b) \quad \frac{4}{5} = 80\% \\
(c) \quad \frac{5}{4} = 125\% \\
(d) \quad \frac{3}{6} = \frac{1}{2} = 50\% \\
(e) \quad \frac{6}{3} = 2 = 200\% \\
(f) \quad \frac{8}{9} = 88.\overline{8}\% \\

Have students order the ratios from smallest to largest.

ANSWERS: $3 : 8 < 3 : 6 < 4 : 5 < 8 : 9 < 5 : 4 < 6 : 3$ \\

Now, explain to students that a rate cannot be thought of as a percent because, unlike a ratio, it does not compare two numbers. Instead, a rate compares two quantities with different units. For example, the rate 200 km / 3 hours does not refer to any percent of anything, because 200 km cannot be thought of as a percent of the 3 hours. Since kilometres and hours are different units, it makes no sense to talk about one as a percent of the other. However, we can still compare different speeds by writing them as unit rates.

Have students explain why the ratio 35 : 120 can be thought of as a percent, but the rate 35 L per 120 km cannot be.

EXTRA PRACTICE:
Either express each given quantity as a percent or explain why you cannot.

(a) 85 : 2 \\
(b) 85 words / minute

The meaning of a/b in context. Tell students that the notation a/b can represent a fraction, a rate, a ratio, a quotient, or a probability. 

ASK: What probability does 3/4 represent for this spinner?

ANSWER: The probability of spinning red.
Have students name some contexts where \( \frac{3}{4} \) represents a fraction. **EXAMPLES:** 3/4 of the pie was eaten; 3/4 of the students are girls; 3/4 of the spinner is red.

Tell students that there are 3 girls for every 4 boys, and write on the board:

\[
\frac{3 \text{ girls}}{4 \text{ boys}}
\]

**ASK:** Are the girls a fraction of the boys? (no, the girls are not part of the set of boys) What does 3/4 represent now? (a ratio)

Remind students that \( \frac{C}{d} \) is a short way of writing \( \frac{\text{Circumference}}{\text{diameter}} \).

Review these terms, and then **ASK:** Is this a fraction? Is the circumference a fraction of the diameter? (no) How do you know? (the circumference of a circle is not part of the diameter; they are two separate lengths) Does this represent a probability? (no) How do you know? (a probability cannot be greater than 1. Is it a rate? (no, the units of the circumference and the diameter are the same and would cancel each other out) What is it then? (a ratio, or a quotient) Explain that the ratio \( \frac{C}{d} = \pi \) means the same thing as the quotient \( C \div d = \pi \), or that \( C = \pi \times d \). In other words, the notation \( C/d \) shows how two lengths compare to each other.

Write on the board:

1. fraction
2. rate
3. ratio
4. quotient
5. probability

Have students signal (by holding up the corresponding number of fingers) whether 4/9 represents a fraction, a rate, a ratio, a quotient, or a probability in different situations, but point out that there are often two correct answers. **EXAMPLES:**

a) 8 is 4/9 as far from 0 as 18 is. (a ratio)

b) When 9 people shared 4 pizzas, everyone got 4/9 of a pizza. (fraction or quotient)

c) I travelled 4 kilometres for every 9 minutes. (rate)

d) I used 4 tablespoons of flour for every 9 tablespoons of milk. (ratio)

e) The chance of spinning red is 4/9. (probability of spinning red)
Extensions

1. Challenge students to draw a spinner where $\frac{4}{9}$ of the spinner is red but the probability of spinning red is not $\frac{4}{9}$.

**SAMPLE ANSWERS:** (students do not have to prove that their answers work)

2. Project idea. Many websites provide estimated walking times from one point to another, but not estimated biking times. If your students can solve proportions, they can use estimated walking times to estimate biking times. First, have students solve the following problem:

   Rani takes 20 minutes to bike to school. An online search tells her that walking to school along the same route will take 45 minutes. The same website tells her that walking to a friend’s place will take 36 minutes. How long will it take Rani to bike to her friend’s place?

**ANSWER:**

   Solve

   \[
   \frac{20 \text{ minutes biking}}{45 \text{ minutes walking}} = \frac{? \text{ minutes biking}}{36 \text{ minutes walking}}
   \]

   The ratio becomes \( \frac{20}{45} = \frac{?}{36} \) or \( \frac{4}{9} = \frac{?}{36} \). Multiplying both terms of the complete ratio by 4 results in \( \frac{16}{36} \), so Rani can bike to her friend’s house in 16 minutes.

   If some students enjoy biking, have them time themselves biking a specific route and then find the estimated walking time for the same route on a “get directions” website. Students can then determine how long it would take those students to bike from school to various locations in your city (**EXAMPLES:** a public library, city hall, a museum, a police station, a hospital, a sports stadium). Students can work in groups of 4 to 7. The biker finds the estimated walking time and determines the actual biking time, and gives this as a unit ratio to the other group members. Each remaining group member chooses a different location and finds the estimated walking time from the school to that location using the same “get directions” website, then uses the unit ratio to determine how long it would take the biker to travel from school to the location. Students can order the trip times to different locations, from longest to shortest.

   Students should not be encouraged to compare times between groups. However, students can share with the class the order of their trips and compare whether the same route took the longest time for all bikers.
3. There are many 4-legged animals and many 2-legged birds in a group. The unit ratio of legs to heads is a whole number.

**QUESTION:** What is the ratio of animals to birds?

**SOLUTION:**

a) What is the unit ratio of legs to heads in the population of birds? 
(2 : 1)

b) What is the unit ratio of legs to heads in the population of animals? 
(4 : 1)

c) Explain why the unit ratio of legs to heads in the total population must be in between the answers to a) and b). (because there are both animals and birds in the group)

d) The ratio of legs to heads is given to be a whole number. What is the ratio of legs to heads? How do you know? (3 : 1 because 3 is the only whole number between 2 and 4)

e) Let \(x\) be the number of animals and \(y\) be the number of birds. Write an expression for:

i) the total number of legs in the group \((4x + 2y)\)

ii) the total number of heads in the group \((x + y)\)

f) Using the variables \(x\) and \(y\), write an expression for the ratio of the total number of legs to the total number of heads. \((4x + 2y : x + y)\)

g) Using your answers to d) and f), write an equation that shows the ratio of total legs to total heads in two ways.

\[
\frac{4x + 2y}{x + y} = \frac{3}{1}
\]

h) Use g) to find the ratio of \(x : y\). (Since the ratio of legs to heads is 3 : 1, the total number of legs, \(4x + 2y\), must be 3 times the total number of heads, \(x + y\), so \(4x + 2y = 3(x + y) = 3x + 3y\). Then \(4x - 3x = 3y - 2y\), so \(x = y\). So the ratio of animals to birds is 1 : 1.)

4. Can 85 : 2 : 17 be thought of as a percent? (No, because it cannot be thought of as a fraction. If there were a percent, there would be a fraction too, and there are no 3-term fractions).
Ratios and rates with fractional terms. Tell students that Jane bikes 10 km in half an hour. **ASK**: How far does she bike in 1 hour if she bikes at the same speed? (20 km) Write on the board:

\[
\begin{align*}
\frac{1}{2} \text{ hours} & \times 2 \\
\frac{10}{\text{km}} & = \frac{20}{? \text{ km}}
\end{align*}
\]

Explain that because 1 is double 1/2, you have to double the kilometres as well. Explain that, although written here as a fraction, 1/2 over 10 is a ratio or rate, not a fraction, because 1/2 is not a whole number.

**ASK**: Do you find fractional terms or whole numbers easier to use? What do you have to do to both terms to keep the ratio the same? (multiply both terms by the same number)

**Review multiplying fractions by a whole number.** You can multiply 5 × 2/3 by adding five two-thirds. Write on the board:

\[
\begin{align*}
\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} & = \\
\frac{10}{3} & = \frac{10}{3}
\end{align*}
\]

Have students multiply:

- a) 3 × 1/4
- b) 7 × 2/5
- c) 4 × 2/3
- d) 5 × 1/2
- e) 2 × 3/5

**ANSWERS**: a) 3/4  b) 14/5  c) 8/3  d) 5/2  e) 6/5

Have students multiply and reduce to lowest terms:

- a) 3 × 1/3
- b) 4 × 3/4
- c) 5 × 2/5
- d) 6 × 5/6
- e) 7 × 3/7

**ANSWERS**: a) 1  b) 3  c) 2  d) 5  e) 3
ASK: What do you notice about the answers to the last set of questions that is different from before? (they are all whole numbers) Tell students that you can always turn a fraction into a whole number by multiplying by the denominator.

ASK: What number can I multiply by to turn the fractional term of each ratio into a whole number?

a) 2/3 : 4  b) 3/5 : 7  c) 6 : 1/2  d) 3/4 : 7

ANSWERS: a) 3  b) 5  c) 2  d) 4

Then have students turn these ratios into ratios where both terms are whole numbers:

a) 1/3 : 5  b) 2 : 1/4  c) 4/5 : 3  d) 3 : 5/6

ANSWERS: a) 1 : 15  b) 8 : 1  c) 4 : 15  d) 18 : 5

Now have students change both terms into whole numbers, and reduce to lowest terms where necessary. Show students how they can change the first fraction to a whole number and then the second fraction in two separate steps. EXAMPLE: 2/3 : 4/5 = 2 : 12/5 = 10 : 12 = 5 : 6

Then have students do these problems:

a) 2/3 : 3/5  b) 3/8 : 3/4  c) 5/8 : 2/3  d) 4/5 : 2/7  e) 2/5 : 3/10

ANSWERS: a) 10 : 9  b) 3 : 6 = 1 : 2  c) 15 : 16  d) 28 : 10 = 14 : 5  e) 20 : 15 = 4 : 3

Remind students that to multiply a fraction by a whole number, you multiply only the numerator by that number. ASK: Is there a way to change both fractions to a whole number in one step? What one number can you multiply the numerators by? (the LCM of the denominators) If students say the product of the denominators, ASK: Is there a smaller number that you can multiply by instead? Have students look at part b above. In this case, multiplying by the product of the denominators gives

\[
\frac{3}{8} : \frac{3}{4} = \frac{3}{8} \times 32 : \frac{3}{4} \times 32 \\
= 3 \times 32 \div 8 : 3 \times 32 \div 4 \\
= 3 \times 4 : 3 \times 8 \\
= 12 : 24 \\
= 1 : 2
\]

This gives the right answer, but we could have multiplied both by 8 (the LCM) instead:

\[
\frac{3}{8} : \frac{3}{4} = \frac{3}{8} \times 8 : \frac{3}{4} \times 8 \\
= 3 \times 8 \div 8 : 3 \times 8 \div 4 \\
= 3 : 6 \\
= 1 : 2
\]
ASK: How does looking at the whole-number ratios make it easier to tell which fraction in each pair is bigger? (The fraction is bigger for whichever whole number is bigger.) Write on the board:

2/3 is ___________ than 3/5
3/8 is ___________ than 3/4
5/8 is ___________ than 2/3
4/5 is ___________ than 2/7
2/5 is ___________ than 3/10

Give students time to answer each question individually. Have students signal their answers by making an L for less or an M for more with their fingers. Alternatively, students can point in the direction of the larger number.

ANSWERS: more, less, less, more, more.

Multiplying decimal tenths by 10 to turn them into whole numbers. Write on the board: 2.7 : 3. ASK: How can I find an equivalent ratio so that both parts of the ratio are whole numbers? What is an easy number to multiply 2.7 by to get a whole number? (multiply 2.7 by 10 to get 27) What do we have to multiply 3 by to keep the ratio the same? (10) Write 2.7 : 3 = 27 : 30. Emphasize that the ratio is easier to work with now that both terms are whole numbers. ASK: Can we reduce this ratio? Yes, 27 : 30 = 9 : 10.

Have students write the following ratios in terms of whole numbers and then reduce where possible.

a) 2.6 to 13  b) 1.4 to 5  c) 6.3 to 3  d) 2.4 to 1.4  e) 1.8 to 2

 Bonus → 1.26 to 2.1

ANSWERS: a) 1 to 5  b) 7 to 25  c) 21 to 10  d) 12 to 7  e) 9 to 10

Word problems with ratios and rates.

1. Lina accidentally added 1/2 cup of water to 1 cup of oatmeal to make hot cereal. The recipe called for the opposite: 1 cup of water and 1/2 cup of oatmeal. How much water should she add to make the ratio correct?

(ANSWER: First solve 1 cup water to 1/2 cup oatmeal = ____ cups water to 1 cup oatmeal. The answer is 2 cups water. Lina already used 1/2 cup of water so she needs to add 1 1/2 cups more.)

PROCESS ASSESSMENT
[CN, PS], 8m1, 8m5

2. Tom is training for a 5-km run. He wants to cover the 5 km in 30 minutes. This is Tom’s training schedule:

Step 1: Run for 1 minute, walk for 4 minutes, repeat 6 times.
Step 2: Run for 2 minutes, walk for 3 minutes, repeat 6 times.
Step 3: Run for 3 minutes, walk for 2 minutes, repeat 6 times.
Step 4: Run for 4 minutes, walk for 1 minute, repeat 6 times.
Step 5: Run for 30 minutes.

PROCESS ASSESSMENT
[CN], 8m5
Tom repeats each step until he is running comfortably at the pace he wants to run at during the race. How far should he be able to run during the 1 minute in Step 1 before he moves to Step 2? How far should he be able to run during the 2 minutes in Step 2 before he moves to Step 3? Repeat for Steps 3 and 4.

ANSWERS:
Step 1: $5000 \text{ m : } 30 \text{ minutes} = x \text{ m : } 1 \text{ minute}$,
so $x = 5000 \div 30 = 166.7 \text{ m}$
Step 2: 333.3 m
Step 3: 500 m
Step 4: 666.7 m

3. Nomi can record 68 hours of video on the 160-gigabyte hard drive of her DVR when it is set to SP mode (a medium- to high-quality recording mode). A better-quality recording mode uses more gigabytes and so reduces the number of hours of video that can be recorded. Nomi also has a 4.7-gigabyte DVD+R disk. Allow students to use a calculator to solve these questions.

a) How many hours of video can Nomi record on her 4.7-gigabyte DVD+R if she uses the same recording mode?

SOLUTION: We have $\frac{68}{160} = \frac{?}{4.7}$
We need to fill the blanks in $4.7 \times \underline{\text{_____}} = 160$ and $? \times \underline{\text{_____}} = 68$
with the same number.

On a calculator: $160 \div 4.7 \approx 34$ so $? \approx 68 \div 34 = 2$. OR Cross multiply to obtain $68 \times 4.7 = 160 \times ?$, so $? = 68 \times 4.7 \div 160 \approx 2$.

NOTE: When a DVD+R claims to be capable of recording 2 hours of video, it is assuming the SP recording mode.

b) Here is a list of some recording modes and the number of hours that can be recorded on Nomi’s DVR.

<table>
<thead>
<tr>
<th>Recording mode</th>
<th>Number of hours of video recording on 160 GB DVR</th>
</tr>
</thead>
<tbody>
<tr>
<td>SP</td>
<td>68</td>
</tr>
<tr>
<td>LSP</td>
<td>84</td>
</tr>
<tr>
<td>ESP</td>
<td>100</td>
</tr>
<tr>
<td>LP</td>
<td>135</td>
</tr>
<tr>
<td>EP</td>
<td>200</td>
</tr>
<tr>
<td>SLP</td>
<td>270</td>
</tr>
<tr>
<td>SEP</td>
<td>340</td>
</tr>
</tbody>
</table>

Nomi wants to put a series of five 1-hour videos on one 4.7-gigabyte DVD+R to give to a friend. What is the highest quality recording mode she can use to do this?
SOLUTION 1: Use the same method as in part a) for each of the recording modes. SP results in 2 hours, LSP results in 2.5 hours, ESP results in 3 hours, LP results in 4 hours, and EP results in 6 hours, so Nomi can use the EP recording mode in order to fit 5 hours of video onto the DVD+R.

SOLUTION 2: Since we need to record two and a half times as much video as in part a), we need a recording mode that allows two and a half times more time. Solve the proportion $68 : ? = 2 : 5$. Since $2 \times 34 = 68$, multiply $5 \times 34 = 170$. Since EP is the highest-quality recording mode that allows at least 170 hours of recording where SP allows 68, Nomi needs to use the EP recording mode.

4. A pack of 24 CDs costs $7.99. A pack of 50 CDs costs $10.45. What is the most economical way to purchase a) 130 CDs? b) 170 CDs?

**ANSWER:** a) 24 is less than half of 50, but $7.99 is more than half of $10.45, so 50 CDs is definitely a better buy than 2 packs of 24 CD. Buying 2 packs of 50 and 1 pack of 24 won’t be enough, so we need to buy 3 packs of 50 (which costs less than 2 packs of 50 and 2 packs of 24).

b) Buy 3 packs of 50 and then 1 pack of 24.

5. You are shopping for cat food in a grocery store, but didn’t bring your calculator to compare unit prices. A 624-g can of Cat Food A costs 95¢, and a 376-g can of Cat Food B costs 55¢. Normally, the larger can is a better buy per unit price, but today the smaller can is on sale, so you wonder if it is a better buy than the larger volume. Estimate which one is a better buy as follows:

a) Estimate the ratio of grams of cat food for Cat Food A to Cat Food B.

b) How many cans of each do you need to buy to get approximately the same amount of cat food?

c) Calculate the costs for the amounts you found in b). Which costs less for the same amount of food? Which cat food is a better buy—A or B?

**ANSWERS:**

a) $625 : 375 = 125 : 75 = 25 : 15 = 5 : 3$

b) 3 cans of Cat Food A is about the same amount as 5 cans of Cat Food B. Indeed, $3 \times 624 \text{ g} = 1872 \text{ g}$ and $5 \times 376 = 1880 \text{ g}$.

c) 3 cans of Cat Food A cost $3 \times 95¢ = $2.95$, whereas 5 cans of Cat Food B cost $5 \times 55¢ = $2.75$. In fact, with Cat Food B, you get 1 880 g for $2.75, whereas for Cat Food A, you get 1 872 g for $2.95. You get more and pay less with Cat Food B, so Cat Food B is a better buy.
Extensions

1. Have students create a scale drawing of their living room on 1-cm grid paper. Mark what measurement each centimetre represents.

2. **Golden Ratio.** The golden ratio is about 1.62 : 1, or just 1.62. This ratio occurs often in human design and architecture and also in nature. Some people believe that rectangles with the ratio of length to width close to the golden ratio are nicer to look at. Do an experiment to see if your class agrees. Give each student the same 5 rectangles (3 cm by 5 cm, 8.5 cm by 11 cm, 3 cm by 12 cm, 8 cm by 10 cm, and 7 cm by 7 cm) and have students choose the two they think are nicest to look at. Do not tell students the reason for the experiment; otherwise, the results will be biased. Count the total number of votes each rectangle got. Then have students calculate and write the ratio of length to width for each rectangle as a unit ratio and put the rectangles in order accordingly, from “closest to golden ratio” to “furthest from golden ratio.” Did the class find the rectangles that were closer to golden ratios nicer to look at?

Have students find the ratios of length to width in the real-life rectangles below. They will have to do some measuring.

a) several picture frames
b) a standard sheet of paper
c) a notebook
d) a television screen
e) a computer monitor
f) a postcard
g) a birthday card
h) a standard sheet of paper folded in half
i) an index card

Which rectangles are close to golden ratios?

b) The Fibonacci sequence starts 1 1 2 3 5 8 13 21. Find the next three terms. The write the ratio and the unit ratio of one term to the previous term for the first 11 terms:

<table>
<thead>
<tr>
<th>ratio</th>
<th>1 : 1</th>
<th>2 : 1</th>
<th>3 : 2</th>
<th>5 : 3</th>
<th>8 : 5</th>
<th>13 : 8</th>
<th>21 : 13</th>
</tr>
</thead>
<tbody>
<tr>
<td>unit ratio</td>
<td>1</td>
<td>2</td>
<td>1.5</td>
<td>1.67</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What ratio do these numbers appear to be getting close to? (the golden ratio)
Percent Strips

0% 10% 20% 30% 40% 50% 60% 70% 80% 90% 100%

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100
### Price List

<table>
<thead>
<tr>
<th>Before Taxes</th>
<th>Cashier 1</th>
<th>Cashier 2</th>
<th>After Taxes</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.99</td>
<td>5.6387</td>
<td>6.6387</td>
<td>5.6387</td>
</tr>
<tr>
<td>5.49</td>
<td>6.2037</td>
<td>7.2037</td>
<td>7.2037</td>
</tr>
<tr>
<td>5.99</td>
<td>6.7687</td>
<td>7.7687</td>
<td>6.6387</td>
</tr>
<tr>
<td>6.49</td>
<td>7.3337</td>
<td>8.3337</td>
<td>8.3337</td>
</tr>
<tr>
<td>6.99</td>
<td>7.8987</td>
<td>8.8987</td>
<td>7.8987</td>
</tr>
<tr>
<td>7.49</td>
<td>8.4637</td>
<td>9.4637</td>
<td>9.4637</td>
</tr>
<tr>
<td>7.99</td>
<td>9.0287</td>
<td>10.0287</td>
<td>10.0287</td>
</tr>
<tr>
<td>8.49</td>
<td>9.5937</td>
<td>10.5937</td>
<td>10.5937</td>
</tr>
<tr>
<td>8.99</td>
<td>10.1587</td>
<td>11.1587</td>
<td>10.1587</td>
</tr>
<tr>
<td>9.49</td>
<td>10.7237</td>
<td>11.7237</td>
<td>10.7237</td>
</tr>
<tr>
<td>9.99</td>
<td>11.2887</td>
<td>12.2887</td>
<td>11.2887</td>
</tr>
<tr>
<td>10.49</td>
<td>11.8537</td>
<td>12.8537</td>
<td>12.8537</td>
</tr>
<tr>
<td>11.49</td>
<td>12.9837</td>
<td>13.9837</td>
<td>13.9837</td>
</tr>
<tr>
<td>12.49</td>
<td>14.1137</td>
<td>15.1137</td>
<td>15.1137</td>
</tr>
<tr>
<td>13.49</td>
<td>15.2437</td>
<td>16.2437</td>
<td>16.2437</td>
</tr>
<tr>
<td>13.99</td>
<td>15.8087</td>
<td>16.8087</td>
<td>16.8087</td>
</tr>
<tr>
<td>14.49</td>
<td>16.3737</td>
<td>17.3737</td>
<td>17.3737</td>
</tr>
<tr>
<td>14.99</td>
<td>16.9387</td>
<td>17.9387</td>
<td>16.9387</td>
</tr>
<tr>
<td>15.49</td>
<td>17.5037</td>
<td>18.5037</td>
<td>18.5037</td>
</tr>
<tr>
<td>15.99</td>
<td>18.0687</td>
<td>19.0687</td>
<td>18.0687</td>
</tr>
<tr>
<td>16.49</td>
<td>18.6337</td>
<td>19.6337</td>
<td>19.6337</td>
</tr>
<tr>
<td>17.49</td>
<td>19.7637</td>
<td>20.7637</td>
<td>20.7637</td>
</tr>
<tr>
<td>17.99</td>
<td>20.3287</td>
<td>21.3287</td>
<td>20.3287</td>
</tr>
<tr>
<td>18.49</td>
<td>20.8937</td>
<td>21.8937</td>
<td>21.8937</td>
</tr>
<tr>
<td>19.49</td>
<td>22.0237</td>
<td>23.0237</td>
<td>23.0237</td>
</tr>
<tr>
<td>19.99</td>
<td>22.5887</td>
<td>23.5887</td>
<td>23.5887</td>
</tr>
<tr>
<td>20.49</td>
<td>23.1537</td>
<td>24.1537</td>
<td>24.1537</td>
</tr>
<tr>
<td>20.99</td>
<td>23.7187</td>
<td>24.7187</td>
<td>23.7187</td>
</tr>
<tr>
<td>21.49</td>
<td>24.2837</td>
<td>25.2837</td>
<td>25.2837</td>
</tr>
<tr>
<td>21.99</td>
<td>24.8487</td>
<td>25.8487</td>
<td>25.8487</td>
</tr>
<tr>
<td>22.49</td>
<td>25.4137</td>
<td>26.4137</td>
<td>26.4137</td>
</tr>
<tr>
<td>22.99</td>
<td>25.9787</td>
<td>26.9787</td>
<td>25.9787</td>
</tr>
<tr>
<td>23.49</td>
<td>26.5437</td>
<td>27.5437</td>
<td>27.5437</td>
</tr>
<tr>
<td>23.99</td>
<td>27.1087</td>
<td>28.1087</td>
<td>28.1087</td>
</tr>
<tr>
<td>24.49</td>
<td>27.6737</td>
<td>28.6737</td>
<td>28.6737</td>
</tr>
<tr>
<td>25.49</td>
<td>28.8037</td>
<td>29.8037</td>
<td>28.8037</td>
</tr>
<tr>
<td>25.99</td>
<td>29.3687</td>
<td>30.3687</td>
<td>29.3687</td>
</tr>
<tr>
<td>26.49</td>
<td>29.9337</td>
<td>30.9337</td>
<td>30.9337</td>
</tr>
<tr>
<td>26.99</td>
<td>30.4987</td>
<td>31.4987</td>
<td>30.4987</td>
</tr>
<tr>
<td>27.49</td>
<td>31.0637</td>
<td>32.0637</td>
<td>32.0637</td>
</tr>
<tr>
<td>27.99</td>
<td>31.6287</td>
<td>32.6287</td>
<td>31.6287</td>
</tr>
<tr>
<td>28.49</td>
<td>32.1937</td>
<td>33.1937</td>
<td>33.1937</td>
</tr>
<tr>
<td>28.99</td>
<td>32.7587</td>
<td>33.7587</td>
<td>32.7587</td>
</tr>
<tr>
<td>29.49</td>
<td>33.3237</td>
<td>34.3237</td>
<td>33.3237</td>
</tr>
</tbody>
</table>
## Three Types of Percent Problems

### What percent is it?

<table>
<thead>
<tr>
<th>Given</th>
<th>the whole and the part</th>
</tr>
</thead>
<tbody>
<tr>
<td>To find</td>
<td>What percent of the whole is the part?</td>
</tr>
</tbody>
</table>
| Equation       | \[
\frac{\text{part}}{\text{whole}} = \frac{?}{100}
\] |

**Examples**
1. What percent of 50 is 10?  
2. 10 is what percent of 50? \[
\frac{\text{part}}{\text{whole}} = \frac{10}{50} = \frac{?}{100}
\]
3. A shirt costs $50.  
   It is $10 off.  
   What percent off is it?  

### How much is the percent?

<table>
<thead>
<tr>
<th>Given</th>
<th>the whole and a percent, (P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>To find</td>
<td>How much is this percent of the whole?</td>
</tr>
</tbody>
</table>
| Equation      | \[
\frac{?}{\text{whole}} = \frac{P}{100}
\] |

**Examples**
1. What is 30% of 50?  
2. 30% of 50 is what number? \[
\frac{\text{part}}{\text{whole}} = \frac{30}{50} = \frac{?}{100}
\]
3. A meal costs $50.  
   Tip and taxes together are 30%.  
   What is the tip and taxes together?  

### How much is the whole?

<table>
<thead>
<tr>
<th>Given</th>
<th>a part that is (P)% of the whole</th>
</tr>
</thead>
<tbody>
<tr>
<td>To find</td>
<td>How much is the whole?</td>
</tr>
</tbody>
</table>
| Equation      | \[
\frac{\text{part}}{?} = \frac{P}{100}
\] |

**Examples**
1. If 5 is 40%, what is the number?  
2. 5 is 40% of what number?  
3. A T-shirt was 40% off.  
   The price was reduced by $5.  
   What was the original price?
Thousandths Grid Practice (1)

1. Each small rectangle is one thousandth of the whole grid. Write how many thousandths are shaded, and then write the amount shaded as a decimal and as a percent.

   a) ______ thousandths
      0.______ = ______% 

   b) ______ thousandths
      0.______ = ______% 

   c) ______ thousandths
      0.______ = ______% 

   d) ______ thousandths
      0.______ = ______% 

2. Write the decimals from Question 1 in order from smallest to largest.
   _________ < _________ < _________ < _________
3. Each small rectangle is one thousandth of the whole grid.
   a) Shade 38.4% of the grid.
   b) Shade 7.5% of the grid.
   c) Shade 80.3% of the grid.
   d) Shade more than 41% but less than 42% of the grid.

   What percent did you shade? ________

4. Write the percents from Question 3 as decimals.
   a) ____________
   b) ____________
   c) ____________
   d) ____________

5. Write the decimals from Question 4 in order from largest to smallest.
   ____________ > ____________ > ____________ > ____________
PS8-2  Searching Systematically I

Teach this lesson after: 8.2 Unit 1

Goals:
Students will use systematic search to solve ratio problems and to find numbers satisfying constraints, including both positive and negative numbers.

Prior Knowledge Required:
Can identify and create equivalent ratios, including three-term ratios
Can add integers
Can multiply integers
Can find the area of a trapezoid (for Problem Bank 3)

Vocabulary: integer, ratio

Review using tables to solve ratio problems. Write on the board:

The ratio of adults to children on a bus is 2 : 5.
There are 35 people on the bus.
How many children are on the bus?

SAY: You can make a table to solve this type of problem. Draw on the board:

<table>
<thead>
<tr>
<th>Adults</th>
<th>Children</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

SAY: You start with two adults and five children. Every time that you add two more adults, you add five more children. Then you calculate the total each time until you get 35 because that’s what the question says to look for. Have volunteers add more rows to the table until the total is 35. (see completed table below)

<table>
<thead>
<tr>
<th>Adults</th>
<th>Children</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>21</td>
</tr>
<tr>
<td>8</td>
<td>20</td>
<td>28</td>
</tr>
<tr>
<td>10</td>
<td>25</td>
<td>35</td>
</tr>
</tbody>
</table>

ASK: So how many children are on the bus? (25)
**Exercises:** Solve the ratio problem using a table.

a) Jack mixed red and blue paint together in the ratio 3 : 4. He used 35 cups of paint altogether. How much red paint did Jack use?

b) Ken used red, blue, and white paint in the ratio 3 : 4 : 2. He used 54 cups of paint altogether. How much of each colour of paint did he use?

c) The ratio of girls to boys in a classroom is 5 : 7. There are six more boys than girls. How many girls and how many boys are in the class?

**Bonus:** Solve the same problems using a tape diagram. Make sure you get the same answer.

**Answers:** a) 15 cups of red paint; b) 18 cups of red paint, 24 cups of blue paint, and 12 cups of white paint; c) 15 girls and 21 boys

**Solving problems where the ratio changes.** SAY: Now let’s make it harder. Write on the board:

The ratio of girls to boys is 3 : 4.

Three more girls joined the class.

Now the ratio of girls to boys is 9 : 10.

How many boys are in the class?

Challenge students to come up with a table to draw for this problem. PROMPT: Think about what headings you need. After a minute, ask students for suggested headings (number of girls, number of boys, new number of girls, new girl : boy ratio). Draw on the board:

<table>
<thead>
<tr>
<th>Number of Girls</th>
<th>Number of Boys</th>
<th>New Number of Girls</th>
<th>New Girl : Boy Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>6</td>
<td>6 : 4</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>9</td>
<td>9 : 8</td>
</tr>
</tbody>
</table>

Have students continue the table until the ratio is 9 : 10. (15 girls and 20 boys becomes 18 girls and 20 boys)

**Exercises:** Draw a table to solve the problem.

a) The ratio of girls to boys in a class is 2 : 3. Then five more girls and two more boys joined the class.

   Now the ratio is 1 : 1. How many girls are now in the class?

b) The ratio of girls to boys is 5 : 3. Five girls moved to another class.

   Now the ratio is 5 : 4. How many girls are now in the class?

**Answers:** a) 11, b) 15

**NOTE:** Students will solve this type of problem using algebra in Lesson PS8-3.

**Finding numbers that satisfy constraints.** Write on the board:

Two numbers that add to 12: Two numbers that multiply to 35:
SAY: I am trying to find two numbers that add to 12 and that multiply to 35. ASK: Do you think it would be quicker to list all the numbers that add to 12 or all the numbers that multiply to 35? (take all predictions) Then have students make both lists on the board. (see below)

Two numbers that add to 12:  Two numbers that multiply to 35:
1 and 11   1 and 35
2 and 10   5 and 7
3 and 9
4 and 8
5 and 7
6 and 6

Have students search the first list for two numbers that multiply to 35 (5 and 7). Then have them search the second list for two numbers that add to 12 (5 and 7). ASK: Which list was quicker to search through? (two numbers that multiply to 35) Why was it quicker to search through? (there are only two pairs of numbers to check)

Exercises: Find all pairs of numbers that satisfy both properties. Start by predicting which list will be shorter and make that list first.

a) Two numbers that add to 8 and multiply to 15
b) Two numbers that add to 9 and multiply to 14

Bonus: Two numbers that add to 30 and multiply to 216

Answers: a) 3 and 5, b) 2 and 7, Bonus: 12 and 18

Finding numbers that satisfy constraints involving negative integers. SAY: Sometimes the numbers you are adding or multiplying are negative. Write on the board:

Two numbers add to $-12$ and multiply to $+35$.

ASK: Do the two numbers being added and multiplied have the same sign or the opposite sign? (the same sign) How do you know? (they multiply to a positive number) Are they both positive or both negative? (they are both negative) How do you know? (they add to a negative number)

What are all the pairs of negative numbers that multiply to 35? ($-1$ and $-35$, $-5$ and $-7$) Write the pairs on the board:

$-1$ and $-35$
$-5$ and $-7$

ASK: Which two numbers add to $-12$? ($-5$ and $-7$)

Exercises: Find the pair of numbers that satisfy both properties.

a) They add to $-15$ and multiply to 44.
b) They add to 7 and multiply to $-44$.
c) They add to $+2$ and multiply to $-35$.
d) They add to $-2$ and multiply to $-35$.

Answers: a) $-11$ and $-4$, b) 11 and $-4$, c) 7 and $-5$, d) $-7$ and 5
SAY: You can also do the same thing with three numbers. I want to find all the ways that three positive integers can multiply to 30. Let’s start by listing the factors of 30. Have a volunteer list the factors of 30 on the board. (see below)

The factors of 30 are: 1, 2, 3, 5, 6, 10, 15, 30.

SAY: The factors of 30 tell us ways that two numbers can multiply to 30: 1 × 30, 2 × 15, 3 × 10 and 5 × 6, but I want to find all the ways that three positive numbers multiply to 30. I can start listing the possible smallest numbers in order and then see what the other two numbers have to be, based on what they have to multiply to. Draw on the board:

<table>
<thead>
<tr>
<th>Smallest Number</th>
<th>Middle Number</th>
<th>Largest Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>30</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

SAY: If the smallest number is 1, I have to find all the pairs that multiply to 30. If the smallest number is 2, I have to list all the pairs that multiply to 15. I listed 3 and 5, but not 1 and 15.

ASK: Why didn’t I list 1 and 15? (2 wouldn’t be the smallest number) SAY: We already listed 2, 1, and 15 as 1, 2, and 15 because that has 1 as the smallest number. Now let’s try 3 as the smallest number. Add another row to the table, as shown below:

3

ASK: What do the other two numbers have to multiply to? (10) What would the other numbers have to be? (1 and 10 or 2 and 5) ASK: Can the other two numbers be 1 and 10? (no) Why not? (3 wouldn’t be the smallest number) Can the other two numbers be 2 and 5? (no) Why not? (3 wouldn’t be the smallest number) If three numbers multiply to 30, can 3 be the smallest number? (no) Can 4 be the smallest number? (no) Why not? (4 is not a factor of 30) Can 5 be the smallest number? (no) Why not? (the other two numbers would have to multiply to 6, so at least one of them would have to be smaller than 5) SAY: These are all the possible ways that three positive integers can multiply to 30.

Exercises: Find all the possible ways that …
a) three positive integers can multiply to 10.
b) three positive integers can multiply to 18.
c) three positive integers can multiply to 24.

Answers:
a) 1, 1, 10; 1, 2, 5
b) 1, 1, 18; 1, 2, 9; 1, 3, 6; 2, 3, 3
c) 1, 1, 24; 1, 2, 12; 1, 3, 8; 1, 4, 6; 2, 2, 6; 2, 3, 4
Now refer to the table on the board with three numbers that multiply to 30. Write on the board:

Three integers multiply to 30 and add to 6.

ASK: Do any of the positive numbers work? (no) SAY: So, some numbers have to be negative.
ASK: How many of the three numbers will be negative? (2) How do you know? (if one or three
of the numbers are negative, the product would be negative) SAY: So, by adding the three
integers, you are really adding one of the positive numbers and subtracting two of them.
The answer is positive 6. Challenge students to find the three numbers by looking at the table.
(add 10 and subtract 1 and 3, so the three numbers are −1, −3, and 10)

**Exercises:** Find three integers that …
a) multiply to 30 and add to −10.  
b) multiply to 30 and add to +10.  
c) multiply to −30 and add to +6.  
d) multiply to 18 and add to −2.  
e) multiply to −18 and add to +6.  
f) multiply to −24 and add to 1.

**Bonus:** Find two ways that three integers can …
g) multiply to 24 and add to 9.  
h) multiply to 30 and add to 0.

**Answers:** a) 1, −5, −6; b) 2, 3, 5; c) −2, 3, 5; d) −2, −3, 3; e) 1, 3, −6; f) 2, 3, −4;
Bonus: g) 2, 3, 4 or −1, −2, 12; h) −1, −5, 6 or −2, −3, 5

**Problem Bank**
1. The ratio of the three sides of a triangle is 2 : 3 : 4. The perimeter of the triangle is 45 cm.
What are the three side lengths?
**Answer:** 10 cm, 15 cm, 20 cm

2. a) The ratio of length to width in a rectangle is 3 : 1. The area is 75 cm². What is the length of
the rectangle?
b) The ratio of length to width of a rectangle is 7 : 5 and its perimeter is 48 cm. What is the area?
c) The ratio of length to width of a rectangle is 4 : 3. The perimeter is 84 cm. What is the area?
d) The ratio of length to width in a triangle is 5 : 3. The width is 8 cm shorter than the length.
What is the area?
**Selected solution:** a) Make a table for length, width, and area with the ratio 3 : 1 between
length and width.

<table>
<thead>
<tr>
<th>Length (cm)</th>
<th>Width (cm)</th>
<th>Area (cm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>48</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
<td>75</td>
</tr>
</tbody>
</table>

The length of the rectangle is 15 cm.
**Answers:** b) 140 cm², c) 432 cm², d) 240 cm²

3. In a trapezoid, the ratio base₁ : base₂ : height = 3 : 5 : 4. If the area of the trapezoid is 144 cm²,
what is the height?
**Answer:** 12 cm
4. The ratio of girls to boys is 5 : 3. There are eight more girls than boys. How many girls and how many boys are there?
Answer: 20 girls and 12 boys

5. The ratio of girls to boys to teachers in a school is 8 : 7 : 2, which means that there are eight girls and seven boys for every two teachers. There are 300 students at the school. How many teachers are at the school? Try to look for a shortcut to solve the problem.
Solution: The ratio of students to teachers is 15 : 2 and there are 300 students. Since 300 = 15 × 20, the number of teachers is 2 × 20 = 40.

NOTE: Students will complete the following type of problem using algebra instead of systematic search in Lesson PS8-3.

6. a) The ratio of girls to boys is 5 : 3. One fifth of the girls are away for a sports event. What is the new ratio of girls to boys? Does the answer depend on the number of girls or boys in the class, or just on the ratio?
b) Sun and Marcel have baseball cards in the ratio 4 : 5. Then Sun gave half her cards to Marcel. Now what is the ratio of Sun’s cards to Marcel’s cards? Does the answer depend on the number of baseball cards they had to start, or just on the ratio?
Solutions:
a) Make a table.

<table>
<thead>
<tr>
<th>Girls</th>
<th>Boys</th>
<th>New Number of Girls</th>
<th>New Girl to Boy Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>4</td>
<td>4 : 3</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>8</td>
<td>8 : 6 = 4 : 3</td>
</tr>
<tr>
<td>15</td>
<td>9</td>
<td>12</td>
<td>12 : 9 = 4 : 3</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>16</td>
<td>16 : 12 = 4 : 3</td>
</tr>
</tbody>
</table>

The ratio is always 4 : 3; it doesn’t depend on the number of girls and boys in the class, only on the ratio of girls to boys in the class.
b) Make a table.

<table>
<thead>
<tr>
<th>Sun</th>
<th>Marcel</th>
<th>New Number of Sun’s Cards</th>
<th>New Number of Marcel’s Cards</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>12</td>
<td>15</td>
<td>6</td>
<td>21</td>
</tr>
<tr>
<td>16</td>
<td>20</td>
<td>8</td>
<td>28</td>
</tr>
</tbody>
</table>

The new ratio of Sun’s cards : Marcel’s cards is 2 : 7 = 4 : 14 = 6 : 21 = 8 : 28. It doesn’t depend on the number of baseball cards they had to start, only on the ratio of baseball cards.
**PS8-3 Using Tape Diagrams and Algebra**

**Teach this lesson after:** 8.2 Unit 1

**Goals:**
Students will use two methods (tape diagrams and algebra) to solve percentage and ratio problems, and they will compare the methods.

**Prior Knowledge Required:**
Can add and subtract linear expressions
Can solve linear equations
Can translate between ratios, fractions, and percentages
Can cross multiply to solve proportions
Can add, subtract, multiply, and divide fractions and decimals

**Vocabulary:** cross multiply, percent, proportion, ratio, tape diagram

**Using tape diagrams to solve “times as many” problems.** Write on the board:

Peter made purple paint by mixing red, blue, and white paint.
He used twice as much blue paint as white paint.
He used three times as much red paint as white paint.
If there are 30 cups of purple paint altogether, how much of each colour paint did he use?

Demonstrate how to use a tape diagram to show that Peter used twice as much blue paint as white paint, as shown below:

```
white
blue
```

Tell students that a diagram like this, where each block represents the same amount, is called a tape diagram. Have a volunteer show how to represent the amount of red paint. Then point out that the 30 cups altogether mean that all six blocks total 30 cups. Show this on the diagram on the board:

```
white
blue
red
```

30
ASK: If all six blocks total 30 cups, what does each block represent? (5 cups) How did you get that? (divided 30 by 6) Write “5” inside each block on the tape diagram, as shown below:

white 5
blue 5 5
red 5 5 5

30

SAY: So Peter used 5 cups of white paint, 10 cups of blue paint, and 15 cups of red paint. Keep the problem on the board for later reference.

Exercises: Draw a tape diagram to solve the problem.
a) John has three times as many hockey cards as baseball cards. He has 10 more hockey cards than baseball cards. How many of each type of card does he have?
b) Sara has four times as many hockey cards as basketball cards and twice as many baseball cards as basketball cards. She has 84 cards altogether. How many of each type of card does she have?
Answers: a) 15 hockey cards and 5 baseball cards; b) 12 basketball cards, 24 baseball cards, and 48 hockey cards

Using algebra to solve “times as many” problems. SAY: Let’s go back to the paint problem. Refer students back to the original problem on the board. SAY: This time, let’s use algebra to solve it. Have a volunteer demonstrate how to use algebra to show that Peter used twice as much blue paint as white paint. (see below)

number of cups of white paint = x
number of cups of blue paint = 2x

Leave the volunteer’s work on the board for students to use in the following exercise.

Exercise: Complete the problem on the board using algebra.
Solution: The number of cups of red paint is 3x, and the total number of cups is 30, so x + 2x + 3x = 30, 6x = 30, x = 5. So Peter used x = 5 cups of white paint, 2x = 10 cups of blue paint, and 3x = 15 cups of red paint.

Comparing using algebra with using tape diagrams to solve “times as many” problems. ASK: What does the x from the algebra solution represent in the tape diagram solution? (what each block is worth) What else is the same in the two methods? (they both solved “6 times what is 30?”; they got the same answer) To summarize, SAY: In both solutions, you determined “6 times what is 30?” to figure out what one block is worth, and you then used it to figure out what all the values were.

Using tape diagrams to solve ratio problems. Write on the board:

There are 10 more boys than girls in a choir. The ratio of girls to boys is 3 : 5. How many girls and how many boys are in the choir?
Give students time to read the problem, then draw a tape diagram to represent the situation, as shown below:

```
girls         boys
   [ ] [ ] [ ] [ ]                 [ ] [ ] [ ] [ ] [ ]
```

SAY: For every three blocks representing girls, there are five blocks representing boys. The problem tells us that there are 10 more boys than girls in the choir. ASK: How can we show that on the tape diagram? (the two extra blocks for boys represents 10 students) Label the extra blocks, as shown below:

```
girls         boys
   [ ] [ ] [ ]                 [ ] [ ] [ ] [ ] [ ] [ ]
```

ASK: What does each block represent? (5) How did you figure that out? (10 ÷ 2) So how many girls and how many boys are there? (15 girls and 25 boys)

**Exercises:** Solve the problem using a tape diagram.

a) The ratio of girls to boys is 4 : 7. There are 9 more boys in the class than girls. How many girls and boys are in the class?
b) The ratio of red paint to yellow paint is 3 : 4. There is twice as much yellow paint as white paint. If 54 cups of paint were used altogether, how much of each colour was used?
c) Yu ran in the school election to be her class representative. When she surveyed her classmates to see who would vote for her, the ratio of Yes to No was 3 : 4. After all the candidates gave a speech to the class explaining what they would do, Yu managed to keep all the Yes voters and change half of the No voters to Yes voters. What is the new ratio of Yes to No votes?
d) The ratio of the number of students in School A to School B was 5 : 8. Then half the students moved from School B to School A. Now what is the ratio of the students in the two schools?

**Answers:** a) 12 girls and 21 boys; b) 18 cups red, 24 cups yellow, and 12 cups white; c) 5 : 2; d) 9 : 4

**Using algebra to solve ratio problems.** SAY: In a tape diagram solution, you can start by figuring out what each block is equal to. You can do that using algebra, too. SAY: Let’s suppose the ratio of girls to boys in a classroom is 3 to 5. Have a volunteer draw the tape diagram on the board, then tell students that each block is \( x \) and write “\( x \)” inside each block, as shown below:

```
girls   boys
   [x] [x] [x]          [x] [x] [x] [x]
```

ASK: What expression would you write for the number of girls, in terms of \( x \)? (3\( x \)) And for the number of boys? (5\( x \)) Write on the board:

```
If one block is \( x \), then girls = 3\( x \) and boys = 5\( x \).
```
You can represent any ratio using algebra in this way. If the ratio of girls to boys is 3 to 5, then the number of girls is three times something and the number of boys is five times the same thing.

**Exercises:** Represent the ratio using variables.

a) The ratio of girls to boys is 4 : 5.
b) The ratio of Yes to No votes is 3 : 7.
c) The ratio of red to yellow to white paint is 3 : 3 : 2.

**Answers:**

a) girls: 4x, boys: 5x; b) Yes votes: 3x, No votes: 7x; c) red: 3x, yellow: 3x, white: 2x

Let’s go back to the problem with girls and boys that you solved with tape diagrams and solve it again using algebra. If the ratio of girls to boys is 3 to 5, then you can say there are 3x girls and 5x boys. We know that there are 10 more boys than girls in the classroom.

**ASK:** How can you write an equation for that using x? (3x + 10 = 5x) Have a volunteer solve the equation on the board.

\[
\begin{align*}
3x + 10 &= 5x \\
10 &= 5x - 3x \\
10 &= 2x \\
5 &= x
\end{align*}
\]

Now you have to remember to substitute 5 for x to get the number of girls and boys, just like with the tape diagrams. Write on the board:

- girls: 3x = 3 × 5 = 15
- boys: 5x = 5 × 5 = 25

**Exercises:** Solve the problems from the previous exercises using algebra instead of a tape diagram. Make sure you get the same answer both ways.

**Choosing between using tape diagrams and using algebra to solve ratio problems.** Write on the board:

The ratio of boys to girls in a class was 3 : 4. When 9 more boys joined the class, the ratio of boys to girls became 6 : 5. How many girls and boys are in the class now?

Read the problem aloud, then draw on the board:

Read the problem aloud, then draw on the board:
ASK: Which tape diagram is correct? (the first one) What is wrong with the second one? (the number of girls should be the same before and after the boys joined) Erase the second diagram. Point to the remaining tape diagram and tell students that this diagram is tricky to use because the white and grey blocks don’t represent the same amount. ASK: Do the white blocks represent more or do the grey blocks represent more? (the white blocks) How do you know? (they are longer) How can I divide the white blocks and the grey blocks into smaller pieces so that there are the same number of each? (divide the white blocks into 5 pieces each and the grey blocks into 4 pieces each) PROMPT: Since the number of girls is the same before and after the change, five grey blocks equal four white blocks.

**Exercises:** Use the tape diagram to solve the problem on the board. Use the parts below to guide you.

a) How many grey blocks are there for the boys if there are 20 grey blocks for the girls?
b) How many white blocks are there for the boys if there are 20 white blocks for the girls?
c) How many more grey blocks than white blocks are there for the boys?
d) What does each extra grey block represent?
e) How many girls and boys are now in the class?

**Answers:** a) 24, b) 15, c) 9, d) 1, e) 24 boys and 20 girls

SAY: This method worked but it was a bit tedious. Let’s try using algebra instead. The ratio of boys to girls was 3 : 4. ASK: How can we write that using algebra? (boys: 3x, girls: 4x) Write on the board:

- boys: 3x
- girls: 4x

SAY: There were 3x boys before nine more joined. ASK: Now how many boys are there? (3x + 9) And how many girls are there? (still 4x) Write on the board:

- new number of boys: 3x + 9
- new number of girls: 4x

ASK: What do we know about the ratio of these new numbers? (the new ratio is 6 : 5) Is that the girls to boys ratio or the boys to girls ratio? (boys to girls) Write on the board:

- boys : girls
- 3x + 9 : 4x
- 6 : 5

SAY: We have two equivalent ratios here. Remember that you can write a ratio using fraction notation. Write on the board:

\[
\frac{3x + 9}{4x} = \frac{6}{5}
\]
ASK: How can you solve this proportion? (cross multiply) Continue writing on the board:

\[(3x + 9)(5) = 6(4x)\]

Have volunteers help you solve for \(x\), as shown below:

\[
15x + 45 = 24x \\
45 = 9x \\
5 = x
\]

SAY: Now that you have \(x\), you have to go back to the original problem and answer the question. Refer students back to the original problem on the board. ASK: What are we being asked to do? (decide how many boys and girls there are now) What is the expression for the new number of boys? (3\(x \) + 9) Write on the board:

\[
3x + 9 = 3(5) + 9 \\
= 15 + 9 \\
= 24
\]

ASK: What is the expression for the new number of girls? (4\(x\)) So how many girls are there now? (20) SAY: So there are 24 boys and 20 girls. ASK: Is the ratio 6 : 5? (yes)

**Exercises:** Use algebra to solve the problem.
a) The ratio of boys to girls in a class was 3 : 4. When 1 more boy and 4 more girls joined the class, the ratio became 2 : 3. How many girls and boys are in the class now?
b) The ratio of boys to girls in a class was 2 : 3. When 2 more boys joined, the ratio became 5 : 6. How many girls and how many boys were there at first?
c) The ratio of boys to girls in a class was 5 : 4. When 3 more boys and 3 more girls joined, the ratio became 6 : 5. How many girls and boys are in the class now?
d) The ratio of boys to girls in a class is 7 : 8. Half the girls in the class are away for the day to play a basketball game. Now there are 6 more boys in the class than girls. How many boys and girls are normally in the class?
**Answers:** a) 16 boys and 24 girls, b) 8 boys and 12 girls, c) 18 boys and 15 girls, d) 14 boys and 16 girls

SAY: Remember, when you are given the question in terms of fractions or percentages, you can change it to a ratio statement.

**Exercises:** Solve the fraction and percentage problems. Did you use algebra or a tape diagram?
a) There are 25% more boys than girls in a class. When 3 girls are away, there are 40% more boys than girls in the class. How many boys and how many girls are usually in the class?
b) Three fifths of the students in the class are girls. Then 4 more girls join the class. Now two thirds of the students in the class are girls. How many girls and boys are in the class now?
**Answers:** a) 35 boys and 28 girls, algebra; b) 16 girls and 8 boys, algebra
Problem Bank
1. Solve the problems in two ways, using algebra and using tape diagrams.
   a) The ratio of red paint to blue paint is 3 : 5. There are 10 more cups of blue paint than red paint. How many cups of paint are there altogether?
   b) The ratio of red paint to blue paint is 5 : 6. Cam added red paint so that there is 60% more red paint than there was originally. What is the new ratio of red paint to blue paint?
   c) The ratio of red paint to blue paint is 3 : 4. When Jen poured 6 cups of blue paint into the mix, the ratio became 1 : 2. How much paint is there altogether?
   d) The ratio of students in two Grade 8 classrooms at a school was 3 : 5. Then one sixth of the students in the larger classroom moved to the smaller classroom. What is the new ratio of students in each classroom?

Solutions:
   a) Using algebra: \(3x + 10 = 5x, 10 = 2x, \) so \(x = 5.\) There are \(8x = 40\) cups of paint altogether.

   Using a tape diagram: Draw the diagram so that red paint has 3 blocks and blue paint has 5 blocks; each block in the tape diagram must be worth 5 cups, since the two extra blocks are worth 10 cups. There are 8 blocks, so there are 40 cups altogether.

   b) Using algebra: There are \(5x\) cups of red paint and \(6x\) cups of blue paint. Since red paint increased by 60% and 60% of \(5x\) is \(3x,\) there are now \(5x + 3x = 8x\) cups of red paint. There are still \(6x\) cups of blue paint, so the new ratio is \(8x : 6x = 4 : 3.\)

   Using a tape diagram: Start with a diagram in the ratio \(5 : 6,\) then add 60% more red blocks (that is, 3 red blocks).

   c) Using algebra: There are \(3x\) cups of red paint and \(4x\) cups of blue paint. When 6 cups of blue paint are added, there are now \(4x + 6\) cups of blue and still \(3x\) cups of red. Now, \(3x : (4x + 6) = 1 : 2,\) so \(3x/(4x + 6) = 1/2.\) By cross multiplying, \(6x = 4x + 6,\) so \(2x = 6\) and \(x = 3.\) There are 9 cups of red paint and now 18 cups of blue paint, which is 27 cups altogether.

   Using a tape diagram: Start with a diagram in the ratio \(3 : 4,\) then add two blue blocks to make the ratio \(1 : 2.\)

   d) Using algebra: The smaller classroom has 3x students and the larger classroom has 5x students. When one sixth of the 5x students move to the smaller classroom, the smaller classroom now has \(3x + 5x/6 = 23x/6\) students, and the larger classroom has \(5x - 5x/6 = 25x/6\) students, so now the ratio is \(23x/6 : 25x/6 = 23 : 25.\)

   Using a tape diagram: Start with a diagram in the ratio \(3 : 5;\) then split each block into sixths:

   Moving one sixth of the larger classroom to the smaller classroom moves 5 small pieces, which makes the larger classroom have 25 equal parts and the smaller classroom have 23 equal parts.
2. For each part in Problem Bank 1, did you prefer using algebra or tape diagrams, or did you have no preference? Explain.

**Answers:** Answers will vary. In parts a), b), and c), both methods are equally easy, but some students might prefer the visual that the tape diagram provides. In part d), students are likely to find the tape diagram method rather tedious.

3. Solve the problems in two ways, using algebra and using tape diagrams. Which method did you like better? Why?

a) Mary has some money saved. She spends $\frac{2}{5}$ of it on clothes. She spends a third of the rest on movies. She has $24 left. How much money did she have at first?

b) Tony has some money. He spends $\frac{3}{5}$ of it on clothes. He spends $\frac{1}{4}$ of the rest on movies. He has $24 left. How much money did he have at first?

c) Hanna has some sports cards. $\frac{1}{4}$ of them are baseball cards and $\frac{2}{5}$ of the remainder are hockey cards. The rest are basketball cards. If Hanna has 18 basketball cards, how many sports cards does she have in total? Hint: Use the two different tape diagrams shown below:

```
<table>
<thead>
<tr>
<th>total</th>
<th>the remainder</th>
</tr>
</thead>
<tbody>
<tr>
<td>baseball</td>
<td>the remainder</td>
</tr>
<tr>
<td>hockey</td>
<td>basketball</td>
</tr>
</tbody>
</table>
```

d) A store has some cans of juice. It sells $\frac{1}{4}$ of them on Friday, $\frac{3}{5}$ of the remainder on Saturday, and $\frac{5}{6}$ of the remainder on Sunday. At the end of the day on Sunday, it still has 30 cans of juice left. How many cans of juice did the store have to start?

e) Greg got some money for his birthday. He spent $\frac{1}{4}$ of it on a T-shirt. He spent $\frac{2}{5}$ of the rest on a book and $\frac{3}{10}$ of the rest on music. He has $9.45 left. How much money did he get for his birthday?

**Selected solution:** c) Using algebra: Let $x$ be the number of sports cards Hanna has in total. $1/4$ of $x$ is the number of baseball cards and $2/5$ of $3/4$ of $x$ is the number of hockey cards. The number of basketball cards is $x - (1/4 of x) - (2/5 of 3/4 of x) = x(1 - 1/4 - 6/20) = 9x/20$. But the number of basketball cards is 18, so $18 = 9x/20$, $x = 20 \times 18 / 9 = 40$, so she has 40 sports cards in total. Using tape diagrams: Since there are 18 basketball cards, each block in the second tape diagram (as shown in the hint) is worth 6, which means the remainder, including both hockey and basketball cards, is worth 30. That means that, in the first tape diagram, each block is worth 10, which means the total is 40. So Hanna has 40 sports cards in total.

**Answers:** a) $60, b) $80, d) 600, e) $30
NOTE: The problems below are taught using algebra in Grade 10. Challenge your students to do them now using tape diagrams.

4. Four children and five adults went to a fair. Their tickets cost $26. One child and one adult went to the same fair, and their tickets cost $5.50. Here are tape diagrams to represent each situation.

a) How much would four children and four adults pay?

b) How much does 1 adult ticket cost? How do you know?
c) How much does 1 child ticket cost? How do you know?

Answers: a) 4 × $5.50 = $22; b) The cost of 4 children and 4 adults is $22. The cost for 4 children and 5 adults is $26. So the cost for 1 adult is $26 − $22 = $4; c) $5.50 − $4 = $1.50.

5. Karen bought 8 pears and 5 apples for $3.40. Carl bought 3 pears and 2 apples for $1.30. How much does each apple and each pear cost? The tape diagram shows what Karen bought:

a) Draw a picture to show what Carl bought.
b) Draw a picture that shows how much 6 pears and 4 apples cost.
c) Draw a picture that shows how much 2 pears and 1 apple cost.
d) Draw a picture that shows how much 8 pears and 4 apples cost.
e) How much does 1 apple cost? How do you know?
f) How much does 1 pear cost? How do you know?

Answers:
a) 1.30  
b) 2 × 1.30 = 2.60

c) 3.40 − 2.60 = 0.80  
d) 4 × 0.80 = 3.20

e) 1 apple costs $3.40 − $3.20 = $0.20, Since 8 pears and 4 apples cost $3.20 and 8 pears and 5 apples cost $3.40.
f) 1 pear costs $0.30; 8 pears and 5 apples cost $3.40 so 8 pears cost $2.40, so 1 pear costs $0.30.
6. Tessa bought a box of pears and a box of apples. When she got home, she counted 50 pieces of fruit altogether. Then she realized that \( \frac{1}{5} \) of the pears were bad and \( \frac{1}{6} \) of the apples were bad. If nine pieces of fruit were bad altogether, how many of each type of fruit were good to eat?

**Solution:**

\[
\begin{array}{c|c|c|c|c}
\text{bad} & \text{good} & \text{bad} & \text{good} \\
\hline
\text{pears} & \text{pears} & \text{apples} & \text{apples} \\
\hline
50 & 9 & 9 & 9 \\
\hline
\end{array}
\]

From the picture, \( ? = 5 \), so there were 25 good apples and 5 bad apples. There were 4 bad pears, so there were 16 good pears. So there were 25 apples and 16 pears that were good to eat.
PS8-4 Using Logical Reasoning II

Teach this lesson after: 8.2 Unit 1

Goals:
Students will solve problems involving both direct and inverse proportions using logical reasoning.
Students will recognize the difference between two quantities being in direct proportional relationship and two quantities being in inverse proportional relationship.

Prior Knowledge Required:
Can solve proportions
Can identify equivalent ratios
Can compare unit rates (for Problem Bank 3)
Can interpret remainders in division problems (for Problem Bank 6)
Can compare fractions (for Extended Problem)
Can multiply a fraction or mixed number by a whole number (for Extended Problem)
Can multiply decimals (for Extended Problem)
Can interpret and calculate percentage increases (for Extended Problem)
Can translate between fractions and ratios (for Extended Problem)
Can interpret three-term ratios (for Extended Problem)

Vocabulary: inverse proportional, proportional, worker-hours

Materials:
BLM Making Punch (pp. M-138–141, see Extended Problem)

Identifying proportional quantities. Write on the board:

<table>
<thead>
<tr>
<th>Amount of Juice (mL)</th>
<th>Price ($)</th>
<th>Amount of Juice (mL)</th>
<th>Price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>2</td>
<td>200</td>
<td>1.50</td>
</tr>
<tr>
<td>400</td>
<td>3.50</td>
<td>400</td>
<td>3</td>
</tr>
</tbody>
</table>

SAY: The bigger the size of the juice, the more you pay, but that doesn’t mean that twice as much juice costs twice as much money. Pointing to the first table, ASK: Does the 400 mL cost twice as much as 200 mL? (no) Is it more or less than twice the cost? (less) Why might a store charge less than twice as much for twice as much juice? (sample answers: they want to encourage you to buy more from them; packaging costs more per mL on smaller amounts)

Now point to the second table and ASK: Does the 400 mL cost twice as much as 200 mL? (yes) SAY: In the second table, the rows are equivalent ratios. When two quantities are always in the same ratio, like they are in the second table, then they are proportional. The amount of juice and the price are not proportional in the first table because they are not always in the same ratio. SAY: You can check whether two quantities are proportional by comparing their values in a table. If the rows are equivalent ratios, then the quantities are proportional, which means that
whatever number you multiply one quantity by, you have to multiply the other quantity by the same number. Continue writing on the board:

\[
\frac{200}{2} \neq \frac{400}{3.5} \quad \frac{200}{1.5} = \frac{400}{3}
\]

<table>
<thead>
<tr>
<th>Amount of Juice (mL)</th>
<th>Price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>2</td>
</tr>
<tr>
<td>400</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Exercise: Are the two quantities proportional?

a) Number of Sheets | Price ($)
-------------------|-----------
100                | 2         
200                | 5         
500                | 15        

b) Number of T-Shirts | Price ($)
----------------------|-----------
1                    | 5         
2                    | 10        
3                    | 15        

c) Distance Run (m) | Time (min)
-------------------|-----------
500                 | 2         
1 000               | 5         
5 000               | 35        

d) Distance Driven (km) | Time (hours)
------------------------|-----------
100                    | 2         
200                    | 4         
500                    | 10        

Answers: a) no, b) yes, c) no, d) yes

SAY: To make the same colour of green paint, the quantities of blue and yellow paint need to be proportional. But if you want to make different shades of green, then the quantities won’t be proportional.

Exercise: Do these mixtures make the same colour of paint?

a) Blue Paint (cups) | Yellow Paint (cups)
---------------------|---------------------
3                    | 2                    
6                    | 5                    
9                    | 8                    

b) Blue Paint (tsp) | Yellow Paint (tsp)
-------------------|---------------------
1                    | 4                    
2                    | 8                    
3                    | 12                   

c) Blue Paint (cups) | Yellow Paint (litres)
---------------------|---------------------
6                    | 3                    
12                   | 6                    
300                  | 15                   

d) Blue Paint (tbsp) | Yellow Paint (cups)
-------------------|---------------------
6                    | 3                    
12                   | 6                    
300                  | 150                  

Answers: a) no, b) yes, c) no, d) yes

Inverse proportional quantities. Write on the board:

Four workers can paint three walls a day.
How many walls can eight workers paint a day?
"A day" means for every one day. Assume that everyone works at the same steady rate.

Draw on the board:

<table>
<thead>
<tr>
<th>Workers</th>
<th>Walls Painted</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>?</td>
</tr>
</tbody>
</table>

Point to the table and ASK: Can we solve this problem with a proportion? (yes) Students can signal their answers with a thumbs up or a thumbs down. Ask a volunteer to find the missing number in the table. (6)

Write on the board:

Four workers can paint an apartment in three hours.
How long will it take eight workers to paint the same apartment?

SAY: Assume that everyone works at the same steady rate. Draw on the board:

<table>
<thead>
<tr>
<th>Workers</th>
<th>Hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>?</td>
</tr>
</tbody>
</table>

Point to the table and ASK: If more people are working, are more hours needed or are fewer hours needed to complete the job? (fewer) Can we solve this problem using a proportion? (no)

SAY: If this was proportional, then twice as many workers would need twice as many hours. But they actually need half as many hours because twice as many workers can get the job done twice as fast, so it will take only 1.5 hours for eight people to paint the apartment. Replace the question mark in the table with “1.5.”

**Exercises:** Find the missing amount. Is it twice as much or half as much?

a) Workers are painting fences.

i) | Workers | Fences Painted |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>?</td>
</tr>
</tbody>
</table>

ii) | Workers | Hours Needed to Paint |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>?</td>
</tr>
</tbody>
</table>

b) A car rental is $12 per hour.

i) | Hours | Total Cost ($) |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>?</td>
</tr>
</tbody>
</table>

ii) | Cars Rented | Cost Per Hour ($) |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>?</td>
</tr>
</tbody>
</table>

iii) | People Sharing a Car Rental | Cost Per Person |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>?</td>
</tr>
</tbody>
</table>

**Answers:** a) i) 12 fences, twice as much; ii) 3 hours, half as much; b) i) $24, twice as much; ii) $24, twice as much; iii) $6, half as much
SAY: If the input (left column) increases and the output (right column) decreases, it’s not proportional, so the rows cannot be equivalent ratios. Using part b) iii) from the previous exercises, draw on the board:

<table>
<thead>
<tr>
<th>People Sharing a Car Rental</th>
<th>Cost per Person</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

SAY: If the rows were equivalent ratios, you could multiply the first row by 2 to get the second row. But, in fact, while you multiply 1 by 2 to get 2, you have to divide 12 by 2 to get 6. So, the quantities are inverse proportional instead of proportional.

**Exercises:** Decide whether the quantities are proportional or inverse proportional. Then answer the question.

a) the number of workers
   the number of fences they can paint in one hour
   If two workers can paint five fences, how many fences can four workers paint?

b) the number of workers
   the number of hours they need to paint a room
   If three workers need five hours to paint a room, how long would it take nine workers?

c) the number of cars rented
   the total cost per hour
   If one car costs $15 per hour, how much would three cars cost for one hour?

d) the number of people sharing the cost
   the cost per person
   If one car costs $15 per hour and three people share the cost, how much does each person pay to rent the car for one hour?

**Answers:** a) proportional, 10; b) inverse proportional, 5/3 hours; c) proportional, $45; d) inverse proportional, $5

**Using the number of hours for one worker to find the number of hours for any number of workers.** Write on the board:

If it takes six workers five hours to paint a room, how long would it take one worker to paint the room?

<table>
<thead>
<tr>
<th>Workers</th>
<th>Hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>?</td>
</tr>
</tbody>
</table>

Have a volunteer answer the question. (30 hours) ASK: Are these quantities proportional or inverse proportional? (inverse proportional) SAY: You are dividing the number of workers by 6, so you need to multiply the number of hours by 6. That’s because one sixth as many workers need six times as long.
Continue drawing on the board:

<table>
<thead>
<tr>
<th>Workers</th>
<th>Hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>?</td>
</tr>
</tbody>
</table>

Write on the board:

If it takes one worker 30 hours to paint the room, how long does it take five workers to paint the room?

Have a volunteer answer the question. (6 hours) SAY: Five times as many workers need one fifth as much time, so divide 30 ÷ 5 = 6.

Exercises:
1. It takes four workers three hours to fill a hole in the road.
   a) How long would it take for one worker to fill the hole?
   b) How long would it take for six workers to fill the hole?
   **Answers:** a) 12 hours, b) 2 hours

2. Three people need one hour to clean a house.
   a) How long would one person need to clean the house?
   b) How long would it take for five people to clean the house?
   **Answers:** a) 3 hours, b) 3/5 hour or 36 minutes

Introduce the concept of worker-hours. Write on the board:

Six workers need five hours. One worker needs 30 hours.
Five workers need six hours. Thirty workers need one hour.

ASK: If six workers each used five hours, how many hours were used in total? (30) SAY: No matter how you look at it, 30 hours were needed in total, whether you spread that between five workers, six workers, 30 workers, or just one worker. You can call this the number of worker-hours needed.

**Exercises:** How many worker-hours are needed?
   a) Three workers need two hours to paint a fence.
   b) Four workers need five hours to move the furniture.
   **Answers:** a) 6, b) 20

SAY: You can use the concept of worker-hours to solve this type of problem easily. Write on the board:

Six workers need 10 hours to paint a fence.
How many hours do 15 workers need?
ASK: How many worker-hours are needed? (60) How did you get that? (6 × 10) If you spread those 60 hours among 15 workers, how many hours will each worker need to work? (4)

**Exercises:**
a) Five workers need eight hours to paint a room. How many hours do four workers need? 
b) Six workers need fifteen hours to paint a room. How many hours do five workers need? 
c) Three workers need eight hours to paint a room. How many hours do four workers need? 

**Answers:** a) 10, b) 18, c) 6

SAY: You can also use this concept to ask how many workers are needed if you know you need to get the job done in a certain number of hours. Write on the board:

Eight workers need 10 hours to paint a fence.

How many workers do you need to hire if you want the fence painted in four hours?

ASK: How many worker-hours are needed to finish the job? (80) If each worker takes four hours, how many workers do you need? (20) Write on the board:

Number of workers × Number of hours each work = Total number of hours worked

**Exercises:**
a) Ten workers need six hours to paint a room. How many workers are needed to complete the job in four hours? 
b) Five workers need 12 hours to complete a job. How many workers do you need if you want the job done in 10 hours? 

**Answers:** a) 15, b) 6

**Problem Bank**
1. The areas of two rectangles are equal. The length of rectangle A is twice the length of rectangle B.
   a) How does the width of rectangle A compare with the width of rectangle B? 
   b) Are the length and width proportional? 
   **Answers:** a) the width of rectangle A is half the width of rectangle B, b) no

2. Can you use a proportion to solve the problem? 
a) Jane is two years old and Yu is five years old. How old will Yu be when Jane is 20 years old? 
b) For every 2 cups of flour, use 1 teaspoon of oil. How many cups of flour would you need for 3 teaspoons of oil? 
c) Three people need one hour to clean the house. How many people would you need if the house needs to be cleaned in 20 minutes? 
   **Answers:** a) no, b) yes, c) no

3. In her first 300 games of solitaire, Lela had 50 wins. After another 50 games (350 in total), her computer recorded 70 wins. Is she improving? Explain how you know. 
   **Answer:** Yes. In 300 games, the ratio of wins to games was 1 : 6. In 350 games, the ratio of wins to games was 1 : 5, which is a better ratio.
4. Five workers need 40 minutes to paint a fence.
   a) How long would it take one worker to paint the fence?
   b) How long would it take five workers to paint two identical fences?
   c) How many workers can paint three fences in 40 minutes?
   d) Are the quantities proportional?
      i) number of workers and number of minutes required
      ii) number of fences painted and number of minutes required
      iii) number of workers and number of fences painted
   Answers: a) 200 minutes or 3 hours and 20 minutes; b) 80 minutes; c) 15 workers;
   d) i) no, ii) yes, iii) yes

5. In two hours, three people can paint a hall with an area of 60 m². How long would it take for
   five people to paint a hall with an area of 75 m² at the same rate of speed?
   Solution: You need 6 worker-hours to paint 60 m². So, you need 7.5 worker-hours to paint
   75 m². So, 5 people would need 1.5 hours to paint 75 m².

6. Five chefs require 20 minutes to prepare 30 appetizers.
   a) How many appetizers can five chefs prepare in 30 minutes?
   b) How many appetizers can five chefs prepare in 50 minutes?
   c) How many chefs can prepare 18 appetizers in 20 minutes?
   d) How many chefs can prepare 48 appetizers in 20 minutes?
   e) How many chefs do you need to hire if you want to prepare 100 appetizers in 20 minutes?
   Selected solution: e) Solving the proportion 100/30 = x/5 gives 30x = 500 or x = 50/3 = 16 2/3,
   so you need to hire 17 chefs.
   Answers: a) 45, b) 75, c) 3, d) 8

7. Four tennis players play a round-robin doubles match (two against two) according to the
   following rules:
   • They play three sets altogether.
   • Each player plays one set with each partner.
   • Each set has one team winning and one team losing (there are no ties).
   Nina says that either one player wins all three sets or one player loses all three sets. Is she
   right? Explain how you know.
   Answer: Nina is right. In the first set, one team wins and the other loses. Now consider the two
   players who won. One of them will win the second set and the other one will lose the second
   set. So, one player won the first two sets. That player’s partner for the third set will be someone
   who lost the first two sets. Either that team will win the third set (and the player who won the first
   two sets will win all three sets) or that team will lose the third set (and the player who lost the
   first two sets will lose all three sets).
Extended Problem: Making Punch

Materials:
BLM Making Punch (pp. M-138–141)

Extended Problem: Making Punch. This extended problem allows students the opportunity to work with ratios, percentages, and markups. Provide students with BLM Making Punch.

Answers:
1. Recipe A is 3 : 5, Recipe B is 3 : 8, Recipe C is 2 : 5
2. a) A, because 3/5 > 2/5 > 3/8; b) B, because 3/8 < 2/5 < 3/5
3. A costs 12¢ per cup, B costs 13 1/8¢ per cup, and C costs 13¢ per cup
4. Yes. This makes sense because using only ginger ale would cost 10¢ per cup and using only cranberry juice would cost 15¢ per cup, so mixing them should cost between 10¢ and 15¢.
5. B. This makes sense because it has the most cranberry taste and cranberry juice is more expensive than ginger ale.
6. A: $1.06; B: $1.15; C: $1.14
7. A: $1.38; B: $1.50; C: $1.48
8. $1.48
9. It will taste more like cranberry than Recipe A but more like ginger ale than Recipe C.
Bonus: Recipe E is 55/111 ginger ale. Recipe F is 1/2 ginger ale. If you mix one recipe of each, you get Recipe G, which is 56/113 ginger ale. Recipe G will taste more like ginger ale than Recipe E but less so than Recipe F, so 56/113 is between 55/111 and 1/2, so 56/113 > 55/111.
Making Punch (1)

You may not use a calculator for this task.

Clara's CranPunch company makes three kinds of cranberry/ginger ale punch:

**Recipe A:** 3 cups of ginger ale and 2 cups of cranberry juice

**Recipe B:** 3 cups of ginger ale and 5 cups of cranberry juice

**Recipe C:** 2 cups of ginger ale and 3 cups of cranberry juice

1. Write the ratio to show the number of cups of ginger ale to the total number of cups of punch for each recipe.
   
   Recipe A: __________
   
   Recipe B: __________
   
   Recipe C: __________

2. Show your reasoning.
   
   a) Which recipe has the strongest ginger ale taste?

   b) Which recipe has the strongest cranberry taste?

3. The company buys ginger ale for 10¢ per cup and cranberry juice for 15¢ per cup. What is the cost, per cup, for each recipe? Write your answer as a whole or mixed number of cents.

   Recipe A: __________

   Recipe B: __________

   Recipe C: __________
Making Punch (2)

4. Are your answers to Question 3 all between 10 and 15? ______
   Why does this make sense?

5. Which recipe is the most expensive, per cup? __________
   Why does this make sense?

6. Clara’s company sells punch in 8-cup bottles. She buys the bottles for 10¢ each.
   How much does she pay for one bottle of each recipe?
   - Recipe A: __________
   - Recipe B: __________
   - Recipe C: __________

7. To decide how much to sell each bottle for, Clara calculates:
   - Total cost of that bottle + 30% of total cost of that bottle
   Then she rounds to the nearest cent. How much does she sell each bottle for?
   - Recipe A: __________
   - Recipe B: __________
   - Recipe C: __________
Making Punch (3)

8. Clara makes a new kind of punch:
   - **Recipe D**: 2 cups of ginger ale, 3 cups of orange juice, and 3 cups of cranberry juice
   
   If orange juice costs 13¢ per cup, how much would she sell each bottle of Recipe D for?

9. Josh made one bottle each of Recipe A and Recipe C. Then he mixed them.
   
   Describe the taste of Josh’s new recipe by comparing it with each of Recipes A and C.
Making Punch (4)

**Bonus** Without doing any multiplication or division, decide which is greater, \( \frac{55}{111} \) or \( \frac{56}{113} \).

Use the following recipes in your explanation.

**Recipe E:** 55 cups of ginger ale and 56 cups of cranberry juice

**Recipe F:** 1 cup of ginger ale and 1 cup of cranberry juice

Hint: Mix the recipes.
Unit 2  Probability and Data Management

Introduction
In this unit students will read, create, and draw conclusions from a variety of graphs, including circle graphs, scatter plots, and histograms. They will use graphs to present primary and secondary data and make convincing arguments about the data.

In order to draw circle graphs, students will need to measure the angles in a circle with a protractor. See lesson G8-17 if you need to review this with your students. You may also choose to do Unit 3 (Geometry) before doing this unit. Reversing the order of the units is easily done because none of the lessons in Unit 2 is a prerequisite for the lessons in Unit 3.

For students studying scatter plots (Ontario), another reason to do Unit 3 before Unit 2 is that there are some nice scatter plots that can be done by comparing geometric properties. Students will have an opportunity to create such scatter plots in lesson PDM8-10 (Extra Practice, Question 3).

Tables and Graphs on BLMs
Several tables and graphs to be used during lessons are provided on BLMs. You may reproduce them on transparencies or scan them into a computer and project them onto the board/wall. You may also distribute photocopies to individuals or pairs.

Meeting your Curriculum
Lessons PDM8-10 through PDM8-13 deal with scatter plots and histograms and so are optional for students working within the WNCP framework. Part of PDM8-14 on comparing graphs is relevant to the WNCP curriculum, but be sure to leave out the references to histograms and scatter plots.

Useful Sources of Data and Graphs

www.statcan.gc.ca/kits-trousses/courses-cours/edu05_0017a-eng.htm
Data, articles, and lessons plans for grades 6 to 8 from Statistics Canada.

www.statcan.ca/english/Estat/licence.htm
E-Stat is a warehouse of statistics about Canada and Canadians, specially designed for use by the educational community. E-Stat is free to registered educational institutions.

www.censusatschool.ca
Census at School is an international online project for students aged 8 to 18. Students complete a brief online survey (questions are non-confidential), analyze their class results, and compare themselves with other students in Canada and in other countries.

www.rom.on.ca/ontario/risk.php
The Royal Ontario Museum provides accessible information about plants and animals at risk in Ontario.

www.cyberschoolbus.un.org
An educational tool from the United Nations. Use Country at a Glance to find physical and population data for UN Member States and InfoNation to produce graphs that compare data for different countries.
The CIA World Factbook provides information about countries around the world.

www.nhlpa.com
The National Hockey League Players’ Association has more than enough data for any conceivable student project.

www.ec.gc.ca
Environment Canada has a wealth of statistical data about weather and the environment.

The Spirit Level: Why Equality Is Better for Everyone, by Richard Wilkinson and Kate Pickett
This book is full of scatter plots about the relationship between income inequality and various factors such as life expectancy, infant mortality rates, educational performance, and so on. The book includes scatter plots where a strong relationship is shown, or a weak relationship, or no relationship. Bar graphs are also used frequently in various contexts.
**Goals**

Students will read, interpret, and create relative frequency tables.

**PRIOR KNOWLEDGE REQUIRED**

- Can convert between tallies and numerals
- Can convert between fractions and percents
- Can find the percent of a number

**Introduce relative frequency tables with fractions.** Tell students that you surveyed 80 students in grade 8 about their favourite type of movie. You tallied the results as follows:

<table>
<thead>
<tr>
<th>Favourite type of movie</th>
<th>Number of People</th>
<th>Fraction of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comedy</td>
<td>12</td>
<td>12/80 = 3/20</td>
</tr>
<tr>
<td>Action</td>
<td>16</td>
<td>16/80 = 1/5</td>
</tr>
<tr>
<td>Horror</td>
<td>32</td>
<td>32/80 = 2/5</td>
</tr>
<tr>
<td>Other</td>
<td>20</td>
<td>20/80 = 1/4</td>
</tr>
</tbody>
</table>

Have students use the data to complete this relative frequency table:

<table>
<thead>
<tr>
<th>Type of Movie</th>
<th>Number of People</th>
<th>Fraction of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comedy</td>
<td>12</td>
<td>12/80 = 3/20</td>
</tr>
<tr>
<td>Action</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horror</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**ANSWERS:**

- Action: 16 people, 16/80 = 1/5
- Horror: 32 people, 32/80 = 2/5
- Other: 20 people, 20/80 = 1/4

Point out that the table shows both the number of times a data value occurs in a set (e.g., comedy is the favourite for 12 people) and the fraction of time each data value occurs (e.g., comedy is the favourite for 12/80 people). (A frequency table shows only the former.)

Encourage students to add the total numbers and fractions. **ASK:** What should the numbers add to? (80) Why? (because you surveyed 80 people) What should the fractions add to? (1) Why? (because 80 out of 80 people is all of them, and 80/80 = 1) Indeed, the numbers add to 80 and the fractions add to 1:
Review converting fractions to percents. Tell students that people often write frequency tables using percents instead of fractions because percents are easier to work with. For example, it is easier to add percents than to add fractions because percents are like fractions that already have the same denominator, namely 100, which means you can treat them as whole numbers. And adding whole numbers, or even decimal numbers, is easier than adding fractions. Remind students how to convert fractions to percents: change the fraction to a fraction with denominator 100 and then write that fraction as a percent (EXAMPLE: \( \frac{3}{20} = \frac{15}{100} = 15\% \)).

Have students convert these fractions to percents:

a) \( \frac{3}{10} \)  
b) \( \frac{4}{5} \)  
c) \( \frac{6}{25} \)  
d) \( \frac{3}{50} \)  
e) \( \frac{9}{20} \)  
f) \( \frac{3}{4} \)  

ANSWERS:  
a) 30\%  
b) 80\%  
c) 24\%  
d) 6\%  
e) 45\%  
f) 75\%

Remind students that sometimes the denominator of the fraction does not divide evenly into 100. In some such cases, students can reduce the fraction so that it does have a denominator that divides evenly into 100. Have students do this to convert these fractions to percents:

a) \( \frac{8}{40} \)  
b) \( \frac{2}{8} \)  
c) \( \frac{18}{75} \)  
d) \( \frac{14}{70} \)  
e) \( \frac{9}{15} \)  
f) \( \frac{36}{48} \)  

ANSWERS:  
a) \( \frac{8}{40} = \frac{1}{5} = 20\% \)  
b) 25\%  
c) 24\%  
d) 20\%  
e) 60\%  
f) 75\%

Introduce relative frequency tables with percents. Have students add a column with heading Percent of People to the relative frequency table above and fill it in.

EXTRA PRACTICE:

Complete the relative frequency tables.

a)  

<table>
<thead>
<tr>
<th>Favourite Type of Television Show</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sitcom</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>Drama</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>Reality TV</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Game show</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

b)  

<table>
<thead>
<tr>
<th>Favourite Type of Video Game</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>Adventure</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>Role-playing</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>Strategy</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>
Review calculating a percent of a number. EXAMPLES:

a) 30% of 50  b) 80% of 60  c) 40% of 80  d) 12% of 40

ANSWERS:

a) $30 \times 50 \div 100 = 15$  b) 48  c) 32  d) 4.8

Then have students complete various frequency tables given the percents and the total. EXAMPLES: Priya surveyed 80 students at a school. Calculate the frequencies. (Demonstrate completing the entry for Sitcom in a) as an example.)

a) **Favourite Type of Television Show**

<table>
<thead>
<tr>
<th>Type</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sitcom</td>
<td></td>
<td>25%</td>
</tr>
<tr>
<td>Drama</td>
<td></td>
<td>30%</td>
</tr>
<tr>
<td>Reality TV</td>
<td></td>
<td>15%</td>
</tr>
<tr>
<td>Game show</td>
<td></td>
<td>10%</td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td>20%</td>
</tr>
</tbody>
</table>

ANSWERS:

a) Sitcom: 20, Drama: 24, Reality TV: 12, Game show: 8, Other: 16

b) **Favourite Type of Video Game**

<table>
<thead>
<tr>
<th>Type</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action</td>
<td></td>
<td>40%</td>
</tr>
<tr>
<td>Adventure</td>
<td></td>
<td>20%</td>
</tr>
<tr>
<td>Role-playing</td>
<td></td>
<td>25%</td>
</tr>
<tr>
<td>Strategy</td>
<td></td>
<td>5%</td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td>10%</td>
</tr>
</tbody>
</table>

ANSWERS:

b) Action: 32, Adventure: 16, Role-playing: 20, Strategy: 4, Other: 8
Curriculum Expectations
Ontario: 7m74, 7m79; 8m1, 8m5, 8m6, 8m7, 8m73, 8m78
WNCP: 7SP3; 8SP1, [R, V, CN, C]

Vocabulary
circle graph
percent
fraction

Goals
Students will use relative frequency tables to draw circle graphs on circles already divided into 100 equal parts. Students will identify conclusions that are inconsistent with a given circle graph and explain the misinterpretation. Students will explain how a given formatting choice can misrepresent the data.

Prior Knowledge Required
Can read, interpret, and create relative frequency tables

Materials
BLM A Large Circle Graph (p N-47)
BLM Small Circle Graphs (p N-48)
BLM Choice of Car Colours in Canada (p N-49)
BLM Students’ Favourite Subjects (p N-50)

Process Expectation
Representing

Introduce circle graphs. Show students some data from the last lesson:

<table>
<thead>
<tr>
<th>Favourite Type of Television Show</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sitcom</td>
<td>16</td>
<td>40%</td>
</tr>
<tr>
<td>Drama</td>
<td>12</td>
<td>30%</td>
</tr>
<tr>
<td>Reality TV</td>
<td>6</td>
<td>15%</td>
</tr>
<tr>
<td>Game show</td>
<td>4</td>
<td>10%</td>
</tr>
<tr>
<td>Other</td>
<td>2</td>
<td>5%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Favourite Type of Video Game</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action</td>
<td>18</td>
<td>30%</td>
</tr>
<tr>
<td>Adventure</td>
<td>12</td>
<td>20%</td>
</tr>
<tr>
<td>Role-playing</td>
<td>21</td>
<td>35%</td>
</tr>
<tr>
<td>Strategy</td>
<td>3</td>
<td>5%</td>
</tr>
<tr>
<td>Other</td>
<td>6</td>
<td>10%</td>
</tr>
</tbody>
</table>

First, check to make sure that the percents add to 100%. Then, explain that we can use a circle divided into 100 equal parts to show the percents. This way of displaying the data is called a circle graph. Project BLM A Large Circle Graph onto the board and demonstrate transferring the first set of percents to the graph. Label each section as you add it. Don’t forget to give the circle graph a title (Favourite Type of Television Show). Emphasize that because the circle is already divided into 100 equal parts, it is easy to create a circle graph. Distribute copies of BLM Small Circle Graphs and have students individually create, label, and title a circle graph for the second set of data.
EXTRA PRACTICE:

<table>
<thead>
<tr>
<th>Favourite Type of Television Show</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sitcom</td>
<td>20</td>
<td>25%</td>
</tr>
<tr>
<td>Drama</td>
<td>24</td>
<td>30%</td>
</tr>
<tr>
<td>Reality TV</td>
<td>12</td>
<td>15%</td>
</tr>
<tr>
<td>Game show</td>
<td>8</td>
<td>10%</td>
</tr>
<tr>
<td>Other</td>
<td>16</td>
<td>20%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Favourite Type of Video Game</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action</td>
<td>32</td>
<td>40%</td>
</tr>
<tr>
<td>Adventure</td>
<td>16</td>
<td>20%</td>
</tr>
<tr>
<td>Role-playing</td>
<td>20</td>
<td>25%</td>
</tr>
<tr>
<td>Strategy</td>
<td>4</td>
<td>5%</td>
</tr>
<tr>
<td>Other</td>
<td>8</td>
<td>10%</td>
</tr>
</tbody>
</table>

Drawing conclusions from circle graphs. Have students transfer the data below (used at the beginning of the previous lesson), about favourite type of movie, to a circle graph. (They can use BLM Small Circle Graphs again.) Have students label the circle graph appropriately and title it Favourite Movies of Grade 8 Students. Students will need to convert all the fractions to percents first. Encourage students to check that their percents add to 100%.

<table>
<thead>
<tr>
<th>Type of Movie</th>
<th>Number of People</th>
<th>Fraction of People</th>
<th>Percent of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comedy</td>
<td>12</td>
<td>12/80 = 3/20</td>
<td>37.5%</td>
</tr>
<tr>
<td>Action</td>
<td>16</td>
<td>1/5</td>
<td>20%</td>
</tr>
<tr>
<td>Horror</td>
<td>32</td>
<td>2/5</td>
<td>25%</td>
</tr>
<tr>
<td>Other</td>
<td>20</td>
<td>1/4</td>
<td>25%</td>
</tr>
</tbody>
</table>

Ensure that students understand the distinction between “most” and “the most.” To make a statement about “most students” is to make a statement about “more than half of the students.” This is different from saying “the most students,” which would refer to the most among several options. For example, from the above frequency chart, the type of movie chosen “the most” is horror because it was picked more than any other, but it is not true that “most students” picked horror because, in fact, less than half of all students picked horror (32/80 is less than 40/80).

Now hide the frequency table. Make the following statements about the data and ask students which ones can be deduced from the circle graph. For statements that are false or that cannot be deduced correctly (we can’t be sure from the data given), have students explain the misinterpretation.

a) Most students in grade 8 like action movies more than comedy movies.

b) More students in grade 8 chose action as their favourite type of movie than chose comedy.

c) Most students in grade 8 chose horror as their favourite type of movie.

d) More than half the students like either horror or action movies.

e) Comedy movies are the favourite type of movie for the fewest students.

ANSWERS:

a) This statement cannot be deduced. For example, many of the people who chose horror or other as their favourite type of movie may prefer comedy to action movies.
b) This is true, because a larger region of the circle graph is dedicated to action than to comedy.

c) This is false: less than half the students chose horror as their favourite type of movie (the horror section is less than 1/2 of the circle). What is true is that among all the types of movies, horror was chosen most often as the favourite (the horror section is the largest section).

d) This is true, because the action and horror sections together cover more than half the circle. The only way the statement could be false would be if someone doesn’t like watching their favourite type of movie!

e) This is false: there might be many types of movies under other and some are likely to be chosen fewer times than comedy. Brainstorm other types of movies that may have been chosen fewer times than comedy (e.g. documentary, drama).

Tell students that students in an elementary school (Grades 1–8) were surveyed about their greatest fear. This circle graph displays the results:

**Greatest Fears**

- Darkness
- Strangers
- Heights
- Spider
- Loud Noises
- Other

**ASK:** Which statements can be deduced correctly from the graph (students can signal thumbs up for correct, thumbs down for incorrect or “we can’t be sure”):

a) Darkness is the second greatest fear among students surveyed.

b) About the same number of students surveyed listed strangers as spiders as their greatest fear.

c) About 25% of 14-year-olds fear darkness the most.

d) Most grade 1 students have a fear of darkness.

Have students write a sentence explaining what is wrong with the statements that are not correct (a, c, and d). Have students focus on answering these questions: What was the person who made the statement thinking? Is it possible the statement is true? Is it likely the statement is true? Students should work with a partner to improve their explanation and then get into a group of four to improve their answers further.
EXPLANATIONS:

a) Darkness is the second largest region, but the largest region is other, which comprises many fears, each with lower percentages. Students could brainstorm what some of those other fears might be (e.g. thunderstorms, snakes). Darkness is very likely the single greatest fear among students surveyed, and if not, the circle graph was done very poorly.

c) This assumes that the same number of people in each grade have a fear of darkness, and this may not be true. In fact, it is unlikely to be true.

d) This may be true if more students in grade 1 have a fear of darkness than students in higher grades, but it cannot be deduced from the graph. More information would be needed.

To show how the conclusion is possible, write on the board:

Percent of students who fear darkness the most
Grade 1: 70%
Grade 2: 60%
Grade 3: 50%
Grade 4: 20%
Grades 5–8: 0%

Now tell students that every grade has the same number of students in it. ASK: What percent of all students in all grades fear darkness the most?

ANSWER: \( \frac{70 + 60 + 50 + 20 + 0 + 0 + 0}{8} = \frac{200}{8} = 25 \), so 25% of all students fear darkness the most. This is exactly what our graph says, so it is possible that most grade 1 students fear darkness the most. However, it is also possible that no grade 1 student fears darkness the most. Perhaps something happened at the school to cause all grade 2s and 3s to fear darkness, while no student in any other grade fears darkness.

Now what is the percent of all students who fear darkness? ANSWER: \( \frac{0 + 100 + 100 + 0 + 0 + 0 + 0}{8} = 25 \). In this case, 25% of all students again fear darkness, but now not a single grade 1 student does! So the conclusion in d) cannot be deduced from the graph. You could discuss which situation students think is more likely and how deciding which situation is more likely can help you decide what to study next.

Comparing data using circle graphs. Tell students that you surveyed a group of 20 students in grade 5 about their favourite type of movie and found the following results:

<table>
<thead>
<tr>
<th>Type of Movie</th>
<th>Number of People</th>
<th>Fraction of People</th>
<th>Percent of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comedy</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Action</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horror</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Have students copy and complete this relative frequency table in their notebooks and then convert the data to a circle graph on BLM Small Circle Graphs. Students can title this one Favourite Movies of Grade 5 Students.

Have students look at Favourite Movies of Grade 5 Students and Favourite Movies of Grade 8 Students and compare the data. Make several statements about the two graphs and ask students which ones are correct. For the incorrect statements, ask students what the misinterpretation is based on.

a) More students in grade 5 chose comedy than did students in grade 8.
b) Horror movies are the favourite type of movie in both grades.
c) Overall, more people liked horror movies than any other type of movie.

ANSWERS:

a) This is incorrect. The fact that a greater region of the grade 5 circle than the grade 8 circle is dedicated to comedy only says that a greater percentage—not number—of grade 5 students chose comedy. In fact, more grade 8 students chose comedy than did grade 5 students (12 vs. 8).

b) No, in grade 5, the comedy section is larger than the horror section.

c) This happens to be true, but you can’t tell this from the graphs only. You would also need to know how many people from each grade were surveyed.

ASK: According to this data, in which grade did a greater percentage of students choose horror as their favourite type of movie? How can you tell from the circle graph? (Grade 8, because the region dedicated to horror is larger on the Grade 8 graph than on the Grade 5 graph)

Circle graphs that look different can still represent the same data.

Draw two circle graphs that show the same data (you can use BLM A Large Circle Graph) but rotate one of the graphs 90° before you label it. Discuss how the graphs are different and how they are the same. Emphasize that two circle graphs can represent the same data even though they look different. Then look at a third circle graph that shows the same data but in a different order and discuss again how the graphs look different though the data is the same. You will find examples of all of the above on BLM Choice of Car Colours in Canada.

Visual representations can misrepresent the data. Remind students that graphs can be visually misleading based on how they are drawn.

Sara wrote three math tests and scored 70%, 75%, and 80% on the first, second, and third tests respectively. She showed her results on the following bar graph:
ASK: Does Sara’s graph show the results accurately? (yes, Test 1 reads as 70, Test 2 as 75, and Test 3 as 80) Even though her graph is accurate, how is it misleading? (It exaggerates her improvement by making the bar for her best test, Test 3, wider than the others.)

Now have students draw a bar graph to represent the same data using this scale, but with all bars equally wide:

ASK: How does this graph make it look like Sara did three times better on Test 3 than on Test 1? How does this formatting choice exaggerate the improvement?
Now show students the following 3-D circle graph (see BLM Students’ Favourite Subjects):

Ask students to rank the subjects listed in order from most chosen to least chosen. Who picked Art as the second most chosen? Who picked Home Ec? Now show students the same graph, but with the percentages labelled.

Discuss how the way the graph was drawn makes it appear as though Art was chosen by more students than Home Ec. For example, Art is more prominent at the front and the 3-D effect makes you able to see more of its curved side than Home Ec’s. Also, the stripes make Art and Other appear wider than they are but the same horizontal stripes make Home Ec look narrower because of its location on the circle. Stripes that are parallel to the radius make a piece look narrower; stripes that are perpendicular to the radius make a piece look wider.

**ASK:** Do you think the graph was more likely drawn by an Art teacher or by a Home Ec teacher? Why? (Art, because he or she wanted to exaggerate Art as the favourite subject of more students) How can you draw a circle graph that is less misleading? (take out the 3-D effect; don’t use stripes for alternating sections) Have students draw a more truthful graph, using a computer program, such as Microsoft Excel. Emphasize that by not using 3-D effects, the region at the front is no longer exaggerated—it looks just like any other.
ACTIVITIES 1–2

1. **Circle graph posters.** Encourage students to look for circle graphs in books, in magazines, on the Internet, on TV (e.g., on the Weather Network) or in brochures (e.g., from financial institutions). Have students record properties that are common to all the circle graphs. How many categories are used in each circle graph? About how many categories do most circle graphs use? Have students cut out various circle graphs and make a poster titled Circle Graphs. Keep these posters for PDM8-14.

2. **Survey your class and compare results.** Survey students about their favourite type of television show (sitcom, drama, reality TV, game show, other) and graph the results. Compare the results for your class with the two sets of results given at the beginning of this lesson. Discuss which data set showed people more like those in your class. Then have students make up four statements, two of which can be correctly deduced from comparing the three graphs and two of which cannot. Students then exchange their statements with a partner and identify the correct statements in each other’s lists.
The total percents must add to 100%. Tell students that you asked students in six different kindergarten classes to name their favourite colour, but you might have made a mistake recording some of the data. Have students translate the data into circle graphs using **BLM Small Circle Graphs** (they should create one graph per class).

<table>
<thead>
<tr>
<th>Class</th>
<th>Red</th>
<th>Blue</th>
<th>Yellow</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class A</td>
<td>35%</td>
<td>25%</td>
<td>10%</td>
<td>35%</td>
</tr>
<tr>
<td>Class B</td>
<td>20%</td>
<td>30%</td>
<td>20%</td>
<td>30%</td>
</tr>
<tr>
<td>Class C</td>
<td>30%</td>
<td>20%</td>
<td>10%</td>
<td>40%</td>
</tr>
<tr>
<td>Class D</td>
<td>15%</td>
<td>40%</td>
<td>15%</td>
<td>40%</td>
</tr>
<tr>
<td>Class E</td>
<td>10%</td>
<td>15%</td>
<td>25%</td>
<td>50%</td>
</tr>
<tr>
<td>Class F</td>
<td>18%</td>
<td>15%</td>
<td>26%</td>
<td>31%</td>
</tr>
</tbody>
</table>

**ASK:** Which data did you have trouble graphing? (data for classes A, D, and F) Why? (because they total either more or less than 100%)

Explain that the percents in a circle graph must always total 100%. This is because when you count all the data, you count 100% of it. So you must have made a mistake when recording the data for classes A, D, and F. Encourage students to use this method to check their work for the relative frequency tables in Question 6, on Workbook page 42, before completing the circle graphs.
Have students draw a circle graph, using **BLM A Large Circle Graph**, for the following data about the percent of people who use each mode of transportation to get to school:

<table>
<thead>
<tr>
<th>Mode of Transportation</th>
<th>Percent</th>
<th>Angle in Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bus</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Bike</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Walk</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Car</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Then, have students use a protractor to measure the angle of each region (or “pie piece”) and fill in the third column. Answers (from top to bottom): $108^\circ$, $90^\circ$, $72^\circ$, $54^\circ$, $36^\circ$.

**The angles in a circle add to 360°.** Have students find the sum of the angles in the chart. What do the angles total? ($360^\circ$) Why? (because the central angles of a circle add to 360°)

**Connect percents to angles.** **ASK:** The percentage of people biking to school was 25% and the angle you found was 90°. **ASK:** Does that make sense? Why? (yes, because 90° is 25% of 360°) Demonstrate this to students. First, remind students how to find different percents of a number. For example, to find 25% of 360, write 25% as a fraction ($\frac{25}{100}$ or $\frac{1}{4}$) and then replace “of” with the multiplication sign:

$$25\% \text{ of } 360^\circ = \frac{25}{100} \times 360^\circ = \frac{1}{4} \times 360^\circ = 360^\circ \div 4 = 90^\circ$$

A quick explanation of why this works: To find 1% of 360°, divide 360° by 100. But 25% of 360° is 25 times more than 1% of 360°, so 25% of 360° is $25 \times 360^\circ \div 100 = 90^\circ$.

Have students use this method to verify the angles they measured: Does the percent of 360° they calculated for each mode of transportation match the measurement? Point out that students no longer need circles divided into 100 equal parts to create circle graphs. Now they can use angles instead!

**Drawing circle graphs using a protractor.** Mark the centre of a circle with a point, then draw a circle around that centre using a compass. Draw a line from the edge of the circle to the centre of the circle (a radius). Demonstrate drawing a circle graph for the data above. Emphasize that this circle is not already divided into 100 equal parts, so students now have to use angles in the circle to draw the regions.
Have students individually draw a circle graph given the angles in the circle for a set of data. Students should use BLM Unmarked Circle Graph (this is a circle not divided into 100 parts, but the same size as the circle on BLM A Large Circle Graph, and with the centre of the circle clearly marked).

**SAMPLE DATA:**

<table>
<thead>
<tr>
<th>Favourite type of snack</th>
<th>Percent</th>
<th>Angle in Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vegetables</td>
<td>20%</td>
<td>72°</td>
</tr>
<tr>
<td>Crackers</td>
<td>15%</td>
<td>54°</td>
</tr>
<tr>
<td>Chips</td>
<td>40%</td>
<td>144°</td>
</tr>
<tr>
<td>Fruit</td>
<td>10%</td>
<td>36°</td>
</tr>
<tr>
<td>Other</td>
<td>15%</td>
<td>54°</td>
</tr>
</tbody>
</table>

Review how to get the number in each column from the number in the other column—how to calculate the angle that corresponds to a percent and vice versa. After students finish drawing the circle graph by measuring the angles, give students a copy of BLM A Large Circle Graph. If students overlay this BLM over their circle graph, they can use the markings to check the accuracy of their measurements—the markings that line up with the lines on their circle graph should correspond to the given percents. It is ideal to photocopy BLM A Large Circle Graph onto transparencies rather than paper. If possible, photocopy the BLM onto enough tranparencies so that groups of 3 or 4 can share one.

Have students draw more circle graphs after completing similar charts in which either the percent or angle for each response is given (see Workbook page 41, Question 3). When they have completed each chart, students should ensure that the percents total 100% and the angles total 360°.

Students should then draw the circle graphs using a compass. Students should be sure to mark the centre point first—they will need the centre point marked when drawing the regions. **EXAMPLE:** (answers are in brackets—do not show them)

<table>
<thead>
<tr>
<th>Favourite type of snack</th>
<th>Percent</th>
<th>Angle in Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vegetables</td>
<td>30%</td>
<td>(108°)</td>
</tr>
<tr>
<td>Crackers</td>
<td>10%</td>
<td>36°</td>
</tr>
<tr>
<td>Chips</td>
<td>25%</td>
<td>90°</td>
</tr>
<tr>
<td>Fruit</td>
<td>15%</td>
<td>(54°)</td>
</tr>
<tr>
<td>Other</td>
<td>20%</td>
<td>(72°)</td>
</tr>
</tbody>
</table>

Review writing fractions as percents. See Workbook page 42, Question 4.

**Writing fractions with denominator 360.** See Workbook page 42, Question 5. Not all fractions can be written this way, but if they can, the corresponding angle in a circle is particularly easy to find—it is just the numerator of the fraction with denominator 360. All fractions that students will find in Question 6 on Workbook p. 42 can be written this way.

**Using technology to draw circle graphs.** Have students use a computer program, such as Microsoft Excel, to draw one of the graphs they have already created by hand. **ASK:** Which was easier to do? What do you have...
to be careful about when you draw a graph using a computer? **PROMPT:**
When you use a calculator to multiply numbers, what do you have to look for in the answer? (you have to make sure the answer makes sense, in case you punch in a wrong number) What kind of things might not make sense in a computer graph? **(EXAMPLE:** if one region takes up more than half the circle when it is only supposed to take up a small portion of the circle, you may have punched in a wrong number)

Show students the following data and circle graph (see **BLM Favourite Car Colour**):

<table>
<thead>
<tr>
<th>Favourite Car Colour</th>
<th>People</th>
</tr>
</thead>
<tbody>
<tr>
<td>White</td>
<td>64</td>
</tr>
<tr>
<td>Black</td>
<td>57</td>
</tr>
<tr>
<td>Blue</td>
<td>18</td>
</tr>
<tr>
<td>Green</td>
<td>13</td>
</tr>
<tr>
<td>Other</td>
<td>12</td>
</tr>
</tbody>
</table>

**ASK:** Is this circle graph drawn correctly? (no) How can you tell? (the section for blue is too big—it should be smaller than the sections for white and black) Have students draw a circle and a rough sketch of what the graph should look like. Then show the correct graph:

Tell students that you had accidentally punched in 81 for blue instead of 18—that’s why the computer made blue so much bigger. In fact, that error made the white and black regions look smaller even though you punched those numbers in correctly, because a circle graph only looks at percentages and not at actual frequencies. Emphasize that just as students check their answers when calculating on a calculator, they need to check their answers when punching in data on a computer.
Goals

Students will read and interpret circle graphs. Students will understand the importance of drawing the angles in circle graphs correctly in order to represent the data accurately.

PRIOR KNOWLEDGE REQUIRED

Can read and draw simple circle graphs
Knows that the total sum of angles in a circle is 360°
Knows that the total sum of percentages in a circle graph is 100%
Can calculate a percentage of a number
Can calculate a fraction of a number
Can convert fractions to decimals, including repeating decimals

What percent of 360° does a given angle represent? See Workbook page 43, Questions 1–3. For Questions 2 and 3, students will need to measure each angle first and then determine what percent of 360° the angle represents.

Sometimes the angle in a circle does not correspond to a whole number percent of 360°. Provide the example shown in the box at the top of Workbook p. 44.

Review changing fractions to decimals. Use long division or estimation, then check your answer on a calculator. See Workbook page 44, Question 4.

What decimal percent of 360° does a given angle represent? Have students round to one decimal place. See Workbook page 44, Question 5.

Show students the steps required to draw a circle graph when the circle is not already divided into 100 parts. EXAMPLE:

In a grade 8 class, 10 students walk to school, 5 travel by bus, 5 bicycle, and 5 skateboard.

Step 1: Find the total number of students. (25)

Step 2: Express each piece of data as a fraction of the total (reduce to lowest terms).

\[
\begin{align*}
\frac{10}{25} &= \frac{2}{5} & \frac{5}{25} &= \frac{1}{5} & \frac{5}{25} &= \frac{1}{5} & \frac{5}{25} &= \frac{1}{5}
\end{align*}
\]

walk  bus  bicycle  skateboard

Step 3: Change each fraction to an equivalent fraction out of 360.

\[
\begin{align*}
\frac{2}{5} &= \frac{?}{360} & \frac{2}{5} \times \frac{72}{72} &= \frac{144}{360}
\end{align*}
\]
The angle for the part of the circle graph that represents the students who walk to school should be 144°.

\[
\frac{1}{5} = \frac{?}{360} \quad \frac{1 \times 72}{5 \times 72} = \frac{72}{360}
\]

The angles for the parts of the graph that represent the students who bicycle, ride the bus, or skateboard to school should each be 72°.

**Step 4:** Mark a point, draw a circle with that point as its centre, and then draw a radius.

**Step 5:** Use a protractor to construct a radius for each of the angles you found in step 3.

**Step 6:** Title and label the circle graph. Include the fraction or percent of the total that each region represents.

*How we get to school*

- **Walk** \(\frac{2}{5} = 40\%\)
- **Bus** \(\frac{1}{5} = 20\%\)
- **Skateboard** \(\frac{1}{5} = 20\%\)
- **Bike** \(\frac{1}{5} = 20\%\)
Have students do Workbook page 44 Question 6. Notice that the total is found to be 100% for part c), but not for part d). Emphasize that this is because rounding makes the results less accurate and hence can make the total appear to be different from 100%.

**Importance of drawing the angles accurately.** Ask students what is wrong with the circle graph below. (Students should be able to estimate what the angles should look like. For instance 3/5 is greater than 1/2, but the part marked H covers less than 1/2 the circle. Also, 1/5 is double 1/10 so the parts marked B and S should each cover twice as much area as the part marked O.) **ASK:** How does this circle graph give the impression that less than half the people chose hockey? Why is it important to measure each angle accurately when drawing a circle graph?

![Circle Graph](image)

Favourite sport
- H: Hockey \( \frac{3}{5} \)
- S: Soccer \( \frac{1}{5} \)
- B: Baseball \( \frac{1}{5} \)
- O: Other \( \frac{1}{10} \)

Point out that you may not always be able to draw a graph precisely, especially if you are drawing it by hand. For example, if the angle of a region is supposed to be 21.6°, you might decide to draw that as 22°. In this case, your graph won’t be misleading because the difference will be too small to be noticeable.

**Recognizing incorrect conclusions.** The following question illustrates an actual reasoning mistake seen on a web page. An opinion poll asked people to strongly agree, agree, disagree, or strongly disagree with an opinion. This graph represents the answers of the people who were surveyed:

![Pie Chart](image)

a) What fraction of people surveyed:
   - Agree? ______
   - Disagree? ______
   - Strongly agree? ______
   - Strongly disagree? ______

b) What fraction of people surveyed either agree or strongly agree? ______

c) The survey concludes: “Not counting the lunatic 1/10 of people who strongly disagree, only 4/10 of people disagree with us.” Is this correct? Explain.
ANSWER: No, this conclusion is incorrect. Out of 10 people, you would expect 3 to agree, 2 to strongly agree, 4 to disagree, and 1 to strongly disagree. If you aren’t going to count the 1 who strongly disagrees, you have to remove 1 from the total: the fraction of people who disagree is 4/9. This is slightly more than 4/10. As well, the data shows that 50% of people “disagree” with the opinion. Point out that the number of people who disagree is the number of people who disagree to any extent, so it’s the number who “disagree” plus the number who “strongly disagree.”

ASK: Does the circle graph show this? (yes, but it’s not obvious) Have students redraw the circle graph so that this fact—that 50% of people disagree—is prominent (put the “disagree” and “strongly disagree” sections next to each other).

EXTRA PRACTICE:

1. The following data shows the number of deaths due to each recreational activity in one year:

   Boating 80    Swimming 40    Biking 36    Jet skiing 4

   a) Make a relative frequency table showing the fraction of deaths due to each recreational activity.

   b) Draw a circle graph showing the data.

   c) Can you conclude that biking is more dangerous than jet skiing? Why or why not? What other information would be relevant? (You want to consider frequency—how often or for how long people participate in each activity. For example, the number of hours people spend biking is likely significantly larger than the number of hours people spend jet skiing, so even if jet skiing is more dangerous per hour, there could still be a lot more deaths from biking than deaths from jet skiing.)

2. Sally surveyed students in her school about whether or not they like her favourite singer. She found that 60% of the students in grades 7 and 8 answered yes, and 40% of the students in grades 1–6 answered yes. She drew a circle graph showing her results:

   Students who like my favourite singer

   Grades 7 and 8
   Grades 1–6
ASK: Is Sally’s circle graph correct? Explain. (Sally has combined the “yes” answers from two different populations on one graph. Her graph makes it look like 60% of the people who like her favourite singer are in Grades 7 and 8! In order to draw the graph correctly, Sally would need to calculate what percent of students who said “yes” are in grades 7 and 8 and what percent are in grades 1–6. To show the data as it is given, Sally would need two circle graphs: one for students in grades 7 and 8—percent who said “yes” and percent who said “no”—and another for students in grades 1–6.)

Circle graphs or bar graphs. Look at the graphs for companies A and B in Question 3 on Workbook page 43. Tell students that a survey shows that employees are more likely to leave a company soon after being hired if they are 34 years old or younger. ASK: Which company has a greater number of employees in this age range? Which company has a greater percentage of such employees? Which graph did you use to answer each question—the double bar graph you drew in part a) or the two circle graphs? Explain your choice. (ANSWER: From the double bar graph, it is easy to see that Company B has higher bars and hence larger numbers of employees in the categories “under 20” and “20–34.” From the circle graphs, it is easy to see that the two categories take up more than half of the circle for Company A, but less than half for Company B, so Company A has a greater percentage of employees 34 and under than does Company B. Notice that the circle graphs give you no information about actual numbers; they display only percentages.)

Extension

What is more dangerous—flying or driving? Students should try to think of arguments to support either answer, then look up data on the Internet and form conclusions. Airlines often cite that flying is safer than driving. Do they have a reason for wanting us to believe that? How should we calculate safety? When judging the fatality rate, should we compare the number of fatalities per hour or would you be more interested in the number of fatalities per trip? If I am travelling from Toronto to Vancouver, it will take me a lot less time to fly than to drive, so this decision could impact the results. Maybe driving is safer on a per-hour basis, but flying is safer on a per-trip basis. Do the research and find out!
Introduce scatter plots. Tell students that scatter plots are used when you have two measurements associated with a single person, object, or event. Display the following chart.

<table>
<thead>
<tr>
<th>Place</th>
<th>Average High Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>January</td>
</tr>
<tr>
<td>Toronto, Canada</td>
<td>−2</td>
</tr>
<tr>
<td>Vancouver, Canada</td>
<td>6</td>
</tr>
<tr>
<td>Canberra, Australia</td>
<td>21</td>
</tr>
<tr>
<td>Montevideo, Uruguay</td>
<td>22</td>
</tr>
<tr>
<td>Santiago, Chile</td>
<td>29</td>
</tr>
<tr>
<td>Inukjuak, Canada</td>
<td>−20</td>
</tr>
</tbody>
</table>

Tell students that this chart is good for answering certain types of questions. Ask students some such questions:

a) What is the warmest place in January? What is the warmest place in July?

b) Which place has the biggest temperature change from January to July?

c) Which places are warmer in January than in July?

All these questions require comparing different places to each other, but sometimes we are not interested in how one place compares to any other, but rather how two sets of measurements—in this case, average high in January and average high in July—relate to each other. If this is what we want to isolate, we can ignore the locations, and only plot the two data values together. Show students how to do this for Toronto, Vancouver, and Canberra, then ask students to copy and complete the resulting scatter plot on grid paper.
Using scatter plots to find relationships. When students finish, **ASK:** Do places with high temperatures in January seem to have higher or lower temperatures in July? (lower) Why is that? (if it is summer in January, it is winter in July, and vice versa) How can you tell from the scatter plot that places with high temperatures in January are likely to have lower temperatures in July? (points on the right are lower than points on the left)

Scatter plots give useful information even when the relationship isn’t perfect. Tell students that the two data values, temperature in January and temperature in July, are related to each other. Places with high temperatures in January will tend to have lower temperatures in July, just because places with high temperatures in January are likely to be in the southern hemisphere and so are more likely to have low temperatures in July. But point out that this isn’t a perfect relationship. Some places will have high temperatures at both times of the year. **ASK:** Can you think of a place that would have high temperatures at both times of the year? Why is that? (places near the equator likely have temperatures that are high all year round) Add a row for Bangkok to the chart (see below) and have students add a point to their scatter plot. Can you think of a place that would have low temperatures at both times of the year? Where could that be? (far north or far south, e.g., Antarctica) Add information for McMurdo Station, Antarctica to both the chart and the scatter plot.

<table>
<thead>
<tr>
<th></th>
<th>January Temperature (°C)</th>
<th>July Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bangkok, Thailand</strong></td>
<td>31</td>
<td>33</td>
</tr>
<tr>
<td><strong>McMurdo Station, Antarctica</strong></td>
<td>0</td>
<td>-22</td>
</tr>
</tbody>
</table>

Also, there are other factors that affect temperature, such as elevation above sea level and proximity to an ocean. The graph doesn’t say that a high January temperature causes a low July temperature, only that places with high January temperatures are more likely to have low July temperatures.

Scatter plots isolate the relationship between measurement values, but can’t answer other types of questions. Point out that by creating a scatter plot, we can answer different types of questions than we could with the chart. We can no longer see which places are warmest or coldest at different times of year, because we didn’t label the points on the scatter.
plot. Labelling the points would have made it harder to see relationships between them. (**NOTE:** If you are looking at a scatter plot on a computer, you may see a label if you hover the cursor over any particular point. You wouldn’t want to see all the labels at the same time because they would make it harder to see any patterns or relationships in the data.)

Practise creating scatter plots and drawing conclusions. Give students more data to transfer onto scatter plots using grid paper. Students will have to add some of the data themselves in b), c), and d). They will also have to decide what scales to use on the axes in each plot.

### a) Person Age Arm span (cm)

<table>
<thead>
<tr>
<th>Person</th>
<th>Age</th>
<th>Arm span (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carene</td>
<td>12</td>
<td>144</td>
</tr>
<tr>
<td>Sara</td>
<td>9</td>
<td>145</td>
</tr>
<tr>
<td>Jason</td>
<td>5</td>
<td>108</td>
</tr>
<tr>
<td>Tom</td>
<td>3</td>
<td>95</td>
</tr>
<tr>
<td>Jo</td>
<td>15</td>
<td>162</td>
</tr>
<tr>
<td>Annette</td>
<td>17</td>
<td>168</td>
</tr>
</tbody>
</table>

### b) Person Age Number of letters in name

<table>
<thead>
<tr>
<th>Person</th>
<th>Age</th>
<th>Number of letters in name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carene</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>Sara</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>Jason</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Tom</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Jo</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Annette</td>
<td>17</td>
<td></td>
</tr>
</tbody>
</table>

### c) Person Number of vowels in name Number of letters in name

<table>
<thead>
<tr>
<th>Person</th>
<th>Number of vowels in name</th>
<th>Number of letters in name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carene</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>Sara</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jason</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tom</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jo</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Annette</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### d) Rectangle Perimeter Area

<table>
<thead>
<tr>
<th>Rectangle</th>
<th>Perimeter</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 × 5</td>
<td>16</td>
<td>15</td>
</tr>
<tr>
<td>1 × 7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 × 4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 × 8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 × 6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 × 3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Have students describe the relationship that each scatter plot shows—if it shows a relationship—by completing these sentences:

- **a)** People who are older are likely to have (larger/smaller) arm span.
- **b)** People who are older are likely to have (more/fewer) letters in their name.
- **c)** People with more vowels in their name are likely to have (more/fewer) letters in their name.
- **d)** Rectangles with larger perimeter are likely to have (larger/smaller) area.

**ANSWERS:**

- **a)** larger
- **b)** can’t tell
- **c)** more
- **d)** larger

Emphasize that sometimes there is no relationship between the two measurements that you are comparing. For example, in b), someone who is older is not any more likely to have more or fewer letters in their name than someone who is younger. **ASK:** How does your scatter plot show that age and number of letters in name are not related? (There is no tendency...
of the dots to go from top left to bottom right or from bottom left to top right—the dots on the left are spread low and high just as much as the dots on the right)

**PROCESS ASSESSMENT**

8m4, [CN]

**Charts vs. scatter plots.** Ask students questions about the data in part a) and have students tell you which way of representing the data, the chart or the scatter plot (or both), they can easily use to answer each question.

i) Who is the oldest?
ii) How many people are over the age of 10?
iii) How many people have arm span over 150 cm?
iv) Who has the smallest arm span?
v) Do older people have larger or smaller arm span compared to younger people?

**ANSWERS:**
i) Annette – chart  ii) three – both  iii) two – both  iv) Tom – chart 
v) Older people tend to have larger arm span compared to younger people – scatter plot

**PROCESS EXPECTATION**

Selecting tools and strategies

**Double bar graphs vs. scatter plots.** Have students plot the data from b) and c) on double bar graphs, using bars for each person. **ASK:** Can you display the data from a) in a double bar graph? Why not? (because the units are different—we can’t measure age in years on the same scale as arm span in centimetres)

Show students the following table of flying distances and flight costs from Toronto to various cities (or, if you have time, make your own table with your nearest airport replacing Toronto). The cost of flying to each city from Toronto was determined by finding the cheapest flight on a given day.

<table>
<thead>
<tr>
<th>City</th>
<th>Flying distance from Toronto (km)</th>
<th>Cost of flying from Toronto (CAD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beijing, China</td>
<td>10 611</td>
<td>$970</td>
</tr>
<tr>
<td>Edmonton, AB</td>
<td>2335</td>
<td>$450</td>
</tr>
<tr>
<td>Halifax, NS</td>
<td>1268</td>
<td>$340</td>
</tr>
<tr>
<td>Orlando, USA</td>
<td>1468</td>
<td>$300</td>
</tr>
<tr>
<td>Ottawa, ON</td>
<td>315</td>
<td>$160</td>
</tr>
<tr>
<td>Quebec, QC</td>
<td>639</td>
<td>$180</td>
</tr>
<tr>
<td>Sudbury, ON</td>
<td>338</td>
<td>$230</td>
</tr>
<tr>
<td>Sydney, Australia</td>
<td>15 562</td>
<td>$1 500</td>
</tr>
<tr>
<td>Vancouver, BC</td>
<td>2905</td>
<td>$450</td>
</tr>
<tr>
<td>Winnipeg, MB</td>
<td>1305</td>
<td>$460</td>
</tr>
</tbody>
</table>

**ASK:** Can you make a double bar graph from this information? (no) Why not? (The scale can either be kilometres or Canadian dollars, but not both.) Can you draw a scatter plot from this information? (yes) What type of information can you get from the scatter plot that is harder to read from the chart? (you can see if there is a relationship between distance and cost of
flying) What type of information can you get from the chart that you lose by making the scatter plot? (information about specific cities, such as which city is farthest from Toronto or which city is cheapest to fly to)

Discuss what relationship you might expect between distance and cost of flying and why (e.g., the greater the distance, the greater the cost, because the cost of fuel and the salaries of flight attendants and pilots are greater for longer flights).

Demonstrate how to enter these data points into a computer program, like Microsoft Excel, to draw the scatter plot. Have students draw the scatter plot on a computer, and discuss how the result shows the relationship.

Tell students that sometimes, even when the units for both data sets are the same, it is still impossible to draw a double bar graph. Remind students that when using double bar graphs to compare two sets of data, the scales for both must be identical. This is why we cannot, for example, compare natural population growth (births minus deaths) to average family size. In Canada, the number of births minus deaths is too large in comparison to the average family size to make it possible to compare them using the same scale. For example, in one year, the average family size was 2.5, whereas the number of births minus deaths was more than 100,000! On a double bar graph, either the bars for average family size would be too small to see or the bars for natural population growth would be too big to fit on the page. But, if we had the data, we could compare natural population growth for various countries to average family size in those countries using a scatter plot. Then we could see whether countries with a larger natural population growth are also more likely to have a larger family size.

Predicting relationships. Have students predict whether there will be a relationship between various pairs of data values. As one increases, will the other increase, decrease, or not be affected? Encourage students to explain their predictions.

a) As the amount a country spends on its Olympic athletes increases, is the number of medals it receives likely to increase, decrease, or not be affected? **SAMPLE ANSWER:** increase, because better training facilities would help the athletes improve their performance.

b) As the height of a person increases, is the time it takes that person to run 100 m likely to increase, decrease, or not be affected? **SAMPLE ANSWER:** decrease, because longer legs allow you to take fewer strides so your legs don’t have to work as hard to do the same thing as when you were shorter.

c) As a person’s resting heart rate increases, is the time it takes that person to run 100 m more likely to increase, decrease, or not be affected? **SAMPLE ANSWER:** people who are in better shape tend to have lower resting heart rates, so as resting heart rate increases, fitness level likely decreases and the time it takes to run 100 m will likely increase too.
d) As a person’s walking speed increases, is that person’s heart rate likely to increase, decrease, or not be affected? **SAMPLE ANSWER**: increase, because exercise gets the heart rate up.

e) As the temperature increases from 0 to 30, is the electricity used in heaters likely to increase, decrease, or not be affected? **SAMPLE ANSWER**: decrease, because you don’t need the heat on as much if it is warmer outside.

f) As the temperature increases from 0 to 30, is the electricity used in air conditioners likely to increase, decrease, or not be affected? **SAMPLE ANSWER**: increase, because you need more air conditioning to keep the air cooler if it is warmer outside.

g) As the temperature increases from 0 to 30, is the electricity used in phones likely to increase, decrease, or not be affected? **SAMPLE ANSWER**: not be affected, because the temperature outside doesn’t affect people’s desire or need to talk on the phone.

Ask students to draw, without the scale, what they think the scatter plot for parts e), f), and g) will look like—will the dots tend to go from bottom left to top right or top left to bottom right or neither?

**ANSWER:**
e) top left to bottom right  f) bottom left to top right  g) neither

**Drawing scatter plots to decide if there is a relationship.**

**EXTRA PRACTICE:**

1. Tell students that a professional tennis player has kept track of her scores and her playing level since she started playing. She has always played the best two sets out of three, and she assigned points to her scores as follows:

   a victory in two sets = 3 points
   a victory in three sets = 2 points
   a loss where she won a set = 1 point
   a loss where she didn’t win a set = 0 points

She recorded her average points per match at different playing levels. She used the standard tennis rating system, which goes from 1.0 (beginner) to 7.0 (world class). Here is her data (some playing levels are repeated because she recorded data from playing on different surfaces separately):

| Playing level | 1.0 | 1.5 | 2.0 | 2.5 | 2.5 | 3.0 | 3.5 | 3.5 | 4.0 | 4.0 | 4.5 | 4.5 | 5.0 | 5.0 | 5.5 | 5.5 | 6.0 | 6.0 | 6.5 | 7.0 |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Points         | 1.8 | 1.8 | 27  | 1.1 | 0.2 | 2.6 | 1.7 | 0.9 | 0.4 | 1.9 | 0.8 | 0.2 | 2.8 | 1.6 | 1.1 | 2.9 | 1.5 | 0.5 | 2.2 | 0.3 |
Have students make a scatter plot. **ASK:** Is there a relationship between the player’s scores and how well she played? (no, there is no relationship) Is it surprising that there is no relationship between the score in a game or match and how well a person played? Would you expect a better score if you play well? Why? Would you feel better about a better score than a worse score? Would you feel better about playing well or not playing well? Why do we tend to feel better about a good score even though it doesn’t reflect how well we’ve played?

Explain to students that a score only tells you how well you played in comparison to someone else; it doesn’t say anything about how well you each played. You might both play well or both not play well. In fact, playing well will likely increase your opponent’s playing level. Expecting a correlation between your score and how well you played is like expecting a correlation between how far apart two points are on the y-axis and how high up or far down they are. Knowing one doesn’t tell you anything at all useful about the other. You might use this to teach students about healthy ways to handle competitive situations. Point out that the best coaches often tell their players to ignore the score, precisely because the score can’t tell you anything useful about how well you are playing. Coach John Wooden, for example, who led the UCLA basketball team to ten NCAA championships in the 1960s and 1970s, told his players not to look at the scoreboard. He said that success is that good feeling you get from knowing that you played your best.

Consider and discuss: If the score in a game is close, you can’t use the information to play better than your best, so just play your best all the time and ignore the score. You can’t control how well the other person plays, so just worry about playing the best you can without worrying about how that compares with how other people are playing. In fact, it is best to not keep score at all, since the information is both distracting and useless.

2. Decide if there is a relationship, and if so what the relationship is, between

   a) the size of the smallest angle in a triangle and the size of the largest angle in the triangle.

   b) the length of the smallest side in a triangle and the length of the largest side in the triangle.

   c) the number of sides in a regular polygon and the length of each side.

   d) the number of sides in a regular polygon and the size of each angle.

   Students can use **BLM Triangles for Scatter Plots** for parts a) and b), and **BLM Regular Polygons of Different Sizes** for c) and d). Students should draw a scatter plot using a graphing program such as Microsoft...
Excel to show their results, and should describe the relationship (or lack of relationship) that their scatter plot shows.

**ANSWERS:**

a) If the smallest angle is large, the largest angle is likely to be smaller.

b) If the length of the smallest side is large, the length of the largest side is not any more likely to be large or small—there is no relationship.

c) If the number of sides is large, the length of each side is not any more likely to be large or small—there is no relationship.

d) If the number of sides is large, the size of each angle is likely to be large as well.

3. Have students compare the resting heart rates and average weights of various adult mammals. Is there a relationship? Students should research three more animals to add to the list below and make a scatter plot with all the data.

<table>
<thead>
<tr>
<th>Adult Mammal</th>
<th>Average Resting Heart Rate (beats per minute)</th>
<th>Average Weight (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beluga Whale</td>
<td>16</td>
<td>7 000</td>
</tr>
<tr>
<td>Cat</td>
<td>120</td>
<td>2.6</td>
</tr>
<tr>
<td>Cow</td>
<td>65</td>
<td>500</td>
</tr>
<tr>
<td>Elephant</td>
<td>30</td>
<td>4 600</td>
</tr>
<tr>
<td>Guinea Pig</td>
<td>280</td>
<td>1.4</td>
</tr>
<tr>
<td>Horse</td>
<td>44</td>
<td>500</td>
</tr>
<tr>
<td>Human</td>
<td>70</td>
<td>80</td>
</tr>
<tr>
<td>Rabbit</td>
<td>205</td>
<td>6</td>
</tr>
<tr>
<td>Rat</td>
<td>328</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Students can also research the life expectancy of these animals and make scatter plots comparing life expectancy to both weight and resting heart rate.

**Plotting identical points on a scatter plot.** Tell students that to do Question 7 on Workbook p. 47 they will need to draw two identical points. As an example, draw students’ attention to the third scatter plot (the one for gender and age). **ASK:** Are there two people on the soccer team that have the same age and gender? (yes, for example, there are two 15-year-old males) Should we show this differently than if there was only one 15-year-old male? (yes we should, to be accurate) Discuss ways of doing so. **(EXAMPLES:** use a different symbol for double points, such as a circle around the dot for each repetition; draw bigger dots) Tell students it doesn’t matter how they do it, as long as you can see that they know which points are doubled (there is no standard way to show duplicate points on a scatter plot).

**Drawing conclusions from scatter plots.** Discuss the relationship that students predicted for Workbook p. 46 Question 4 f). What might explain the relationship? (Possible answer: cities that are closer to the United States
tend to have better weather, so more people are motivated to live there.)
Have students research to find the data to complete the chart below.

<table>
<thead>
<tr>
<th>Canadian city</th>
<th>Population (thousands)</th>
<th>Distance from US border</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cornwall, ON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Edmonton, AB</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Espanola, ON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kingston, ON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sudbury, ON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Toronto, ON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vancouver, BC</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Windsor, ON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Winnipeg, MB</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then have students use a computer program to create the scatter plot.
Is there a relationship? (yes, Canadian cities that are closer to the border are more likely to have a larger population) Tell students that next year, Sara predicts that Espanola will have more people. **ASK:** Does this mean that she thinks that it will be closer to the US border next year? (no, you can’t move a city!) Tell students that just because two measurements are related, it doesn’t mean that you can change one by changing the other. You can sometimes, but not always.

After students finish Workbook pages 45–47, ask them if changing one of the factors in the problems below will change the other for a given data point. Students can signal thumbs up for yes and thumbs down for no.

a) If Sara (eight years old) gets older, is she likely to get taller? In other words, will she be taller a year from now than she is right now? (yes)

b) If a house price drops, is the house likely to move farther from the subway line? (no)

c) If home owners expand their house by adding a bedroom, is the house likely to increase in price? (yes)

d) If you increase the denominator of a fraction, is the fraction likely to have a smaller value? (yes)

e) If Finland’s population grows, is Finland’s area likely to become larger? (no)

Discuss with students their answers to Workbook Questions 7 and 8 in particular. Was there anything surprising about them? Discuss how, even though age and gender are both related to height, they are not related to each other. **ASK:** Do you see any similar relationships in Question 8? (distance from the subway line and number of bedrooms are both related to price, but might not be related to each other—we don’t have the data to check)

**PROCESS EXPECTATION**

Using logical reasoning
Extensions

1. Age of children and family size.
   a) Is there a relationship between the age of a child and the size of the child’s family (number of people that live in the same household including adults and children)? Design an experiment to find out. Show the results using a scatter plot.
   b) Cindy thinks that older children tend to have more siblings because there is more time for their parents to give them siblings. Since students in a grade 8 classroom are older than children on average (if you assume that children on average are about 9 years old), you would expect that, on average, students in a grade 8 classroom have more siblings than children on average, and hence a larger average family size. Check if this is true in your class. Survey the class: How many people live in your home? In Canada, the average size of families with children is 3.5. Is your class’s average higher or lower than the national average? Why do you think that is? (Hint: Use your answer to part a) to help you explain.)

2. The scatter plot students drew in the lesson for flights from a given city to various other cities showed that larger flying distances cost more. Cindy thinks that flights between the same two destinations (e.g., Vancouver and Beijing) will follow the opposite trend—longer trips (with more stops and time between connections) will cost less rather than more. What reasons might Cindy have for thinking this? (Longer trips are more inconvenient and hence it is less likely that people will want to pay for them. To entice people to suffer through longer trips, airlines make them cheaper.) Predict whether the prices for longer trips between the same cities will be lower. Then research the price of different flights between two cities of your choice to check your prediction.

3. Investigate whether there is a relationship between the natural population growth in a country and its average family size. NOTE: Students will likely find data on the Internet in the form of birth rate per 1000 people and death rate per 1000 people. If this is the case, students will need to multiply the birth rate by the population in thousands to determine the total number of births, and similarly for deaths, then subtract to find the natural population growth. The average family size will likely be given as an average and so won’t need manipulation.
Comparing scatter plots and line graphs. Have students individually draw the data from Workbook p. 48, Questions 1 and 2, as both scatter plots and line graphs. **ASK:** Did you have any problems drawing a line graph for one of the data sets? (yes) Which one? (the second one) What made it more difficult? (some cars were tested at the same speed, so there were two data values on the same vertical line)

Tell students that in both cases, the data measures efficiency of cars at different speeds. **ASK:** What is the difference between the two data sets? (In Question 1, the same car is tested at different speeds. In Question 2, different cars are tested at different speeds.) Point out that the data in Question 2 does not lend itself well to a line graph, because there is no natural “next” data value as there was for the data in Question 1.

Tell students that when we take measurements about one car at different speeds it makes sense to use a line graph, because the speeds are naturally increasing over time, so we can decide if there is a trend or not. Have students look at their graph for Question 1. **ASK:** Is there a trend? (yes: as the speed increases up to about 70, the car efficiency improves—gas used/100 L gets smaller—but efficiency worsens as the speed increases above 70) When we look at a single car over different speeds, we can look to see if that car behaves differently at different speeds. When we look at several cars over different speeds, we want to know if, in general, cars use gas more efficiently at lower or higher speeds. One car may be more efficient at 50 km/h than another car is at 70 km/h, even if both cars are more efficient at 70 km/h than at 50 km/h.

Have students finish Workbook Questions 1 and 2. Then have students decide between a scatter plot and a line graph for the following data:

<table>
<thead>
<tr>
<th>John’s height (cm)</th>
<th>80</th>
<th>82</th>
<th>85</th>
<th>90</th>
<th>95</th>
<th>100</th>
<th>110</th>
<th>116</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>John’s age (years)</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>
b) The ages and heights of various children living on the same street:

<table>
<thead>
<tr>
<th>height (cm)</th>
<th>89</th>
<th>86</th>
<th>88</th>
<th>95</th>
<th>90</th>
<th>98</th>
<th>96</th>
<th>105</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>age (years)</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>

**ASK:** Would you use a scatter plot or line graph to graph

- a) the number of words in various children’s vocabulary against their ages.
- b) the number of words in Sara’s vocabulary as she gets older.
- c) the average number of words in a child’s vocabulary against age.
  (Remind students that the average number of words in a 7-year-old’s vocabulary is the sum of all the values for all 7-year-olds divided by the number of 7-year-olds.)

**ANSWERS:**
- a) scatter plot
- b) line graph
- c) line graph

**Turn a scatter plot into a line graph by averaging.** For example, suppose the graph is height vs. age for ages 1–10. You might average all the heights for 1-year-olds, then average all the heights for 2-year-olds, and so on. Have students do this for the age-height scatter plot in Question 7 on Workbook page 47. Get them started by finding the average height for all 12-year-olds in the data together as a class. When students finish, **ASK:** What trend does the line graph show?

Have students turn the data from Workbook page 48 Question 2 into a line graph by averaging the data for the two cars at 30 km/h and then for the two cars at 100 km/h. Students can also change the scatter plot from Tasfia’s data on Workbook page 47, Question 8, into a line graph.

**ACTIVITY**

**Graph posters.** Encourage students to look for scatter plots and line graphs in books, in magazines, on the Internet, on television (e.g., on the Weather Network) or in brochures (e.g., from financial institutions).

Have students record properties that are common to all the scatter plots and all the line graphs. What scales and units does each graph use? What scales are more common? Have students cut out various line graphs and scatter plots and make two posters, one titled Line Graphs and the other Scatter Plots. Keep these posters for PDM8-14.

**Extension**

Change the scatter plot from Ahmed’s data on Workbook page 47, Question 8, into a line graph by creating intervals and averaging the data values in each interval. For example, create intervals for distances of 1–5 km, 6–10 km, and 11–15 km. Students can do the same for the scatter plots on Workbook page 46, Question 2a). **ASK:** What trend does the line graph show?
Limitations of bar graphs. Present the data table below (see BLM Canadian Incomes) and tell students that you want to make a graph to show this data. **ASK**: What type of graph would you use?

<table>
<thead>
<tr>
<th>Income of Canadians in 2008</th>
<th>Number of persons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under $5,000</td>
<td>2,037,550</td>
</tr>
<tr>
<td>$5,000 to $9,999</td>
<td>2,063,570</td>
</tr>
<tr>
<td>$10,000 to $14,999</td>
<td>2,445,550</td>
</tr>
<tr>
<td>$15,000 to $19,999</td>
<td>2,482,960</td>
</tr>
<tr>
<td>$20,000 to $24,999</td>
<td>1,983,710</td>
</tr>
<tr>
<td>$25,000 to $34,999</td>
<td>3,348,590</td>
</tr>
<tr>
<td>$35,000 to $49,999</td>
<td>3,974,750</td>
</tr>
<tr>
<td>$50,000 to $74,999</td>
<td>3,523,180</td>
</tr>
<tr>
<td>$75,000 to $99,999</td>
<td>1,528,680</td>
</tr>
<tr>
<td>$100,000 to $149,999</td>
<td>862,480</td>
</tr>
<tr>
<td>$150,000 to $199,999</td>
<td>223,700</td>
</tr>
<tr>
<td>$200,000 to $249,999</td>
<td>90,840</td>
</tr>
<tr>
<td>$250,000 and over</td>
<td>165,910</td>
</tr>
</tbody>
</table>

Tell students that you want to make a bar graph using the given income ranges as categories. Discuss what scale you should use. (The data values range from about 90,000 to about 4,000,000. Or, in thousands, from 90 to 4000. Because the smallest data value is so much smaller than the largest data value, we might as well go from 0 to 4000 instead of from 90 to 4000. About 8 intervals of size 500,000 would work.)

Show students the finished graph (see BLM Canadian Incomes on a Bar Graph). Discuss with students whether this is a good way to present the data. What problems does this graph have? (It has too many bars,
there is not enough space to write the categories on the horizontal axis, some bars are very small and hard to see.) Which interval has the largest number of people (the tallest bar)? ($35,000 to $49,999) Compare this interval to the interval before it. Is this a fair comparison? (no) Why not? (The previous interval is smaller than this interval: $10,000 vs. $15,000). Part of the reason there might be more people in the second bar is that the interval is larger.) Repeat with the interval after the tallest bar. Why do you think that the categories for higher income have increments of 50,000 and the categories for low income have increments of only 5000? (There are few people earning the larger amounts. The bars would become very small and there would be too many of them if we used increments of 5000 for the larger amounts. Using smaller increments at the low end of the scale makes the bars there shorter, which gives the impression that there are fewer low-income earners than there actually are.)

Suppose somebody earns $24,999.50 in a year. Can we add this person to the graph? (no) How can we modify the graph to allow for people with incomes that are not whole numbers of dollars? (one answer could be to make the categories go to a number that ends in 999.99)

**Introduce histograms.** Explain that *histograms* are similar to bar graphs, but they have categories that are numerical intervals of the same size. The next interval starts where the previous interval ends, so there are no spaces between the bars. Also, instead of labelling the range for each category (e.g., 0–5000) we use a regular number line. A bar for the category 0–5000 will take up the space between zero and 5000 on the horizontal number line. Data values that are on the border between intervals are counted in the bar for the higher interval.

**ASK:** What size interval would make sense for the data above? Can we use intervals of 5000? Why not? (we only have data in intervals of 5000 for the low end of the scale) Does it make sense to use intervals of 50,000? Why not? Prompt students to think about what the first and last data values will be. (If the first interval is “below $50,000,” the bar will need to be about 18,000,000 units tall, and the last bar, “$250,000 and over,” will be only about 165,900 units tall, or about 100 times smaller than the first bar!) Lead students to using intervals of 25,000 and making the last category “$100,000 or more.” Have students make a new frequency table for the data:

<table>
<thead>
<tr>
<th>Income of Canadians in 2008</th>
<th>Number of persons</th>
</tr>
</thead>
<tbody>
<tr>
<td>under $25,000</td>
<td>11,013,340</td>
</tr>
<tr>
<td>$25,000–$49,999</td>
<td>7,323,340</td>
</tr>
<tr>
<td>$50,000–$74,999</td>
<td>3,523,180</td>
</tr>
<tr>
<td>$75,000–$99,999</td>
<td>1,528,680</td>
</tr>
<tr>
<td>$100,000 and over</td>
<td>1,342,930</td>
</tr>
</tbody>
</table>

As a class, discuss the scale that should be used (e.g., count by millions, from 0 to 12 million), and then show students a histogram for the data (see BLM Canadian Incomes on a Histogram).
Advantage of a histogram. Point out how the last two bars are easy to compare because they are side by side. The last bar is only about 2/15ths shorter than the previous bar. It would be hard to see this difference if the bars had a gap between them.

Analyze the histogram. What can students tell about the data from the histogram? Ask them to make a list of conclusions individually then compare and combine their lists with a partner. You can use the Scribe, Stand, Share strategy to check the results. (Answers could include: the largest number of people have income below $25,000; the higher the income, the fewer people earn that much; more than half of all Canadians earn less than $50,000 in a year; the total number of Canadians with any kind of income is about 24 million people; only about one 1/16 of the whole population with income get more than $100,000; a very small number of Canadians get a lot of money)

Discrete and continuous data. Review what it means for data to be continuous (it is possible to have values between the increments of the scale). If the categories have no order to them, then the data is definitely discrete (not continuous) because there can be nothing between the data values. Have students decide whether the data in the following cases is discrete or continuous:

a) shoe size (discrete, there cannot be shoe size 9 3/4)
b) weekly salary in dollars (continuous, $234.6 makes sense)
c) car colour (discrete, there is no natural order)
d) height of a person in centimetres (continuous, a person can be 156.9 cm tall)
e) favourite rainbow colour (discrete – although the colours of a rainbow are always in the same order and there are colours that we would consider to be between green and blue for example, the colours between green and blue are not called rainbow colours)

A bar graph or a histogram? Explain that when data is continuous, it makes sense to use a histogram; when it is discrete, a bar graph should be used. Have students decide whether they would use a histogram or a bar graph to display the following data:

<table>
<thead>
<tr>
<th>Distance from school (km)</th>
<th>0–1</th>
<th>1–2</th>
<th>2–3</th>
<th>3–4</th>
<th>4+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of students</td>
<td>98</td>
<td>123</td>
<td>314</td>
<td>154</td>
<td>201</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time spent exercising (hours per week)</th>
<th>0–2</th>
<th>2–4</th>
<th>4–6</th>
<th>6–8</th>
<th>8+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of people</td>
<td>4</td>
<td>6</td>
<td>12</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of pets owned</th>
<th>0–1</th>
<th>2–3</th>
<th>4–6</th>
<th>7–8</th>
<th>9+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of people</td>
<td>18</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>
d) |
<table>
<thead>
<tr>
<th>Favourite type of music</th>
<th>rock</th>
<th>pop</th>
<th>jazz</th>
<th>hip hop</th>
<th>other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of people</td>
<td>6</td>
<td>8</td>
<td>4</td>
<td>12</td>
<td>10</td>
</tr>
</tbody>
</table>

ANSWERS:

a) histogram  
b) histogram  
c) bar graph  
d) bar graph

Point out that marks on a test are often regarded as continuous data because even though most teachers don’t hand out marks like 39.5 out of 50, such marks are possible. Also, if a test is out of 18, you won’t have whole number percents, and marks are often recorded as percents; hence marks on tests are often displayed on a histogram. The marks definitely have a natural order to them.

Show the table below and ask students to decide whether a bar graph or a histogram is more appropriate to display this data. (a bar graph, because the categories are discrete) What if a store is open 24 hours? In this case, a histogram will make sense as well, though we could still create a bar graph by placing all the sales that were made before midnight into one day and all the sales made after midnight into the next day.

<table>
<thead>
<tr>
<th>Total Sales ($)</th>
<th>800</th>
<th>1500</th>
<th>900</th>
<th>2000</th>
<th>600</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day</td>
<td>Monday</td>
<td>Tuesday</td>
<td>Wednesday</td>
<td>Thursday</td>
<td>Friday</td>
</tr>
</tbody>
</table>

**ACTIVITY**

**Graph posters.** Encourage students to look for histograms and bar graphs in books, in magazines, on the Internet, on television (e.g., on the Weather Network) or in brochures (e.g., from financial institutions). Have students record properties that are common to all the histograms or bar graphs. How many categories are used in each graph? About how many categories do most histograms use? Is it different for bar graphs? Have students cut out various bar graphs and histograms and make two posters, one titled Histograms and the other Bar Graphs. Keep these posters for PDM8-14.
Choosing a scale for a histogram. Present the following set of ages of people responding to a survey.

18 18 19 19 20 20 21 21 21 22 25 28 28 29 30 33 35 36 39 42
42 43 44 45 47 48 48 48 50 51 53 54 54 55 58 68 69 71 72

Explain that you would like to make a histogram to show this data, but you need to decide what the intervals should be and where the scale should start and end. ASK: Does it make sense to start at 0? (No, the smallest data value is 18, so any bar for people below that age will be empty. Also, young children usually do not respond to surveys.) Remind students that when the scale does not start at zero, we use a jagged line to show that. Does it make sense to have the bars go to 100? (no, the highest data value is 72, so bars for higher ages will be empty as well) This means that we have to concentrate on the range of the data—we should start close to 18 and end close to 72. Have students find the range of the data. (72 − 18 = 54) Remind students that the intervals should be equal in length. ASK: If we divided the range into intervals of length 3, how many intervals would we need? (54 ÷ 3 = 18) Is that a good number of intervals? (no, it’s too many) How can we divide the range into fewer intervals? (use larger intervals) Tell students that you want to try making intervals of length 5. Divide the students into two groups to make the histogram: one group uses intervals of 5 starting at 18 (18–23, 23–28, 28–33, ..., 68–73) and the other uses intervals of 5 starting at 15 (15–20, 20–25, 25–30, ..., 70–75). When students finish, SAY: I noticed that the students that used intervals starting at 15 finished more quickly. Is this because they had fewer intervals? (no, they actually had one more interval) Why do you think they finished more quickly? (because multiples of 5 are easier to work with) Tell students that you think this is still a lot of intervals though. What other interval length can we use to make the histogram easy to work with? (intervals of 10, starting
at 10 and finishing at 80). **ASK:** How many intervals would there be then?

(7) Have students make the histogram using these intervals. Then have students make a histogram using these intervals: 0–50 and 50–100. **ASK:** Are these intervals convenient to display the results? (no, because there are now too few intervals; the data is only divided into two groups)

**Comparing histograms with different intervals.** Display four computer generated histograms for the above data and intervals side by side (see BLM Ages of People Responding to a Survey). The histograms use these intervals:

- **A** 18–23, ..., 68–73
- **B** 15–20, ..., 70–75
- **C** 10–20, ..., 70–80
- **D** 0–50, 50–100

Have students compare the histograms. **ASK:** Do the histograms with the same interval length have bars of the same height? (no) Do they all have an empty bar? (no) Which histogram has the smallest number of bars? (the histogram with the widest intervals) Which histogram has the largest number of bars? (the histogram that started below the lowest data value and ended above the highest data value, using the smallest intervals) Which histogram had more people in each bar? (the histogram with the widest intervals) The first bar is the tallest on Histogram A. Is the first bar the tallest on all the histograms? (no) Why not? (when a histogram starts below the first data value, some of the values that would fall into the first interval in other histograms now fall into the second bar, and then the second bar is one of the tallest)

Have students work through Questions 1–4 on Workbook p. 52.

**EXTRA PRACTICE:**

An organization wants to know the ages of the people who responded to their survey. Here are the data:

29  37  41  40  38  42  49  54  57  42  61  28  34  44  44  58  63  88  64  59

a) Organize the data into intervals and make a frequency table.

b) Make a histogram using your intervals. Title your histogram and label both axes.

**REMEMBER:** If the data does not start at 0, you will need a jagged line to show this.

**Relative frequency histograms.** Review converting fractions to percents. Have students convert to percents the following fractions:

a) $\frac{4}{10}$  b) $\frac{4}{40}$  c) $\frac{1}{4}$  d) $\frac{10}{40}$  e) $\frac{6}{40}$  f) $\frac{2}{40}$

**ANSWERS:**

a) 40%  b) 10%  c) 25%  d) 25%  e) 15%  f) 5%

Review what frequency tables and relative frequency tables are, and have students convert the frequency table for the ages of survey respondents given below to a relative frequency table that includes the percent of respondents. To guide students, **ASK:** How many respondents are there
altogether? \((4 + 10 + 8 + 8 + 6 + 2 + 2 = 40)\) (The answers for the percents row are in brackets.)

<table>
<thead>
<tr>
<th>Ages of survey respondents</th>
<th>15–20</th>
<th>20–30</th>
<th>30–40</th>
<th>40–50</th>
<th>50–60</th>
<th>60–70</th>
<th>70+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of respondents</td>
<td>4</td>
<td>10</td>
<td>8</td>
<td>8</td>
<td>6</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Percent of respondents</td>
<td>(10)</td>
<td>(25)</td>
<td>(20)</td>
<td>(20)</td>
<td>(15)</td>
<td>(5)</td>
<td>(5)</td>
</tr>
</tbody>
</table>

Explain that people whose age is at the border of an interval are placed in the higher interval. Ask students to draw a histogram of the data, using the frequency table. Which of the intervals should be changed, and how? (15–20 should be changed to 10–20) Then draw a template for the relative frequency histogram: use the same intervals on the horizontal axis but explain that you want to see the percents on the vertical axis instead of the number of respondents. **ASK:** What scale should we use? (All percents in the table are multiples of 5, so it makes sense to count by 5s, from 0 to 25.) Explain that a histogram that shows percents instead of frequencies is called a **relative frequency histogram.**

Have students compare the two histograms. How are they the same? (they have the same intervals on the horizontal axis and the same shape) How are they different? (the vertical axis shows percents on the relative frequency histogram and the actual number of respondents on the regular histogram)

Have students answer the following questions about both histograms. They should say which histogram is more convenient to use to answer the question.

a) Which age group was the most represented among the respondents? (20–30, both histograms work well)

b) Is the following statement true: As age increases, the number of survey respondents decreases? Is there a certain age from which this statement seems true? Which age? Modify the statement to make it true. (Starting from age 20, the number of survey respondents decreases. As a matter of fact, it is more accurate to say that the number does not increase (it levels off at the end), but the previous statement is a good answer as well. Both histograms can be used.)

c) About 12% of the Canadian population over age 15 is older than 65. Is this group proportionally represented in the survey results? (Using the relative frequency histogram, we can see that people over 60 represent only 10% of the respondents, so people over 65 represent at most 10% of the number of respondents. This is definitely lower than 12%, so this group is under-represented. Relative frequencies are more convenient to use here.)

Have students do Question 5 on Workbook page 53.
EXTRA PRACTICE:

Here are two histograms that use the same data as in Question 5.

![Histogram 1]

![Histogram 2]

a) Does one of the graphs look like the histogram you drew in Question 5? (the first one has a similar shape)

b) Are both graphs consistent with each other? How do you know? (yes—for example, the totals for 0–6 and 6–12 add to the totals for 0–12 (5 + 5 = 10) and this can be checked for the other pairs of intervals as well)

c) Why do the two graphs look so different? (because they use different intervals)

PROCESS EXPECTATION

Reflecting on the reasonableness of the answer

ACTIVITY

Have students brainstorm data that would best be presented as a histogram. Students should think about data that they can collect themselves or data they can research. (EXAMPLES: how many hours/wk students spend playing video games; how many hours/wk Canadians spend playing video games) Then have students survey the class or search the Web to find the data. Students should present their data in the form of a histogram and write a paragraph about the conclusions they can draw from the data.
Goals
Students will distinguish between the purpose of the various types of graphs and decide which graph is most appropriate to display given data.

PRIOR KNOWLEDGE REQUIRED
Can read and draw circle graphs, line graphs, bar graphs, double bar graphs, scatter plots, and histograms
Can calculate the mean of a given data set

MATERIALS
BLM Matching Data to Graphs (p N-60)
posters from lessons PDM8-7, 10, 12

Compare situations when each type of graph is used. If students made posters in lessons 7, 10, and 12, bring them out and have students compare when each type of graph is used. ASK: In which situations are bar graphs used? Repeat for line graphs, circle graphs, histograms, and scatter plots. ASK: When are double bar graphs used? Can you use a double line graph? When would you do so? (to compare trends) When would you use a scatter plot? (to compare two characteristics) Could you use circle graphs or histograms to compare data? (yes, but you would need two graphs side by side)

EXTRA PRACTICE: BLM Matching Data to Graphs
Review the purpose of each type of graph. Have students individually do Workbook p. 55 Question 3, then take up the answers as a class.

Line graphs vs. bar graphs. A line graph is used when you are looking for a trend and when the data can be divided into categories that have a natural order to them. A bar graph is used when the data is divided into categories that do not have a natural order to them. For example, days of the week and times of day have a natural order to them; favourite types of movies do not. ASK: Which type of graph should a mall use to determine:

a) the number of people who enter the mall between 9 a.m. and 10 a.m., between 10 a.m. and 11 a.m., and so on, to between 5 p.m. and 6 p.m.
b) the number of people who enter each store in the mall between 9 a.m. and 10 a.m.

ANSWER:
a) line graph     b) bar graph
Both bar graphs and line graphs can be used to study frequency of results, although line graphs can also be used to study data where frequency doesn't make sense (e.g., the temperatures at different times of day).

**Scatter plots.** Tell students that the management of a mall wants to know whether there is a connection between the number of people who enter clothing stores and the time of the day. They record the number of people that enter each clothing store in the mall between 9 a.m. and 10 a.m., between 10 a.m. and 11 a.m., and so on, to between 5 p.m. and 6 p.m. **ASK:** Which type of graph should they use to determine if there is a relationship? (scatter plot) Will a number of line graphs (one for each store) make sense? Why? (The answer depends on the number of stores. If there are two or three, a double or triple line graph will make sense, but if there are more stores, the graph will become very messy.)

Remind students that sometimes it makes sense to take the mean of some data and to convert a scatter plot into a line graph. This is convenient when we work with two scatter plots. For example, if the same mall wants to compare the relationship they found for clothing stores with a similar relationship for restaurants, they could split the data into groups showing how many people enter each store (or restaurant) in a given hour, take the mean of each group, and plot the means on a line graph. The data for each particular store would be lost, but it would be easy to compare restaurants on average to clothing stores on average.

**Bar graphs vs. histograms.** Review the fact that when the data has a natural order and is continuous, a histogram makes more sense than a bar graph. Remind students that you cannot use a histogram when the data cannot be split into intervals of equal size. **ASK:** Would you use a histogram or a bar graph to show the following sets of data? Have students explain their choices.

- a) Sales of tickets for the most popular movies of the season
- b) Number of families in Canada by number of children
- c) Number of families in Canada by income
- d) Number of families in Canada by type of family (single parent, two parents, extended family living together, etc)
- e) Sales of tickets for [insert a popular movie] during its first weeks when the data is grouped into two intervals, “Monday to Friday” and “Saturday to Sunday”

**ANSWERS:** a) and d) bar graph—data is categorical, not numerical; b) bar graph—data is numerical, but discrete; c) histogram—data is continuous; e) bar graph—Though the data is continuous, the time intervals are unequal, so you have to use a bar graph. However, if you have data for several weeks, you might group the data by weeks and use a histogram.

Have students determine which type of graph (line graph, bar graph, double line or bar graph, scatter plot, histogram) should be used in the following situations:
a) To see the progress of a particular student in mathematics over the year. (looking for trends, line graph)

b) To check the results of all students in grade 8 in the school on a particular unit. (continuous data, histogram)

c) To compare how two classes did on a particular French test. (double bar graph or two histograms)

d) To compare the progress of two classes in French over the year. (take the average of each class and use a double line graph)

e) To determine whether there is a relationship between class average in social studies and the experience of the teacher. (scatter plot)

Circle graphs. Note that circle graphs and bar graphs are both used with categories. Use circle graphs if you are more interested in percentages or relative frequencies, and use bar graphs if you are more interested in absolute frequencies. For example, if you want to know how much one company spends as compared to another, use a bar graph. If you want to know how the distribution of spending compares (i.e., does one company spend a greater proportion of its budget on advertising than the other), then comparing bar heights won’t tell you anything; you need a circle graph with advertising as one of the regions. Have students answer the following questions individually.

Would you use a circle graph or a bar graph to compare

a) how many Canadian stamps each of three friends has

b) what percent of each friend’s stamp collection is Canadian

c) the population of various Canadian cities

d) what percent of the population in Toronto and Vancouver are various minority groups

ANSWERS: Use a bar graph for a) and c), a circle graph for b) and d).

EXTRA PRACTICE:
Here is some data from Statistics Canada:

Population by ages in 2006

<table>
<thead>
<tr>
<th></th>
<th>0–14</th>
<th>15–64</th>
<th>65+</th>
<th>total</th>
<th>85+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canada</td>
<td>5 579 835</td>
<td>21 697 805</td>
<td>4 335 255</td>
<td>31 612 895</td>
<td>1 167 310</td>
</tr>
<tr>
<td>Toronto</td>
<td>949 945</td>
<td>3 556 180</td>
<td>607 025</td>
<td>5 113 150</td>
<td>157 650</td>
</tr>
<tr>
<td>Vancouver</td>
<td>345 745</td>
<td>1 499 370</td>
<td>271 465</td>
<td>2 116 580</td>
<td>76 135</td>
</tr>
</tbody>
</table>

Which types of graphs could be used to present and analyze the data? If you use a bar graph, what categories will you use? (0–14, 15–64, 65–84, 85+) How will you get the data for each? (for 0–14, 15–64, and 85+, get the data directly from the table; for 65–84, subtract the 85+ entries from the 65+ entries)
Can you present the data for one location using a histogram? (no, because the intervals are not equal) A circle graph? (yes, because you can ask which percent of the total population each age group is—you would need three circle graphs, one for each location, and you would need to first make a relative frequency table) Will a scatter plot make sense? (No, a scatter plot only makes sense when each individual object has two sets of data measurements for it, e.g., a person’s age and the same person’s height.)

To compare age groups in Vancouver and Toronto by size, which graph makes the most sense? (double bar graph) To compare distributions (e.g., which location has the largest proportion of people 85+ years old) which type of graph would you use? (circle graph)

### Activity

Have students find their own data and then choose the kind of graph they want to use to represent it. (See the Introduction for online sources of data on different topics. Consider cross-curricular connections: Is there any data from another subject that students can graph?) Topics students can research and compare include:

- prices of hotel rooms in a particular location
- prices of plane tickets to a given destination across various companies
- average temperatures and precipitations at different locations
- world records or events (e.g., number of goals per game) in a particular sport over a period of time
- information about different countries (e.g., infant mortality rate)
- information about different provinces or communities in Canada (e.g., age distribution)
- the spending distribution of different organizations
- relationship between infant mortality rate and gross domestic product (GDP) per capita or income inequality

Students should formulate a question about the data. **Examples:**

- How have world record times for running the 100 m dash changed over time?
- How many participants in the 100 m dash at the last Olympics would have broken the world record 50 years ago if they ran their times then?
- How has the average number of goals per game in the NHL changed over time?
- How do different charities spend their money?
- As a country’s GDP per capita increases, does that country’s infant mortality rate increase or decrease?
A Large Circle Graph
Small Circle Graphs
Choice of Car Colours in Canada

black 18%
silver 17%
grey 15%
other 18%
red 11%
white 21%
black 18%
other 18%
red 11%
grey 15%
white 21%
black 18%
other 18%
red 11%
silver 17%
Students’ Favourite Subjects

Students’ Favourite Subjects

- Phys Ed: 40%
- Art: 18%
- Music: 10%
- Home Ec: 21%
- Math: 5%
- Other: 6%
Unmarked Circle Graph
## Favourite Car Colour

<table>
<thead>
<tr>
<th>Favourite Car Colour</th>
<th>People</th>
</tr>
</thead>
<tbody>
<tr>
<td>White</td>
<td>64</td>
</tr>
<tr>
<td>Black</td>
<td>57</td>
</tr>
<tr>
<td>Blue</td>
<td>18</td>
</tr>
<tr>
<td>Green</td>
<td>13</td>
</tr>
<tr>
<td>Other</td>
<td>12</td>
</tr>
</tbody>
</table>

- **White** 28%
- **Black** 25%
- **Blue** 36%
- **Green** 6%
- **Other** 7%

**Favourite Car Colour People**

- **White** 64 people
- **Black** 57 people
- **Blue** 18 people
- **Green** 13 people
- **Other** 12 people

- **White** 39%
- **Black** 35%
- **Blue** 11%
- **Green** 8%
- **Other** 7%
Triangles for Scatter Plots
Regular Polygons of Different Sizes
## Canadian Incomes

<table>
<thead>
<tr>
<th>Income of Canadians in 2008</th>
<th>Number of persons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under $5,000</td>
<td>2,037,550</td>
</tr>
<tr>
<td>$5,000 to $9,999</td>
<td>2,063,570</td>
</tr>
<tr>
<td>$10,000 to $14,999</td>
<td>2,445,550</td>
</tr>
<tr>
<td>$15,000 to $19,999</td>
<td>2,482,960</td>
</tr>
<tr>
<td>$20,000 to $24,999</td>
<td>1,983,710</td>
</tr>
<tr>
<td>$25,000 to $34,999</td>
<td>3,348,590</td>
</tr>
<tr>
<td>$35,000 to $49,999</td>
<td>3,974,750</td>
</tr>
<tr>
<td>$50,000 to $74,999</td>
<td>3,523,180</td>
</tr>
<tr>
<td>$75,000 to $99,999</td>
<td>1,528,680</td>
</tr>
<tr>
<td>$100,000 to $149,999</td>
<td>862,480</td>
</tr>
<tr>
<td>$150,000 to $199,999</td>
<td>223,700</td>
</tr>
<tr>
<td>$200,000 to $249,999</td>
<td>90,840</td>
</tr>
<tr>
<td>$250,000 and over</td>
<td>165,910</td>
</tr>
</tbody>
</table>

Data adapted from Statistics Canada.
Canadian Incomes on a Bar Graph

Income of Canadians in 2008

Income ($)

Under $5,000
$5,000 to $9,999
$10,000 to $14,999
$15,000 to $19,999
$20,000 to $24,999
$25,000 to $34,999
$35,000 to $49,999
$50,000 to $74,999
$75,000 to $99,999
$100,000 to $149,999
$150,000 to $199,999
$200,000 to $249,999
$250,000 and over

Number of Persons

4,500,000
4,000,000
3,500,000
3,000,000
2,500,000
2,000,000
1,500,000
1,000,000
500,000
### Canadian Incomes on a Histogram

<table>
<thead>
<tr>
<th>Income of Canadians in 2008</th>
<th>Number of persons</th>
</tr>
</thead>
<tbody>
<tr>
<td>under $25 000</td>
<td>11 013 340</td>
</tr>
<tr>
<td>$25 000-$49 999</td>
<td>7 323 340</td>
</tr>
<tr>
<td>$50 000-$74 999</td>
<td>3 523 180</td>
</tr>
<tr>
<td>$75 000-$99 999</td>
<td>1 528 680</td>
</tr>
<tr>
<td>$100 000 and over</td>
<td>1 342 930</td>
</tr>
</tbody>
</table>

**Income of Canadians in 2008**

**NOTE:** The income interval for the right hand bar is open-ended to include all incomes over $100 000. Therefore, it does not have a finite length like the other intervals.
Ages of People Responding to a Survey

<table>
<thead>
<tr>
<th>Age</th>
<th>Number of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>10</td>
</tr>
<tr>
<td>23</td>
<td>9</td>
</tr>
<tr>
<td>28</td>
<td>8</td>
</tr>
<tr>
<td>33</td>
<td>7</td>
</tr>
<tr>
<td>38</td>
<td>6</td>
</tr>
<tr>
<td>43</td>
<td>5</td>
</tr>
<tr>
<td>48</td>
<td>4</td>
</tr>
<tr>
<td>53</td>
<td>3</td>
</tr>
<tr>
<td>58</td>
<td>2</td>
</tr>
<tr>
<td>63</td>
<td>1</td>
</tr>
<tr>
<td>68</td>
<td></td>
</tr>
<tr>
<td>73</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Age</th>
<th>Number of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>4</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
</tr>
<tr>
<td>25</td>
<td>4</td>
</tr>
<tr>
<td>30</td>
<td>4</td>
</tr>
<tr>
<td>35</td>
<td>3</td>
</tr>
<tr>
<td>40</td>
<td>4</td>
</tr>
<tr>
<td>45</td>
<td>5</td>
</tr>
<tr>
<td>50</td>
<td>5</td>
</tr>
<tr>
<td>55</td>
<td>2</td>
</tr>
<tr>
<td>60</td>
<td>1</td>
</tr>
<tr>
<td>65</td>
<td>1</td>
</tr>
<tr>
<td>70</td>
<td></td>
</tr>
<tr>
<td>75</td>
<td></td>
</tr>
</tbody>
</table>
Matching Data to Graphs

Match the data with its graph.

1. Favourite Sport | Frequency | Fraction of Total
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Hockey</td>
<td>42</td>
<td>( \frac{3}{10} )</td>
</tr>
<tr>
<td>Soccer</td>
<td>35</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>Baseball</td>
<td>28</td>
<td>( \frac{1}{5} )</td>
</tr>
<tr>
<td>Volleyball</td>
<td>28</td>
<td>( \frac{1}{5} )</td>
</tr>
<tr>
<td>Other</td>
<td>7</td>
<td>( \frac{1}{20} )</td>
</tr>
</tbody>
</table>

2. Martha’s math test scores (out of 10):

<table>
<thead>
<tr>
<th>Test#</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

3. Month | Jan | Feb | Mar | Apr | May | Jun
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Company A’s home sales</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Company B’s home sales</td>
<td>10</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

4. Class marks on science test:

| 29 | 84 | 47 | 69 | 71 | 72 | 88 | 64 | 38 | 42 |
| 56 | 51 | 77 | 79 | 80 | 80 | 43 | 81 | 76 | 77 |

5. Families are surveyed about how many cars they have:

<table>
<thead>
<tr>
<th>Number of cars</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>12</td>
<td>15</td>
<td>11</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>
## Unit 3  Geometry

In this unit, students will use various tools and strategies to

- investigate geometric properties related to parallel and perpendicular lines, angles, and sides of shapes
- sort and classify triangles and quadrilaterals by their geometric properties
- investigate properties of similar and congruent shapes

### Protractors

If you need additional protractors for individual students, you can photocopy a protractor (or **BLM Protractors**, p O-89) onto a transparency and cut it out. Such protractors are also convenient to use on an overhead projector.

### Paper Folding

Many Activities in these lessons involve paper folding. Unless otherwise noted, the starting shape is a regular 8 1/2” × 11” sheet of paper. Sometimes the starting shape is an oval or a cloud, to make sure there are no angles for students to start with or refer to.

### Technology: Geometer’s Sketchpad

Students are expected to investigate geometric properties of lines, angles, triangles, and quadrilaterals using dynamic geometry software. Many activities in this unit use a program called Geometer’s Sketchpad, and some are instructional—they help you teach students how to use the program. If you are not familiar with Geometer’s Sketchpad, the built-in Help Centre provides explicit instructions for many constructions. Search the Index using phrases such as “How to construct congruent angles” or “How to construct a line segment of given length.”

Please note: dynamic geometry software is not required to complete the unit.

### Summary BLMs

Step-by-step instructions for constructions used in the unit are summarized on BLMs, for easy reference. This chart lists the summary BLMs available and the lessons they relate to.

<table>
<thead>
<tr>
<th>Summary BLM</th>
<th>Lesson(s)</th>
<th>Constructions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measuring and Drawing Angles and Triangles (p O-90)</td>
<td>G8-17</td>
<td>Measuring an angle&lt;br&gt;Drawing an angle&lt;br&gt;Drawing lines that intersect at an angle&lt;br&gt;Drawing a triangle</td>
</tr>
<tr>
<td>Drawing Perpendicular Lines and Bisectors (p O-91)</td>
<td>G8-19 G8-29</td>
<td>Drawing a line segment perpendicular to AB through point P (using a set square, using a protractor)&lt;br&gt;Drawing the perpendicular bisector of line segment AB</td>
</tr>
</tbody>
</table>
### Quadrilaterals and Special Quadrilaterals: A Summary

Quadrilaterals are two-dimensional shapes with four straight sides. Special quadrilaterals share certain properties. There are three broad groups of special quadrilaterals:

**Parallelograms** have two pairs of parallel sides. Parallelograms include rectangles, squares, and rhombuses.

**Trapezoids** have one pair of parallel sides. In isosceles trapezoids, the non-parallel sides are equal in length. In right trapezoids, there are two right angles.

**Kites** have two pairs of equal adjacent sides and no indentation. Kites include rhombuses and squares.

In Lesson G8-22, Extension 2, you can build a diagram that shows the relationships between different special quadrilaterals.

<table>
<thead>
<tr>
<th>Special Quadrilateral</th>
<th>Sides</th>
<th>Angles</th>
<th>Diagonals</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>opposite</td>
<td>adjacent</td>
<td>opposite</td>
</tr>
<tr>
<td>Parallelogram</td>
<td>parallel, equal</td>
<td>equal</td>
<td>add to 180°</td>
</tr>
<tr>
<td>Rectangle</td>
<td>parallel, equal</td>
<td>perpendicular</td>
<td>equal, 90°</td>
</tr>
<tr>
<td>Square</td>
<td>parallel, equal</td>
<td>perpendicular, equal</td>
<td>equal, 90°</td>
</tr>
<tr>
<td>Special Quadrilateral</td>
<td>Sides</td>
<td>Angles</td>
<td>Diagonals</td>
</tr>
<tr>
<td>----------------------</td>
<td>-------</td>
<td>--------</td>
<td>-----------</td>
</tr>
<tr>
<td></td>
<td>opposite</td>
<td>adjacent</td>
<td>opposite</td>
</tr>
<tr>
<td>Rhombus</td>
<td>parallel, equal</td>
<td>equal</td>
<td>equal</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kite</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trapezoid</td>
<td>one pair of parallel sides</td>
<td>not equal</td>
<td>angles at non-parallel sides add to 180°</td>
</tr>
<tr>
<td>Isosceles trapezoid</td>
<td>one pair of parallel sides, the other sides are equal</td>
<td>add to 180°</td>
<td>angles at non-parallel sides add to 180°, angles at parallel sides are equal</td>
</tr>
<tr>
<td>Right trapezoid</td>
<td>one pair of parallel sides</td>
<td>not equal</td>
<td>angles at non-parallel sides add to 180°, two right angles</td>
</tr>
</tbody>
</table>
Review the concepts of a point, line, line segment, and ray.

A dot represents a point. A point is an exact location. It has no size—no length, width, or height. The dot has size, or you couldn’t see it, but real points do not.

A line extends in a straight path forever in two directions. It has no ends. Lines that are drawn have a thickness, but real lines do not.

A line segment is the part of a line between two points, called endpoints. It has a length that can be measured.

A ray is part of a line that has one endpoint and extends forever in one direction.

Review naming a point, line, ray, and line segment.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>A</th>
<th>B</th>
<th>A</th>
<th>B</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Point</td>
<td>A point is</td>
<td></td>
<td>To name</td>
<td></td>
<td>To name</td>
<td></td>
<td>To name</td>
</tr>
<tr>
<td></td>
<td>named with</td>
<td></td>
<td>a line,</td>
<td></td>
<td>a ray,</td>
<td></td>
<td>a line</td>
</tr>
<tr>
<td></td>
<td>a capital</td>
<td></td>
<td>give the</td>
<td></td>
<td>give the</td>
<td></td>
<td>segment</td>
</tr>
<tr>
<td></td>
<td>letter.</td>
<td></td>
<td>names of</td>
<td></td>
<td>name of</td>
<td></td>
<td>, give</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>any two</td>
<td></td>
<td>any point</td>
<td></td>
<td>the names</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>points on</td>
<td></td>
<td>on the</td>
<td></td>
<td>of the</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>the line.</td>
<td></td>
<td>endpoint and any point on the line.</td>
<td></td>
<td>endpoints.</td>
</tr>
</tbody>
</table>

Point out that you don’t need to draw arrows and/or dots at the ends of lines, rays, or line segments unless you especially want to show that something is a line, ray, or line segment.

Introduce the intersection point—a point that lines, line segments, or rays have in common. Draw several examples of intersecting line segments, including the cases below. Point out the intersection point in each picture.
**Review intersecting lines and line segments.** Draw the picture in the margin on the board and **ASK:** Does \( AB \) intersect \( CD \)? (no) Explain that the answer is correct if we regard these as line segments, but the general answer depends on whether \( AB \) and \( CD \) are lines, line segments, or rays. Check all possible combinations of lines and line segments, extending the lines to show intersection. (If either one of these is a line segment, they do not intersect. If both are lines, they intersect.)

Ask students to check all four possible rays, as well as all combinations of rays and lines, or rays and line segments.

**ANSWERS:** Out of four possible rays (\( AB, BA, CD, DC \)), only \( BA \) and \( DC \) intersect. Also, line \( AB \) intersects with ray \( DC \) and ray \( BA \) intersects with line \( CD \). **EXAMPLES:**

- Rays \( AB \) and \( DC \) do not intersect
- Line \( AB \) and ray \( DC \) intersect
- Line \( AB \) and ray \( CD \) do not intersect

**ACTIVITIES 1–2**

1. Teach students to draw lines, line segments, and rays using the line tool of Geometer’s Sketchpad. Then ask them to draw a line and an independent point. Ask them to move the point so that it looks like it is on the line. Then have them modify the line. Does the point stay on the line? (no) Now show students how to construct a line through two given points and a point on the line, so that modifying the points keep the line and the points together.

2. a) Ask students to draw a line segment using Geometer’s Sketchpad and to measure it. Then ask them to try to move the endpoints so that the length of the segment becomes, say, 3 cm. Is it easy or hard to do? If you drag the line segment around, does its length change? (no) If you move the endpoints around, does the length of the line segment change? (yes) Show students how to draw a line segment of fixed length (using the circle tool). Will moving the endpoints change the length of the line segment now? (no)

b) Ask students to draw a line segment and label it \( AB \). Challenge students to draw a line segment \( CD \) that is exactly the same length as \( AB \). **PROMPT:** Measure the length of \( AB \). You can use the length of \( AB \) as the parameter when drawing a line segment of fixed length.
Review the concept of an angle and how to name it by following the progression on the worksheet.

An angle is formed by two rays with the same endpoint.

The endpoint is the vertex of the angle. The two rays are the arms of the angle.

Point out that the letter for the point at the vertex has to be in the middle of the name for an angle.

**EXTRA PRACTICE:**

1. Name the angle in all possible ways.

2. a) Which of the following are possible names for this angle?

   \[ \angle CAT, \angle CTP, \angle PTA, \angle UTA, \angle UTC, \angle TCP \]

   b) Write three more different names for the angle.

   Point out that sometimes, when there is no chance of confusion, only
   the vertex letter is used to name an angle. For example, in the rectangle
   at left there is only one possible \( \angle D \), but there are three possibilities for
   \( \angle A: \angle BAD \) or \( \angle BAC \) or \( \angle DAC \).
Review the concept of a polygon. A polygon:

- is a closed 2-D (flat) shape
- has sides that are straight line segments
- has each side touching exactly two other line segments, one at each of its endpoints

The point where two sides of a polygon meet is called a vertex. (The plural of vertex is vertices.)

Have students explain (using the definition above) why each of the shapes below is not a polygon.

Review naming polygons.

1. Start at any vertex. Choose a letter to name the vertex.
2. Go around the polygon clockwise or counter-clockwise, labelling the other vertices. To name the polygon, write the letters at the vertices in order. For example, you could name this rhombus $ABCD$ or $BADC$ or $DCBA$, but not $DBAC$.
3. You can choose any sequence of letters.

EXTRA PRACTICE:

Which triangles are shown in this picture?

$ABC \ AB\ D \ ACE \ BDE \ ECD \ EBA$

Name another triangle shown in this picture. ($ACD$)

ACTIVITY

Teach students to construct polygons using Geometer’s Sketchpad. Have them measure the lengths of the sides of the polygons.
**Goals**

Students will measure and draw angles.

**PRIOR KNOWLEDGE REQUIRED**

Knows what an angle is
Can name an angle and identify a named angle

**MATERIALS**

- dynamic geometry software (optional)
- dice
- geoboards and elastics
- grid paper

**CURRICULUM EXPECTATIONS**

Ontario: 6m48, 6m49; 7m46; review, 8m4
WNCP: 6SS1; optional, [T]

**VOCABULARY**

angle
vertex
arms
acute
obtuse

---

**Review the concept of an angle’s size.** Draw two angles:

_ask:_ Which angle is smaller? Which corner is sharper? The diagram on the left is larger, but its corner is sharper, and mathematicians say that this angle is smaller. The distance between the ends of the arms in both diagrams is the same, but this does not matter; angles are made of rays and these can be extended. What matters is “sharpness”: The sharper the corner on the “outside” of the angle, the narrower the space between the angle’s arms. Explain that the size of an angle is the amount of rotation between the angle’s arms. The smallest angle is closed; both arms are together. Draw the following picture to illustrate what you mean by smaller and larger angles.

![Smaller and Larger Angles](image)

You can show how much an angle’s arm rotates with a piece of chalk. Draw a line on the board then rest the chalk along the line’s length. Fix the chalk to one of the line’s endpoints and rotate the free end around the endpoint to any desired position.

![Rotating Chalk](image)

You might also illustrate what the size of an angle means by opening a book to different angles.

Draw some angles and ask your students to order them from smallest to largest. **EXAMPLE:**
Define acute and obtuse angles in relation to right angles. Obtuse angles are larger than a right angle but smaller than the angle that makes a straight line; acute angles are smaller than a right angle.

EXTRA PRACTICE:

1. Copy the shapes onto grid paper and mark any right, acute, and obtuse angles. Which shape has one internal right angle? What did you use to check? (e.g., a corner of a sheet of paper)

2. Which figures at left have
   a) all acute angles?
   b) all obtuse angles?
   c) some acute and some obtuse angles?

Introduce protractors. On the board, draw two angles that are close to each other—say, 50° and 55°—without writing the measurements and in a way that makes it impossible to compare the angles visually. ASK: How can you tell which angle is larger? Invite volunteers to try different strategies they suggest (such as copying one of the angles onto tracing paper and comparing the tracing to the other angle, or creating a copy of the angle by folding paper). Lead students to the idea of using a measurement tool.

Explain that to measure an angle, we use a protractor. A protractor has 180 subdivisions around its curved edge. These subdivisions are called degrees (°). Degrees are a unit of measurement, so just as we write cm or m when writing a measurement for length, it is important to write the degree symbol (°) for angles.

Using protractors. Show your students how to use a protractor, on the board or on the overhead projector. Identify the origin (the point at which all the degree lines meet) and the base line (the line that goes through the origin and is parallel to the straight edge). When using a protractor, students must
   • place the vertex of the angle at the origin;
   • position the base line along one of the arms of the angle.

You could draw pictures (see below) to illustrate incorrect protractor use, or demonstrate it using an overhead projector and a transparent protractor.
Introduce the degree measures for right angles (90°), acute angles (less than 90°), and obtuse angles (between 90° and 180°). Point out that there are two scales on a protractor because the amount of rotation can be measured clockwise or counter-clockwise. Students should practice choosing the correct scale by deciding whether the angle is acute or obtuse, then saying whether the measurement should be more or less than 90°. (You may want to do some examples as a class first.)

Then have students practice measuring angles with protractors, using Questions 2 and 3 on Workbook page 61. For extra practice, ask students to measure the angles in Question 1, where the arms of the angles have to be extended first.

Introduce angles in polygons, then have students measure the angles in several polygons. Students can draw polygons (with both obtuse and acute angles) and have partners measure the angles in the polygons.

**Drawing angles.** Model drawing angles step by step (see page 62 in the Workbook or BLM Measuring and Drawing Angles and Triangles, p O-90). Emphasize the correct position of the protractor. Have students practice drawing angles. You could also use Activity 2 for that purpose.

Have students practice drawing lines that intersect at a given angle. Another way to practice drawing angles is to construct triangles with given angle measures.

### ACTIVITIES 1–4

1. Students can use geoboards and elastics to make right, acute, and obtuse angles. When students are comfortable doing that, they can create polygons satisfying different criteria. **EXAMPLES:**
   - a triangle with 3 acute angles
   - a quadrilateral with 0, 2, or 4 right angles
   - a quadrilateral with 1 right angle
   - a polygon with 3 right angles
   - a quadrilateral with 3 acute angles

2. Students will need a die, a protractor, and a sheet of paper. Draw a starting line on the paper. Roll the die and draw an angle of the measure given by the die; use the starting line as your base line and draw the angle counter-clockwise. Label your angle with its degree measure. For each next roll, draw an angle in the counter-clockwise direction so that the base line of your angle is the arm drawn at the previous roll. The measure of the new angle is the sum of the result of the die and the measure of the angle in the previous roll. Stop when there is no room to draw an angle of the size given by the roll.
1. **Angles on an analogue clock.** Draw an analogue clock that shows 3:00 on the board. Ask your students what angle the hands create. What is the measure of that angle? If the time is 1:00, what is the measure of the angle between the hands? Do you need a protractor to tell? Ask volunteers to write the angle measures for each hour from 1:00 to 6:00. Which number do they skip count by? (30)

   An hour is 60 minutes and a whole circle is 360°. What angle does the minute hand cover every minute? (6°) How long does it take the hour hand to cover that many degrees? How do you know? (12 minutes, because the hour hand covers only one twelfth of the full circle in an hour, moving 12 times slower than the minute hand)

   If the time is 12:12, where do the hour hand and the minute hand point? What angle does each hand make with a vertical line? What is the angle between the hands? *(Answer: The hour hand points at one minute and the angle that it makes with the vertical line is 6°. The minute hand points at 12 minutes and the angle that it makes with the vertical line is 12 \times 6 = 72°. The angle between the hands is 72° − 6° = 66°.)*

   What is the angle between the hands at 12:24? 1:36? 3:48? **Answers:** 132°, 168°, 176°. Students can check their answers by precisely drawing the hands on a large piece of a clock and measuring the angle with a protractor.

2. Some scientists think that moths travel at a 30° angle to the sun when they leave home at sunrise. Note that the sun is far away, so all the rays it sends to us seem parallel.

3. Teach students to draw and measure angles using Geometer’s Sketchpad. Then ask them to try moving different points (on the arms of the angle, or its vertex) so that the size of the angle becomes, say, 50°. Is it easy or hard to do? When you move the line segments, does the angle change? (yes) When you move the vertex or other point on the arms, does the angle change? (yes) Show students how to draw an angle of fixed measure (using menu options). Will moving the endpoints change the size of the angle now? (no) Show students how to draw angles equal to a given angle.

4. Have students draw polygons in Geometer’s Sketchpad and measure the size of the angles and the length of the sides of these polygons. Have students check that the angle measures they obtain make sense. For example, clicking on three vertices of a quadrilateral and then using menu options to measure the angle might produce different angles, depending on the order in which the vertices were selected. Also, the software sometimes measures angles in the wrong direction, producing an answer more than 180°.
a) The sun rises in the east and sets in the west. What angle do the moths need to travel at to find their way back at sunset? (30°)

b) A moth sees the light from the candle flame and thinks it's the sun. The candle is very near to us, and the rays it sends to us go out in all directions. Where does the moth end up? Draw the moth's path.
Goals
Students will use logic, symbols, and all of the geometric properties and relationships they have learned to date to make quick and accurate sketches.

PRIOR KNOWLEDGE REQUIRED
Can identify equal sides, angles, and right angles
Familiar with standard markings for equal sides, equal angles, and right angles
Understands properties of triangles and quadrilaterals based on number of sides, angles, and parallel sides
Can name angles and polygons; can identify a named angle or polygon
Can find the perimeter of a polygon
Can solve problems using the Pythagorean Theorem
Can solve equations of the form \(x^2 + b = a\)

MATERIALS
BLM Sum of the Angles in a Triangle (pp O-94–O-95)
Dynamic geometry software (optional)

Review. Remind students how to mark equal sides, equal angles, and right angles. Review with students the properties of special quadrilaterals related to sides and angles (see table) and the classification of triangles according to the number of equal sides and the size of the angles. Remind students that the angles of an equilateral triangle are equal, and the base angles of an isosceles triangle are equal.

<table>
<thead>
<tr>
<th>Quadrilateral</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>parallelogram</td>
<td>2 pairs of equal parallel sides</td>
</tr>
<tr>
<td>rectangle</td>
<td>2 pairs of equal sides, 4 right angles</td>
</tr>
<tr>
<td>square</td>
<td>4 equal sides, 4 right angles</td>
</tr>
<tr>
<td>rhombus</td>
<td>4 equal sides</td>
</tr>
<tr>
<td>trapezoid</td>
<td>exactly 1 pair of parallel sides</td>
</tr>
</tbody>
</table>

Sum of the angles in a triangle. Have students complete the Investigation on BLM Sum of the Angles in a Triangle. What is the sum of the angles in a triangle? Then have students draw several triangles, measure their angles, and add the measures. Did they get the same sum every time? Explain that the sums might not add to 180° due to mistakes in measurement. The protractor, though a convenient tool, is imprecise. If Geometer’s Sketchpad is available, students can use it to measure the angles of a triangle with great precision and modify the triangle to see that the angles always add to 180°.
You will need to review the Pythagorean Theorem before students start working on problems on Workbook page 65. Students need to remember to convert the equation of the form \( x^2 + b = a \) to \( x^2 = c \) and to find the square root, so you need a table of squares of whole numbers to 20 visible somewhere in the classroom.

**EXTRA PRACTICE:**

1. Find the missing side of a right triangle:
   - a)  
   - b)  
   - c)  

   **ANSWERS:** a) 15  b) 24  c) 6

2. Use the Pythagorean Theorem to check whether these triangles are right triangles.
   - a)  
   - b)  
   - c)  

   **ANSWERS:** a) no  b) yes  c) no

Making a sketch. Explain to your students that a sketch is a quick drawing made without using tools such as a ruler or protractor. Knowing how to make a sketch is an important math skill. Sketches can help us organize information, see relationships, and solve problems.

Have students complete the worksheets. You can use the questions below as extra practice for students who work more quickly than others (to keep the class working through the questions at approximately the same pace) or for students who are struggling with a particular concept or step.

**EXTRA PRACTICE for page 63:**

1. Sketch an obtuse isosceles triangle and an acute isosceles triangle.

2. Circle the better sketch.
   - a)  
   - b)  
   - c)  
   - d)  

3. Sketch the figures.
   - a) a line segment \( AC \) with midpoint \( B \)
   - b) a rectangle 2 cm by 6 cm
   - c) a rhombus with angles 20° and 160°
   - d) \( \triangle KLM \) with sides 5 cm, 3 cm, and 3 cm

**PROCESS ASSESSMENT**

Worksheets:
- [R, V], 8m3, 8m6
  - Questions 9, 10
- [CN], 8m5
  - Question 13
- [R, C], 8m2, 8m7
  - Question 14
EXTRA PRACTICE for page 64:

1. A spinner is made of six triangles, three red and three blue, sharing a common vertex. The red and blue triangles alternate. All triangles are equilateral with sides of 3 cm each. What is the shape of the spinner? Solve by making a sketch. (The spinner is a hexagon.)

2. Make a sketch for the problem (without solving it). Ignore the unnecessary information.

A traffic island has the shape of a right trapezoid with one of the angles 65°. The island contains three shrubs and a circular flowerbed 1 m wide. What are the sizes of the angles of the traffic island?

**Bonus** The sum of the angles in the trapezoid in 2 is 360°. Solve the problem. (90°, 90°, 65°, 115°)

**Bonus** Solve the problem below by making a sketch.

John climbs a ladder to the attic window. The ladder is propped against the wall, and the foot of the ladder is 1 m from the wall. The ladder is made of two pieces and is 4.5 m long in total. The window is 7 m from the ground. Can John reach the window from the ladder? (Note that this problem does not require the use of the Pythagorean Theorem. Students only need to figure out that the height the ladder reaches is less than the length of the ladder, 4.5 m. The sketch will clearly show that John has to be over 2.5 metres tall to reach the window.)

EXTRA PRACTICE for page 65:

1. In a quadrilateral $ABCD$, $AC = 5$ cm. Two sides of $ABCD$ are 3 cm, and two other sides are 4 cm. What special quadrilateral could it be? Make two sketches to show that order of sides matters to the answer.

   **(ANSWER:** If the equal sides are adjacent, the shape is a kite. If they are not, sides 3 and 4 and the diagonal form a triangle, and $3^2 + 4^2 = 5^2$, so the triangle is a right triangle and the shape is a rectangle.)

2. Add the information that is not on the sketch.

   a) $BD$ is perpendicular to $AC$ in triangle $\triangle ABC$ with $\angle A = 50^\circ$.

   b) $E$ is the midpoint of $AD$. $BE$ is perpendicular to the side $AD$ of parallelogram $ABCD$. $BE = 3$ cm.

3. Add other information you can deduce to the sketch. Then solve the problem.

   $BD$ is perpendicular to $AC$ in isosceles triangle $\triangle ABC$, so that $AD = CD = 8$ cm. $BD = 15$ cm. What is the perimeter of $\triangle ABC$?
A city tower is a rectangular prism completely covered with glass panels. The base of the building is a rectangle 35 m by 24 m. To clean the glass panels, workers use a platform that is 6 m long. How many times will the workers have to move the scaffold up and down to clean the entire building?

**NOTE:** Explain to students that the platform can be shifted sideways at any height without moving the platform up or down, but not around the corner.

**More practice.** Solve each problem by making a sketch.

1. The shorter side of a parallelogram is 5 cm. The longer side is 2 cm longer than the shorter side. What is the perimeter of the parallelogram? (24 cm)

2. ABC is an isosceles triangle with one of the angles 100°. What are the sizes of the other angles? (40°, 40°)

3. A square is cut into two identical parts and rearranged to make a rectangle. The short side of the rectangle is 6 cm. How long is the long side of the rectangle? (24 cm)

4. A square is cut into two identical parts and rearranged to make a triangle. What are the angles of the triangle? (45°, 45°, 90°)

5. A rectangle is cut along a diagonal into two identical parts and rearranged to make a parallelogram. Sarah thinks the parallelogram is a rhombus. Steven thinks it cannot be a rhombus. Who is correct—Sarah or Steven? Explain. (Steven, because two sides of the parallelogram are also sides of the rectangle, and the other two sides are the diagonals of the rectangle. Diagonals are always longer than the sides of a rectangle, so the four sides cannot be equal, which they would be in a rhombus.)

**Extension**

A rectangle has sides 4 cm and 5 cm. It is cut (not along a diagonal) into two identical parts and rearranged to make a rhombus. Use grid paper to show how it could be done.

**Answer:**
**Goals**

Students will identify and draw perpendicular lines, identify complementary angles, and solve problems involving measures of complementary angles.

**PRIOR KNOWLEDGE REQUIRED**

Knows what a right angle is
Can name an angle and identify a named angle
Is familiar with variables
Can solve equations of the form $ax = b$, $ax + bx = c$

**Review perpendicular lines** and how to mark them with a square corner. Draw several pairs of intersecting lines on the board and have students identify the perpendicular lines. Include pairs of lines that are not horizontal and line segments that intersect in different places and at different angles (see examples below). Invite volunteers to check whether the lines are perpendicular using a corner of a page, a protractor, and/or a set square.

Ask students where they see perpendicular lines or line segments—also called **perpendiculars**—in the environment (sides of windows and desks, intersections of streets, etc.).

**EXTRA PRACTICE:**

Which lines look like they are perpendicular?

a) $\text{A}$ $\text{B}$ $\text{C}$ $\text{D}$

b) $\text{F}$ $\text{G}$ $\text{H}$ $\text{E}$

c) $\text{J}$ $\text{K}$ $\text{L}$ $\text{M}$

d) $\text{O}$ $\text{P}$ $\text{Q}$ $\text{R}$

**A perpendicular through a point.** Explain that sometimes we are interested in a line that is perpendicular to a given line, but with an additional condition: the perpendicular should pass through a given point. In each diagram below, have students identify:

a) the lines that are perpendicular to the line segment $AB$,
b) the lines that pass through point $P$,
c) the one line that satisfies both conditions.
Constructing a perpendicular through a point. Using a set square, model how to construct a perpendicular through a point that is not on the line. Emphasize the correct position of the set square (one side coinciding with the given line, the other touching the given point). Repeat with a protractor (the given line should pass through the origin and through the 90° mark). Have students practise the constructions both ways. Circulate among the students to ensure that they are using the tools correctly. Then invite volunteers to model the construction of a perpendicular through a point that is on the line. (Emphasize the difference in the position of the set square: the square corner is now at the point, though one arm still coincides with the given line.) Then have students practise this construction as well.

EXTRA PRACTICE:

Draw a pair of perpendicular slant lines (i.e., lines that are neither vertical nor horizontal) and a point not on the lines. Draw perpendiculars to the slant lines through the point. What quadrilateral have you constructed? (rectangle)

**Bonus**

Draw a slant line and a point not on the line. Using a protractor and a ruler or using a set square, draw a square that has one side on the slant line and one of the vertices at the point you drew. (ANSWER: Draw a perpendicular through the given point to the given line. Measure the distance from the point to the line along the perpendicular. Then mark a point on the given line that is the same distance from the intersection as the given point. Draw a perpendicular to the given line through this point as well. Finally, draw a perpendicular to the last line through the given point. Alternatively, when you draw the second last perpendicular, mark a point on that perpendicular that is the same distance from the given line as the given point is, and on the same side of the given line as the given point. Join this point to the given point.)

**Why perpendiculars are important.** Discuss with students why perpendiculars are important and where are they used in real life. For example, if construction workers need to cut wooden floorboards into perfect rectangles, they can measure the necessary length on one side of the board and make the cut at a right angle to the side of the board.

**Angle measures in adjacent angles add to the measure of the large angle.** Draw a line segment divided into two smaller segments. **ASK:** What is the length of the whole line segment? How do you know? What do you do with the lengths of the smaller line segments to obtain the length of the whole line segment?
Point out that the same procedure applies to capacities, volumes, and areas.

\[
\frac{1}{3} \text{ cup} + \frac{1}{2} \text{ cup} = \frac{5}{6} \text{ cup} \quad 4 \text{ cm}^3 + 2 \text{ cm}^3 = 6 \text{ cm}^3
\]

What happens with angles? Their measures are also added: the measure of the large angle at left is \(30^\circ + 45^\circ = 75^\circ\).

**Complementary angles.** Introduce *complementary angles*: two angles that add to \(90^\circ\). If \(b\) and \(c\) add to \(90^\circ\), we say that \(b\) complements \(c\). Have students identify complementary angles in pictures like the one in the margin. **ANSWERS:** \(b\) and \(c\), \(d\) and \(f\), \(e\) and \(g\). Point out that complementary angles do not have to be adjacent. Have students identify all pairs of complementary angles in the diagram of a quadrilateral at left. **ANSWERS:** \(\angle AED\) and \(\angle BAD\); \(\angle ADE\) and \(\angle ABD\).

**Acute angles in a right triangle are complementary angles.** Students can discover this fact using Activity 2 below. After the activity, students can identify complementary angles in pictures, as in Question 8 on Workbook page 68.

**Use complementary angles to find angle measures.** Have students identify complementary angles in the pictures below, then have them write an equation for each picture. Have students solve the equations to identify the angle measures represented by variables.

\[
\begin{align*}
\angle a &= 35^\circ \\
\angle y &= 23^\circ \\
\angle x &= 2x
\end{align*}
\]

Proceed to word problems, such as:

a) In a right triangle, one acute angle is 8 times larger than the other acute angle. Sketch the triangle and use equations to find both angles. (Equation: \(x + 8x = 90^\circ\), \(x = 10^\circ\), the other angle is \(80^\circ\))

b) In a right triangle one acute angle is \(40^\circ\) smaller than the other acute angle. Sketch the triangle and use equations to find both angles. (Equation: \(x + (x - 40^\circ) = 90^\circ\), \(x = 65^\circ\), the other angle is \(25^\circ\))

**ACTIVITIES 1–3**

1. Have students use Geometer’s Sketchpad to:
   a) Draw a line. Label it \(m\).
   b) Mark a point \(A\) on the line \(m\).
   c) Draw another line through point \(A\).
   d) Measure the angle between the two lines.
e) Try to make the angle a right angle by moving the points around. Is it easy or hard to do?

f) Check whether the lines stay perpendicular when you move any of the points in the picture.

g) Repeat parts a) through f) with a point not on the line. Note that when point A is not on the line, the second line might be modified so that it does not intersect the line \( m \), and the angle you measure disappears.

Explain that you need a method to draw perpendicular lines that will keep them perpendicular even if the points are moved around. Teach students to draw perpendicular lines using the perpendicular line command from the menu. Do these lines stay perpendicular to the given line even if points are moved around? (yes)

2. Have students draw a triangle using the polygon tool in Geometer’s Sketchpad. Ask them to move the points around to make it look like a right triangle. Then ask them to measure the angles of the triangle and to check whether it is indeed a right triangle. Is it easy to draw a perfect right triangle this way? (no) If you move the points around, does the triangle remain a right triangle? (no) Ask students to think about how they draw a right triangle on paper. What tools do they use and why? (a protractor or a set square) What tools could we use instead of protractors and set square in Geometer’s Sketchpad? (perpendicular lines) Then have students draw a right triangle in Geometer’s Sketchpad. Ask them to check that the triangle remains a right triangle even if points are moved around.

Students can also add the measures of the acute angles in the right triangle they created, and check that the sum remains 90° even when the triangle is modified.

3. Ask students to draw three different right triangles and measure the acute angles in each one. Ask them to add the measures of the acute angles in each triangle. What do they notice? Have students cut out the triangles carefully and fold them across the short sides so that the acute angles meet at the vertex of the right angle (see picture).

What should the acute angles in a right triangle add to? Compare the results in pairs, then in groups of four. Did everybody have the same triangles? (no) Did everybody get the same sum of the angles? (yes) What do we call angles that add to 90°? (complementary angles)
Extension

Discuss with students whether there can be more than one line perpendicular to a given line through a given point, and whether such a perpendicular always exists. You can use the diagrams below to help students visualize the answers.

PROCESS EXPECTATION

Visualizing
**Goals**

Students will identify supplementary and opposite angles and will solve problems that involve finding the measures of supplementary and opposite angles.

**PRIOR KNOWLEDGE REQUIRED**

- Knows what a right angle is
- Knows what a straight angle is
- Can name an angle and identify a named angle
- Is familiar with variables
- Can solve equations of the form \( ax = b, \ ax + bx = c \)
- Knows that \( a - (b - c) = a - b + c \)

**MATERIALS**

- transparency
- protractors

**Introduce supplementary angles.** Review: **straight angles** have arms that make a straight line, and the measure of a straight angle is \( 180^\circ \). Then define **supplementary angles** as a pair of angles that add to \( 180^\circ \). Point out that if \( \angle A \) and \( \angle B \) are supplementary angles, we can say that \( \angle A \) supplements or is a supplementary angle to \( \angle B \). Have students identify pairs of supplementary angles in pictures, as in Question 1 on Workbook page 69. Repeat with the picture at left.

**Finding the measure of a supplementary angle.** Ask students to identify the angles that supplement \( \angle a \). (\( \angle b, \angle c \)) Then ask students to write an equation that shows what it means that \( \angle a \) and \( \angle b \) are supplementary. (\( \angle a + \angle b = 180^\circ \)) Mark \( \angle a \) in the picture as \( 115^\circ \) and have students rewrite the equation using the measure of \( \angle a \). (\( 115^\circ + \angle b = 180^\circ \)) Then ask them to solve the equation. What is the measure of \( \angle b \)? (\( 180^\circ - 115^\circ = 65^\circ \)) Mark the measure in the picture, and have students find the measure of \( \angle c \) the same way. Then challenge them to find the measure of \( \angle d \) and to explain the solution.

Have students find the measures of all four angles in more such pictures, but provide different measures for \( \angle a \), including both acute and obtuse angles. Then ask students to look at the pictures and to tell which angles are always the same. (the opposite angles) Then say that the measure of \( \angle a \) is \( x^\circ \), and have students write expressions for the measures of the other four angles. (in picture above: 

\[
\angle b = \angle c = 180^\circ - x^\circ; \quad \angle d = 180^\circ - (180^\circ - x^\circ) = 180^\circ - 180^\circ + x^\circ = x^\circ
\]

**Introduce adjacent and opposite angles** as in the box at the top of Workbook page 70. Then have students practise identifying opposite angles.
in the pictures you used for the previous exercise. **ASK:** What do you notice about the measures of opposite angles? (they are always equal) Work through Question 3 on page 70 together as a class.

**NOTE:** There are no opposite angles in the figures below because there are no intersecting lines (only intersecting rays).

There are no intersecting lines in this figure, but ∠1 and ∠2 are opposite angles.

Use rotations to show that opposite angles are equal. Project the picture in the margin using an overhead projector, and trace the lines and mark the angles on the board. Press a pencil tip to the intersection point on the transparency to create a pivot for turning the transparency. Then rotate the overhead 180° to show how the rotated image coincides exactly with the original image. (Or use two identical transparencies: keep one fixed and rotate the other.) Have students identify the amount of rotation. Show the rotation several times, if necessary. Which angles are rotated onto which angles? (∠1 becomes ∠2, and ∠3 becomes ∠4) Do angles change their measure in rotation? (no) What does this tell us about the sizes of ∠1 and ∠2? What about ∠3 and ∠4? (they are equal)

Another way to look at that is to think of an angle as the amount of rotation you need to get one arm from the other. The amount of rotation you need to get the upper arm of ∠1 from its lower arm is the same as the amount of rotation you need to get the lower arm of ∠2 from its upper arm, because the arms of both angles are parts of the same lines.

**Finding angle measures using both supplementary and opposite angles.**
Work through the following problems as a class, and then have students practise with problems like those in Workbook page 70 Question 5.

Find the measures of all angles in the picture using the given angle measures.

**Extension**
Find x. (**ANSWER:** x = 45°)
**G8-21 Parallel Lines**

**Goals**
Students will identify parallel lines.

**PRIOR KNOWLEDGE REQUIRED**
- Can identify and construct a perpendicular using a set square or a protractor
- Can identify and mark right angles
- Can name a line segment and identify a named line segment
- Can draw and measure with a ruler

**CURRICULUM EXPECTATIONS**
- Ontario: 7m46; 8m1, 8m2, 8m7, review
- WNCP: 7SS3; optional, [C, R, PS]

**VOCABULARY**
- line segment
- parallel
- right angle
- perpendicular

**Introduce parallel lines** — straight lines that never intersect, no matter how much they are extended. Show how to mark parallel lines with the same number of arrows.

Have students identify parallel lines in several diagrams showing intersecting and parallel lines and rays. Then have students identify and mark parallel sides of polygons. Include polygons that have pairs of parallel sides that are neither horizontal nor vertical.

Introduce the symbol (||) for parallel lines, and have students write which sides are parallel using the new notation in the polygons used above.

Ask students to think about where they see parallel lines. Some examples of parallel lines in the real world are a double center line on a highway and the edges of construction beams.

**Distance between parallel lines.** Students can investigate distances between parallel lines with BLM Distance Between Parallel Lines. (They will use the fact that parallel lines are the same distance apart everywhere in the next lesson, to check that lines they have drawn are parallel.) Explain to students that they can use the distance between lines to check whether two lines are parallel. All they need to do is measure the distance between the two lines in question (along a common perpendicular) in several places, and compare the distances; if they are equal, the lines are parallel. Have students draw two pairs of lines, one parallel and the other not (without indicating which is which), and swap their drawings with a partner. Then ask students to predict which pair of lines is parallel and use this method to verify the prediction.

**ACTIVITY**

Have students use Geometer's Sketchpad to:

a) Draw a line. Label it \(m\).

b) Mark a point \(A\) not on the line \(m\).
1. A plane is a flat surface. It has length and width, but no thickness. It extends forever along its length and width. Parallel lines in a plane will never meet, no matter how far they are extended in either direction. Can you find a pair of lines not in a plane that never meet, do not intersect, but are not parallel?

2. Draw:
   a) a hexagon with three parallel sides
   b) an octagon with four parallel sides
   c) a heptagon with three pairs of parallel sides
   d) a heptagon with two sets of three parallel sides
   e) a polygon with three sets of four parallel sides
   f) a polygon with four sets of three parallel sides

SAMPLE ANSWERS:

---

Extensions

1. A plane is a flat surface. It has length and width, but no thickness. It extends forever along its length and width. Parallel lines in a plane will never meet, no matter how far they are extended in either direction. Can you find a pair of lines not in a plane that never meet, do not intersect, but are not parallel?

2. Draw:
   a) a hexagon with three parallel sides
   b) an octagon with four parallel sides
   c) a heptagon with three pairs of parallel sides
   d) a heptagon with two sets of three parallel sides
   e) a polygon with three sets of four parallel sides
   f) a polygon with four sets of three parallel sides

SAMPLE ANSWERS:
Introduce corresponding angles. Draw a pair of lines that will intersect if extended and a third line intersecting them. Label the angles. Introduce corresponding angles as angles that create a pattern like in the letter F. Copy the picture three more times and have students find all four pairs of corresponding angles and trace the letter F in each picture, as in the box at the top of Workbook page 72. (If students do not see that the upside-down or reflected pattern resembles an F, copy the picture to a transparency, highlight the pattern in question and turn it over or rotate it so that students see that it resembles the letter F.) Then have students identify pairs of corresponding angles in other pictures like the one in the margin or those in Question 1 on Workbook page 72.

Corresponding angles for parallel lines are equal. Have students perform Investigation 1 on Workbook page 72 individually, then ask them to check their conjecture by drawing another pair of parallel lines with a line intersecting them and measuring the angles. Repeat with a line intersecting the parallel lines at a different angle.

**ASK:** Are corresponding angles always equal? Have students perform Investigation 2 on Workbook page 72 individually.

**Is the reverse true?** Write on the board:

> When lines are parallel, corresponding angles are equal.

Underline the parts of the statement as shown. Remind students that when we want to reverse a statement, we need to reverse the order of the parts that make it. Review **EXAMPLES:** “All boys are people” has the reverse “All people are boys”; “When it rains, the sky is cloudy” has the reverse “When the sky is cloudy, it rains.” **ASK:** What would be the reverse statement for
the statement on the board? (When corresponding angles are equal, the lines are parallel.) Do you think the reverse statement is true? How can we check it? (Draw a pair of lines with equal corresponding angles and check whether they are parallel.) How did you check that lines are parallel in the last lesson? (using distance between lines measured along common perpendiculars) You can use Activity 1 to check the reverse statement.

**When the reverse is true, statements are equivalent.** Explain to students that when a statement and its reverse are both true, you can say that the parts of the statement mean the same thing. For example, the statements “All mammals are animals that feed their young milk” and “All animals that feed their young milk are mammals” are both true, and they mean the same thing — mammals are exactly those animals that feed their young milk. “All squares are quadrilaterals with four equal sides and four right angles” is true, and “All quadrilaterals with four right angles and four equal sides are squares” is true as well. We use the equality of sides and angles to check whether the shape is a square: if sides are equal and angles are all right angles, the shape is a square. If even one angle is not a right angle, or even one pair of sides is not equal, we know right away the shape is not a square. Similarly, we can use the equality of corresponding angles to check whether lines are parallel.

**Practice identifying parallel lines using the equality between corresponding angles.** Use Activity 2.

**Find measures of angles between intersecting lines using corresponding, opposite, and supplementary angles.** Review what opposite angles are, and the fact that they are equal. Remind students what supplementary angles are. Then have students use these concepts to find measures of angles and to identify parallel lines in Questions 2 to 4 on Workbook page 73. Have students explain their solution to Question 4f).

**EXTRA PRACTICE:**

Find all the angles in the pictures:

![Angles](image)

Which lines are parallel in c)? (b \(\parallel\) c)

**ACTIVITIES 1–3**

1. a) Draw a pair of intersecting lines.
   b) Measure the angle between the lines.
PROCESS EXPECTATION

Technology

1. a) Draw a third line that meets one of the lines at the same angle. Try to make the third line look parallel to one of the lines you started with. (The equal angles should be corresponding angles, i.e., make a pattern like in the letter F.)

Like this: \[\text{Not like this:}\]

b) Draw a line perpendicular to one of the lines that look parallel. Extend it so that it intersects the second line. Check that it is perpendicular to the second line.

c) Measure the distance between the lines along the perpendicular you drew. Distance 1 = ______

d) Repeat steps d) and e) with a different perpendicular. Distance 2 = ______

e) Are the distances equal? _____ Are the lines parallel? _____

2. a) Draw two pairs of lines, one parallel and the other not but looking like it might be. (For example, draw a line, then place a ruler along the line as if to draw another line along the parallel side of the ruler. Then rotate the ruler very slightly.) Do not indicate which pair is which.

b) For each pair of lines draw a third line intersecting both.

c) Swap your paper with a partner. Use corresponding angles to identify the pair of parallel lines.

3. Students can use Geometer’s Sketchpad to investigate the properties of corresponding angles. Part e) encourages students to pay close attention to what they see, and to what changes, on the screen.

a) Draw a line. Mark a point not on the line, and construct a line parallel to the first one through the new point.

b) Draw a third line, intersecting both.

c) Mark a pair of corresponding angles and measure them. What do you notice? Make a conjecture.

d) Move the lines or the points on the lines around. Are the corresponding angles always equal?

e) Move a point on the third line so that the angles you measured are not equal (e.g., move the point that is used to name both angles to be between the parallel lines). Look at the pattern the angles create. Are they corresponding angles? Does this create a counter-example to your conjecture from c)? Explain.
Extension

Another way to look at the equality between corresponding angles is to think in terms of transformations. Translation doesn’t change angles, and corresponding angles are just translations of each other. If you slide the line $KQ$ along the line $KL$, to the position of $LR$, then $\angle MKQ$ slides to $\angle MLR$. This means $\angle MKQ = \angle MLR$.

70°

Complete the statements.

$KQ$ is parallel to ____ and $\angle RLK$ and $\angle QKM$ are ___________ angles, so $\angle QKM = \angle RLK = 70^\circ$.

$\angle QKM$ and $\angle TQS$ are corresponding angles, and $\angle QKM = 70^\circ = \angle TQS$, so ____ is parallel to ____.
Introduce alternate angles. Draw a pair of lines that will intersect if extended and a third line intersecting them. Number the angles in the picture. Introduce alternate angles as angles that create a pattern like in the letter Z. Copy the picture and have students find the other pair of alternate angles and trace the letter Z in the second picture, as in the box at the top of Workbook page 74. (If students do not see that the reflected pattern resembles a Z, copy the picture to a transparency, highlight the pattern in question, and turn it over or rotate it so that students see that it resembles the letter Z.) Then have students identify pairs of alternate angles in other pictures, like the one in the margin or those in Question 1 on Workbook page 74.

Alternate angles for parallel lines are equal. Have students perform steps A and B of the Investigation on Workbook page 74 individually, then ask them to check their conjecture by drawing another pair of parallel lines with a line intersecting them and measuring the angles. Repeat with a line intersecting the parallel lines at a different angle.

ASK: Are alternate angles always equal? Work through part C of the Investigation as a class, then have students do part D individually.

Reverse the statement. Remind students what reverse statements are. Ask them to give an example of a pair of reverse statements. Then write on the board: When lines are parallel, _______. Have students complete the statement according to the results of the Investigation. (alternate angles are equal) Point out that this statement is not a conjecture; we know it is true for all angles because we proved it using logic. (You might point out that statements that have been proven to be true using logic are called theorems.) Underline the parts of the statement to help students reverse it. Ask students to write a reverse statement to this statement. (When alternate
angles are equal, the lines are parallel.) Say that you want to investigate whether this statement is true. What do you need to do? (Draw a pair of lines with equal alternate angles and check whether they are parallel.)

Prove the reverse. Draw a picture on the board showing two equal alternate angles of 73° (see margin). Point out that you do not know whether the lines are parallel, so you do not know anything about the distance between the lines, and you do not know anything about the corresponding angles yet. Ask students to find the measure of \( \angle EGF \). How do they know what it is? (73°, because \( \angle EGF \) and \( \angle AGC \) are opposite angles; write that on the board and continue recording the steps of the argument as you continue below) Ask students to name all pairs of corresponding angles they see in this picture. ASK: What do you notice about the pair of corresponding angles \( \angle EGF \) and \( \angle DCF \)? (both are 73°) What do we know about equal corresponding angles and parallel lines? (Parallel lines have corresponding equal angles. Corresponding equal angles mean lines are parallel.) Which of these statements can help us to decide whether the lines are parallel? (the second one)

ASK: What did we prove now? Students are likely to think that this argument proves that when alternate angles are equal, lines are parallel. Explain that you have only checked one case—when the angles are 73°. Will the same argument work for angles of 80°? (yes) Why? (because we never used the fact that the angle is precisely 73°) Will the same argument work for any other angle measure? (yes) Have students rewrite the same argument using 80° instead of 73°, but writing 80° in pencil each time. Then have them erase the 80° and write \( x° \) instead. Does the proof still work? Point out that \( x \) is a variable, so it can represent any relevant number. This means the argument now works for any angle measure and students have proved the reverse statement: When alternate angles are equal, the lines are parallel.

Practise identifying parallel lines using the equality between alternate angles. You can do an activity similar to Activity 2 from Lesson G8-22, but using alternate angles instead of corresponding angles.

Find measures of angles between intersecting lines using alternate, corresponding, opposite, and supplementary angles. Review what opposite angles are, and the fact that they are equal. Remind the students what supplementary angles are. Then have students use these concepts to find measures of angles, using Questions 3 to 5 on Workbook page 75.

EXTRA PRACTICE:

Find all the angles in the pictures.

a) 

b) 

c)
Which lines are parallel in c)? \( b \parallel c \)

Note that students can answer Question 7c) on Workbook page 75 in different ways, using alternate, corresponding, or supplementary angles.

**Extension**

Another way to look at the equality between alternate angles is to think in terms of transformations. In this picture, \( \angle NLK \) and \( \angle MKQ \) are alternate angles, and translation doesn’t change them. If you slide the line \( KQ \) along the line \( KL \), to the position of \( LR \), then \( \angle MKQ \) slides to \( \angle MLR \). This means \( \angle MKQ = \angle MLR \). The angles \( \angle MLR \) and \( \angle NLK \) are opposite angles, so they are equal. But \( \angle MLR = \angle MKQ \), so \( \angle MKQ = \angle NLK \).

[Diagram of parallel lines and angles]

**PROCESS EXPECTATION**

- **Technology**
  
  Repeat Activity 3 from Lesson G8-22 using alternate angles instead of corresponding angles. In e), one way to make the angles unequal is to pull one of the points on the parallel lines (but not on the line intersecting them) to the other side of the line intersecting the parallel lines.

**ACTIVITY**

Repeat Activity 3 from Lesson G8-22 using alternate angles instead of corresponding angles. In e), one way to make the angles unequal is to pull one of the points on the parallel lines (but not on the line intersecting them) to the other side of the line intersecting the parallel lines.
Review the sum of the angles in a triangle.

**Using the sum of the angles in a triangle.** Draw a triangle on the board and write the measure of two of the angles in the triangle. **ASK:** How can I find the measure of the third angle? (180° minus the sum of the other two angles) Have students find the measures of the angles in several problems of this sort, then proceed to more complicated questions, such as:

- All the angles in a triangle are equal. What is the size of each angle?
- One angle of a triangle is 30°. The other two angles are equal. What is the size of these angles?
- A triangle has two equal angles. One of the angles in this triangle is 90°. What are the sizes of the other two angles?

With the last question, ask whether the equal angles can be 90° each. Have students explain why this is not possible, using a sketch. (The two angles would already add to 180°, leaving no room for the third angle.) Invite a volunteer to draw the correct triangle on the board and mark the measures of the angles. Then present a similar problem:

- A triangle has two equal angles. One of the angles in this triangle is 50°. What could the sizes of the other two angles be?

**ASK:** How is this problem different from the previous problem? (The given angle is an acute angle, not a right angle.) Can a triangle have two angles of 50°? What is the third angle then? (80°) Draw an acute isosceles triangle on the board and ask volunteers to mark the angles on the picture. Then draw another acute isosceles triangle, mark the base angles as equal, and
mark the unequal angle as 50°. Can this situation happen? What are the measures of the other two angles? (65°)

**EXTRA PRACTICE:**

<table>
<thead>
<tr>
<th>a) If half an angle is 20°, the whole angle is _____°.</th>
<th>b) If one-third of an angle is 30°, the whole angle is _____°.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="" alt="20° angle" /></td>
<td><img src="" alt="30° angle" /></td>
</tr>
<tr>
<td>c) What is half of 90°? _____°.</td>
<td>d) If one-quarter of an angle is 25°, the whole angle is _____°.</td>
</tr>
<tr>
<td><img src="" alt="90° angle" /></td>
<td><img src="" alt="25° angle" /></td>
</tr>
</tbody>
</table>

**Bonus** What is one-third of 120°? _____°

Review alternate, supplementary, and opposite angles. **SAMPLE PROBLEMS:**

Find the missing angles:

- ![48° angle]()
- ![38° angle]()

Find angle measures using supplementary, alternate, opposite angles and the sum of the angles in a triangle. Have students copy the following diagrams, then fill in all the missing angle measures as a class. Start by marking the equal angles. Use Workbook Questions 2 and 3 for more practice. Have students explain how they found the measures of the angles in the triangle in Question 3. Encourage students to use a sketch in their explanation.

**PROCESS ASSESSMENT**

8m7, [C]

**Proving the Sum of the Angles Theorem.** Draw the picture from Question 4 on Workbook page 76 on the board and have students use it to mark the pairs of alternate angles that they know are equal. Then work through Question 4 together as a class.
Exterior angles. Draw a triangle on the board, extend one of the arms beyond the vertex and mark the exterior angle as shown in the margin. Ask students if anyone knows what “exterior” means (outer, on the outside). Explain that this angle is called an exterior angle of the triangle because it is outside the triangle. Then mark the measures of $\angle a$ and $\angle b$ in this triangle (e.g., $\angle a = 50^\circ$, $\angle b = 57^\circ$) and ask students to find the measure of $\angle c$. $\angle c = 73^\circ$)

ASK: What do you know about $\angle c$ and $\angle x$? (They are supplementary angles; they add to $180^\circ$) Have students find the measure of $\angle x$. $\angle x = 107^\circ$

Start a table with headings $\angle a$, $\angle b$, $\angle x$ and fill in the first row. Repeat with several other triangles labelled the same way. Then ask students to look for a pattern and have them formulate a conjecture about the sizes of the angles.

Finally, add a row to the table with variables $a$ and $b$ for $\angle a$ and $\angle b$, and have students find the measure of angle $c$ ($180^\circ - a - b$). Then ask them to write the equation showing that $\angle c$ and $\angle x$ are supplementary angles, and to find the measure of the angle $c$ this way. $(c + x = 180^\circ$, so $x = 180^\circ - c = 180^\circ - (180^\circ - a - b) = 180^\circ - 180^\circ + a + b = a + b)$ Point out that students have now proved their conjecture using logic, so they can call this conjecture the Exterior Angle Theorem.

Use the Exterior Angle Theorem to find missing angles. Have students practise using the Exterior Angle Theorem with problems such as those in Question 5 on Workbook page 77.

EXTRA PRACTICE:

a) $x = 54.5^\circ$, $y = 109^\circ$

b) $x = 36^\circ$, $y = 144^\circ$

c) $x = 128^\circ$, $y = 93^\circ$
Classifying triangles by the number of equal sides and equal angles.

Do the Investigation on Workbook page 78 as a class. Then have students work in groups of three. Each student has to draw three triangles using a protractor:

<table>
<thead>
<tr>
<th>Triangle</th>
<th>Student 1</th>
<th>Student 2</th>
<th>Student 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle 1</td>
<td>two angles of 60°</td>
<td>two angles of 60°</td>
<td>two angles of 60°</td>
</tr>
<tr>
<td>Triangle 2</td>
<td>two angles of 30°</td>
<td>two angles of 40°</td>
<td>two angles of 25°</td>
</tr>
<tr>
<td>Triangle 3</td>
<td>two angles of 55°</td>
<td>two angles of 75°</td>
<td>two angles of 70°</td>
</tr>
</tbody>
</table>

Ask students to find the measure of the third angle in each triangle and to classify the triangles they created by their angle measures. Then ask students to cut the triangles out and fold them, so as to check whether they have any equal sides. Compare the findings in each group. Students will discover that when two angles are equal, there will be two equal sides, regardless of the measure of the third angle. If all three angles are equal, there will be three equal sides. **ASK:** What should the measure of the two equal angles be in order for the triangle to be an isosceles right triangle? (45°)

**Isosceles Triangle Theorem.** Explain to the students that in an isosceles triangle the unequal side is often called the base, and the angles adjacent to the base are called base angles. Point out that we use the word “base” here in a different way than we did in Measurement. In Measurement, any side to which we can draw a perpendicular could be called a base, but here we mean specifically the unequal side of an isosceles triangle. Ask students to say what they found out in the course of their investigations about the base angles of an isosceles triangle. Summarize by writing on the board: The base angles in an isosceles triangle are equal. Explain that
this conjecture can actually be proved using logic, and students will do so later, but for now they can use this conjecture when working with isosceles triangles. For example, have students find the angle measures in the triangles in the margin.

**Drawing a triangle with given side lengths using a compass.** Model the construction step by step:

<table>
<thead>
<tr>
<th><strong>Step 1:</strong> Construct a line segment $AC$.</th>
<th><strong>Step 2:</strong> Set your compass to the width of $AB$. Construct an arc centred at $A$.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Step 1 diagram" /></td>
<td><img src="image" alt="Step 2 diagram" /></td>
</tr>
<tr>
<td><strong>Step 3:</strong> Set your compass to the width of $BC$. Construct an arc centred at $C$ with radius $BC$, so that it intersects the first arc at one point.</td>
<td><strong>Step 4:</strong> Label the intersection point of the arcs $B$. Use a straightedge to join $B$ to $A$ and $C$.</td>
</tr>
<tr>
<td><img src="image" alt="Step 3 diagram" /></td>
<td><img src="image" alt="Step 4 diagram" /></td>
</tr>
</tbody>
</table>

Have students practise constructing triangles with sides of different lengths using a compass and a ruler. **EXAMPLES:**

a) 3 cm, 4 cm, 5 cm  
b) 8 cm, 8 cm, 12 cm  
c) 5 cm, 5 cm, 8 cm  
d) 6 cm, 6 cm, 6 cm

Students might have trouble creating the triangle on Workbook page 79, Question 7a) i). In this case have students try to make a model of the triangle with straws cut to the exact lengths first, and then copy it onto paper. Students will realize that the only way to create this triangle is to have two sides of 6 cm and one side of 3 cm; two sides of 3 cm and one side of 6 cm will not produce a triangle. Students can investigate this further in Activity 4.

**ACTIVITIES 1–4**

1. Each student will need a die, a ruler, a protractor, and a spinner divided into three parts and labelled with the types of triangles they have been studying (acute triangle, obtuse triangle, right triangle). The object of the game is to build a set of triangles around any vertex that fills all 360°.

Students start with a horizontal line of 5 cm. Students roll the die and spin the spinner each turn. The spinner gives the type of triangle to be constructed. The result of the die multiplied by 10 gives the
size of one angle of the triangle. (The other angles should also be multiples of 10°.) Students have to draw the first triangle using the 5 cm line as the base. Each new triangle should use one of the sides of an existing triangle as a side. The position of the angles is up to the student. The triangles cannot overlap. Students may find it convenient to write the sizes of the angles on the triangles they draw.

Let students know that there is one combination of the results of the spinner and the die that makes it impossible to draw a triangle with angles that are all multiples of 10°. Ask students to figure out what combination that is. To lead students to the answer, **ASK:** What could be the largest angle in an acute triangle, if all angles are multiples of 10°? (80°) Have students give an example of an acute triangle with at least one angle 80°. What could the smallest angle in an acute triangle be? If students answer 10°, have them try to find the possible measures of the other two angles. Students will see that an acute triangle with one angle 10° is impossible (the other two angles should add to 170°, but 80° + 80° = 160° only), so 1 and acute is the problematic combination. If this combination occurs, students have to roll again.

**SAMPLE GAME:** This game started with a roll of 5 and a spin of “acute triangle,” which gave Triangle 1. The next roll was 2 and the next spin was acute triangle, so Triangle 2 had to be isosceles with angles of 80° at the base. The student decided to put the 80° angle next to the 50° angle. The next two turns were 6, right triangle (Triangle 3) and 1, right triangle (Triangle 4), and the game ended with 6, acute triangle (Triangle 5). In the last turn, the 60° angle did not fit in the remaining 50° gap, but a triangle with 50° and 60° angles is still acute, so the student used the 50° angle to fill the gap and end the game. If the final spin had given obtuse triangle instead of acute triangle, the student would have had a choice of drawing the next triangle either around a different vertex or placing the smallest angle of the triangle (which could be only 10° or 20°) in the remaining gap. In any case the game would have continued.

**VARIATION:** Students can play this game in pairs. Player 1 rolls the die, Player 2 spins the spinner. Player 1 tells Player 2 which choices of triangles are available, and Player 2 decides which triangle to use and how and where to place it. When a game is finished, players switch roles.

2. **Paper folding**

First, create a square from a rectangular sheet of paper:

a) Fold the short side of the paper down onto the long side to create a right isosceles triangle. The extra part will be a rectangle.

b) Fold the extra part of the page—a rectangle—over the triangle.

c) Unfold the paper and cut off the rectangle.
Now, create a special triangle as follows:

a) Fold the square in half (vertically, not diagonally) so that a crease divides the square into two rectangles.

b) Fold the top right corner of the square down so that the top right vertex meets the crease. The vertex should be slightly above the bottom edge.

c) Mark the point where the corner meets the crease and trace a line along the folded side of the square with a pencil.

d) Unfold.

e) Repeat steps b) to d) with the top left corner of the square. The corner will meet the crease at the same point, which is the vertex of your triangle.

f) Cut the triangle out along the traced lines. Which triangle have you created? Explain.

ANSWER: You started with a square and ensured in b) that the sides of the triangle are equal to the side of the square. It is an equilateral triangle.

3. Students can draw triangles with sides of given measures using Geometer’s Sketchpad. Discuss with students how the method of drawing a triangle with three given side lengths in Geometer’s Sketchpad is similar to the process of using a compass and a ruler. (To draw a line segment of given length in Geometer’s Sketchpad, you use a circle tool, similar to using a compass in the pencil-and-paper construction.)

4. The triangle inequality. Give students a ruler, scissors, and some straws. Have students measure and cut a set of 10 straws:

- one straw of each of the following lengths: 10 cm, 9 cm, 8 cm, 6 cm, 3 cm
- two straws of 4 cm
- three straws of 5 cm
Questions:

a) How many (distinct) right triangles can you make using the straws? **Answer:** 2 distinct triangles with sides of length 3, 4, 5 and 6, 8, 10.

b) How many isosceles triangles can you make? **Answer:** 9 triangles with the following side lengths (in cm):

3, 4, 4  3, 5, 5  4, 5, 5  5, 5, 5  5, 4, 4  
6, 4, 4  6, 5, 5  8, 5, 5  9, 5, 5

Note that equilateral triangles are also isosceles – this is a special case.

c) Why can you not make triangles with side lengths 8, 4, 4; 9, 4, 4; 10, 4, 4; and 10, 5, 5? If you had two straws of length 3 cm, could you make a triangle with sides 3 cm, 3 cm, and 6 cm? If you had two straws of length 6 cm, could you make a triangle with sides 3 cm, 6 cm, and 6 cm?

Now discuss this question: What could be a rule for determining the sets of straws that will make a triangle and those that won’t?

The rule is known as the triangle inequality. It says that for three lengths to make a triangle, the sum of the lengths of the two smallest sides must be greater than the length of the largest side.

\[ a + b > c \]
Review angle properties learned to date. Have students make a list of the properties of special angles that they investigated in the last lessons (e.g., corresponding angles at parallel lines are equal). Then have them work in pairs to check and add to their lists. Use the Scribe, Stand, Share strategy to list all the properties that students should have learned. Explain to the students that the conjectures about properties of special angles that they made in previous lessons can be proved using logic. They only used logic to prove some of these conjectures, and checked several cases to persuade themselves that the conjectures are true. Still, these properties are actually theorems (statements that are true and can be proved using logic) and some of these theorems have special names. Display the theorems for future reference, and draw a picture for each as on Workbook pages 80 and 81.

- **Supplementary Angle Theorem (SAT):** \( \angle 1 + \angle 2 = 180^\circ \)
- **Corresponding Angle Theorem (CAT):** Corresponding angles at parallel lines are equal.
- **Alternate Angle Theorem (AAT):** Alternate angles at parallel lines are equal.
- **Opposite Angle Theorem (OAT):** Opposite angles are equal.
- **Sum of Angles in a Triangle Theorem (SATT):** \( \angle a + \angle b + \angle c = 180^\circ \)
- **Exterior Angle Theorem (EAT):** \( \angle x = \angle a + \angle b \)
- **Isosceles Triangle Theorem (ITT):** The base angles in an isosceles triangle are equal.

Find missing angle measures using the angle properties. Go through the first two problems below as a class, and have students do the rest of the problems individually. Ask them to explain what theorem they used to find each angle, and to write the abbreviation for the theorem used at each step. Start with problems where the missing angles are numbered in the order they should be found, and finish with problems where all missing angles
should be found and students themselves decide the order in which they will find them.

a)  b)  c)

\[
\begin{align*}
45° & \quad 1 \quad 2 \\
& \quad 53° \quad 1 \quad 42° \quad 3 \\
& \quad 35° \quad 55° \quad 3
\end{align*}
\]

d)  e)  f)

\[
\begin{align*}
41° & \quad \downarrow \quad 111° \\
& \quad \downarrow \quad 63° \\
& \quad \downarrow \quad 51° \quad 97°
\end{align*}
\]

Find angles in triangles using equations. Work through the first problem below as a class to review solving equations, then have students work on the rest of the problems independently.

Write an equation for each picture. Solve the equation and find all the angles in the triangle.

a)  b)  c)

\[
\begin{align*}
x + 5° & \quad 2x + 5° \\
53° & \quad 2x - 15° \\
5x & \quad 63° \quad 2x + 12°
\end{align*}
\]

Use Workbook page 81 Question 5 for more practice. Note that to solve Question 5 g), students need to solve the equation \(4x - 30 = x + 5 + 40\). To do so, they need to have done either Extension 2 of Lesson PA8-15 or Lesson PA8-32. You might return to this question later and regard it as an Extension for now.
### Goals

Students will investigate properties of quadrilaterals connected to parallel sides and right angles.

### Prior Knowledge Required

- Is familiar with special quadrilaterals
- Can identify and mark equal sides and angles
- Can identify and mark right angles
- Can identify and construct parallel lines
- Can identify and knows the properties of supplementary, corresponding, alternate, and opposite angles

### Materials

- straws
- protractors

#### Introduce special quadrilaterals. SAY:

Quadrilaterals are shapes with four sides. Some quadrilaterals are special, and they have special names. For example, a square is a special quadrilateral. **ASK:** What are some of its special properties? (all sides are equal, all angles are equal, all angles are right angles, it has two pairs of parallel sides) What other special quadrilaterals do you know? What special properties do they have? Introduce any special quadrilaterals that are not mentioned (see Vocabulary and chart in Introduction). Make sure students understand the meaning of opposite sides and adjacent sides. Use Questions 1–8 on Workbook pages 82 to review the properties of quadrilaterals.

#### Checking for parallel opposite sides. Review with students the methods for checking whether two lines are parallel (distances between lines measured along perpendicularrays are equal, alternate angles are equal, corresponding angles are equal). Then draw the picture in the margin on the board and ask students what they can tell about \( \angle a \) and \( \angle b \). Ask students to explain why the angles add to 180°. (Sample **ANSWER:** \( \angle b = \angle d \) (alternate angles) and \( \angle d + \angle a = 180° \) (supplementary angles), so \( \angle b + \angle a = 180° \)) Remind students that if alternate angles are equal, lines are parallel. We could work backwards in the argument above: If \( \angle b + \angle a = 180° \), then \( \angle b = \angle d \), and the lines are parallel. This method of checking for parallel lines is particularly useful when dealing with quadrilaterals. Give students protractors and have them work through Question 9 on Workbook page 83. Check answers as a class, and make sure that multiple correct answers are shown for Questions 9a) and c). (For example, in a) both \( \angle ABC \), \( \angle BCD \) and \( \angle BAD \), \( \angle ADC \) are good answers.) Then have students decide which sides are parallel in the quadrilaterals below before finishing the worksheet.

---

**CURRICULUM EXPECTATIONS**

Ontario: 7m47, 7m48; 8m1, 8m5, 8m7, 8m43

WNCP: 7SS3; optional, [C, CN, R]

**VOCABULARY**

- parallel
- corresponding angles
- opposite angles
- supplementary angles
- alternate angles
- theorem
- parallelogram
- trapezoid
- rectangle
- rhombus
- square
- opposite sides
- adjacent sides
Making connections between properties. Have students look at the tables they filled in when doing Question 10 on Workbook page 83. Ask students to make statements of the form: All shapes with [side property from right table] have [angle property from left table]. Students should either explain why the statement will be always true, or find a counter-example to explain why it is only sometimes true. EXAMPLES:

1. All shapes with 4 right angles have 2 pairs of parallel sides. (PROMPT: I see that shapes A and C are together in the right table. Where do they appear together in the left table?) True, because the adjacent angles in the shape then add to 180°.

2. All shapes with 1 right angle have no equal sides. False: the shape in the margin has 1 right angle and 2 equal sides.

Extensions

1. Special Quadrilaterals and Number Squares

Draw a square on grid paper and number the grid squares in sequence as follows: Start at the top left corner with any number and fill in each row from left to right before continuing to the next row. The example at left is a 4 by 4 square with the numbers starting at 5.

Draw several quadrilaterals with vertices in four of the grid squares and then add the numbers in the grid squares of opposite vertices.

EXEMPLARY:

\[
\begin{array}{cccc}
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
17 & 18 & 19 & 20 \\
\end{array}
\]

\[
\begin{align*}
6 + 19 &= 25 \\
9 + 16 &= 25
\end{align*}
\]

\[
\begin{align*}
5 + 15 &= 20 \\
6 + 19 &= 25 \\
9 + 15 &= 24
\end{align*}
\]
Complete the following chart for several quadrilaterals of each type:

<table>
<thead>
<tr>
<th>Vertex numbers</th>
<th>6, 9, 16, 19</th>
<th>5, 6, 15, 19</th>
<th>6, 9, 15, 18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum of opposite vertex numbers</td>
<td>25, 25</td>
<td>20, 25</td>
<td>24, 24</td>
</tr>
<tr>
<td>Square</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rhombus</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Parallelogram</td>
<td>×</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>Rectangle</td>
<td>×</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trapezoid</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What patterns do you notice? Try a grid of a different size.

**Bonus**

Instead of counting by 1s to fill in the grid, skip count by a different number.

Students might create new charts to look for other patterns in the vertex numbers, e.g., the difference between opposite vertex numbers, the sum of all vertex numbers. Encourage students to be creative in looking for patterns.

2. Arrange circular blocks in the shape shown. How many squares (with the centres of the circles as vertices) can you find in the figure? (There are 19 squares altogether.)

**NOTE:** Show your students the bold square as an example.

9 like this 4 like this 4 like this 2 like this
**Goals**

Students will investigate properties of quadrilaterals related to angles, sides and diagonals, and sort quadrilaterals according to these properties.

**PRIOR KNOWLEDGE REQUIRED**

- Is familiar with special quadrilaterals
- Can identify equal sides and angles
- Can identify and mark parallel lines
- Can identify and mark right angles
- Can find missing sides in a right triangle using the Pythagorean Theorem
- Is familiar with square roots of non-perfect squares

**MATERIALS**

- BLM Quadrilaterals (pp O-96–O-97)
- BLM Sorting Quadrilaterals (p O-98)
- dynamic geometry software (optional)

**Introduce diagonals.** Review with students the meaning of the terms adjacent and opposite (as they apply in polygons). Explain that a line segment that joins the opposite vertices in a quadrilateral is called a diagonal. How many diagonals does each quadrilateral have? (2) Draw several quadrilaterals, label the vertices, and ask students to name the diagonals in them. Then invite volunteers to draw the diagonals.

**Kites.** Review the names of special quadrilaterals (see Vocabulary). Remind students that kites are quadrilaterals with two pairs of equal adjacent sides, without indentations. Have students draw several kites. Remind students that some quadrilaterals fit several definitions. For example, a square is also a rectangle and a parallelogram. What other special quadrilaterals are also kites? (rhombus, square)

**Sort quadrilaterals by properties of sides, angles, and diagonals.** Draw a parallelogram $ABCD$ on the board, and mark the measures of the angles (e.g., $\angle A = \angle C = 45^\circ$, $\angle B = \angle D = 135^\circ$). Ask students which pairs of angles on this shape add to $180^\circ$. ($\angle A$ and $\angle B$, $\angle A$ and $\angle D$, $\angle C$ and $\angle B$, $\angle C$ and $\angle D$—four pairs in total) Then ask students to sketch a square $KLMN$ and to list all possible pairs of angles that add to $180^\circ$. How many pairs are there? (6; any pair of angles adds to $180^\circ$) You can use the Scribe, Stand, Share strategy to check the answers for both questions.

Give students copies of BLM Quadrilaterals. Have them draw diagonals on the shapes. Then ask them to sort the quadrilaterals into the table below (an empty copy is provided in the top part of BLM Sorting Quadrilaterals; the bottom part will be used in Lesson G8-29), using rulers to check lengths and protractors to check angles.

**VOCABULARY**

- acute, obtuse, right angle
- equilateral
- special quadrilaterals: trapezoid, parallelogram, rectangle, rhombus, square, kite, right trapezoid, isosceles trapezoid
- opposite sides/angles
- adjacent sides/angles
- diagonal
- Pythagorean Theorem
## Properties of sides

<table>
<thead>
<tr>
<th>Properties of sides</th>
<th>Shapes with the property</th>
<th>Properties of angles</th>
<th>Shapes with the property</th>
</tr>
</thead>
<tbody>
<tr>
<td>No equal sides</td>
<td>3, 4, 6, 7, 11</td>
<td>No equal angles</td>
<td>3, 4, 5, 7</td>
</tr>
<tr>
<td>1 pair of equal sides</td>
<td>5, 9</td>
<td>1 pair of equal angles</td>
<td>2, 5, 11</td>
</tr>
<tr>
<td>2 pairs of equal sides</td>
<td>2, 10, 12</td>
<td>2 pairs of equal angles</td>
<td>1, 9, 12</td>
</tr>
<tr>
<td>Equal sides are adjacent</td>
<td>1, 2, 5, 8</td>
<td>Equal angles are adjacent</td>
<td>5, 8, 9, 10, 11</td>
</tr>
<tr>
<td>Equal sides are opposite</td>
<td>1, 8, 9, 10, 12</td>
<td>Equal angles are opposite</td>
<td>1, 2, 8, 10, 12</td>
</tr>
<tr>
<td>4 equal sides</td>
<td>1, 8</td>
<td>4 equal angles</td>
<td>8, 10</td>
</tr>
<tr>
<td>1 pair of parallel sides</td>
<td>4, 5, 9, 11</td>
<td>No pairs of angles add to 180°</td>
<td>2, 3</td>
</tr>
<tr>
<td>2 pairs of parallel sides</td>
<td>1, 8, 10, 12</td>
<td>2 pairs of angles add to 180°</td>
<td>4, 5, 11</td>
</tr>
<tr>
<td>Perpendicular diagonals</td>
<td>1, 2, 7, 8</td>
<td>4 pairs of angles add to 180°</td>
<td>1, 9, 12</td>
</tr>
<tr>
<td>Equal diagonals</td>
<td>6, 8, 9, 10</td>
<td>6 pairs of angles add to 180°</td>
<td>8, 10</td>
</tr>
</tbody>
</table>

### Relationships between properties

Ask students to look at the table closely. Can they find a pair of properties that are matched with exactly the same shapes? (yes: “4 equal angles” and “6 pairs of angles add to 180° match with rectangles and squares) Why did that happen? (Students might remember from Grade 7 that the sum of the angles in a quadrilateral is 360°, in which case all angles—if they are equal—have to be 90°. If they do not remember this fact, they might notice that all the angles in both shapes are right angles. With four right angles, every pair of angles adds to 180°, and there are six possible pairs.)

Ask students to look for groups of shapes fulfilling one property that all fall into another category. For example, in the completed table, students will see that all shapes with no equal angles also have no equal sides. Does this mean “All shapes with no equal angles also have no equal sides” or “All shapes with no equal sides have no equal angles”? (All shapes with no equal angles have no equal sides.) Have students find the shape that is a counter-example for the other phrase. (shape 11—a right trapezoid) Is the first statement true for all shapes? Ask students to look for a counter-example (see Example in margin). Have students repeat the exercise with two properties of their own, including at least one not given in the table. Debrief as a class.
EXAMPLE: “Two pairs of angles add to 180°” (shapes 4, 5, and 11) might seem to imply “one pair of parallel sides”; however, a kite with two right angles as shown has two pairs of angles adding to 180° and no parallel sides.

Diagonals of kites are perpendicular. In order to do Question 4 on Workbook page 85, students need to review the Pythagorean Theorem. Using the same method as in Question 4, check as a class whether the quadrilateral at left is a kite. (no: sides $a$ and $b$ are equal, but sides $c$ and $d$ are not)

Ask students to draw several kites and to draw diagonals in them. Do all kites have perpendicular diagonals? (yes) Ask students to draw two of each other type of special quadrilaterals and check whether they have perpendicular diagonals. Which other shapes also have perpendicular diagonals? (squares and rhombuses) ASK: Are these also kites? (yes) Then ask students to look at the table they filled out earlier. Are all shapes with perpendicular diagonals kites? (no) Which shape is a counter-example? (shape 7) Have students look for another counter-example to the statement “All shapes with perpendicular diagonals are kites” among the shapes in Question 7 (shape F).

ACTIVITY

Students can use Geometer’s Sketchpad to construct a kite (make sure that the shape has two pairs of equal adjacent sides), draw diagonals in it, and measure the angles between the diagonals. To ensure students have done the construction correctly, ask them to modify their kite to check that the angle between the diagonals does not change. Note that there is more than one way to perform the construction.

Extensions

1. a) The sum of the angles in a quadrilateral is always 360°. Draw a quadrilateral with exactly three equal angles (see two samples in margin). Is it a special quadrilateral? (most likely no)

b) Can a quadrilateral with exactly three equal angles be a parallelogram? (no) A trapezoid? (no) Use what you know about the sum of adjacent angles in a parallelogram or a trapezoid to answer this question. (Explanation: Adjacent angles in a parallelogram or a trapezoid add to 180°. At least two of the three equal angles have to be adjacent, so they have to be 90°. The three equal angles are all right angles, so the fourth angle is $360° − 3 \times 90° = 90°$. This shape has four equal angles and not three, so a quadrilateral with exactly three equal angles cannot be a parallelogram or a trapezoid.)

c) A special quadrilateral has exactly three equal angles. What type of special quadrilateral is it? (a kite)
2. You can build a diagram that shows relationships between different special quadrilaterals. If a shape has all the properties of some other shape, it is drawn inside a more general shape.

Start with a rectangle. All squares are rectangles, so we can draw a square as part of a rectangle:

```
   or even
```

All squares are also rhombuses. Let’s draw a square as part of a rhombus:

```
```

Can you find a shape that is a rectangle and a rhombus at the same time but is *not* a square? No. Let’s show that—a square is the intersection of a rhombus and a rectangle:

```
```

All rectangles and rhombuses are also parallelograms, so we can draw the shape from the last step inside a parallelogram:

```
```

A rhombus is a parallelogram that is also a kite. Can you find a parallelogram that is also a kite but *not* a rhombus? No, so the rhombus is the intersection of a kite and a parallelogram.
**G8-29  Perpendicularrays and Bisectors**

**Pages 86–87**

**CURRICULUM EXPECTATIONS**
Ontario: 7m48, 8m1, 8m2, 8m3, 8m5, 8m7, 8m43
WNCP: 6SS1; 7SS3; optional, [C, CN, R, V]

**VOCABULARY**
line segment
midpoint
bisector
diagonal
reverse statement
perpendicular bisector

**Goals**
Students will identify and draw perpendicular bisectors.

**PRIOR KNOWLEDGE REQUIRED**
Can identify and construct a perpendicular using a set square or a protractor
Can identify and mark right angles
Can name line segments and identify named line segments
Can draw and measure with a ruler
Can find sides of a right triangle using the Pythagorean Theorem

**MATERIALS**
BLM Triangles (p O-99)
BLM Quadrilaterals (pp O-96–O-97)
BLM Sorting Quadrilaterals (p O-98)
protractors, rulers

**Introduce the midpoint.** Model finding the midpoint of a segment using a ruler and marking the halves of the line segment as equal, then have students practise this skill.

**Introduce bisectors.** Explain that a bisector of a line segment is a line (or ray or line segment) that divides the line segment into two equal parts. There can be many bisectors of a line segment. Ask students to draw a line segment with several bisectors.

**ASK:** Can a line have a bisector? What about a ray? (no, because lines and rays have no midpoints)

**EXTRA PRACTICE:**
Draw a scalene triangle using a ruler. (Be careful! It is hard to draw a triangle that is not a right or an isosceles triangle.) Choose a side and draw a bisector to that side that passes through the vertex of the triangle that is opposite that side. Repeat with the other sides. What do you notice? (ANSWER: All three bisectors pass through the same point.) Make a conjecture and check it using triangles from BLM Triangles.

**Introduce perpendicular bisectors.** Of all the bisectors of a line segment only one is perpendicular to the line segment, and it is called the perpendicular bisector. The perpendicular bisector of a line segment

- divides the line segment into two equal parts AND
- intersects the line segment at right angles (90°).

Point out that there are two parts in the definition, and both must be satisfied. **ASK:** How can we draw a perpendicular bisector? How is that
problem similar to constructing a bisector? Constructing a perpendicular?
Lead students to the idea that they should first find the midpoint of the line
segment, then construct a perpendicular through that point. Have students
practise drawing perpendicular bisectors using set squares and using
protractors and rulers.

Present the diagram at left and have students identify equal segments, then
find perpendicular lines and bisectors. (EXAMPLE: Find a bisector of $EG$. Is it
a perpendicular bisector?) Ask other questions about the diagram, such as:

- $J$ is the midpoint of what segment? ($CG$) Is it also the midpoint of
$AD$? (no) How do you know? (We do not have any information about
the lengths of $DJ$ and $AJ$. They look equal, but might have
different lengths.)

- Name three line segments $GJ$ is perpendicular to. ($EG$, $FG$, $FH$, $AJ$, etc.)

Finally, have students identify all the perpendicular bisectors and the line
segments they bisect. (ANSWERS: $AD$, $JA$, and $JD$ bisect line segment $CG$;
line $CJ$ and line segments $JG$ and $CG$ bisect line segment $FH$)

Ask students to find examples of equal line segments, midpoints, and
perpendicular bisectors in the classroom or elsewhere, such as in letters
of the alphabet, in pictures or photographs, and so on.

Sort quadrilaterals by properties of diagonals. Give students copies of
BLM Quadrilaterals. For all the quadrilaterals pictured, have students draw
and measure the diagonals (REMEMBER: a diagonal is any line that joins
vertices that are not adjacent. Diagonals can be outside a shape!); find the
midpoints of the diagonals; measure the angles between the diagonals and
the sides, and mark equal angles. Have students sort the quadrilaterals
into the table below (an empty copy is provided in the bottom part of
BLM Sorting Quadrilaterals). (You may choose to assign different shapes
to small groups of individuals and pool their results to complete the table
as a class.)

<table>
<thead>
<tr>
<th>Properties</th>
<th>Shapes with the property</th>
<th>Properties</th>
<th>Shapes with the property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perpendicular diagonals</td>
<td>1, 2, 7, 8</td>
<td>Equal diagonals</td>
<td>6, 8, 10</td>
</tr>
<tr>
<td>One of the diagonals is also</td>
<td>2, 3, 5</td>
<td>Exactly one diagonal bisects</td>
<td>2</td>
</tr>
<tr>
<td>an angle bisector</td>
<td></td>
<td>the other</td>
<td></td>
</tr>
<tr>
<td>Both diagonals are angle bisectors</td>
<td>1, 8</td>
<td>Both diagonals bisect each other</td>
<td>1, 8, 10, 12</td>
</tr>
</tbody>
</table>

Ask students to identify and name the shapes that are special quadrilaterals
(all but 3, 6, and 7 are special). Discuss the properties they share. For
example, which property do all parallelograms share? (both diagonals
bisect each other) Rectangles are also parallelograms—do both of
their diagonals bisect each other? (yes) Is there another property that all rectangles have? (equal diagonals) Where are all rhombuses? (both diagonals are angle bisectors, both diagonals bisect each other) Squares are also rhombuses—does the square share these properties as well? (yes)

**Perpendicular diagonals that bisect each other mean the shape is a rhombus.** Write on the board:

All rhombuses have perpendicular diagonals that bisect each other.

**ASK:** Does our table support this statement? Invite a volunteer to rewrite the statement in the form “If —, then —.” (If a shape is a rhombus then its diagonals are perpendicular and bisect each other.) Ask students to draw two different rhombuses to check whether this statement is true all the time.

Now ask students to reverse the statement. (All shapes with perpendicular diagonals that bisect each other are rhombuses.) Is this a true statement? Ask students to construct a line segment $EG$ that is an even number of centimetres long, find its midpoint $O$, and construct a perpendicular bisector of $EG$. Then ask them to find points $F$ and $H$ on the perpendicular bisector so that $OF = OH$ and $FH$ is an even number of centimetres long but not the same length as $EG$. **ASK:** Is $EG$ a perpendicular bisector of $FH$? (yes) Now have students construct quadrilateral $EFGH$. What are $EG$ and $FH$ for this quadrilateral? (diagonals) Does $EFGH$ look like a special quadrilateral? Which one? (a rhombus)

Review the Pythagorean Theorem with students, then ask them to use it to find the length of the sides of $EFGH$. For example, if students constructed $FH = 8$ cm and $EG = 10$ cm, then they would get:

$EO = 5$ cm, and $OH = 4$ cm, so by the Pythagorean Theorem for $\triangle EOH$,

$EH^2 = 5^2 + 4^2 = 25 + 16 = 44$, so $EH = \sqrt{44}$.

$GO = 5$ cm, and $OH = 4$ cm, so by the Pythagorean Theorem for $\triangle GOH$,

$GH^2 = 5^2 + 4^2 = 25 + 16 = 44$, so $GH = \sqrt{44}$.

Similarly, $EF = GF = \sqrt{44}$.

**ASK:** Did you prove that $EFGH$ is a rhombus? (yes) Did you prove that any quadrilateral that has perpendicular diagonals that bisect each other is a rhombus? (no) Why not? (we checked only one case, not all possible cases) Have students sketch the same situation but assign different lengths to the diagonals and work out the side lengths of the new quadrilateral.

**ASK:** Did the proof work for the different numbers? (yes) Will it work for any lengths of line segments? (yes) In your calculations, did you use the whole length of the diagonal or half the length of the diagonal? (half the length) Would it make sense to denote the whole length of the diagonal as a variable or just half of it? Have students repeat the calculation using variables, setting the length of one diagonal as $2a$ and the length of the other diagonal as $2b$. (each side of the rhombus has length $\sqrt{a^2 + b^2}$)
1. **Paper folding and line segments**

   Draw a line segment $AB$ dark enough that you can see it through the paper. Fold the paper so that $A$ meets $B$. What line has your crease made? *(Answer: a perpendicular bisector)* Use a ruler and protractor to check your answer.

2. **Divide students into groups of three, and give each group a copy of BLM Triangles.** Each student should get four paper triangles from the same set (two copies of an isosceles triangle and two copies of a scalene triangle).

   Each student should try to create as many shapes as possible from two copies of the same triangle by joining the triangles along a pair of sides of the same length. Ask students to trace the shapes and to identify them if possible. In their groups, students should discuss the shapes they produced. How many shapes does each triangle make? Are all the shapes quadrilaterals? When is the resulting shape a triangle? What types of quadrilaterals can be created and what special features do they have?

   **Answers:** Each scalene triangle produces 6 shapes, each isosceles triangle produces 3 shapes.

   Set 1 has two pairs of right triangles. The isosceles triangles produce a right isosceles triangle, a square, or a parallelogram. The scalene triangles create two different isosceles triangles, two different parallelograms, a kite, or a rectangle.

   Set 2 has two pairs of acute triangles. The isosceles triangles produce a rhombus, a parallelogram, or a kite. The scalene triangles produce three different kites or three different parallelograms.

   Set 3 has two pairs of obtuse triangles. The isosceles triangles produce a rhombus, a parallelogram, or a shape with two pairs of equal adjacent sides and an indentation. The scalene triangles produce a kite, two shapes with two pairs of equal adjacent sides and an indentation, and three different parallelograms.

   All resulting shapes in all three groups have either a line of symmetry or rotational symmetry.

   Ask students to think about the shapes produced from the same pair of triangles. What can they say about the areas of those shapes? *(The areas are the same)* For each pair of triangles, what quadrilateral is the easiest for finding the area? *(rectangle or parallelogram)* Have students find the area of the parallelograms and triangles.
Extensions

1. a) Use the fact that a kite can be split into two identical triangles to find a formula for the area of a kite using its diagonals.

**ANSWER:** A kite has perpendicular diagonals, and at least one of the diagonals bisects the other. Let \(a\) and \(b\) be the lengths of the diagonals as shown.

Divide the kite into two identical triangles. Take the common side of these triangles \((a)\) to be the base. The base is one of the diagonals of the kite. The other diagonal is the height for both triangles. The area of each triangle is base \(\times\) height \(\div\) 2, with the height being half of diagonal \(b\), so the area of each triangle is

\[
\text{area} = a \times \left(\frac{b}{2}\right) \div 2 \\
= a \times \frac{b}{2} \div 2 \\
= a \times b \div (2 \times 2) \\
= a \times b \div 4
\]

The area of the kite is twice the area of the triangle, or \(a \times b \div 4 \times 2 = a \times b \div 2\).

b) For which special quadrilaterals will the formula above work as well? Why?

i) rhombus ii) rectangle iii) parallelogram iv) square

**ANSWER:** rhombus and square, because they are both kites.

2. Is the origin the midpoint of the x-axis? Explain. (no, a line has no midpoint, because it has no endpoints and infinite length)

3. Start with a paper circle. Choose a point \(C\) on the circle and draw a right angle so that its arms intersect the circle. Label the points where the arms intersect the circle \(A\) and \(B\) and draw the line segment \(AB\). Repeat with several circles to produce different right triangles. (You can use BLM Circles for this Extension.) Make at least one triangle with arms of different lengths.

Fold the circle in two across the side \(AC\) so that \(A\) falls on \(C\) (creating a perpendicular bisector of \(AC\)). Mark the point where \(B\) falls on the circle. Repeat with the side \(BC\), marking the point where \(A\) falls on the circle. What do you notice? (The image of \(A\) should be the same as the image of \(B\).) What type of special quadrilateral have you created? (a rectangle or a square, which is a type of rectangle)

Repeat the exercise starting with an obtuse or an acute angle \(C\). Do the images of \(A\) and \(B\) coincide? (no)
Assess prior knowledge. Start the lesson with a diagnostic test: ask students to draw a triangle. Then ask them to draw another triangle that is different in some way, and have them explain how it is different. What changed in the second triangle compared to the first? Ask them to draw a third triangle, different from the other two, and explain the differences again. Students whose descriptions focus on size, orientation, and other non-geometric attributes (e.g., this triangle is pointing down, this triangle is long and thin) should be prompted to remember and use geometric attributes. Give these students a copy of BLM Triangles for Sorting and have them sort the triangles in different ways, such as number of equal sides (or angles), whether a triangle has a right (or obtuse) angle, or number of lines of symmetry. Start with sorting by one attribute, then continue to two attributes using Venn diagrams. Then have students test ideas about their triangles, e.g., Can there be a triangle with two right angles? Can you draw a triangle that has no right angles? Can you draw a triangle with all acute angles?

Review the notation for equal sides (equal number of markings) and equal angles (equal number of arcs) in polygons.

Introduce congruent shapes. Explain that congruent shapes have the same size and shape, so if you put one shape on top of the other, they should match exactly. This means that the sides and the angles of congruent shapes have to match exactly as well, and they have to match in order. You could use Activity 1 to illustrate the fact that all sides and angles of two non-congruent shapes can be equal, just not in the same order. See also Extensions 1 and 2.

In triangles, the order of sides does not matter, because there are only three sides. Give your students three straws of different lengths and ask
them to mark the ends of the largest straw with different colours. Have students make a triangle with their three straws and trace it on a sheet of paper. Then ask them to change the order of the straws (the straw that touched the blue end of the longest straw will now touch the red end). Look at the new triangle. Is it the same triangle or a different one? If students do not see that they created identical triangles, they can trace the new triangle again, cut out the tracing and compare it to the tracing of the first triangle. Have students combine the straws with a partner and try to see whether they can create different shapes using four of their six straws. Does the order you place the straws in matter? (yes) Explain that in triangles the order of the sides does not matter - if you try to change it, you produce a reflection of the first triangle.

**Corresponding sides and angles.** Explain that when we want to check whether shapes are congruent, we pretend we can place the shapes one on top of the other. Which side of one shape will go over which side of the other shape? The sides (or angles) of different shapes that will go one on top of the other are called **corresponding sides** (angles). If the shapes are congruent, corresponding sides and angles will be equal. Have students identify corresponding sides and angles on triangles such as in the Workbook, Questions 3 through 6.

Return to the pentagons A and B from Activity 1 below. Point out that when we try to place these pentagons one on top of the other, we define correspondence – the order in which we check the angles and the sides. If we place the pentagons so that at least one side matches, we might make some of the angles match, but we cannot make the other angles match – for example, if the obtuse angles match and at least one side adjacent to them matches and are equal, as shown at left, the other sides adjacent to the obtuse angle will not be equal, because the order of the sides is different.

**More than one way to find corresponding sides and angles.** Show your students a regular sheet of paper and have them identify the shape. What properties does it have? (You might use Scribe, Stand, Share to check the answers.) Trace the rectangle on the board and label the vertices (say, \(ABCD\)). Then mark each corner of your paper rectangle with a different letter (say, \(EFGH\)). **ASK:** Are these rectangles congruent? (yes) Explain that when you place the rectangle over the tracing, you create correspondence – \(E\) corresponds to vertex \(A\), \(F\) to \(B\), and so on (have students write down which angles and sides correspond and are equal: \(\angle A = \angle E\), \(AB = EF\), etc). However, you can place the rectangle onto the tracing in a different way (show how to do so) and then \(E\) will correspond to \(C\), \(F\) will correspond to \(D\) and so on. Have students decide which angles and sides are equal in the new way of correspondence. **ASK:** Why were we able to do this? (Since a rectangle has equal angles and two pairs of equal sides, you can place two rectangles one on top of the other in different ways. Another way to say it is that a rectangle is a symmetrical shape; if students use this term (being familiar with it from the earlier grades), invite them to explain how symmetry helps here.) To let students practise writing pairs of equal corresponding...
angles, draw two isosceles congruent triangles and have students mark
the equal angles, equal sides and then to write the equalities between the
corresponding sides and angles in different ways.

**Congruence symbol.** Explain that when two shapes are congruent, we
have a special symbol for that: $\cong$. Similar to writing that two sides have the
same length, $AB = CD$ instead of writing $AB$ is equal to $CD$, we can write
$ABCD \cong EFGH$ instead of writing $ABCD$ is congruent to $EFGH$.

**Congruence statements.** Explain that the congruence symbol means
more than plain equality sign. When mathematicians write a congruence
statement, (such as $ABCD \cong EFGH$) they agreed to write the statements so
that one would be able to say which side is equal to which side and which
angle is equal to which angle. **EXAMPLE:** $ABCD \cong EFGH$, but not $ABCD \cong FGHE$, because $AB = EF$, but $AB \neq FG$. Teach students writing congruence
statements using the same procedure as on Workbook page 89.

**EXTRA PRACTICE:**

After students finish Workbook Questions 2 and 3, have them write
congruence statements for triangles in these questions.

**PROCESS ASSESSMENT**

8m3, [C, R]

Have students explain what Tom from Workbook Question 12 did wrong.

**ACTIVITIES 1–2**

1. Give students a copy of BLM Two Pentagons. Have students cut
out the pentagons, compare their sides and angles, and answer the
questions on the BLM individually. (Instead of using the pentagons on
the BLM, students can use Extension 2 to create their own. Students
may retain more from this Activity if they create their own shapes.)
When students are finished, have them signal the number of right
angles on each polygon to check the answer to Question a), then
ask them to hold up the folded shapes so that they can show that the
remaining four angles are all equal. Students can signal thumbs up
if the answer is “yes” and thumbs down if the answer is “no” as you
check the remaining yes-no questions on the BLM as a class. Then
pair students up (pair the weakest students with the strongest ones so
that the stronger students can coach the weaker ones) and have them
compare their answers for Question i) and come up with a common
answer. Repeat with groups of four and groups of eight and then have
groups share their answers with the whole class.

**ANSWERS:**

a) How many right angles does each pentagon have? (3)

b) Does each side on pentagon A have a side of the same length
on pentagon B? (yes)

d) $KL = PT$, $TS$  $LM = QR$, $MN = TS$, $PT$
$NO = PQ$, $RS$  $OK = RS$, $PQ$
Extensions

1. Look at the pentagons from BLM Two Pentagons. Which one has greater area? Take the necessary measurements to check. (**Answer:** Pentagon A)

2. Pentagons similar to those on BLM Two Pentagons can be created by paper folding. Here are the folding instructions.

   Start with a regular sheet of paper. Fold it in two and cut along the fold to create two long and thin rectangles. (One rectangle will be enough for both pentagons.)

   Pentagon A:

   Create a square from your rectangle by folding the short side down to the long side of the rectangle, and cut the residue off to use for Pentagon B. The result is a square with one diagonal fold.
Fold the square and unfold it again to create the second diagonal. Fold one of the corners of the square onto the centre of the square. Cut the new small triangle off.

Pentagon B:

Use the residue (a rectangle) from making Pentagon A. Place Pentagon A on top of the rectangle as shown and mark the length of the “cut off” side of Pentagon A on the short side of the rectangle.

Place the rectangle with the long side down, so that the mark you’ve made is in the top half of the left side edge. Fold the top left corner of the rectangle down to the bottom edge so that the left side falls onto the bottom edge, as if you were making a square again. Fold the residue on your right side over the triangle. Unfold.

Fold the top edge of the rectangle down through the mark, so that the side edges fold onto themselves, to create a narrower rectangle.

Fold the top right corner of the narrower rectangle diagonally down, so that it falls over the vertical crease (it will not reach all the way to the bottom). Fold the bottom right corner diagonally up so that it falls over the same vertical fold, and you are done!
G8-31 Congruence Rules

Pages 91–93

**Goals**

Students will identify congruent triangles using congruence rules.

**PRIOR KNOWLEDGE REQUIRED**

- Can measure angles and sides of polygons
- Is familiar with notation for equal sides and angles
- Can name angles and polygons
- Knows the symbols for angle (\(\angle\)), triangle (\(\triangle\)), congruent (\(\cong\))

**MATERIALS**

- protractors
- rulers
- dynamic geometry software (optional)
- BLM Investigating Congruence (p O-102)
- BLM Congruence Rules on Geometer’s Sketchpad (pp O-103–O-105)

**CURRICULUM EXPECTATIONS**

- Ontario: 7m50; 8m2, 8m4, 8m6, 8m7, 8m45, 8m50
- WNCP: optional, [C, R, T, V]; 9SS9

**VOCABULARY**

- triangle
- corresponding angles
- corresponding sides
- right, isosceles, equilateral
- triangle congruence rule
- SAS (side-angle-side)
- ASA (angle-side-angle)
- SSS (side-side-side)
- SAA (side-angle-angle)
- congruent triangles
- conjecture

**PROCESS EXPECTATION**

Making and investigating conjectures

**Congruence and triangles.** Remind students that when checking for congruence we select the order in which we want to check that the sides and angles are equal. This order is called *correspondence*, and we check that corresponding sides and angles are equal. Tell students that you will start by looking at triangles. They have only 3 sides and 3 angles, so they are the simplest possible polygons, and this might make checking for congruence easier. Explain that you will be looking for “shortcuts,” or *congruence rules*. For example, if we checked two pairs of angles, would we need to check the third one or is there a shortcut? What is the sum of the angles in any triangle? If we know the measures of two angles in a triangle, how can we find the measure of the third angle? (by subtracting the sum of the measures of the other two angles from 180°) This means we do not have to check all 6 elements of a pair of triangles (3 sides and 3 angles); checking 3 sides and 2 angles will be enough. Can we do better?

**SSS, SAS, ASA rules.** Have students investigate one of the congruence rules (SSS, SAS, or ASA) using Geometer’s Sketchpad (see Activity 1). Alternatively, have students investigate all three rules using BLM Investigating Congruence.

**SAA rule.** Review the fact that the sum of the angles in a triangle is 180°. Present the picture in the margin and ask students whether they think the triangles are congruent. **ASK:** Is there a congruence rule that tells us that the triangles are congruent? (no) What elements are marked as equal in these triangles? (How many pairs of angles? How many pairs of sides?) Are these equal angles and side in the same order in both triangles? Are the equal sides opposite equal angles? Can we say that we have two pairs of equal corresponding angles and a pair of equal corresponding sides? What congruence rule uses two pairs of equal corresponding angles and a pair...
of equal corresponding sides? (ASA) What pair of sides would we need to know are equal to use the ASA congruence rule? \( AC = DF \) Do we know that these sides are equal? (no) What pair of angles would we need to know are equal to use the ASA congruence rule? \( \angle B = \angle E \)

**ASK:** How can you find the measure of \( \angle B \) from the rest of the angles of the triangle? Have students write down the expression for the measure of the angle. \( \angle B = 180^\circ - \angle A - \angle C \) Repeat for \( \angle E \). **ASK:** Are \( \angle B \) and \( \angle E \) equal? (yes) How do you know? (The other two pairs of angles are equal. Because the angles in a triangle add to 180°, the remaining angles have to be equal.)

Write on the board:

\[
\angle B = 180^\circ - \angle A - \angle C \\
\angle E = 180^\circ - \angle D - \angle F
\]

Since \( \angle A = \angle D \) (underline them, or circle them in one colour), and \( \angle C = \angle F \) (underline these angles in a different way, or circle them in a different colour), we have \( \angle B = \angle E \). Now mark angles \( \angle B \) and \( \angle E \) as equal on the picture and **ASK:** Are the triangles congruent? (yes) By which rule? (ASA)

Have students repeat the argument for another pair of triangles satisfying SAA, such as the pair at left.

Explain that if two triangles satisfy SAA, they will always satisfy ASA, so they are congruent, and we can say directly that two triangles that satisfy SAA are congruent.

Emphasise the necessity of having the equal sides opposite a pair of equal angles. Look at the pictures in the box on Workbook page 92 together: the triangles on the right are not congruent, though they have two pairs of equal angles and a pair of equal sides, because the equal sides are opposite different angles. Finally, write out the SAA congruence rule as in the box on Workbook page 92, and have students practise with questions such as Questions 1–9 on Workbook pages 91–93.

**EXTRA PRACTICE:**

1. Which congruence rule tells that the two triangles are congruent? Write the congruence statement.

   a)  
   
   ![Diagram A]

   b)  
   
   ![Diagram B]

   c)  
   
   ![Diagram C]

   **ANSWERS:**
   
   a) ASA, \( \triangle GHI \cong \triangle JLK \)
   
   b) SAS, \( \triangle MOQ \cong \triangle PRN \)
   
   c) SAA, \( \triangle ABC \cong \triangle FDE \)
2. Sketch a counter-example to show why each statement is false.
   a) If two triangles have three pairs of corresponding equal angles, the triangles are congruent.
   b) \( \triangle ABC \) has \( AB = BC = 7 \text{ cm} \) and \( \triangle DEF \) has \( DE = EF = 7 \text{ cm} \). Then \( \triangle ABC \cong \triangle DEF \).

Another hint for Workbook page 93, Question 10. Have students sketch an acute isosceles triangle with one angle 30°, and an obtuse isosceles triangle with one angle 30°. Ask them to label the triangles \( \triangle ABC \) and \( \triangle DEF \), so that \( \angle A \) and \( \angle D \) are both 30°. **ASK:** Can you make one of the triangles larger so that \( AB = DE \)?

**SSA is not a congruence rule.** Have students do the Investigation on Workbook page 93. (*NOTE:* The markings on the triangles in the Workbook should show \( AB = DE, BC = EF \). If they do not, ask students to correct them.) Students can also use the paper folding activity below (Activity 2) to see that SSA is not a congruence rule.

**Using the Pythagorean Theorem to show congruence.** Review the Pythagorean Theorem with students. Remind them how to find a missing side, both in the case when the missing side is a hypotenuse and in the case when the missing side is a leg of a right triangle.

Have students work through Questions 11–13 on Workbook page 93. Note that there is more than one solution possible for Questions 11 and 12: once students find the missing sides of the triangles using the Pythagorean Theorem, they can use either SSS or SAS congruence rule, depending on which sides and angles they decide to use.

### ACTIVITIES 1–3

1. Divide the students into groups of three. Students will work on the construction individually using **BLM Congruence Rules on Geometer’s Sketchpad** (page 1, 2, or 3—each student in a group works on a different construction) and share the results. Students should tell which elements of the construction could be modified and by using which transformation. (**EXAMPLE:** I could modify the first triangle any way I want by moving any of the vertices. I could only translate the second triangle by moving the vertex \( E \), and I could only rotate the triangle by moving the vertex \( D \). When I tried to move vertex \( D \), it would only go along a circle, because it was constructed so that \( ED \) has fixed length.)

   Have students, in their groups of three, match each BLM with the congruence rule it seems to be showing (SSS, SAS, or ASA). For example, page 2 shows SAS by keeping two side lengths and the angle between them constant, which forces the triangle to be fixed.

2. Start with a regular sheet of paper. Fold it diagonally and cut along the create to create two congruent right scalene triangles. Set aside
one of the triangles. Fold the other triangle through the right angle so as to make the fold perpendicular to the longest side of the triangle. Part of the longest side will fall on top of the other part of the same side. Cut off the part that “sticks out”—this part, in the shape of scalene obtuse triangle, is the triangle you need. You can discard the isosceles triangle that is left.

Look at the right triangle you set aside at the beginning and the small scalene obtuse triangle. Compare the sides and the angles. Label the pairs of equal sides with the same number. Mark the equal angles. Are the equal angles opposite a pair of corresponding equal sides? (yes) Are the triangles congruent? (no) How many pairs of equal sides do you have? (2) How many pairs of equal angles? (1) Which description identifies the equal elements, in order, in your triangles: side-angle-side (SAS) or side-side-angle (SSA)? Which one of these is an acronym for congruence rule? (SAS) Is this statement true or false: SSA is a congruence rule. (false) How do the triangles you’ve made help you to decide? (They are a counter-example to this statement.)

3. A restaurant has many windows of unusual shapes. One of them is the trapezoid shown below. You call the company to order a replacement for this window. You tell the person that the window is a trapezoid with AB parallel to CD, but you need to give more information to ensure that the replacement window is the exact shape and size.

Is it enough to give:

a) the four sides, AB, BC, CD, DA?
b) the angles A, B and sides AB, DA?
c) the angles A, B and sides AB, CD?
d) the angles A, B and sides BC, DA?
e) the sides AB, BC, CD and angle B?
f) the sides AB, BC, CD and angle A?
g) the sides AB, BC, CD, DA and angle A?

If yes, explain your answer. If no, draw two different trapezoids with the given sides and angles equal. Investigate other combinations.

Extensions

1. On grid paper, draw two non-congruent figures with:

   a) the same perimeter  
   b) the same area  
   c) the same shape

   **Bonus** same perimeter and the same area.

2. Students can prove the Pythagorean Theorem the same way Euclid proved it 2300 years ago, using congruence and area of triangles. See BLM Proving the Pythagorean Theorem (pp O-106–O-107).
Introduce angle bisectors. An angle bisector is a ray that cuts an angle exactly in half, making two equal angles. Point out that an angle has only one bisector. Ask students to think about where they see angle bisectors in the real world. For example, angle bisectors are often seen in corners on furniture and picture frames. Remind students that a line (or line segment, or ray) that splits a line segment into two equal parts is called a bisector as well.

Explain that we can use an equal number of small lines to show that the angles in a diagram are equal, as in the pictures on Workbook page 94. Draw several bisected angles on the board and give the measure of one angle (as in Question 1 on Workbook page 94). Have students determine the measure of the second angle and of the whole (unbisected) angle. Then do the reverse: provide the measure for the whole angle and have students determine the measures of the parts.

EXTRA PRACTICE: Find the size of each of the equal angles.

a) b)

Constructing angle bisectors. Draw an angle on the board, then have a volunteer measure it and write the measurement. **ASK:** If you were to draw an angle bisector, what would be the degree measure of each half? Model
drawing the bisector. Emphasize that the bisector lies between the arms of the angle. Have students practise bisecting different angles, including obtuse, right, and acute angles. (You might use Activity 1 at this point.) Ask students to identify both the angles they drew and the halves as acute or obtuse angles. Can they get two obtuse angles after bisecting an angle? (no) Why not? (Because double any obtuse angle is more than 180°. You might point out that such angles are called reflexive angles.) If students need a prompt to see this, have them bisect a straight angle or draw two equal obtuse angles with a common arm.

**EXTRA PRACTICE:** Draw a scalene triangle and bisect each of the angles. If the bisection is performed correctly, the bisectors should meet at the same point.

**Identifying congruent triangles and explaining why they are congruent when triangles have a common side.** Explain that sometimes you see triangles that share a side. Draw the picture at left to illustrate this. To prove that such triangles are congruent, you could use the fact that the common side belongs to both triangles and therefore makes a pair of equal sides. For example, in the picture at left, the triangles $KMN$ and $LMN$ have two pairs of equal corresponding angles. They also have a common side $MN$, which is opposite equal angles, $\angle K$ and $\angle L$. Is there a congruence rule that applies to $\triangle KMN$ and $\triangle LMN$? (yes, SAA) So triangles $KMN$ and $LMN$ are congruent. Which sides are then equal in $\triangle KMN$ and $\triangle LMN$? What does this say about $\triangle KML$? (it is isosceles) Have students practise identifying which congruence rules prove that triangles are congruent using Question 3 on Workbook page 94.

**Introduce the median.** Explain to students that a line segment that joins a vertex in a triangle to the midpoint of the opposite side is called a median. A median is always a bisector of the opposite side of the triangle. Ask students to draw three copies of a triangle of each of the different types (isosceles, equilateral, and scalene) on grid paper and to draw a different median in each. **ASK:** In which triangles does the median splits the triangle into what looks like two congruent triangles? (all three medians of an equilateral triangle, and the median to the unequal side of an isosceles triangle) Invite volunteers to sketch these situations on the board. Remind students to mark equal sides. **ASK:** Are there triangles where a median does not split the triangle into two congruent triangles? Invite volunteers to sketch again, and mark the equal line segments. Make sure you use as many different volunteers as possible.

**Perpendicular to the opposite side and median.** Repeat the exercise above with perpendiculars from a vertex to the opposite side. Then **ASK:** Are there triangles where the perpendicular from a vertex to the opposite side was also a median? Invite volunteers to again sketch these situations on the board. Ask students what they notice. (In an equilateral triangle, a median is also perpendicular to the opposite side. The same happens with isosceles triangle, where the median is drawn to the unequal side.) To emphasize that the side to which the median or the perpendicular is drawn in the isosceles triangle is important, ask students to draw two
copies of an acute isosceles triangle and to draw in one of them the median
to one of the equal sides, and in the other the perpendicular to one of the
equal sides, passing through the opposite vertex. Do they split the triangles
into two congruent triangles?

Ask students to look at the picture where the median is perpendicular to
the opposite side. **ASK:** What do we call lines that are both perpendicular
to a line segment and also bisect the same line segment? (perpendicular
bisector)

**If a median is also a perpendicular bisector, the triangle is isosceles.**
Have students draw triangle ABC following these steps:

**Step 1:** Draw line segment AC.

**Step 2:** Find the midpoint of AC and label it D. Mark the equal line segments.

**Step 3:** Draw a perpendicular bisector to AC.

**Step 4:** Choose any point (different from D) on the perpendicular bisector
of AC and label it B. Draw line segments AB and BC.

The resulting picture will look like the picture at left. Explain that in this
triangle BD is both a median and a perpendicular bisector. **ASK:** What
type of triangle have you drawn? (isosceles) Have students measure the
sides to check. Did anybody draw a triangle that is not isosceles? Point
out that students have made a conjecture, a statement that they think
is true, but to make sure it is true we would like to prove it using logic.
State the conjecture on the board: **If a triangle has a median that is also a
perpendicular bisector of the opposite side, the triangle is isosceles.**

**ASK:** What triangle does ∆ABD look congruent to? (∆CBD) Remind
students of the need to write the order of the letters correctly in congruence
statements, and ask them to write the correct congruent statement. (∆ABD
≅ ∆CBD) **ASK:** If we can prove that, will that be enough to prove that ∆ABC
is isosceles? (yes) Why? (because ∆ABD ≅ ∆CBD would mean AB = CB)

Have students try to prove that ∆ABD ≅ ∆CBD individually, then check the
proof as a class: **ASK:** What congruence rule did you use? (SAS) **SAY:** But
I see only one pair of equal sides marked in the diagram: AD = CD. How
do you know there is another pair of equal sides? (BD is the common side,
so it is the same in both triangles) Summarize by saying that we now know
AD = CD and BC = BC. Are the angles between the pairs of equal sides
equal too? How do you know? (AC ⊥ BD, so ∠ADB = ∠CDB = 90°) This
means ∆ABD ≅ ∆CBD by SAS, and AB = CB, so ∆ABC is isosceles. Point
out that students have now proved their conjecture using logic.

**Is the reverse statement true?** Write on the board:

One median is also a perpendicular bisector of the side it bisects. The triangle is isosceles.
SAY: We proved that one of these statements follows from the other. Draw an arrow pointing from the left statement to the right statement. ASK: Does it work the other way? Does the first statement follow from the second one? Draw the arrow pointing from the statement on the right to the statement on the left and write the reverse statement: If a triangle is isosceles, then one median is also a perpendicular bisector. Remind students that each statement in such a pair is the reverse of the other. Have students do Question 4 on Workbook page 95 to check the reverse statement.

Point out that the medians to the other two sides of an isosceles triangle are not perpendicular bisectors of the sides – the statement is only true for the median to the uneven side, as the statement in Question 4 says.

Present the three statements below and tell students they are all true. ASK: Which one of them have you proved in this lesson? Give students some time to think (they can check the statement in the workbook), then ask them to hold up the number of fingers equal to the number of the statement that they proved. (ANSWER: 2)

1. In an isosceles triangle $ABC$ with $AB = BC$, a bisector of $\angle B$ is the perpendicular bisector of side $AC$.
2. In an isosceles triangle $ABC$ with $AB = BC$, a median to $AC$ is the perpendicular bisector of side $AC$.
3. In an isosceles triangle $ABC$ with $AB = BC$, a perpendicular bisector of $AC$ bisects $\angle B$.

**Using congruent triangles to verify diagonal properties of quadrilaterals.** Work through Question 5 on Workbook page 95 as a class. After doing part a), have students identify what special quadrilateral $ABCD$ is. (a rhombus) Students can also identify all pairs of congruent triangles in the question at this time. When working with a particular pair of triangles, as in parts b) and c), have students trace or shade the congruent triangles with the same colour, to make it easier to distinguish that pair from other pairs of congruent triangles. If necessary, remind students of the information given in the question: the circles have equal radii (that’s how we know $\triangle ABC$ is isosceles.) Note that the explanation for part e) will use the Isosceles Triangle Theorem from Question 4: since the triangles $ABE$ and $CBE$ are congruent, $AE = CE$, so $BE$ is a median of $\triangle ABC$, and therefore is a perpendicular bisector of $AC$. (Another way to look at it is to say that angles $AEB$ and $CEB$ are equal supplementary angles, so they have to be right angles, using the same logic as in the Isosceles Triangle Theorem, as in Question 4.) When students have finished, point out that they have now proved that the diagonals of a rhombus are perpendicular bisectors of each other.

**Bonus** Use $\triangle ABE$ and $\triangle ADE$ to prove that $AC$ is a perpendicular bisector of $BD$.

Have students work individually through Question 6 on Workbook page 95.
Review alternate angles. Remind students what alternate angles are, and that when lines are parallel, the alternate angles are equal. Then draw a parallelogram $ABCD$ with a diagonal $AC$ and have students find all pairs of alternate equal angles in that picture, for each pair of parallel sides. Then have students finish the worksheets.

### ACTIVITIES 1–2

1. Students will each need two dice of different colours. If two dice are unavailable, students can roll one die twice, taking the first roll as the result on the red die ($r$) and the second roll as the result on the blue die ($b$). Students roll the dice and draw an angle with the degree measure equal to $25r + b$. **(EXAMPLE:** If you roll 2 on the red die and 6 on the blue die, you draw an angle of $56^\circ$.) Then they bisect the angle.

   **ASK:** What is the largest angle you can create with this rule? ($156^\circ$) What is the smallest angle? ($26^\circ$) How many ways can you create an angle that is a multiple of 5? (6 ways: $25r$ is a multiple of 5 for any $r$, but to get the whole sum to divide by 5 you need $b$ to be 5.)

2. **Paper folding**

   Make an angle using folds. To bisect your angle, fold the paper through the point of intersection so that the creases that form the two arms of the angle fall one on top of the other.

   Students can also use paper folding to check their answers in Question 2 on Workbook page 94.

### Extensions

1. You are given an angle and a transparent mirror (a Mira). How can you find the angle bisector? **HINT:** An angle bisector is a line of symmetry. **(ANSWER:** Place the Mira across the angle so that it passes through the vertex of the angle. Rotate the Mira around the vertex of the angle so that the reflection of one arm coincides with the other arm (behind the Mira). Use the Mira as a ruler to draw the bisector.)

2. Draw a parallelogram. Bisect each angle of the parallelogram and extend each bisector so that it intersects two other bisectors. What geometric shape can you see in the middle of the parallelogram? (a rectangle) Use the sum of the angles in the shaded triangle in the picture to confirm that the shape in the middle is indeed a rectangle. **(ANSWER:** The acute angles of the triangle are both half of the angles of a parallelogram. The adjacent angles of a parallelogram add to $180^\circ$, so their halves add to $90^\circ$. By the sum of the angles in a triangle, the third angle is a right angle. Since there are three other triangles like the shaded triangle in the parallelogram, the shape in the middle has four right angles and must be a rectangle.)
3. Draw a slant line segment $AB$ and a point $C$ not on $AB$. Reflect point $C$ in the line $AB$ (by drawing a line through $C$ perpendicular to $AB$). Call the image point $C'$.

a) Look at $AB$ and $CC'$. Do you see a perpendicular bisector anywhere? If so, where?

b) Draw $AC$, $AC'$, $BC$, and $BC'$. Is triangle $ABC$ congruent to triangle $ABC'$? How do you know?
Review perpendicular bisectors and circles. Remind students what perpendicular bisectors are, and how to construct them using a ruler and a protractor. As well, review with students how to construct circles and arcs using a compass, and the basic geometric properties of a circle (all points on a circle are at the same distance from its centre, and when we want to construct several points at the same distance from a given point, we use a circle). Remind students how this property of a circle is used in Geometer’s Sketchpad to construct line segments of given length (see Lesson G8-15).

**Introduce the term equidistant.** Write *equidistant* on the board. Ask students to identify the parts of the word (equi, distant), and to look for other words that start with “equ.” (equilateral, equal, equation, equality, etc.) What do you think “equ” means when it’s at the beginning of a word? (equal, the same) What should equidistant mean? (the same distance) For example, points on a circle are equidistant—or the same distance—from its centre.

**Constructing perpendicular bisectors using a compass and a straightedge.** Present the following problem:

a) Draw a line segment $PQ$. Set a compass to a width that is more than half the line segment $PQ$. Construct an arc centred at $P$ and an arc centred at $Q$ without changing the settings of the compass. Extend the arcs so that they intersect, and label the intersection point $R$. What type of triangle is $\triangle PQR$? (isosceles)
b) How can you add a point S to the diagram so that \( PR = QR = PS = QS \)?

(PROMPT: Extend one of the arcs to the other side.)

Ask students to join points \( R \) and \( S \) with a line. What special properties does this line appear to have? Why could that be? Prompt students to draw line segments that connect \( R \) and \( S \) to \( P \) and \( Q \). Ask them to mark the equal line segments in the picture. What type of quadrilateral is \( PRQS \)? (rhombus) \( PQ \) and \( RS \) are the diagonals of this rhombus. What do we know about the diagonals of a rhombus? (they are perpendicular bisectors of each other)

Point out to students that they have constructed a perpendicular bisector to the line segment \( PQ \). Ask them to repeat the construction using the same line segment \( PQ \), but a different setting for the compass. **ASK:** Did you get the same perpendicular bisector? (yes) Would the same construction produce the same perpendicular bisector for any other setting of the compass? (yes)

**Review congruence rules.** Draw several pairs of triangles as in the margin and ask students which congruence rule will help them to explain why the triangles are congruent. As well, you might ask students to sketch two triangles (sharing a side) that could be congruent by the following congruence rule: a) SAS, b) SSS, c) ASA.

After that have students do the Investigations on Workbook page 97. Through the Investigations, students will learn that a point is on a perpendicular bisector of \( AB \) if and only if it is equidistant from \( A \) and \( B \).

Write on the board:

Point \( E \) is on \( CD \), the perpendicular bisector of \( AB \)

Point \( E \) is equidistant from \( A \) and \( B \)

Which statement follows from the other? Ask students to signal which direction they proved in Investigation 1 (left to right). Repeat with Investigation 2. (right to left). Students can point their arms or fingers in the right direction.

**Perpendicular bisectors to line segments between points on a circle meet at the centre of the circle.** Have students do Activity 1 below. Then have them draw the radii to points \( A \), \( B \), and \( C \) on the circles they used in the Activity, and look for pairs of congruent triangles that these lines and the folds create. Ask students to pick a pair of congruent triangles and explain why they are congruent. Which congruence rule are they using? (SAS)

Draw a circle on the board, but do not mark a centre. Ask students how they could find the centre of that circle. (PROMPT: Draw a triangle as in Activity 1 and SAY: If a circle was a paper circle, you could fold it, as you did in the Activity. What did you create when you folded the circle? (a perpendicular bisector of one of the sides of the triangle)) So you can find the centre of a circle by drawing a triangle with vertices on the circle and constructing perpendicular bisectors of its sides. **ASK:** Do you need to find all three perpendicular bisectors? (no, two is enough) Why? (because all three perpendicular bisectors of sides will meet at the same point, but only two are needed to find that point) For practice, students can find the centres
of some circles on **BLM Circles** by drawing triangles with vertices on the circle and finding perpendicular bisectors of the sides, and then check the answer by folding.

**Circumcircles.** Tell students that it is easy to draw a triangle with vertices on a given circle, but the opposite problem isn’t so easy. **ASK:** If I am given a triangle, how can I find a circle touching all three vertices? Explain that such a circle is called a **circumcircle.** Ask students to sketch what the triangle and a circle look like. How could they find the centre of the circle? (by finding the intersection point of perpendicular bisectors of the three sides) How should you set the width of the compass after you find the centre? (put the point on the centre and the pencil on any vertex of the triangle) Finally, have students construct circles through three given points. (**PROMPT:** How can we turn this problem into the previous one?)

**ACTIVITIES 1–2**

1. **Paper folding and circles**

   Give each student a circle (you can use **BLM Circles**) and ask students to draw and label a scalene triangle on their circle such that the vertices of the triangle fall on the circumference of the circle.

   a) Fold the circle in half so that A meets B.

      Look at the line that the crease in your fold makes. Is it a bisector of angle C? (no) Is it a perpendicular bisector of line segment AB? (yes)

   b) Fold the circle in half again, this time making A meet C. What two properties will the crease fold have? (perpendicular bisector of AC and diameter of the circle) **PROMPT:** When you fold the circle exactly in half, what does the fold form in the circle?

   c) Repeat, making B meet C. At what point in the circle will all three perpendicular bisectors meet? How do you know? (the centre, because all three perpendicular bisectors are diameters.)

2. **Arcs and circles in Geometer’s Sketchpad.** Have students work through the following steps:

   a) Draw a line segment AB and measure its length.

   b) Mark a point C not on AB.

   c) Using the arrow tool, select the points in order A, C, B and construct an arc through three points using the Construct menu option.

   d) Move the points around. How can you make the arc look like a circle? (move the two endpoints really close together)
Review with students how to construct a perpendicular bisector to a line segment (find a midpoint first, and construct a perpendicular line through the midpoint). Then have them construct circles through three points using the perpendicular bisectors and the circle tool.

**Extensions**

1. Using Geometer’s Sketchpad, construct three points $A$, $B$, $C$ on the same line. Try to construct an arc through these points. What happens? (The menu option of constructing an arc through three points on the same line does not exist.) Draw perpendicular bisectors to $AB$ and $BC$. What do you notice? (they are parallel) Explain why it is impossible to construct a circle through three points.

2. **The incentre of a triangle**
   a) Draw a triangle.
   b) Draw the angle bisectors for all the angles of the triangle. What do you notice? (The bisectors all pass through the same point, called the **incentre** of the triangle.)
   c) Label the point where the bisectors intersect $O$. Draw perpendiculars through $O$ to each of the sides of the triangle. Measure the distances along the perpendicular lines from $O$ to the sides of the triangle. What do you notice? (the distances are all the same)
   d) Draw a circle with centre $O$ and radius equal to the distance from $O$ to any side of the triangle. In how many points does the circle meet the triangle?
3. Explain why the circumcentre of a right triangle is always in the middle of the hypotenuse by using the steps below:

**Step 1:** Draw a right triangle $\triangle KLM$ with right angle $K$. Draw a triangle $\triangle LMN$ congruent to $\triangle KLM$ so that both triangles together form a rectangle $KLNM$.

**Step 2:** Draw the second diagonal $KN$. Label the intersection point of the diagonals $O$. What do you know about the diagonals of a rectangle? (they bisect each other and are equal) What can you say about the distances $KO, LO, MO, NO$? ($KO = LO = MO = NO$)

**Step 3:** Is it possible to draw a circle through $KLNM$? Where is its centre? ($O$) *(PROMPT: On grid paper, draw a circle of radius 5 clearly marking the centre, and draw a rectangle inscribed in it. Draw the diagonals in the rectangle. What do you notice? (diagonals intersect at the centre of the circle) Check your conjecture on another circle and rectangle.)* What is its radius? ($KO$) Construct the circle.

**Step 4:** Look at the initial triangle and the circle you constructed. Is the circle the circumcircle for the triangle? (yes) Where is the circumcentre? At the middle of the hypotenuse, just as suggested!

4. **Constructing perpendicular and parallel lines using a compass and a straightedge.**

   **Perpendicular line:**
   a) Draw a line and a point $A$ not on the line.
   b) Draw an arc centred at $A$ so that it intersects the line in two points. Label the points $B$ and $C$.
   c) Construct a perpendicular bisector of $BC$. Does it pass through $A$? Explain why this happens.

   **Parallel line:**
   d) Draw a line and mark a point $P$ not on the line.
   e) Mark any two points on the line and label them $Q$ and $R$.
   f) Construct a perpendicular bisector to $QR$. Label it $m$.
   g) Using steps a) to c), construct a line $n$ through $P$ perpendicular to $m$.
   h) Explain why the line $n$ you constructed is parallel to the line $QR$. 
Produce similar shapes with a flashlight. Start the lesson with an experiment: you will need a darkened room with a single source of light, such as a small flashlight. Show your students several paper shapes (you could use triangles, rectangles, parallelograms, or trapezoids). Hold a shape (say, a right-angled scalene triangle) parallel to the floor and hold the light above it, to create a clear, sharp shadow on the floor. Have students trace the shadow with masking tape. (NOTE: To produce a clear shadow, shapes should be larger than the source of light.)

Let your students compare the shape and the shadow. ASK: What is the shape of the shadow? Are the triangles of the same kind? Invite volunteers to measure the sides and the angles of both triangles and record them in a table. If one side of the triangle on the floor is, say, two times larger than the side of the paper triangle it is the shadow of, what happens with the other sides? (If you hold the triangle parallel to the floor, the other sides should also be twice as large as the sides of the paper triangle they are a shadow of.) What do students notice about the angle measures in both triangles? (they are the same) Repeat the exercise holding the triangle at a different height.

Define similar shapes. Explain to students that similar shapes have the same shape, but not necessarily the same size. ASK: What are shapes that have the same size and shape called? (congruent) When we talked about congruence, what did we say about the sides and the angles of the shapes? (sides are equal, and angles are equal) Remind students about the pentagons they saw in Lesson G8-13 (see BLM Two Pentagons), the ones with equal sides and equal angles that did not have the same shape. Show the pentagons again if possible. What was wrong with them? (the order of the equal sides was different) Point out that in similarity we also need to take care of the order. What term did we use to say which sides and which
angles should be compared? (corresponding sides/angles) So two shapes are similar if their corresponding angles are equal and their corresponding sides are proportional. In the examples above, the side on the floor and the side it is the shadow of are corresponding sides.

**Review the meaning of proportion.** Remind students that proportion is equality between ratios. In case of proportional sides this means that the ratios of the corresponding sides are equal. Draw two right scalene triangles, one having sides twice as large as the other, and label the vertices as shown. Explain that in these two triangles, the proportion between the sides looks like this:

\[
AB : DE = AC : DF = BC : EF = 2 : 1
\]

What does this tell us about the ratio of the sides \(AB : DE\)? \((AB : DE = 2 : 1)\) Which side is larger? How many times larger? \((AB\) is twice as large as \(DE)\) Repeat with the other pairs of sides. Point out that the longest side of \(\triangle ABC\) should correspond to the longest side of \(\triangle DEF\). **ASK:** Why does this make sense? (If you multiply three numbers by 2, you do not change which number is the largest.) What if I multiply all three numbers by 3? by 1.7? by \(\pi\)? No matter what you multiply the numbers by, the largest number will stay the largest, and the longest side should correspond to the longest side.

**Checking for similarity.** **ASK:** Which angles do we need to check to make sure these triangles are similar? (all of them) Point out that it makes sense to start with the largest angles. If the shapes are similar, the largest angle in one should be equal to the largest angle in the other. If the largest angles are not equal, the shapes are not similar. If the largest angles are equal, we continue to check the other angles in the order they appear on the shape. Have volunteers check the angles in triangles \(\triangle ABC\) and \(\triangle DEF\) in clockwise order, starting at the largest angle. Students will find that the angles are equal, and so the triangles \(\triangle ABC\) and \(\triangle DEF\) are similar. Then **ASK:** Suppose I checked that \(\angle A = \angle D\), and then I checked angles \(\angle B\) and \(\angle F\), because they are both adjacent to the largest angles. I see that \(\angle B \neq \angle F\). Does this mean that the triangles are not similar? (no) What do I have to do? (go in the other direction on one of the triangles and check again)

Have students check whether the triangles below are similar. Does it make sense to go around both shapes in clockwise order in this case? (no)

![Diagram](image)

Ask students to copy the following pairs of triangles to grid paper and to say which sides and angles they are going to compare in order to check if the triangles are similar. Then ask them to compare the sides and the angles. Students should compare the shortest sides, the middle sides, and the longest sides in pairs, and the angles between them.
Repeat with rectangles $4 \times 8$ and $6 \times 12$, then $5 \times 6$ and $6 \times 7$. **ASK:** Do we need to check the equality between the angles for the rectangles? Why not? (all the angles in the rectangles are $90^\circ$, so they are already equal)

Repeat with the shapes below.

**Emphasize that one way to prove that two shapes not similar is to show that a pair of corresponding sides in the two shapes is not in the same ratio as another corresponding pair of sides.** These shapes are not similar because the side marked with an X in B is 3 times longer than the corresponding side in A, whereas the side marked with a circle in B is only 2 times longer than the corresponding side in A (it should be 3 times longer).

**Which sides to compare?** Present the two rectangles at left and have students decide which ratios of sides they should check to determine if the rectangles are similar:

- a) $\frac{8}{12}$ and $\frac{16}{10}$
- b) $\frac{8}{16}$ and $\frac{10}{12}$
- c) $\frac{8}{12}$ and $\frac{10}{16}$
- d) $\frac{8}{10}$ and $\frac{16}{12}$

Ask students to explain their choice, and then check whether the rectangles are similar. (**ANSWER:** If the shapes are similar, the longest side will correspond to the longest side, and the shortest side will correspond to the shortest side. So the ratios we need to compare are $\frac{8}{12} = \frac{2}{3} = \frac{10}{15} = \frac{10}{16}$, so the rectangles are not similar.)

**Practice checking for similarity.**

Check whether these triangles are similar. How did you know which pairs of sides to check?

a) 

b)
**ANSWERS:** a) similar; b) not similar; \( \frac{3}{9} = \frac{4}{12} \neq \frac{6}{20} \), taking the ratio of the longest side to the longest side, the shortest to the shortest, and the ratio of the medium sides.

**Bonus** Are these shapes similar? (yes)

**Using similarity to find missing sides.** Draw two rectangles and tell the students that they are similar. Give the dimensions of two sides of one rectangle, and one side of the other, and have students find the length of the other side using proportions. **EXAMPLES:**

a) Width of A: 3 cm, length of A: 7 cm. Width of B: 9 cm. What is the length of B?

b) Width of A: 3 cm, length of A: 8 cm. Length of B: 16 cm. What is the width of B?

c) Width of A: 6 cm, length of A: 8 cm. Width of B: 3 cm. What is the length of B?

d) Width of A: 4 cm, length of A: 12 cm. Length of B: 3 cm. What is the width of B?

Repeat with triangles. This time mark the missing sides with variables, and have students write the ratios between the corresponding sides. Have students find the missing sides. **EXAMPLE:**

\[
x : 5 = 60 : 12 = y : 13, \text{ so } x = 25 \text{ and } y = 65.
\]

**Finding missing sides when the order of sides is not given.** Present the following problem:

Sarah built two similar triangles with toothpicks. One has sides 3 toothpicks, 4 toothpicks, and 6 toothpicks, and the other has one side of 15 toothpicks and another side of 30 toothpicks. What is the length of the third side of the second triangle?

Ask students to make a sketch and to label the triangles. Which sides are given in the second triangle? Could they be the smallest side and the middle side? Ask students to write the proportion, assuming that this is the correspondence. Are the proportions the same? (no, 3 : 15 is not 4 : 30) Which sides are given? (the smallest and the largest sides, because 3 : 15 = 6 : 30) Have students find the third side. (20 toothpicks)
Have students solve a harder problem:

Sarah built two similar triangles. One has sides 9 cm, 12 cm, and 16 cm, and the other has one side of 36 cm and another side of 48 cm. Steven said the third side of the second triangle is 64 cm long. Sarah measured the third side and said that the length of the third side is a whole number of centimetres, but Steven is wrong. How can that be?

**ANSWER:** Sarah did not say which side corresponds to which. Steven thought that the sides given in the second triangle corresponded to the first two sides of the first triangle, in which case the lengths are multiplied by four and the length of the third side has to be $16 \text{ cm} \times 4 = 64 \text{ cm}$. However, if the sides in the second triangle correspond to the second and the third sides in the first, the lengths are multiplied by three, and the missing side, corresponding to the first side, is $9 \text{ cm} \times 3 = 27 \text{ cm}$.

**Similarity in real life.** Ask your students to find examples of similar shapes in real life. **EXAMPLES:** shades produced by finger bunnies on a wall, shapes projected by an overhead projector (shapes on transparency are small, shapes on the screen are large), toy dinosaurs that grow when you put them in water, photos developed from negatives, scale drawings.

### ACTIVITIES 1–3

1. **Students can measure various dimensions of a dry toy that grows in water** (length, circumference of various parts, and so on) and compare them with the dimensions of the grown wet toy to check whether the toys are similar.

2. **Give students a set of paper rectangles that can be divided in three sets of at least three similar rectangles.** (There are three such sets on BLM Rectangles.) Ask students to draw diagonals on all of the rectangles. Then have students measure the sides of the rectangles and divide them into sets, so that each set contains only similar rectangles. Ask students to order the rectangles in each set from smallest to largest and to arrange the rectangles in each set into a pile, the largest rectangle on the bottom and the smallest on the top, so that the rectangles in each pile share a common vertex.

   What do students notice about the diagonals of the similar rectangles? (the diagonals coincide; the opposite vertex of each smaller rectangle sits on the diagonal of a larger rectangle) Does this hold for diagonals of non-similar rectangles? (No. Students will have to move the rectangles around to see this. They can mark the rectangles in each set with a symbol or letter in order to regroup the similar rectangles afterwards.) A diagonal in a rectangle divides the right angle into two angles. What can your students say about these angles in similar rectangles? (The angles between the diagonals and the corresponding sides of similar rectangles are equal.)

3. **Use newsprint or scrap paper to create pairs of shapes,** such as rectangles, triangles, and parallelograms, some similar and some
Extension

Rita wants to estimate the height of a tree growing in the middle of a park. Rita uses a pencil and a measuring tape. Rita holds the pencil vertically in her outstretched hand so that the point of the pencil is level with the top of the tree. Rita moves far enough away from the tree so that the tree appears smaller than the pencil.

Rita holds the pencil so that her fingers are level with the bottom of the tree and the point of the pencil is still level with the top of the tree. She compares the size of the pencil and the tree. For example, the tree appears to be as long as three quarters of the pencil. Rita measures the pencil and finds that three quarters of the pencil is 15 cm. She also measures her arm and finds that her arm is 45 cm. Rita says that the image of the tree is three times shorter than her arm. After that, Rita measures the distance between herself and the tree with giant steps. She says that the tree is three times shorter than the distance to it. So she divides the distance by 3 and obtains the height of the tree in giant steps. (A giant step is close to 1 m.) Can you explain why Rita’s method works?

EXPLANATION: Rita uses two similar right-angled triangles. The smaller triangle has Rita’s eye, her fingers, and the point of the pencil as vertices. The larger triangle’s vertices are Rita’s eye, and the bottom and the top of the tree. The triangles are similar because they have the same angles (see the picture). The ratio between the corresponding sides is the same in both triangles.

As an activity, students can use Rita’s method to measure objects outside the class.
Review similar shapes. Remind the students that similar shapes have equal corresponding angles, and proportionate corresponding sides. To make sure that two shapes are similar we need to check all the angles and all the sides of the shapes in question, in order. Explain that the ratio between the corresponding sides is often expressed as a number, and is called scale factor. For example, if in two triangles we have $A'B' : AB = C'B' : CB = A'C' : AC = 3 : 1$, we can write the proportions in fractional form, $\frac{A'B'}{AB} = \frac{C'B'}{CB} = \frac{A'C'}{AC} = \frac{3}{1}$, and say that the scale factor is 3.

This means that $A'B' = 3 \times AB$, $C'B' = 3 \times CB$, and $A'C' = 3 \times AC$.

Remind students that when investigated congruent triangles they found shortcuts, which were called congruence rules. These rules show that it is enough to check only some pairs of sides and angles to determine whether two triangles are congruent. Explain that today they will be looking for shortcuts that will help them to check whether two triangles are similar. These shortcuts will be called similarity rules.

Investigating similarity rules. Review constructing triangles with given side lengths with a compass and a ruler. Students can then investigate similarity rules on paper only (see the Investigation on Workbook pages 101–102), or both on paper and on computer (see Activities 1 and 2 below—Activity 1 is preparation for Activity 2). If you are going to use the Activities, review constructing triangles given fixed side lengths and fixed angles using dynamic geometry software. The parts of BLM Similarity Rules Using Geometer's Sketchpad match the exercises done on paper.
in the Investigation in the Workbook; you can choose which parts students do on paper, on computer, or both. For example, students could investigate the SSS similarity rule using Geometer’s Sketchpad and then do the Investigation in the workbook, so that SSS is covered twice. Make sure students cover all of the similarity rules. Then have students work through Questions 1 through 4 on Workbook pages 102–103.

Similar triangles and parallel lines. Review the properties of angles created when a line intersects two other lines. (Corresponding and alternate angles at parallel lines are equal, and if they are equal, the lines are parallel. Opposite angles are equal.) Then work through Question 5 on Workbook page 103 as a class. Point out to the students that they have just learned an important fact: when you have an angle and you draw parallel lines that intersect both sides of the angle, these parallel lines create similar triangles. We can use this fact in problems such as this:

An engineer plans a bridge that will be 120 m long and will be shaped like two right triangles (as shown), with vertical poles to support the top beams. The tallest pole is in the middle of the bridge, and it is 10 m tall. The rest of the poles are at equal distances from each other. What are the heights of the rest of the poles?

Ask students to sketch the bridge and to mark all the parallel lines, right angles, and equal line segments that they see. Then ask students to identify triangles similar to $\triangle ABC$ in the sketch. How do we know that they are similar? (because the poles are all vertical, so they are parallel lines intersecting the sides of $\angle BAC$) Have students find the length of the sides $AG$, $AH$, $AI$, and $AC$. Which side in $\triangle ADG$ corresponds to the side $AC$ in $\triangle ABC$? (AG) What is the scale factor between these two triangles?

$AC : AG = 60 : 15 = 4 : 1$, or $\frac{AC}{AG} = \frac{60}{15} = \frac{4}{1} = 4$. Ask students to write the solution in both forms and express the ratio as a number.) Ask students to write the ratio between the sides $BC$ and $DG$, and find the height of the shortest pole. (2.5 m) Repeat with the other triangles to find the height of the rest of the poles.

Have students complete the worksheet.

**ACTIVITIES 1–2**

1. Review with students how to draw a line segment of fixed length using dynamic geometry software, such as Geometer’s Sketchpad. Then teach students how to draw a line segment that is, say, 3 times longer than the initial segment. Have them practise the construction:
   a) Draw a line segment $AB$. Measure its length.
   b) Draw a line segment that is twice as long as $AB$.
   c) Draw a line segment that is three times as long as $AB$.
   d) Draw a line segment that is 2.5 times as long as $AB$.
Extension

Dividing a segment into a number of equal line segments using a compass and a straightedge.

a) Look at the following diagram. $BC \parallel DE \parallel FG$. What can you tell about $\triangle AFG$, $\triangle ADE$, and $\triangle ABC$? (they are similar)

b) $AB = BD = DF$.

i) Express $AD$ and $AF$ in terms of $AB$.

ii) Find the scale factor between $\triangle AFG$ and $\triangle ABC$ and the scale factor between $\triangle ADE$ and $\triangle ABC$.

iii) Using similarity, find the ratios $AG : AC$ and $AE : AC$. Express $AE$ and $AG$ in terms of $AC$.

iv) What can you say about $AC$, $CE$, and $EG$? Explain.

ANSWERS:

i) $AD = 2AB$, $AF = 3AB$.

ii) $AD : AB = 2$, $AF : AB = 3$.

iii) $\triangle ABC$ and $\triangle ADE$ are similar with scale factor 2, so $AE : AC = 2$, and $AE = 2AC$.

$\triangle ABC$ and $\triangle AFG$ are similar with scale factor 3, so $AG : AC = 3$, and $AG = 3AC$.

iv) Since $AE = 2AC$, and $AE = AC + CE$, then $CE = AC$.

Since $AG = 3AC$, and $AG = AE + EG = 2AC + EG$, then $EG = AC$.

c) Draw a line that is about the same width as a regular sheet of paper.

Mark a line segment $AB$ at one end of your line, so that it is about a quarter of the line you drew.

Set your compass to the width of $AB$. Use the compass to construct two more line segments along the same line, so that $AB = BC = CD$.

Without using a ruler, try to set your compass to the width that is exactly $1/3$ of the width of $AB$. Try to construct three line segments along the line, so that $AE = EF = FB$. 
NOTE: Students will need to construct parallel lines using a compass and a straightedge. See Extension 5 of Lesson G8-33 for instructions on how to do that.

What is easier to do using a compass and a straightedge only: to construct a line segment that is three times as long as a given line segment, or to divide a given line segment into three equal parts?

d) In b) you proved that if $AF$ is divided into 3 equal parts, then $AG$ is also divided into 3 equal parts. This gives us a method to divide any line segment into three equal parts using a straightedge and a compass:

Given a line segment $AG$, draw a ray at any angle starting at $A$.

Set your compass to any width you like and mark three equal segments $AB = BD = DF$ along the ray you drew.

Construct a line segment $FG$.

Construct lines parallel to $FG$ through points $B$ and $D$.

The lines intersect $AG$ at points $C$ and $E$.

$AC = CE = EG$ as in b).
Review similar shapes. Corresponding sides of similar shapes have the same ratio, and this ratio is called the scale factor. Review also the formulas for the area of a parallelogram and the area of a triangle. You can distribute BLM Area of Parallelogram and Triangle for reference.

When shapes are similar with scale factor $s$, the ratio of heights is also $s$. Draw two triangles, as shown, and ask students what can they say about triangles $ABC$ and $DFE$. (they are similar) How do students know? (SSS similarity rule) What is the scale factor? ($DF : AB = 2$)

Ask students to sketch the triangles in their notebooks, to mark the equal angles in the triangles, and to draw the heights $AG$ (from $A$ to $BC$) and $DH$ (from $D$ to $EF$). **ASK:** What will the ratio of the heights be? Have students explain their thinking. Lead them to compare $\triangle ACB$ and $\triangle DHE$. These triangles have two equal pairs of angles, $\angle C = \angle E$ and $\angle G = \angle H = 90^\circ$, so by AA similarity rule, the triangles are similar. Since the triangles are similar, their corresponding sides have the same ratio, so


**ASK:** If we had a pair of similar triangles with different side lengths but with the same scale factor, 2, what do you think the ratio of the heights would be? Did we use the exact side lengths or just the ratio between the corresponding sides? (just the ratio of the sides) What if we had a different scale factor? Students can use dynamic geometry software, such as Geometer's Sketchpad, to check the conjecture: construct a pair of similar
triangles with ratio \( s \) between the corresponding sides, construct a pair of corresponding heights and measure them, and find the ratio of the heights; modify the triangles and the scale factors to check that the ratio of the heights is always the same as the ratio between the sides. If files from the previous lesson were kept, students could use them for this exercise. Have students save the files for later use.

**The ratio between the areas.** Tell students that the height \( AG = 4.1 \) cm, and have them find \( DH \). (6.2 cm) Then ask them to find the area of both triangles and the ratio between the areas. (4) Write on the board:

\[
4 = 2 + 2 \quad 4 = 2 \times 2 \quad 4 = 2^2
\]

**ASK:** Suppose I have two similar triangles, with scale factor 3. What will the ratio between the areas be? Write on the board:

\[
3 + 3 = 6 \quad 3 \times 3 = 9 \quad 3^3 = 27
\]

**SAY:** These are some ways of generalizing how to get 4 from 2, but I get three different answers. Which way do you think gives the ratio of the areas? Students can predict and try to explain their prediction, then check it. If students worked on computers in the previous part of the lesson, they can now find the ratio between the areas and see that the ratio of the areas is the scale factor squared.

Finally, have students look at the formula for the area of triangles. If we multiply both the base and the height by 3 (the scale factor), then the area is multiplied by \( 3^2 \): \( 3 \times 3 = 9 \). Therefore, the ratio between the areas should be the square of the scale factors. Repeat the argument with scale factor 4, then ask students to write down the argument for scale factor 5 (writing 5 in pencil), and then replace it in the argument with \( s \).

Draw two similar polygons on the board and divide them into triangles in the same way. Tell students that the ratio between the sides of the polygons is 5 (the sides of the large polygon are 5 times larger than the sides of the small polygon). What will the ratio between the areas of the polygons be? Ask students to explain their reasoning. Summarize that the ratio of the areas is the square of the scale factor for any polygon, because any polygon can be divided into triangles.

**PROCESS ASSESSMENT**

[PS], 8m1

**PROCESS EXPECTATION**

Connecting

Point out that the ratio rule for any polygon actually follows from the rule for triangles using the distributive law. For example, for the quadrilaterals at left and scale factor 5 we have:

\[
\text{Area of } A' = \text{Area of } A \times 25 \quad \text{Area of } B' = \text{Area of } B \times 25
\]

\[
\text{Area of large quadrilateral} = \text{Area of } A' + \text{Area of } B' \\
= \text{Area of } A \times 25 + \text{Area of } B \times 25 \\
= (\text{Area of } A + \text{Area of } B) \times 25 \\
= \text{Area of small quadrilateral} \times 25
\]

**Using the ratio between the areas.** Review solving proportions.

Present the following problem:
\( \triangle ABC \) is similar to \( \triangle DEF \), with \( AB : DE = BC : EF = AC : DF = 5 \). The area of \( \triangle ABC = 325 \text{ cm}^2 \). What is the area of \( \triangle DEF \)?

**ASK:** Which triangle is larger: \( \triangle ABC \) or \( \triangle DEF \)? \( (\triangle ABC) \) How do you know? (scale factor is more than 1) What will the ratio between the areas be? \( (25) \) Ask students to write the ratio between the areas:

\[
\text{Area of } \triangle ABC : \text{Area of } \triangle DEF = 25 : 1
\]

Have students circle the value they do not know. \( (\text{area of } \triangle DEF) \) Then ask them to rewrite the ratio using the information they know—the area of \( \triangle ABC \). \( (325 \text{ cm}^2 : \text{Area of } \triangle DEF = 25 : 1) \) Have students solve the proportion to find the area of \( \triangle DEF \). Students will need plenty of practice to become comfortable with this type of question. When working with decimals, have students estimate the answers before doing any calculations, regardless of whether or not they use a calculator.

**EXTRA PRACTICE:**

a) \( \triangle ABC \) is similar to \( \triangle DEF \), with \( AB : DE = BC : EF = AC : DF = 2 : 1 \). The area of \( \triangle ABC = 3.6 \text{ cm}^2 \). What is the area of \( \triangle DEF \)? \( (3.6 \text{ cm}^2 \div 4 = 0.9 \text{ cm}^2) \)

b) \( \triangle ABC \) is similar to \( \triangle DEF \), with \( AB : DE = BC : EF = AC : DF = 4 : 1 \). The area of \( \triangle DEF = 3.5 \text{ cm}^2 \). What is the area of \( \triangle ABC \)? \( (56 \text{ cm}^2) \)

c) \( \triangle ABC \) is similar to \( \triangle DEF \), with \( AB : DE = BC : EF = AC : DF = 15 \). The area of \( \triangle ABC = 5 \text{ m}^2 \). What is the area of \( \triangle DEF \) in \( \text{cm}^2 \)? \( (\text{about } 222.2 \text{ cm}^2 \). You will need to review conversion between units of area to get to this answer. Otherwise, the answer will be in \( \text{m}^2 \).)

d) \( \triangle ABC \) is similar to \( \triangle DEF \), with \( AB : DE = BC : EF = AC : DF = 1.5 \). The area of \( \triangle DEF = 3.5 \text{ m}^2 \). What is the area of \( \triangle ABC \)? \( (7.875 \text{ m}^2) \)

e) \( \triangle ABC \) is similar to \( \triangle DEF \), with \( AB : DE = BC : EF = AC : DF = 0.4 \). The area of \( \triangle ABC = 16 \text{ cm}^2 \). What is the area of \( \triangle DEF \)? \( (100 \text{ cm}^2) \)

**Ratio between perimeters.** Remind students that the perimeter of a shape is the sum of all the side lengths, and ask them what they think will be the ratio between the perimeters of similar shapes. Have students explain their thinking. Then have students do Investigation 2 on Workbook page 105. If dynamic geometry software is available, students can check the ratio between perimeters of similar triangles, using the same files they used in the previous part of the lesson.

**A height in a right triangle drawn from the right angle to the hypotenuse splits the right triangle into two similar triangles, and they are similar to the initial right triangle.** Draw a right triangle with a horizontal hypotenuse on the board, and draw the height as shown. Have students identify all the right triangles they see in the picture. Then tell them that the measure of \( \angle UTW \) is \( x \). What should the measure of the rest of the angles in the picture be? Then ask students to mark all the equal angles on the sketch. Finally, ask them to identify all pairs of similar triangles they see in the picture. How do they know the triangles are similar? \( (\text{AA similarity rule}) \)
Workbook Questions 4–9 on Workbook pages 105–106 combine the material learned to date, using the properties of parallel lines, similarity, right triangles, and the Pythagorean Theorem. You can use them for review.

Extension

a) Show that $\triangle ABC$ and $\triangle CAD$ are similar. Use the ratio between the medium sides of the triangles and the ratio between the shortest sides of the triangles. What similarity rule would you use?

b) What kind of shape is $ABCD$? How do you know?

**ANSWERS:**

a) $\triangle CAD$ is a right triangle, so by the Pythagorean Theorem, $AC = 156$ cm.

The ratio of the medium sides is:

\[
\frac{\text{medium side of } \triangle ABC}{\text{medium side of } \triangle CAD} = \frac{AC}{CD} = \frac{156}{144} = \frac{13}{12}
\]

The ratio of the shortest sides is:

\[
\frac{\text{short side of } \triangle ABC}{\text{short side of } \triangle CAD} = \frac{BC}{AD} = \frac{65}{60} = \frac{13}{12}
\]

and $\angle ADC = \angle BCA = 90^\circ$.

By the SAS similarity rule, $\triangle ABC$ and $\triangle ADC$ are similar.

b) Since $\triangle ABC$ and $\triangle CAD$ are similar, $\angle DCA = \angle CAB$.

$\angle DAC + \angle CAB = \angle DAC + \angle DCA = 90^\circ$, by the Sum of the Angles in a Triangle Theorem.

Therefore $AB$ and $CD$ are both perpendicular to $AD$, which means $AB \parallel CD$, and $ABCD$ is a trapezoid.
Protractors
Measuring and Drawing Angles and Triangles

Measuring an angle

*Step 1:* Place the origin of the protractor over the vertex of the angle.

*Step 2:* Rotate the protractor so the base line is exactly along one of the arms of the angle.

*Step 3:* Look at that arm of the angle and choose the scale that starts at 0°.

*Step 4:* Use that scale to find the measurement.

Drawing an angle

*Step 1:* Draw a line segment.

*Step 2:* Place the protractor with the origin on one endpoint. This point will be the vertex of the angle.

*Step 3:* Hold the protractor in place and mark a point at the angle measure you want.

*Step 4:* Draw a line from the vertex through the angle mark.

Drawing lines that intersect at an angle

*Step 1:* Draw a line. Mark a point P on the line.

*Step 2:* Draw an angle of the given measure using P as vertex.

*Step 3:* Extend the arms of your angle to form lines.

Drawing a triangle

*Step 1:* Sketch the triangle you want to draw.

*Step 2:* Use a ruler to draw one side of the triangle.

*Step 3:* Use a protractor to draw the angles at each end of this side. Extend the arms until they intersect.

*Step 2:* Erase any extra arm lengths.
Drawing Perpendicular Lines and Bisectors

Drawing a line segment perpendicular to $AB$ through point $P$

Using a set square

Using a protractor

Drawing the perpendicular bisector of line segment $AB$

Using a set square or protractor

Using a compass and a straightedge

Drawing Parallel Lines

Drawing a line parallel to \( AB \) through point \( P \)

Using a set square

**Step 1:** Line up one of the short sides of the set square with \( AB \).

**Step 2:** Use the set square and a straightedge to draw a perpendicular to \( AB \).

**Step 3:** Draw a line perpendicular to the new line that passes through \( P \).

**Step 4:** Erase the line you no longer need.

Using a protractor

**Step 1:** Line up the 90° line on the protractor with \( AB \). Use the straight side of the protractor to draw a line segment perpendicular to \( AB \).

**Step 2:** Line up the 90° line on the protractor with the line segment drawn in Step 1 and the straight side of the protractor with point \( P \). Draw a line parallel to \( AB \). Erase the first perpendicular you drew.

Using a compass and a straightedge

**Step 1:** Mark any two points \( A, B \) on the line. Construct the perpendicular bisector of \( AB \).

**Step 2:** Mark any two points \( C, D \) on the line you drew. Construct the perpendicular bisector of \( CD \). Label it \( m \). Line \( m \) is parallel to \( AB \).
Distance Between Parallel Lines

A. Measure the line segments with endpoints on the two parallel lines with a ruler. Write the lengths of the line segments on the picture.

B. Use a square corner to draw at least three perpendiculars from one parallel line to the other, as shown.

Measure the distance between the two parallel lines along the perpendiculars. What do you notice?

C. Explain why all the perpendiculars you drew in part B are parallel.

D. A parallelogram is a 4-sided polygon with opposite sides parallel. You can draw parallelograms by using anything with parallel sides, like a ruler. Place a ruler across both of the parallel lines and draw a line segment along each side of the ruler. Use this method to draw at least 3 parallelograms with different angles.

E. Measure the line segments you drew between the two given parallel lines in part D. What do you notice?

F. To measure the distance between two parallel lines, draw a line segment perpendicular to both lines and measure it. Does the distance between parallel lines depend on where you draw the perpendicular?
**Sum of the Angles in a Triangle (1)**

**INVESTIGATION** ▶ What is the sum of the angles in a triangle?

**A.** Circle the combinations of a 70° angle and another angle that will make a triangle. (Hint: Imagine the sides of the triangle extended. Will they ever intersect?)

Circle the combinations of a 50° angle and another angle that will make a triangle.

Circle the combinations of a 90° angle and another angle that will make a triangle.

---

**Make a prediction:**
To make a triangle, the total measures of any two angles must be **less than** ____°.

**B.** List the sum of the measures of the angles in each triangle.

---

What do you notice about the sums of the angles?

Do you think this result will be true for all triangles?

**Make a conjecture:** The sum of the three angles in any triangle will always be ____°.
C. Calculate the sum of the angles.

\[
\begin{align*}
70^\circ + 90^\circ + 20^\circ &= \_\_\_\_^\circ \\
57^\circ + 69^\circ + 54^\circ &= \_\_\_\_^\circ \\
41^\circ + 116^\circ + 23^\circ &= \_\_\_\_^\circ \\
24^\circ + 24^\circ + 132^\circ &= \_\_\_\_^\circ \\
\end{align*}
\]

What do you notice about the sums of the angles? ____________________________

D. Cut out a paper triangle and fold it as follows:

**Step 1:** Find the midpoints of the sides adjacent to the largest angle (measure or fold). Draw a line between the midpoints.

**Step 2:** Fold the triangle along the new line so that the top vertex meets the base of the triangle. You will get a trapezoid.

**Step 3:** Fold the other two vertices of the triangle so that they meet the top vertex.

The three vertices folded together add up to a straight angle.

What is the sum of the angles in a straight angle? _____°

So \( \angle A + \angle B + \angle C = \_\_\_\_\_\_ \)°

E. Could you fold the vertices of any triangle as you did in part D and get a straight angle? Do the results of the paper folding support your conjecture in part B? Explain.

F. In fact, it has been mathematically proven that...

**The sum of the angles in a triangle is _____°.**
Quadrilaterals (1)

1

2

3

4

5

6
Quadrilaterals (2)

7

8

9

10

11

12
# Sorting Quadrilaterals

<table>
<thead>
<tr>
<th>Properties of sides</th>
<th>Shapes with the property</th>
<th>Properties of angles</th>
<th>Shapes with the property</th>
</tr>
</thead>
<tbody>
<tr>
<td>No equal sides</td>
<td></td>
<td>No equal angles</td>
<td></td>
</tr>
<tr>
<td>1 pair of equal sides</td>
<td></td>
<td>1 pair of equal angles</td>
<td></td>
</tr>
<tr>
<td>2 pairs of equal sides</td>
<td></td>
<td>2 pairs of equal angles</td>
<td></td>
</tr>
<tr>
<td>Equal sides are adjacent</td>
<td></td>
<td>Equal angles are adjacent</td>
<td></td>
</tr>
<tr>
<td>Equal sides are opposite</td>
<td></td>
<td>Equal angles are opposite</td>
<td></td>
</tr>
<tr>
<td>4 equal sides</td>
<td></td>
<td>4 equal angles</td>
<td></td>
</tr>
<tr>
<td>1 pair of parallel sides</td>
<td></td>
<td>No pairs of angles add to 180°</td>
<td></td>
</tr>
<tr>
<td>2 pairs of parallel sides</td>
<td></td>
<td>2 pairs of angles add to 180°</td>
<td></td>
</tr>
<tr>
<td>Perpendicular diagonals</td>
<td></td>
<td>4 pairs of angles add to 180°</td>
<td></td>
</tr>
<tr>
<td>Equal diagonals</td>
<td></td>
<td>6 pairs of angles add to 180°</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Properties of sides</th>
<th>Shapes with the property</th>
<th>Properties of angles</th>
<th>Shapes with the property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perpendicular diagonals</td>
<td></td>
<td>Equal diagonals</td>
<td></td>
</tr>
<tr>
<td>One of the diagonals is also an angle bisector</td>
<td></td>
<td>Exactly one diagonal bisects the other</td>
<td></td>
</tr>
<tr>
<td>Both diagonals are angle bisectors</td>
<td></td>
<td>Both diagonals bisect each other</td>
<td></td>
</tr>
</tbody>
</table>
Triangles

Set 1

Set 2

Set 3
Triangles for Sorting
Two Pentagons

a) How many right angles does each pentagon have? _____

b) Cut the pentagons out. Fold the pentagons so that you can see that the remaining angles are all equal.

c) Does each side on pentagon A have a side of the same length on pentagon B? _____

d) Match each side from pentagon A with a side of the same length from pentagon B. Do not use the same side twice.

\[ KL = \quad LM = \quad MN = \quad NO = \quad OK = \]

e) If you place one pentagon on top of the other, do they match? _____

f) Are they of the same shape? _____

g) Can we say that these pentagons have the same sides and angles? _____

h) Are the pentagons congruent? _____

i) Explain why these pentagons have the same sides and angles but aren’t congruent.
Investigating Congruence

INVESTIGATION ► Do you have to check that all 3 pairs of sides and all 3 pairs of angles are equal to determine if two triangles are congruent?

3 sides

Conjecture: If two triangles have 3 pairs of equal sides, then the triangles are congruent.

Test the conjecture: Take 3 straws of different lengths. Make a triangle with them and trace it. See if you can make a different triangle with the same three straws. Try with a different set of three straws.

Are your triangles congruent? ______

Do you think this result will be true for all triangles? ______

2 sides and the angle between

Conjecture: If 2 sides and the angle between them in one triangle are equal to 2 sides and the angle between them in another triangle, the triangles are congruent.

Test the conjecture: Draw 3 triangles, each with one side 3 cm long, one side 5 cm long, and a 45° angle between these sides.

Are the triangles congruent? ______

Do you think this result will be true for all triangles? ______

Pick three measurements: \( a = \) ______ cm, \( b = \) ______ cm, \( C = \) ______°

Draw three triangles, each with one side \( a \) cm long, one side \( b \) cm long, and a \( C \)° angle between these sides. Are your triangles congruent?

2 angles and the side between them

Conjecture: If 2 angles and the side between them in one triangle are equal to 2 angles and the side between them in another triangle, the triangles are congruent.

Test this conjecture: Draw 3 triangles, each with a 60° angle, a 45° angle, and a side 5 cm long between these angles.

Are the triangles congruent? ______

Do you think this result will be true for all triangles? ______

Pick three measurements: \( a = \) ______ cm, \( B = \) ______°, \( C = \) ______°

Draw three triangles, each with a \( B \)° angle, a \( C \)° angle, and a side \( a \) cm long between these angles. Are your triangles congruent? ______
1. Using the polygon tool, construct a triangle, $ABC$. Measure the sides and the angles of your triangle.

2. a) Construct a point $D$. Using a command for constructing circles and the length of $AB$ as the radius, construct a circle with centre $D$. Construct line segment $DE = AB$. Hide the circle.
   
   b) Using a command for constructing circles and the length of $BC$ as the radius, construct a circle with centre $E$.
   
   c) Using a command for constructing circles and the length of $AC$ as the radius, construct a circle with centre $D$.
   
   d) Construct a point that is on both circles. Label it $F$. Use the polygon tool to construct triangle $DEF$. Hide the circles.

3. a) Which sides are equal in $\triangle ABC$ and $\triangle DEF$?
   
   $\quad = \quad$, $\quad = \quad$, $\quad = \quad$

   b) Measure the angles of $\triangle DEF$. What can you say about $\triangle ABC$ and $\triangle DEF$?

4. Try to move the vertices of $\triangle DEF$ around.
   
   a) How does your triangle change? Which transformations can you make: rotations, reflections, translations?

   b) Can you move $\triangle DEF$ onto $\triangle ABC$ to check whether they are congruent? ______
   
   Do you need to reflect the triangle to do that? ______

5. Move $\triangle DEF$ away from $\triangle ABC$. Try to move the vertices of $\triangle ABC$ around.
   
   a) Are the changes you can make in this triangle different from the changes you could make to $\triangle DEF$? Why?

   b) What happens to $\triangle DEF$ when you modify $\triangle ABC$? What can you say about the triangles $\triangle ABC$ and $\triangle DEF$?

6. Are three side lengths enough to determine a unique triangle? ______
**Congruence Rules on Geometer’s Sketchpad (2)**

1. Using the polygon tool, construct a triangle, \( \triangle ABC \). Measure the sides and the angles of your triangle.

2. a) Construct a point \( D \). Using a command for constructing circles and the length of \( AB \) as the radius, construct a circle with centre \( D \). Construct line segment \( DE = AB \). Hide the circle.
   
   b) Using a command for constructing circles and the length of \( BC \) as the radius, construct a circle with centre \( E \).
   
   c) Select \( E \) as a centre of rotation. Use a command in the Transformation menu to mark \( \triangle ABC \) as the angle of rotation. Rotate point \( D \) around \( E \) by the angle chosen. Construct a ray from \( E \) through the image of \( D \).
   
   d) Construct a point that is on the ray and the circle. Label it \( F \). Use the polygon tool to construct triangle \( \triangle DEF \). Hide the circle and the ray \( EF \).

3. a) Which sides and angles are equal in \( \triangle ABC \) and \( \triangle DEF \)?
   
   \[
   \text{_____} = \text{_____},\quad \text{_____} = \text{_____},\quad \text{_____} = \text{_____}
   \]
   
   b) Measure the rest of the sides and angles of \( \triangle DEF \). What can you say about \( \triangle ABC \) and \( \triangle DEF \)?

4. Try to move the vertices of \( \triangle DEF \) around.
   
   a) How does your triangle change? Which transformations can you make: rotations, reflections, translations?
   
   b) Can you move \( \triangle DEF \) onto \( \triangle ABC \) to check whether they are congruent? _____
   
   Do you need to reflect the triangle to do that? _____

5. Move \( \triangle DEF \) away from \( \triangle ABC \). Try to move the vertices of \( \triangle ABC \) around.
   
   a) Are the changes you can make in this triangle different from the changes you could make to \( \triangle DEF \)? Why?
   
   b) What happens to \( \triangle DEF \) when you modify \( \triangle ABC \)? What can you say about the triangles \( \triangle ABC \) and \( \triangle DEF \)?

6. Are two sides and the angle between them enough to determine a unique triangle? _____
Congruence Rules on Geometer’s Sketchpad (3)

1. Using the polygon tool, construct a triangle, \(ABC\). Measure the sides and the angles of your triangle.

2. a) Construct a point \(D\). Using a command for constructing circles and the length of \(AB\) as the radius, construct a circle with centre \(D\). Construct line segment \(DE = AB\). Hide the circle.

b) Select \(E\) as a centre of rotation. Use a command in the Transformation menu to mark \(\angle ABC\) as the angle of rotation. Rotate point \(D\) around \(E\) by the angle chosen. Construct a ray from \(E\) through the image of \(D\).

c) Select \(D\) as a centre of rotation. Mark \(\angle BAC\) as the angle of rotation. Rotate point \(E\) around \(D\) by the angle chosen. Construct a ray from \(D\) through the image of \(E\).

d) Construct a point \(F\) that is the intersection point of both rays. Use the polygon tool to construct triangle \(DEF\). Hide the rays.

3. a) Which sides and angles are equal in \(\triangle ABC\) and \(\triangle DEF\)?

   \[ \text{____}_1 = \text{____}_2, \quad \text{____}_3 = \text{____}_4, \quad \text{____}_5 = \text{____}_6 \]

b) Measure the rest of the sides and angles of \(\triangle DEF\). What can you say about \(\triangle ABC\) and \(\triangle DEF\)?

4. Try to move the vertices of \(\triangle DEF\) around.
   a) How does your triangle change? Which transformations can you make: rotations, reflections, translations?

   b) Can you move \(\triangle DEF\) onto \(\triangle ABC\) to check whether they are congruent? ______
      Do you need to reflect the triangle to do that? ______

5. Move \(\triangle DEF\) away from \(\triangle ABC\). Try to move the vertices of \(\triangle ABC\) around.
   a) Are the changes you can make in this triangle different from the changes you could make to \(\triangle DEF\)? Why?

   b) What happens to \(\triangle DEF\) when you modify \(\triangle ABC\)? What can you say about the triangles \(\triangle ABC\) and \(\triangle DEF\)?

6. Are two angles and a side between them enough to determine a unique triangle? ______
1. **PQRS** is a rectangle.
   a) Find the areas of the shapes:
      i) **PQRS**  
      ii) **ΔPRS**  
      iii) **ΔPTS**  
   b) Describe how the three areas in a) are related.

2. **CDEA, AFGB, and CBHJ** are squares.
   a) The base of the two shapes is given. Find the heights of the shapes.
      i) **ΔCBD**  
         **CDEA**  
         base: **DC**  
         height: **AC**  
      ii) **ΔCBG**  
         **AFGB**  
         base: **BG**  
         height: **AC**  
      iii) **ΔCAJ**  
         **LKJC**  
         base: **JC**  
         height: ________  
      iv) **ΔHBA**  
         **BHKL**  
         base: **BH**  
         height: ________
   b) Write the area of each triangle in terms of another shape.
      i) Area of **ΔCBD** = \(\frac{1}{2}\) of area of **CDEA**  
      ii) Area of **ΔCBG** = \(\frac{1}{2}\) of area of ________
      iii) Area of **ΔCAJ** = \(\frac{1}{2}\) of area of ________  
      iv) Area of **ΔHBA** = \(\frac{1}{2}\) of area of ________

---

**REMINDER** ► Area of triangle = base \(\times\) height \(\div\) 2
Proving the Pythagorean Theorem (2)

3. Fill in the blanks to finish proving the Pythagorean Theorem the same way Euclid did it 2300 years ago.

In a triangle ABC with \( \angle A = 90^\circ \), \( AB^2 + AC^2 = BC^2 \).

a) The picture at right shows the Pythagorean Theorem.
   
   What special quadrilateral is each of these quadrilaterals?
   
   \[ \text{CDEA} \quad \text{square} \quad \text{AFGB} \quad \text{__________} \]
   
   \[ \text{CBHJ} \quad \text{__________} \quad \text{LKJC} \quad \text{__________} \]
   
   \[ \text{BHKL} \quad \text{__________} \]

b) Area of \( \triangle CBD \) = _____ of area of square CDEA, because they both have base _____ and height _____.

c) Area of \( \triangle CAJ \) = _____ of area of rectangle _________, because they both have base CJ and height _____.

d) Mark the equal line segments in the picture.

e) \( \triangle CBD \cong \triangle CAJ \) by ______ congruence rule, because
   
   \( DC = _____ \), \( CB = _____ \)
   
   \( \angle DCB = 90^\circ + \angle _____ = \angle _____ \)

f) Area of CDEA = area of LKJC because ____________

g) Area of \( \triangle CBG \) = _____ of area of square _________, because they both have base _____ and height _____.

h) Area of \( \triangle HBA \) = _____ of area of rectangle _________, because they both have base _____ and height _____.

i) Mark the equal line segments in the picture.

j) \( \triangle CBG \cong _____ \) by ______ congruence rule, because
   
   \( BG = _____ \), \( CB = _____ \)
   
   \( \angle CBG = 90^\circ + \angle _____ = \angle _____ \)

k) Area of AFGB = area of BHKL because ____________

j) \( AB^2 + AC^2 = BC^2 \), because ________________________________
Angle Properties

Supplementary angles are angles that add to 180°.

Corresponding angles make a pattern similar to the letter F.

Corresponding Angle Theorem:
When lines are parallel, corresponding angles are equal.
The reverse statement is true as well:
When corresponding angles are equal, the lines are parallel.

Alternate angles make a pattern similar to the letter Z.

Alternate Angle Theorem:
When lines are parallel, alternate angles are equal.
The reverse statement is true as well:
When alternate angles are equal, the lines are parallel.

Opposite angles are created when two lines intersect.

Opposite Angle Theorem:
Opposite angles are equal: \( \angle 1 = \angle 2 \) and \( \angle 3 = \angle 4 \)

Complementary angles are angles whose sum is 90°.
Properties of Angles in a Triangle

**Sum of Angles in a Triangle Theorem**
The sum of the angles in a triangle is $180^\circ$.

$$\angle A + \angle B + \angle C = 180^\circ$$

**Exterior Angle Theorem**
The measure of an exterior angle in a triangle is equal to the sum of the opposite angles in the triangle.

$$\angle x = \angle a + \angle b$$

**Isosceles Triangle Theorem**
The base angles in an isosceles triangle are equal.

The reverse statement is true as well:

If two angles in a triangle are equal, then the triangle is isosceles, with the equal sides adjacent to the third angle in the triangle.
Rectangles
INVESTIGATION ▶ Which information about a pair of triangles will ensure they are similar?

A. SSS (side-side-side)

a) Draw a triangle, \(ABC\). Measure the sides.

b) Create a new parameter, \(s\), to be the scale factor. Set \(s = 2\).

c) Construct a triangle \(A'B'C'\) with sides \(A'B' = sAB, A'C' = sAC, B'C' = sBC\).

d) Measure the angles of both triangles. What do you notice?

Are the triangles similar? ____

e) Modify \(\triangle ABC\) by moving the vertices.

Does \(\triangle A'B'C'\) stay similar to \(\triangle ABC\)? ____

f) **Conjecture**: SSS is a similarity rule:

\[
\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC},
\]

then \(\triangle ABC\) is similar to \(\triangle A'B'C'\).

Change the scale factor \((s)\) to three different values, including two decimals, one of them more than 1 and the other less than 1.

Does \(\triangle A'B'C'\) stay similar to \(\triangle ABC\)? ____

Is the conjecture true? ____

B. SAS (side-angle-side)

a) Draw a triangle, \(ABC\). Measure the sides and the angles.

b) Create a new parameter, \(s\), to be the scale factor. Set \(s = 3\).

c) Construct a triangle \(A'B'C'\) with sides \(A'B' = sAB, A'C' = sAC\) and \(\angle A' = \angle A\).

d) Measure the side \(B'C'\) and the angles \(\angle B', \angle C'\). Find the ratio \(B'C' : BC\). What do you notice?

Are the triangles similar? ____

e) Modify \(\triangle ABC\) by moving the vertices. Does the triangle \(\triangle A'B'C'\) stay similar to \(\triangle ABC\)? ____

f) **Conjecture**: SAS is a similarity rule:

\[
\frac{\angle A}{\angle A'} = \frac{A'B'}{AB} = \frac{A'C'}{AC},
\]

then \(\triangle ABC\) is similar to \(\triangle A'B'C'\).

Change the scale factor \((s)\) to three different values, including two decimals, one of them more than 1 and the other less than 1.

Does \(\triangle A'B'C'\) stay similar to \(\triangle ABC\)? ____

Is the conjecture true? ____
Similarity Rules Using Geometer’s Sketchpad (2)

C. ASA (angle-side-angle)
   a) Draw a triangle, \(ABC\). Measure the sides and the angles.
   b) Create a new parameter, \(s\), to be the scale factor. Set \(s = 4\).
   c) Construct a triangle \(A'B'C'\) with sides \(A'B' = sAB\), \(\angle A' = \angle A\) and \(\angle B' = \angle B\).
   d) Measure the sides \(B'C', A'C'\) and the angle \(\angle C'\). Find the ratios \(B'C' : BC\) and \(A'C' : AC\). What do you notice?

Are the triangles similar? ______

e) Modify \(\triangle ABC\) by moving the vertices.
   Does the \(\triangle A'B'C'\) stay similar to \(\triangle ABC\)? ______

D. The ratios between corresponding sides of similar triangles are the same. In which parts—A, B or C—did you construct the triangles so that at least two pairs of corresponding sides had the same ratios? ______
   How was the construction in the other part different?

   a) Draw a triangle, \(ABC\). Measure the sides and the angles.
   b) Construct a triangle \(A'B'C'\) with \(\angle A' = \angle A\) and \(\angle B' = \angle B\). What do you know about the angles \(C\) and \(C'\)? ______
   Explain. __________________________________________________________________________________________

   c) Measure the sides of \(\triangle A'B'C'\). Find the ratios \(\frac{A'B'}{AB}, \frac{C'B'}{CB}, \frac{A'C'}{AC}\). What do you notice?

   Are the triangles similar? ______

   d) Modify \(\triangle ABC\) by moving the vertices.
   Does \(\triangle A'B'C'\) stay similar to \(\triangle ABC\)? ______

   e) Conjecture: AA (angle-angle) is a similarity rule:
   If \(\angle A = \angle A'\) and \(\angle B = \angle AB'\), then \(\triangle ABC\) is similar to \(\triangle A'B'C'\).
   Modify \(\triangle A'B'C'\) by moving the vertices. What do you notice about the ratios between the corresponding sides?

   Does \(\triangle A'B'C'\) stay similar to \(\triangle ABC\)? ______
   Is the conjecture true? ______
Area of Parallelogram and Triangle

Parallelogram

A parallelogram is a quadrilateral with two pairs of parallel sides.
Any pair of parallel sides can be chosen to be the bases.
The distance between these two parallel sides is the height.
The height is measured along a line perpendicular to the bases. This line can be drawn anywhere. In these pictures, the thick black line is one of the bases and the dashed line is the height.

\[ \text{Area of a parallelogram} = \text{base} \times \text{height} \]

Triangle

A triangle is a polygon with three sides.
Any side of a triangle can be the base. Draw a perpendicular from the vertex opposite the base to the base. The distance from the vertex to the base along that perpendicular is the height.
Sometimes the height is outside the triangle. In these pictures, the thick black line is the base and the dashed line is the height.

Any triangle is half of a parallelogram with the same base and height.

\[ \text{Area of a triangle} = \text{base} \times \text{height} \div 2 \]
Unit 4  Patterns and Algebra

In this unit, students will solve equations with integer coefficients using a variety of methods, and apply their reasoning skills to find mistakes in solutions of these equations. They will also investigate patterns and their equivalent linear relations, using tables of values, graphs, and formulas. Students will represent patterns and linear relations using a variety of tools and strategies and make connections between representations. They will use and improve their reasoning skills while analyzing patterns, formulas, and graphs.

Materials
In many lessons you will need a pre-drawn grid on the board. If such a grid is not already available, you can photocopy BLM 1-cm Grid Paper (p U-27) onto a transparency and project it onto the board. This will allow you to draw and erase points and lines on the grid without erasing the grid itself.

Meeting Your Curriculum
Teachers following the Ontario curriculum are not required to cover expressions of the form $\frac{x}{a} + b$, so certain questions in Lessons PA8-16 through PA8-18 in the Workbook will be optional for Ontario students.

Lesson PA8-23 is optional for students following the WNCP curriculum.

Lesson PA8-31 takes the concepts learned in Grade 6 to a Grade 8 level, using expressions with more than one variable and integers. It prepares students for the work in PA8-32, so we recommend that both Ontario and WNCP teachers teach it.
Review basic operations with integers. You can use Workbook page 107 Questions 1 and 2 as a diagnostic test.

Substituting integers for variables. Begin with expressions that require only one operation. **EXAMPLES:**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $3x$, $x = -2$</td>
<td>b) $v - 2$, $v = -7$</td>
</tr>
</tbody>
</table>

Then continue with expressions that require two operations. **EXAMPLES:**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $3t + 2$, $t = -4$</td>
<td>b) $8m - 1$, $m = -3$</td>
</tr>
</tbody>
</table>

**EXTRA PRACTICE:**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $9x$, $x = -2$</td>
<td>b) $r + 6$, $r = -3$</td>
</tr>
<tr>
<td>d) $4u - 2$, $u = -56$</td>
<td>e) $5w + 1$, $w = -2$</td>
</tr>
<tr>
<td>g) $4v + 2$, $v = -56$</td>
<td>h) $2 + 7m$, $m = -2$</td>
</tr>
</tbody>
</table>

**Looking for equivalent answers.** Tell students that among extra practice questions d) through i) there are two questions with the same answer, and ask students to find them. (f and h) Can students explain why the answers are the same? ($7r + 2$, $r = -2$ and $2 + 7m$, $m = -2$ both give the answer $-12$ because the expressions have the same meaning and we substitute the same number into each one)

Substituting integers for two variables. Have students substitute integers into expressions with two variables. **EXAMPLES:**

Find the value of each expression for $x = -3$ and $y = -2$.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $5x + 4y$</td>
<td>$(-23)$</td>
</tr>
</tbody>
</table>

**Bonus**

Evaluating expressions with negative coefficients. Have students evaluate $-3a$ and $3a$ for positive and negative values of $a$ by completing the chart below. What do students notice? ($3a$ and $-3a$ are opposite integers: $3a = a + a + a$ and $-3a = 0 - a - a - a$)
Have students evaluate expressions with integer coefficients, first substituting positive numbers only, and then substituting both positive and negative numbers. Start with expressions with one operation, and continue to expressions with two operations. Finish with expressions with two variables.

<table>
<thead>
<tr>
<th>a</th>
<th>−3</th>
<th>−2</th>
<th>−1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3a</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>−3a</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a + a + a</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 − a − a − a</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**NOTE:** Workbook Question 8 parts p) through u) are optional for Ontario students.

Review the fractional notation for division before assigning the questions below. Students can convert fractions into division statements while evaluating the equations. **EXAMPLE:**

a) \( \frac{x}{3}, x = −12 \)  \( \text{SOLUTION: } \frac{x}{3} = \frac{-12}{3} = -4 \)

b) \( \frac{d}{5}, d = -30 \)

c) \( \frac{b}{3}, b = 15 \)

d) \( \frac{c}{3} + (-5), c = -18 \)

e) \( \frac{z}{5} - (-6), z = -20 \)

f) \( \frac{t}{-3} + (-6), t = -45 \)
Goals

Students will solve equations with integer coefficients by guessing integer values for $x$, checking by substitution, and then revising their answer.

PRIOR KNOWLEDGE REQUIRED

Can substitute integers for variables in an expression
Can add, subtract, multiply, and divide integers
Is familiar with variables

NOTE: If you are following the WNCP curriculum, review using fractional notation for division before assigning Question 2 and part d) of Question 4. If you are following the Ontario curriculum, these questions are optional for your students.

Use a chart to solve equations. Have students copy and complete the chart below in their notebooks.

<table>
<thead>
<tr>
<th>$n$</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-5)n$</td>
<td>25</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-3 + (-5)n$</td>
<td>22</td>
<td>17</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then have students use the chart to solve for $n$:

a) $-3 + (-5)n = -23$  
b) $-3 + (-5)n = 7$

c) $-3 + (-5)n = 17$  
d) $-3 + (-5)n = -8$

Repeat with these charts and equations:

<table>
<thead>
<tr>
<th>$x$</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x - 2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(-4) \times (x - 2)$</td>
<td>28</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) $(-4) \times (x - 2) = 20$  
b) $(-4) \times (x - 2) = 8$

c) $(-4) \times (x - 2) = 12$  
d) $(-4) \times (x - 2) = -8$

<table>
<thead>
<tr>
<th>$a$</th>
<th>-30</th>
<th>-24</th>
<th>-18</th>
<th>-12</th>
<th>-6</th>
<th>0</th>
<th>6</th>
<th>12</th>
<th>18</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{a}{-6}$</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{a}{-6} - (-7)$</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) $\frac{a}{-6} - (-7) = 6$  
b) $\frac{a}{-6} - (-7) = 8$  
c) $\frac{a}{-6} - (-7) = 10$  
d) $\frac{a}{-6} - (-7) = 3$
Introduce the guess and check method to solve equations. Show the equation \((-7)h - 2 = -51\). Tell students that you are going to solve this equation by guessing and checking. Start by guessing \(h = 5\). **ASK:** If \(h = 5\), what is \((-7)h - 2\)? \((-37)\) Then guess \(h = 6\). **ASK:** If \(h = 6\), what is \((-7)h - 2\)? \((-44)\) What does this tell me? Which number is closer to \(-51\): \(-37\) or \(-44\)? \((-44)\) Should my next guess be smaller than 5 or larger than 6? (larger than 6) What would your next guess be? Continue in this way until students find that \(h = 7\).

Repeat with equation \((-2)(t - 3) = 14\). Start by guessing \(t = 3\) and then \(t = 4\). Make sure students realize that the next guess should be smaller than 3. **ANSWER:** \(t = -4\).

Repeat with equation \((-7)(t - 23) = 28\). Start by guessing \(t = 0\) (which produces \((-7)(t - 23) = 161\)) and then \(t = 1\) (which produces \((-7)(t - 23) = 154\)). **ASK:** Should my next guess be larger than 1 or smaller than 0\? (larger than 1) Does it make sense to guess 2\? (no) Why not? (The change from 161 to 154 is less than 10, and we need a change of about 120, so adding 1 to the possible answer will not change the answer enough.) Next, guess 10 (which produces \((-7)(t - 23) = 91\)). **ASK:** Should the next guess be a lot larger than 10 or a little larger than 10\? (a lot larger) Guess 20, which produces \((-7)(t - 23) = 21\). Should the next guess be larger than 20 or smaller than 20\? (smaller) A lot smaller than 20 or a little smaller than 20\? (a little) Have students continue guessing and checking until they find the answer, \(t = 19\).

**Compare the two methods of solving equations.** **ASK:** Which method takes less work? Which method is quicker? (the guess and check method is quicker)

Have students practise solving equations by guessing and checking, always explaining the choice for each next guess.

\[
\begin{align*}
\text{a)} & \quad (-3) \times (x - 4) = 27 \\
\text{b)} & \quad (-4)y - 2 = -14 \\
\text{c)} & \quad 2 - (-4) \times z = 18 \\
\text{d)} & \quad 7 - 5u = -8 \\
\text{e)} & \quad (-5) \times (3 - v) = 25 \\
\text{f)} & \quad \frac{a}{-6} + 4 = -14 \\
\text{g)} & \quad 2 - \frac{w}{3} = 17 \\
\text{h)} & \quad 7 - \frac{p}{-4} = -1
\end{align*}
\]

**ANSWERS:** a) \(-5\) b) 3 c) 4 d) 3 e) 8 f) 108 g) \(-45\) h) \(-32\)
PA8-18  Solving Equations
Page 110

**Goals**
Students will solve equations with integer coefficients using preservation of equality.

**PRIOR KNOWLEDGE REQUIRED**
- Can substitute integers for variables in an expression
- Can add, subtract, multiply, and divide integers
- Is familiar with variables
- Can solve equations with whole number coefficients using preservation of equality

**NOTE:** If you are following the WNCP curriculum, students need to be familiar with fractional notation for division to do Workbook Questions 1g), 3i) and j), and 6e) and f). These questions are optional for students in Ontario.

Review solving equations with whole number coefficients using preservation of equality. Do the first several questions below as a class, and have students do a few more similar questions independently.

a) \(4x + 2 = 14\)  
b) \(6a - 9 = 21\)  
c) \(10(x + 4) = 180\)  
d) \(15 + 30y = 105\)  
e) \(3b - 4 = 14\)  
f) \(11c - 9 = 57\)  
g) \(12 + 15d = 102\)  
h) \(2(m - 4) = 18\)

**ANSWERS:** a) 3  
b) 5  
c) 14  
d) 3  
e) 6  
f) 6  
g) 6  
h) 13

Solving equations with whole number coefficients and decimal answers. Present the next set of problems and have students solve them. Tell students that the answers will be fractions or decimals.

a) \(3x + 2 = 15\)  
b) \(6a - 7 = 26\)  
c) \(10(x - 4) = 18\)  
d) \(5 + 4y = 10\)  
e) \(3b - 4 = 10\)  
f) \(10c - 9 = 52\)  
g) \(12 + 5d = 19\)  
h) \(2(m + 4) = 27\)

**ANSWERS:** a) \(13/3\)  
b) 5.5  
c) 5.8  
d) 1.25  
e) 14/3  
f) 6.1  
g) 1.4  
h) 9.5

Have students check their answers by substitution.

Ask students to try to solve the equation \(3x - 4 = 15\) using guess and check. Give students a few minutes to start solving the equation, then **ASK:** Are you finding it easy to solve this equation using guess and check? Are you having any problems? (We usually guess whole numbers, and \(3x - 4 = 14\) for \(x = 6\) and \(17\) for \(x = 7\). So the answer is between 6 and 7. Even if we try to guess \(6\frac{1}{3}\), which is the correct answer, substituting this number is hard work.) Then ask students to solve the same equation using preservation of equality. Which method is more convenient? Why? (preservation of equality because it let’s us work with whole numbers to start—fractions don’t appear until the end—and it’s faster)

**PROCESS EXPECTATION**
Reflecting on what made the problem easy or hard

**PROCESS EXPECTATION**
Reflecting on other ways to solve a problem
Undoing operations with integers. Ensure that students are comfortable with undoing operations when integers are involved. Again, solve several questions together as a class, then have students solve equations independently.

1. Write the number that makes each equation true.
   a) \( a + (-4) \) _______ = \( a \)  
   b) \( a \times (-3) \div _______ = a \)  
   c) \( a \div (-2) \times _______ = a \)  
   d) \( a - (-5) + _______ = a \)  
   e) \( \frac{a}{-2} \times _______ = a \)  
   f) \( (-6) + a - _______ = a \)  
   g) \( -6a \div _______ = a \)  
   h) \( (-7)a \div _______ = a \)  
   i) \( (-12) \times a \div _______ = a \)

2. Write the operation and number that make each equation true.
   a) \( x + (-2) _______ = x \)  
   b) \( y \div (-2) _______ = y \)  
   c) \( (-2) + z _______ = z \)  
   d) \( (-2)u _______ = u \)  
   e) \( v \times (-2) _______ = v \)  
   f) \( w - (-2) _______ = w \)  
   g) \( -2a _______ = a \)  
   h) \( \frac{m}{-2} _______ = m \)  
   i) \( (-2) \times n _______ = n \)

Ask students to look at the equations in question 2 above and identify those that have the same answers. (b and h; a and c; d, e, g, i) Why do these questions have the same answers? (because the expressions on the left side of the equations in each group mean the same thing, so you need to perform the same operation to get back to the variable) Then have students identify the expressions in each of these groups that are always equal, for any number \( p \):

a) \( 5p \quad p + 5 \quad -(-5)p \quad p \times (-5) \quad p \times 5 \)

b) \( 7 + p \quad p - 7 \quad p \div (-7) \quad -7p \quad 7 - p \)

c) \( \frac{p}{-3} \quad p \div (-3) \quad p - 3 \quad (-3) \div p \quad \frac{3}{p} \quad -\frac{p}{3} \)

d) \( 8p \div 8 \quad p \div (-8) \times (-8) \quad \frac{p}{-8} \times (-8) \quad 8p - 8 \quad p - 8 \quad -(-8) \)

Solving equations with one operation and integer coefficients.

EXAMPLES:

a) \( x + (-2) = 12 \)  
   b) \( y \div (-2) = 34 \)  
   c) \( (-4) + z = 14 \)  
   d) \( (-3)u = 15 \)  
   e) \( v \times (-4) = -12 \)  
   f) \( w - (-5) = 10 \)  
   g) \( -6a = 216 \)  
   h) \( \frac{m}{-2} = 35 \)  
   i) \( (-7) \times n = 35 \)  
   j) \( x + (-3) = -11 \)  
   k) \( y \div (-3) = -13 \)  
   l) \( r - (-4) = -67 \)

ANSWERS: a) 14  b) -68  c) 18  d) -5  e) 3  f) 5  g) -36  h) -70  i) -5  j) -8  k) 39  l) -71
Have students check their answers by substitution.

**Solving equations with two operations and integer coefficients.**

**EXAMPLES:**

a) \(3x + (-5) = 13\)  
b) \((-4)y - (-2) = 34\)  
c) \((-4) + 9z = 14\)

d) \(12 + (-3)u = -15\)  
e) \(v \times (-4) - 6 = 14\)  
f) \(-11w - (-5) = 115\)

g) \(12 - 6a = 216\)  
h) \(\frac{m}{-2} + (-5) = 35\)  
i) \((-7) \times n - 16 = 33\)

j) \((-2)x + (-3) = -11\)  
k) \(y \div (-3) - (-1) = -11\)  
l) \(-2 \times r - (-4) = 66\)

**ANSWERS:** a) 6  
b) -8  
c) 2  
d) 9  
e) -5  
f) -10  
g) -34  
h) -80  
i) -7  
j) 4  
k) 36  
l) -31

Have students check their answers by substitution.
Patterns and Algebra 8-19

Review modelling equations with whole number coefficients and solving equations given by models. Using the same notation as in the Workbook (triangle = variable, circle = 1), the expression $3a = 2$ can be represented by three triangles on one scale balanced by two circles on the other. Have students draw the scales representing the following equations:

a) $4x + 2 = 10$  

b) $6a + 3 = 9$  

c) $10 + 2x = 14$  

d) $1 + 3y = 10$

Remind students of the steps to solve equations, below, and have students use Steps 2 and 3 to solve the equations they modelled.

**Step 1:** Write the equation that represents the model.

**Step 2:** Remove all circles from the side that has the triangle(s) and remove the same number of circles from the other side. Write the new equation.

**Step 3:** Divide the circles into the number of groups given by the number of triangles. Keep only one group of circles and one triangle. Write the new equation. This will be the solution!

**Solving equations that have fractional solutions.** Present the model at left, and have students write the equation. **SAY:** To solve this equation, you would divide the circles into the number of groups given by the number of triangles, so you would need to divide the circles into three groups. **ASK:** How is this equation different from the previous ones? (you cannot evenly divide the circles into the number of groups given by the triangles) Have students solve the equation using preservation of equality. **ASK:** How is the answer different from the answers to the previous equations? (it’s a fraction) How could we draw the solution using the model? (draw a triangle on one side of the balance and one and one third of a circle on the other side)
Have students solve the equations with fractional solutions given by balance models. **EXAMPLES:**

a) ![Balance Model 1](image1.png)

b) ![Balance Model 2](image2.png)

**Review and use the distributive law.** Remind students of the distributive law and how we can use it to rewrite expressions with variables. For example, we can rewrite the expression \((-2) (x + 5)\) as \((-2)x + (-2) \times 5\), which is equal to \(-2x - 10\). Have students rewrite the following expressions using the distributive law:

a) \((-6) (x + 5)\)  
b) \((-3) \times (x - 4)\)  
c) \((x + 1) (-3)\)  
d) \(2(x + (-8))\)

**ANSWERS:**

a) \(-6x + (-30) = -6x - 30\)  
b) \((-3) x - (-12) = -3x + 12\)  
c) \(-3x + (-3) = -3x - 3\)  
d) \(2x + (-16) = 2x - 16\)

Then have students rewrite the constant term in several expressions as a product of the coefficient of the variable and another number. **EXAMPLE:**

\((−2)a + 14 = (−2)a + (−2)(−7)\)

**ANSWERS:**

a) \((-2)x + 6\)  
b) \((-3) \times x - 24\)  
c) \(3x - 6\)  
d) \(2x + (-8)\)  
e) \((-3)x + (-15)\)  
f) \((-4)x - 4\)  
g) \(5x + (-35)\)  
h) \(2x - (-10)\)

Next, have students use the distributive law to rewrite the expressions above with brackets. **EXAMPLE:** \((-2)a + (-2)(-7) = (−2)(a + (-7))\). Students can also rewrite this expression as \((-2)(a - 7)\).

**ANSWERS:**

a) \((-2)x + (-2)(-3)\)  
b) \((-3) \times x - (-3)(-8)\)  
c) \(3x - 3 \times 2\)  
d) \(2x + 2 \times (-4)\) or \(2x + 2(-4)\)  
e) \((-3)x + (-3)(-5)\)  
f) \((-4)x - (-4)(-1)\)  
g) \(5x + 5(-7)\)  
h) \(2x - 2(-5)\)

Have students practise rewriting expressions using the distributive law:

a) \((-4)x + 12\)  
b) \((-5) \times x - 25\)  
c) \(7x - 63\)  
d) \(4x + (-8)\)  
e) \((-5)x + (-15)\)  
f) \((-8)x - 48\)  
g) \(15x + (-135)\)  
h) \(5x - (-10)\)

**ANSWERS:**

a) \((-4)(x - 3)\)  
b) \((-5)(x + 5)\)  
c) \(7(x - 9)\)  
d) \(4(x - 2)\)  
e) \((-5)(x + 3)\)  
f) \((-8)(x + 6)\)  
g) \(15(x - 9)\)  
h) \(5(x + 2)\)
**A shortcut for solving equations.** Remind students that sometimes people don’t write out every step when they solve equations. For example, the solution

\[-3x + (-12) = 33\]
\[(-3)(x + 4) = 33\]

Rewrite the left side using the distributive law
\[(-3)(x + 4) ÷ (-3) = 33 ÷ (-3)\]
\[x + 4 = -11\]
\[x + 4 - 4 = -11 - 4\]
\[x = -15\]

Divide both sides
Subtract 4 from both sides
Rewrite both sides

Can be shortened to

\[-3x + (-12) = 33\]
\[-3(x + 4) = 33\]
\[x + 4 = -11\]
\[x = -15\]

Rewrite both sides
Divide both sides by -3
Rewrite both sides
Subtract 4 from both sides

Instead of rewriting both sides, we can do that step mentally. Put the following solution on the board and **ASK:** How would you rewrite this solution?

\[3x + (-12) = -33\]
\[3x + (-12) - (-12) = -33 - (-12)\]
\[3x = -21\]
\[3x ÷ 3 = -21 ÷ 3\]
\[x = -7\]

Subtract 12 from both sides
Rewrite both sides
Divide both sides by 3
Rewrite both sides

**ANSWER:**

\[3x + (-12) = -33\]
\[3x = -21\]
\[x = -7\]

Subtract 12 from both sides
Divide both sides by 3

Emphasize that if students can subtract -12 from both sides and rewrite them mentally, then students can skip writing that step. However, they might be more likely to make mistakes when they don’t write down a step, so they have to be more careful.

Have students write the missing steps in each of these solutions:

a) \[3x + 4 = -23\]
   \[3x = -27\]
   \[x = -9\]

b) \[-2(x + 5) = 24\]
   \[x + 5 = -12\]
   \[x = -17\]

c) \[(-4)x - 3 = 25\]
   \[(-4)x = 28\]
   \[x = -7\]

d) \[5x - 3 = -18\]
   \[5x = -15\]
   \[x = -3\]

**SAMPLE ANSWER:**

a) \[3x + 4 = -23\]
   \[3x + 4 - 4 = -23 - 4\]
   \[3x = -27\]
   \[3x ÷ 3 = -27 ÷ 3\]
   \[x = -9\]

Subtract 5 from both sides
Rewrite both sides
Divide both sides by 3
Rewrite both sides

Tell students that, when solving problems, they will never be expected to skip steps—in fact, they should always write out all steps so that you can...
check their understanding. Even if they tend to skip steps with equations involving whole numbers, integers are trickier, and writing out all the steps helps to avoid mistakes. Also, students need to know how to read solutions that have skipped steps.

**Identifying mistakes by using substitution.** Tell students that you saw two different solutions to the same equation. Write these on the board.

<table>
<thead>
<tr>
<th>Solution 1</th>
<th>Solution 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3x + 6 = 18$</td>
<td>$-3x + 6 = 18$</td>
</tr>
<tr>
<td>$-3x + 6 - 6 = 18 - 6$</td>
<td>$-3(x + 2) = 18$</td>
</tr>
<tr>
<td>$-3x = 12$</td>
<td>$-3(x + 2) \div (-3) = 18 \div (-3)$</td>
</tr>
<tr>
<td>$-3x \div (-3) = 12 \div (-3)$</td>
<td>$x + 2 = -6$</td>
</tr>
<tr>
<td>$x = -4$</td>
<td>$x + 2 - 2 = -6 - 2$</td>
</tr>
<tr>
<td></td>
<td>$x = -8$</td>
</tr>
</tbody>
</table>

**ASK:** These students are both trying to find the $x$ that satisfies $-3x + 6 = 18$, so they should get the same answer. Did they? (no) How can we decide who is right? (we could check each step of both solutions) Is there a way to know if the answer is right before checking so that we can know ahead of time that we are looking for a mistake? (yes, substitute the answer into the original expression and check if you get 18)

Have students substitute both of the answers into the equation to determine which answer is correct. ($-3(-4) + 6 = 12 + 6 = 18$, but $-3(-8) + 6 = 24 + 6 = 30$, so $x = -4$ is correct and $x = -8$ is incorrect)

Have students individually describe each step of Solution 1.

To find the mistake in Solution 2, have students substitute the right answer into every step of the solution. They will see that the second line is the first line where the correct answer does not work. ($-3(-4 + 2) = 6 \neq 18$)

**PROCESS EXPECTATION**

Reflecting on the reasonableness of an answer

**PROCESS EXPECTATION**

Reflecting on other ways to solve the problem

**PROCESS EXPECTATION**

Working backwards

Suggest that students work backwards. Substituting 8 into the second-last line gives $8 - 2 + 2 = 8$ and $6 + 2 = 8$, so the mistake is above that. Continue to the line before it: $8 - 2 = 6$, and not $-6$. **ASK:** What does this mean? (the mistake was made between this line and the next one) Have students correct the mistake. ($x - 2 + 2 = -6 + 2$) Have students solve the equation correctly from that point. Point out that when a mistake was made close to the beginning of the solution and you know the right answer, substituting the right answer from the beginning is the quicker way to find
the mistake. If the mistake was made close to the end and you do not know what the right answer is, working backwards will give you the answer faster.

Now show students these solutions to the same problem.

### Solution 4

\[-3x + 6 = 18\]  
\[-3(x - 2) = 18\]  
\[-3(x - 2) \div (-3) = 18 \div (-3)\]  
\[x - 2 = -6\]  
\[x = -4\]

### Solution 5

\[-3x + 6 = 18\]  
\[-3(x + 6) = 18\]  
\[-3(x + 6) \div (-3) = 18 \div (-3)\]  
\[x + 6 = -6\]  
\[x = -12\]

### Solution 6

\[-3x + 6 = 18\]  
\[-3x + 6 - 6 = 18 - 6\]  
\[-3x = 12\]  
\[-3x \div (-3) = 12 \div 3\]  
\[x = 4\]

### Solution 7

\[-3x + 6 = 18\]  
\[-3(x - 2) = 18\]  
\[x - 2 + 2 = 18 + 2\]  
\[x = 20\]

**ASK:** Which solution is correct? (Solution 4) How do you know? (it gets the right answer) Have students go through each incorrect solution line by line to find the mistake.

**ANSWERS:**

**Solution 5:** wrote \(-3x + 6\) as \(-3(x + 6)\)—this is not true since

\[-3(x + 6) = -3x + (-18), \text{ not } -3x + 6\]

**Solution 6:** divided the left side by \(-3\) but the right side by 3, so the two sides are no longer equal

**Solution 7:** if \(-3 \times (x - 2)\) is 18, then \(x - 2\) isn’t equal to 18—forgot to divide 18 by \(-3\)

Have students identify and correct any mistakes below. If the solution is correct, they should write “correct.” If no correct solution is given for one of the equations (and indeed there is none for 5\((x + 2) = -30\)), students should also produce a correct solution.

### a)

\[5(x + 2) = -30\]  
\[5x + 2 = -30\]  
\[5x + 2 - 2 = -30 - 2\]  
\[5x = -32\]  
\[5x \div 5 = -32 \div 5\]  
\[x = -6.4\]

### b)

\[5(x + 2) = -30\]  
\[5(x + 2) \div 5 = -30 \div 5\]  
\[x + 2 = -30\]  
\[x + 2 - 2 = -30 - 2\]  
\[x = -32\]

### c)

\[5(x + 2) = -30\]  
\[5(x + 2) \div 5 = -30 \div 5\]  
\[x + 2 = 6\]  
\[x + 2 - 2 = 6 - 2\]  
\[x = 4\]

### d)

\[2x + (-8) = -22\]  
\[2x - 8 + 8 = -22 - 8\]  
\[2x = -30\]  
\[2x \div 2 = -30 \div 2\]  
\[x = -15\]
e) \(2x + (-8) = -22\)  
\[2(x - 4) = -22\]  
\[x - 4 = -11\]  
\[x - 4 + 4 = -11 + 4\]  
\[x = -7\]

f) \(2x + (-8) = -22\)  
\[2(x + (-4)) = -22\]  
\[x + (-4) = -11\]  
\[x + (-4) - (-4) = -11\]  
\[x = -11\]

g) \(-3x + 9 = -27\)  
\[-3(x + 3) = -27\]  
\[x + 3 = 9\]  
\[x + 3 - 3 = 6\]  
\[x = 6\]

h) \(-3x + 9 = -27\)  
\[-3x + 9 - 9 = -27 - 9\]  
\[-3x = -18\]  
\[-3x \div (-3) = -18 \div (-3)\]  
\[x = 6\]

i) \(-3x + 9 = -27\)  
\[-3(x + (-3)) = -27\]  
\[x + (-3) = 9\]  
\[x + (-3) - (-3) = 12\]  
\[x = 12\]

**ANSWER:** Only e) and i) are correct, and the right answer for \(5(x + 2) = -30\) is \(x = -8\).

**Solving problems, checking the answer using substitution, and identifying and correcting their own mistakes.** Have students use the distributive law (see Method 2 from Workbook page 111 Question 2) to solve these equations and say what they do at each step. Students should check their answer by substituting it into the original expression. If incorrect, have students find their own mistake!

a) \(3x + (-9) = 15\)  
b) \(-4x + 20 = 36\)  
c) \(2x + 8 = -20\)

d) \((-3)x + 12 = 27\)  
e) \(-2x + 4 = -12\)  
f) \(-3x - 6 = 15\)

g) \(5x + (-10) = -25\)  
h) \(-5x - (-5) = 20\)

**ANSWERS:** a) \(x = 8\)  b) \(x = -4\)  c) \(x = -14\)  d) \(x = -5\)  e) \(x = 8\)  
f) \(x = -7\)  g) \(x = -3\)  h) \(x = -3\)

**Word problems practice:** Remind the students how to find the mean of a set of numbers before assigning this question.

a) Find the average temperature in Yellowknife, NWT, over a week in October:  
\[-1°C, 2°C, -1°C, 0°C, -2°C, -2°C, -3°C (ANSWER: -1°C)\]

b) Find the average temperature in Labrador City, NL, over the same week in October:  
\[-1°C, 0°C, 1°C, 0°C, 0°C, -1°C, 0°C (ANSWER: -1/7°C \approx -0.14°C)\]

c) Which city was colder on average during that week? (Yellowknife was colder)
How formulas help with patterns. Draw a simple geometric design, like the one in the margin. **ASK:** How many pentagons did I use? How many triangles? How many triangles and how many pentagons will I need for two such designs? For three designs? Remind your students that in earlier grades they used T-tables to solve this type of question (see also PA8-3). Ask students to draw a T-table and to fill it in for five designs. **ASK:** I want to make 20 such designs. Should I continue the table to check how many pentagons and triangles I need? Can you think of a more efficient way to find the number of pentagons and triangles? How many triangles are needed for each pentagon in the design? (5) What do you do to the number of pentagons to find the number of triangles in any number of designs? (multiply by 5)

Remind students that mathematicians often use letters instead of numbers to represent a changing quantity. These letters are called *variables*, because the quantity they represent can vary (change). The quantity that does not change (the number the variable is multiplied by) is called a *coefficient*. For example, mathematicians could use $p$ for the number of pentagons and $t$ for the number of triangles. We have a verbal rule for the number of triangles used in a design: Multiply the number of pentagons by 5 to get the number of triangles. What algebraic equation does this rule produce? $(5 \times p = t \text{ or } 5p = t)$ Explain that an equation that shows how to calculate one quantity from another is called a *formula*. Write the term on the board beside the formula itself. **ASK:** In which other contexts have we seen and used formulas? (area, volume) For example, a formula for the area of a circle ($A = \pi r^2$) shows us how to find the area of a circle from its radius. Both the area and the radius are represented by variables.

**Producing a table of values for a formula.** Write another formula, such as $8 \times s = t$. Explain that $s$ represents the number of squares and $t$ is the number of triangles in a pattern, as before. What rule does the formula express? (The number of triangles is 8 times the number of squares, or multiply the number of squares by 8 to get the number of triangles.) Remind students how to make a table of values that matches the formula.
by substituting actual numbers of squares for \( s \) and multiplying. The result of the multiplication is the number of triangles. Draw the table of values:

<table>
<thead>
<tr>
<th># of Squares (s)</th>
<th>Formula ((8 \times s = t))</th>
<th># of Triangles (t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(8 \times 1 = 8)</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>(8 \times 2 = 16)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(8 \times 3 = )</td>
<td></td>
</tr>
</tbody>
</table>

Ask students to copy the table and to fill in the missing numbers. Then have them add two more rows to the table. **ASK:** Do we need to extend the table to find how many triangles will be needed for 25 squares? (no) How will you find how many triangles are needed for 25 squares? (substitute 25 for \( s \))

Ask students to find the number of triangles for 25 squares (200). Let your students practise drawing tables for more formulas, such as \( 3 \times t = s \), \( 6 \times s = t \) (\( t \) = number of triangles, \( s \) = number of squares)

**Producing a formula of the type \( t = a \times s \) for a T-table.** Tell students that the T-tables below were created using a formula of the same type as above. Now, instead of creating the table for a formula, students will do the opposite: produce a formula for the table. First identify the type of formula. **ASK:** How are all the formulas for the tables below the same? (all formulas are of the sort “number \( \times s = t \)”)

How could you find the coefficient for each formula from the table? (e.g., look at the number of triangles in the row with \( s = 1 \) or divide the number of triangles in any row by the number of squares)

Point out to students that it is essential to check that the formula they produced works for all rows of the table. Include several more challenging examples, such as the last two below.

<table>
<thead>
<tr>
<th>Squares (s)</th>
<th>Triangles (t)</th>
<th>Squares (s)</th>
<th>Triangles (t)</th>
<th>Squares (s)</th>
<th>Triangles (t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>2</td>
<td>8</td>
<td>2</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>3</td>
<td>12</td>
<td>3</td>
<td>36</td>
</tr>
<tr>
<td>Squares (s)</td>
<td>Triangles (t)</td>
<td>Squares (s)</td>
<td>Triangles (t)</td>
<td>Squares (s)</td>
<td>Triangles (t)</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>2</td>
<td>8</td>
<td>2</td>
<td>21</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>4</td>
<td>16</td>
<td>4</td>
<td>27</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
<td>6</td>
<td>24</td>
<td>6</td>
<td>48</td>
</tr>
</tbody>
</table>

Students can practise creating T-tables for multiplicative rules (rules that involve only multiplication) and finding formulas for them using the Activity below.

**Formulas for patterns of the type \( n + a \).** Start with a simple problem: Rose invites some friends to a party. She needs one chair for each friend and one for herself. Can you give Rose a formula or equation for the number of chairs she will need?

Ask your students to suggest a variable for the number of friends and a variable for the number of chairs. Given the number of friends, how do you find the number of chairs? Ask students to write a formula for the number of
chairs. Suggest that students make a T-table similar to the one they used for multiplicative rules. They should start at 1 and fill the table in for a few rows.

Have students practise producing T-tables for formulas using addition or subtraction.

Tell students that the following tables were made by adding or subtracting a number from the variable, and have students find the formulas that show how to get the second number \( t \) from the first number \( s \). **EXAMPLES:**

\[
\begin{array}{c|c|c}
\text{a)} & s & t \\
1 & 6 \\
2 & 7 \\
3 & 8 \\
\end{array}
\begin{array}{c|c|c}
\text{b)} & s & t \\
1 & 0 \\
2 & 1 \\
3 & 2 \\
\end{array}
\begin{array}{c|c|c}
\text{c)} & s & t \\
1 & 12 \\
2 & 13 \\
3 & 14 \\
\end{array}
\begin{array}{c|c|c}
\text{d)} & s & t \\
1 & 10 \\
2 & 11 \\
3 & 12 \\
\end{array}
\begin{array}{c|c|c}
\text{e)} & s & t \\
21 & 15 \\
22 & 16 \\
23 & 17 \\
\end{array}
\begin{array}{c|c|c}
\text{f)} & s & t \\
12 & 8 \\
14 & 10 \\
16 & 12 \\
\end{array}
\begin{array}{c|c|c}
\text{g)} & s & t \\
7 & 11 \\
9 & 13 \\
16 & 20 \\
\end{array}
\begin{array}{c|c|c}
\text{h)} & s & t \\
7 & 1 \\
9 & 3 \\
16 & 10 \\
\end{array}
\]

**ANSWERS:**

\[
\begin{array}{c|c}
\text{a)} & t = s + 5 \\
\text{b)} & t = s - 1 \\
\text{c)} & t = s + 11 \\
\text{d)} & t = s + 9 \\
\text{e)} & t = s - 6 \\
\text{f)} & t = s - 4 \\
\text{g)} & t = s + 4 \\
\text{h)} & t = s - 6 \\
\end{array}
\]

**Tables with input and output.** Explain to students that the number that you put into a formula in place of a letter is often called the input. The result that the formula provides—the number of chairs, for instance—is called the output. Write these terms on the board and ask volunteers to circle the input and underline the output in the formulas you have written on the board.

**EXAMPLE:** \( r = 3 + t \)

Draw several T-tables with headings Input and Output on the board, provide a rule for each and the input numbers, and ask your students to find the output numbers. Start with simple inputs like 1, 2, 3 or 5, 6, 7 and continue to more complicated combinations like 6, 10, 14. Provide all types of rules: additive (add 4 to the input), multiplicative (multiply the input by 5), and subtractive (subtract 3 from the input).

Suggest that students try a more complicated task: produce a rule and a formula for a given table. Ask students to think about what was done to the input to get the output. Remind them to check that the formula works for all rows. For example, for the first table below, if you look only at the first row, the formula could be Output = Input \( \times 5 \) or Output = Input \( + 4 \), so you need to check the other rows. Give students several simple tables to work with. **EXAMPLES:**

\[
\begin{array}{c|c}
\text{Input} & \text{Output} \\
1 & 5 \\
2 & 6 \\
3 & 7 \\
\end{array}
\begin{array}{c|c}
\text{Input} & \text{Output} \\
2 & 4 \\
3 & 6 \\
4 & 8 \\
\end{array}
\begin{array}{c|c}
\text{Input} & \text{Output} \\
8 & 5 \\
7 & 4 \\
6 & 3 \\
\end{array}
\begin{array}{c|c}
\text{Input} & \text{Output} \\
2 & 6 \\
4 & 12 \\
6 & 18 \\
\end{array}
\]

Output = Input \( + 4 \) Output = Input \( \times 2 \) Output = Input \( - 3 \) Output = Input \( \times 3 \)
Students work in pairs. Each student decides on a formula (such as $s = 3 \times t$) and makes a T-table of values for it, with three rows. Students exchange their T-tables. They have to find the formula the T-table was made with and then check each other’s answers. They can also try to produce a design that will go with the formula they found.

Variation: Use a spinner as shown and a die to randomize the formulas students produce. Students spin the spinner and roll the die. They should write a formula for the rule given by the spinner and the die. For example, if a student spins Multiply and rolls 3, the rule is “Multiply the input by 3” and the formula is “$3 \times \text{Input} = \text{Output}.”

**Extension**

Tell students that a family is having a party. The formula for the number of chairs they will need for the party is $g + 4 = c$. **ASK:** If $g$ is the number of guests and $c$ is the total number of chairs needed, how many people are in the family? (4) Point out that any change in the number of guests produces a change in the total number of chairs needed. For example, if there are two guests, $g = 2$ and the family will need 6 chairs; if there are three guests, $g = 3$ and the family will need 7 chairs; and so on. The number of family members is always 4, and it does not change.

Next, show a different formula for the number of chairs: $g + f - 1 = c$. Say that $f$ represents the number of family members, and “− 1” represents a baby in the family who does not need a chair. This time, the number of family members can change, too. What other quantities can change? (the number of guests, the number of chairs) If the family has 10 chairs, how many guests and how many family members could be at this party? (There are different solutions to this problem. Students should find them systematically.)

Have students write a formula that shows how to find the number of chairs if, among the guests and the family, there are $b$ babies that do not need chairs. (**ANSWER:** $c = f + g - b$)
Goals
Students will identify sequences (presented numerically or geometrically) that vary directly with the term number and find formulas for such sequences.

Prior Knowledge Required
Can create and extend a T-table for a pattern
Is familiar with variables
Can identify increasing and decreasing sequences
Can find the gaps in a sequence

Review finding formulas of the type \( t = an \) for tables and patterns.
Show students several sequences made of blocks with a multiplicative rule, such as:

ASK: How do we obtain each new figure from the previous one? (by adding two squares and four triangles) Which rule is this rule similar to, “Start at 2 and add 2 each time” or “Multiply the term number by 2”. (Start at 2 and add 2 each time) Can you describe how to get a next term in a different way, using only the first figure? (repeat the first figure \( n \) times to get the \( n \)th figure: 1 for the first term, 2 for the second term, etc.) Point out that the first figure is repeated the number of times that is equal to the figure number.

Have students draw T-tables for the number of triangles and the number of squares, and fill in the numbers for 3 figures in the sequence.

<table>
<thead>
<tr>
<th>Figure Number ( (f) )</th>
<th>Number of Squares ( (s) )</th>
<th>Figure Number ( (f) )</th>
<th>Number of Triangles ( (t) )</th>
<th>Number of Squares ( (s) )</th>
<th>Number of Triangles ( (t) )</th>
</tr>
</thead>
</table>

Ask your students to write a formula for each table. (If necessary, prompt students to use the general rule they figured out for the whole pattern: the first figure is repeated term number times. There are 2 squares in the first figure, so the \( n \)th figure there will have \( 2n \) squares.) **ANSWERS:** \( s = 2 \times f \), \( t = 4 \times f \), \( t = 2 \times s \).

Direct variation. Remind students of the meaning of the terms input and output. Explain to students that when the rule is “Multiply the input by ____,” we say that the output varies directly with the input. So in this pattern the Number of Squares varies directly with the Figure Number, and the Number of Triangles varies directly with both the Figure Number and the Number of Squares.
Present several tables and have students say whether the output varies directly with the input or not. **EXAMPLES:**

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>8</td>
<td>18</td>
</tr>
<tr>
<td>10</td>
<td>24</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

**Bonus**

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

Draw a sequence of squares with side lengths 1, 2, 3, etc. Ask students to find the areas and the perimeters of the squares. Ask them to make a T-table for both and to check which quantity varies directly with the side length. (perimeter) Ask students to write a formula for the perimeter and for the area of the square.

**Bonus**

a) The number of feet, $f$, varies directly with the number of people, $p$. (2 people, 4 feet; 3 people, 6 feet; 4 people, 8 feet; $f = 2 \times p$). Does the number of paws vary directly with the number of cats? What is the formula? ($c = 4p$)

b) A cat has five claws on each front paw and four claws on each back paw. Make a T-table showing the number of cats and the number of claws and another T-table showing the number of paws (add one paw at a time in the same order, e.g., right front, left front, right hind, left hind—then go to the next cat—right front, left front, and so on) and the number of claws. Does the total number of claws vary directly with the number of paws or with the number of cats? (The total number of claws varies directly with the number of cats ($claws = 18 \times cats$), but not with the number of paws.)

**Formulas that do not vary directly with the figure number.** Draw the sequence of figures and shade the growing towers of blocks in each figure, as shown.

![Figure 1](image1.png)  
![Figure 2](image2.png)  
![Figure 3](image3.png)

Ask students to draw two T-tables, one for the figure number and the number of shaded blocks, and the other for the figure number and the total number of blocks. **ASK:** In which T-table does the number of blocks
in the output column vary directly with the figure number: the T-table that shows the number of shaded blocks or the one that shows the total number of blocks?

Have students produce a formula for the number of shaded blocks and explain their reasoning. Then ASK: Does the number of unshaded blocks change from figure to figure? To get the total number of blocks, what do you have to add to the number of shaded blocks? (the number of unshaded blocks) Have students write the formula for the total number of blocks. Then have students practise finding formulas for the number of shaded blocks and the total number of blocks using Question 4 on Workbook page 116.

Have students compare the formulas they produced. How are the formulas (and rules) that show direct variation different from the formulas that do not show direct variation? (the formulas for direct variation are all of the type “Multiply the figure number by a number”, the formulas that do not show direct variation involve addition as well) Then ask students whether the formulas below show direct variation. Have students signal “yes” and “no” to assess the whole class at a glance. Then ask students to pick a formula that does not show direct variation and explain how they know it does not show direct variation. Which part of the formula does show direct variation?

EXAMPLES:

a) 3 \times \text{Figure Number} 

b) 3 \times \text{Figure Number} - 5

c) \text{Figure Number} \times 4 - 5

d) \text{Figure Number} \times 6

e) \text{Figure Number} \times 7.5

f) 12 + 3 \times \text{Figure Number}

Word problems practice:

A cab charges a flat rate of $4 (you pay this just for using the cab) and $2 for every minute of the ride. Write a formula for the price of a cab ride. How much will you pay for a 4-minute cab ride? For a 5-minute ride?

Extensions

What does each formula below help you to find? Which of the formulas show direct variation? Which quantity varies directly with which?

a) \( C = 2\pi r \) 

b) Area = width \times length 

c) Perimeter = 4 \times \text{side length}

Bonus: Area = \( \pi r^2 \)

ANSWERS: a) Circumference of a circle varies directly with the radius of a circle. b) Area of a rectangle does not vary directly with width or length, because neither of these is a constant. c) Perimeter of a square or a rhombus varies directly with side length. Bonus: Area of a circle varies directly with the square of the radius, but not with the radius itself.

Find a sequence of rectangles in which area will vary directly with length. Does the perimeter vary directly with the length? (ANSWER: Use rectangles of the same width, say 5. The area will be 5l, so it will vary directly with the length. The perimeter in this case will be 2l + 10 and will not vary directly with the length.)
Review finding formulas for patterns of the type $n + a$. Present several geometric patterns where a fixed number of blocks is added to the term number of blocks to produce each term, and have students identify the formulas for the patterns. **EXAMPLES:**

a) \[
\begin{array}{ccc}
\,
\end{array}
\]  \quad b) \[\begin{array}{ccc}
\triangle
\end{array}\]

**ANSWERS:** a) $n + 3$  \quad b) $n + 2$

Finding formulas for patterns in various ways. Present the following pattern and explain that you would like to find a formula for it, but you want to use a different method from the one you used in the previous lesson.

Ask students to look at the unshaded blocks in each figure. What do they notice? (there are 6 in all figures) Does the number of unshaded blocks vary directly with the term number? (no) Ask students to find the formula for the number of shaded blocks in each circled group of shaded blocks. (Figure Number + 1) Then ask them to find the formula for the total number of shaded blocks. (The formula is $4 \times$ (Figure Number + 1). **PROMPT:** How many times does the circled group of shaded blocks repeat in each figure?) Finally, ask students to find the formula for the total number of blocks in the pattern. ($4 \times$ (Figure Number + 1) + 6)

Redraw the pattern on the board but don’t shade or circle any of the blocks. Ask students to copy the pattern and to pretend that they are making the
figures from cubes. They should build the first figure, then add some blocks to it to make the second figure, then add more blocks to make the third figure from the second, and so on. How many blocks will students need to add each time? (4) Ask them to shade the blocks that were added in the second figure. In the third figure, ask them to shade both the blocks that were added when making the second figure and the blocks added to make the third figure. Have students find the formula for the number of shaded blocks in each figure. If students need a prompt, shade the blocks on the board, so that there are four separate groups of shaded blocks in both second and third figures, and circle one of the groups in each.

Have students find the formula for the number of shaded blocks in the circle first. (Figure Number − 1 in the circle, so the number of shaded blocks is 4 × (Figure Number − 1))  

**ASK:** What is the number of unshaded blocks in each figure? (the number of blocks in the first figure, 14) What is the formula for the total number of blocks? (4 × (Figure Number − 1) + 14)

**Comparing the formulas. SAY:** We found the number of blocks in two patterns in two different ways, and obtained two different formulas. What should we expect from the formulas we got? (they should be the same) Have students use the distributive law to write both formulas without the brackets, and compare the results. Do both formulas give the same result? Yes, they do:

\[
\begin{align*}
4 \times \text{Figure Number} + 1 + 6 &= 4 \times \text{Figure Number} - 4 + 14 \\
4 \times \text{Figure Number} + 4 + 6 &= 4 \times \text{Figure Number} - 4 + 14 \\
4 \times \text{Figure Number} + 10 &= 4 \times \text{Figure Number} + 10
\end{align*}
\]

Have students compare all three formulas: \(4 \times \text{Figure Number} + 1 + 6\), \(4 \times (\text{Figure Number} - 1) + 14\), and \(4 \times \text{Figure Number} + 10\). How are the formulas the same? (all have coefficient 4, neither shows direct variation) How are they different? (different expressions are multiplied by 4, the number added to the multiple of 4 is different)

Ask students to draw another copy of the pattern and to shade \(4 \times \text{Figure Number}\) blocks in each figure. What is the number of unshaded blocks in each figure? (10) Does this fit the last formula? (yes) Have students look at the part of each formula that represents the number of shaded blocks. How is the number of shaded blocks the same in each formula? (it shows direct variation, with coefficient 4) What does the number of shaded blocks vary directly with in each formula? (Figure Number + 1 in the first, Figure Number − 1 in the second, and Figure Number in the third) Point out that the fact that the number of shaded blocks was different and varied directly with different quantities is responsible for the different additive constant (the number of unshaded blocks) in each formula. Emphasize that all three
ways of finding the number of blocks in each pattern should produce the same formula when it is simplified.

**Using the formula to find terms or term numbers.** Ask students how they could use the formula to find which figure in the pattern will have 50 blocks. (by using equations) Have students solve the equation $4 \times \text{Figure Number} + 10 = 50$. (Figure Number $= 10$) Then have students predict which figure will have: 150 blocks ($50^{th}$ figure), 210 blocks ($50^{th}$ figure), 1010 blocks ($250^{th}$ figure). Then ask which figure will have 10,000 blocks. Have students solve the equation $4 \times \text{Figure Number} + 10 = 10000$. (Figure Number $= 2497.5$)

**ASK:** What does this mean? (no figure in this pattern can have this number of blocks)

Have students check whether there is a figure in this pattern that will have the number of blocks given, and, if yes, which figure it will be:

- a) 325 (none)  
- b) 326 ($79^{th}$ figure)  
- c) 3456 (none)  
- d) 5678 ($1,417^{th}$ figure)

**EXTRA PRACTICE:**
Use the patterns in Questions 2 and 3 on Workbook page 117. In each pattern, is there a term that has 100 shapes? 1000 shapes? 10,001 shapes? If yes, which term is that?

**ANSWERS:**
- 100 shapes:  
  2a) no  
  3) $47^{th}$  
- 1000 shapes:  
  2a) no  
  3) $497^{th}$  
- 10,001 shapes:  
  2a) $4999^{th}$  
  2b) no  
  3) no
Review using formulas for linear relations. Tell students that you have to pay $30 a month for up to 600 text messages plus $2 for each additional text message. **ASK**: How much do each of the first 600 text messages cost? ($30 \div 600 = 0.05 = 5\text{¢}$) If I send 610 text messages this month (i.e., 10 additional text messages), how much will my cellphone bill be? Have students write the expression using the quantities $30, $2, and 10.

**ANSWER**: $30 + 2 \times 10. \textbf{ASK}: Which of these amounts is most likely to change from month to month? (the 10 additional text messages) What do we use to represent an amount that changes? (a variable) What number in the expression $30 + 2 \times 10$ should we replace with a variable? (10) Why? (because that is what changes) Write on the board:

\[
\text{monthly cost of cellphone} = 30 + 2n, \text{ where } n \text{ is the number of additional text messages above 600}
\]

Remind students that an expression of the sort they have just written can be converted into a verbal rule, such as “Multiply the number of text messages above 600 by 2 and add 30.”

**ASK**: How much would it cost if I sent 625 text messages? ($30 + 2 \times 25 = 30 + 50 = 80$) Remind students that replacing a variable in an expression with a number and evaluating it is called substitution. They substituted $n = 25$ into the formula for the cost, $30 + 2n$.

**Making a sequence from a formula.** Remind students that a formula tells you how to calculate the term from the term number. That is, you would substitute 1 into the formula to get the first term, substitute 2 into the formula to get the second term, and so on.

Demonstrate with this formula: Term = $2 \times \text{Term Number} + 1$

<table>
<thead>
<tr>
<th>Term Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Model substituting 1 and 2 into the expression and add the results to the table of values, then have students do 3, 4, and 5 individually. Leave the table on the board (to refer to during the discussion below). Point out that the terms now form a sequence.

Have students convert these formulas to sequences:

a) $4 \times \text{Term Number} - 4$

b) $15 - 2 \times \text{Term Number}$

c) Multiply the Term Number by 3 and add 5

d) Subtract the Term Number from 12

e) $3n + 1$

f) $(-2)n + 7$

g) $2n - 2$

h) $12 - 4n$

**Bonus** Subtract the Term Number multiplied by $(-3)$ from 21

**ANSWERS:**

a) 0, 4, 8, 12, 16  
b) 13, 11, 9, 7, 5  
c) 8, 11, 14, 17, 20  
d) 11, 10, 9, 8, 7  
e) 4, 7, 10, 13, 16  
f) 5, 3, 1, $-1$, $-3$

g) 0, 2, 4, 6, 8  
h) 8, 4, 0, $-4$, $-8$  

**Bonus** 24, 27, 30, 33, 36

Point out that we substitute numbers into the formula in place of Term Number. This means Term Number is a quantity that changes. What do we call a quantity that changes in a formula? (a variable) So Term Number is a variable.

**Converting a formula into a general rule and vice versa.** Tell students that the formula $\text{Term} = 2 \times \text{Term Number} + 1$ can be converted to a verbal rule: Multiply the term number by 2 and add 1. Then look at an example with a negative coefficient. For example, the formula $\text{Term} = 3 - 2 \times \text{Term Number}$ can be converted to a verbal rule in two ways. Multiply the Term Number by 2 and subtract the result from 3 is one way. The other way is to think of subtraction as adding a negative number:

\[
3 - 2 \times \text{Term Number} = 3 + (-2) \times \text{Term Number} = (-2) \times \text{Term Number} + 3.
\]

Now we can translate the formula to a verbal rule slightly differently: Multiply the Term Number by $(-2)$ and add 3. Have students check that both rules produce the same sequence. Then ask students to convert several formulas to verbal rules and vice versa. Use the same formulas as above:

b) $15 - 2 \times \text{Term Number}$  
f) $(-2)n + 7$  
h) $12 - 4n$

**ANSWERS:** b) Multiply the term number by $(-2)$ and add 15, f) Multiply the term number by $(-2)$ and add 7, h) Multiply the term number by $(-4)$ and add 12.

**General and stepwise rules.** Write the two types of rules for the sequence 3, 5, 7, 9, 11 next to the table of values already on the board. (Start at 3 and add 2 each time; Multiply the term number by 2 and add 1) Look at the rules and the table of values side by side, and **ASK:** How are these
rules different? (one uses the term number, the other gives directions on how to get the next number from the previous one; one has a variable, Term Number, in it, and the other does not.) Explain that the rule that uses a variable is called a general rule. The rule that tells you how to get the sequence starting from the first term, step by step, is called a stepwise rule. Which of the rules is easy to convert to a formula? (the general rule) Why? (it already contains a variable) Emphasize that a general rule is just a verbal form of a formula.

Producing stepwise rules from sequences. Remind students that the terms form the sequence 3, 5, 7, 9, 11. Write the sequence on the board, leaving enough room to add the circles for the gaps, as in Question 1 on Workbook page 118. Demonstrate how to find the stepwise rule for the sequence from the gaps. (The gaps are always +2, so the rule is “Start at 3 and add 2 each time.”) Have students find the stepwise rules for the sequences they found above.

ANSWERS:
a) Start at 0, add 4 each time  b) Start at 13, subtract 2 each time
c) Start at 8, add 3 each time    d) Start at 11, subtract 1 each time
e) Start at 4, add 3 each time    f) Start at 5, subtract 2 each time
g) Start at 0, add 2 each time    h) Start at 8, subtract 4 each time

Bonus  Start at 24, add 3 each time (or Start at 24, subtract –3 each time)

Connecting sequence properties to the formula. Which part or parts of the stepwise rule do you see in the formula? What numbers are the same? (the number that you add or subtract each time and the number in front of n) Remind students that the number in front of the variable is called the coefficient of the variable. ASK: Do you think the coefficient of the variable will always be the number that you add or subtract each time? Let students’ share their predictions, and then have them check their predictions on the sequences above and by completing Workbook page 118 Question 1.

SAY: Look at the stepwise rules that tell you to subtract each time instead of to add each time. How are the formulas for those sequences different from the formulas for other sequences? (there is a minus sign in front of the coefficient; the coefficient is negative; the gap is negative)

ASK: Where does the gap in the sequence appear in the stepwise rule? Where does it appear in the general rule? How can you find the gap in the sequence from the formula? (The gap is what you add or subtract each time, so it is the coefficient of the variable, i.e., the number that you multiply by the Term Number.)

ASK: How can you find the first term in the sequence from the stepwise rule? (It’s the number you start at) How can you find the first term in the sequence from the formula? PROMPT: What is the term number for the first term? (1, so find the first term by substituting 1 for the term number)
Challenge students to write the rule for the sequence given each formula, without producing a table of values first:

a) \( \text{Term} = 3 \times \text{Term Number} + 2 \)
b) \( \text{Term} = 20 - 3 \times \text{Term Number} \)
c) \( \text{Term} = 2 \times \text{Term number} - 1 \)
d) \( \text{Term} = 11 + (-2) \times \text{Term number} \)

**ANSWERS:**
a) The first term is \( 3(1) + 2 = 5 \), so start at 5. The coefficient is 3, so add 3 each time. The rule is Start at 5, then add 3 each time.
b) Start at 17, then subtract 3 each time. (Subtract because there is a minus sign in front of the coefficient.)
c) Start at 1, then add 2 each time.
d) Start at 9, then subtract 2 (or add \(-2\)) each time.

Repeat for formulas written in terms of \( n \). **EXAMPLES:**
a) \( 4n - 1 \)  b) \( 17 - 2n \)  c) \( 3n + 3 \)  d) \( 22 + (-3)n \)

**ANSWERS:** a) Start at 3, add 4 each time. b) Start at 15, subtract 2 each time. c) Start at 6 and add 3 each time. d) Start at 19, subtract 3 (add \(-3\)) each time.

The advantage of the formula over the rule for finding the term for large term numbers. Write the first five terms of a sequence and its corresponding formula. (**EXAMPLE:** \( 34 \times \text{Term Number} + 7 \); Sequence: 41, 75, 109, 143, 177) Verify that the formula is correct for each of the first five terms. Then challenge students to find the 6th term of the sequence. (211) Ask students for the strategies they used. (**EXAMPLES:** I found the gap in the sequence and added it to, or subtracted it from, the 5th term. I substituted the term number 6 into the formula.) Have students solve the problem again using whichever strategy they didn’t use the first time, to verify that they get the same answer both ways. Discuss which way was faster. (Using the gap will likely be faster because it only requires adding or subtracting, whereas the formula requires multiplication.) Now tell students you want to find the 60th term. **ASK:** Should we keep adding the gap until we find the 60th term, or should we substitute 60 into the formula? Why?

**Extension**

For some sequences, it is much easier to produce a stepwise rule than a general rule. For other sequences, a general rule will come more easily than a stepwise rule.

a) Fibonacci numbers are produced using a more complicated stepwise rule. Instead of giving a single number to start with, you have to give the first two terms: Start at 1 and 1. The rule uses the previous two terms instead of just the previous one term: Add the two previous terms to
get the next term. Write the first eight terms of the Fibonacci sequence. **(NOTE: A general formula for this sequence requires high school math.)**

b) Here is a rule that seems simple: Start at 1 and add the gap each time. But instead of telling you what the gap is, there is a rule to find the gap: Start at 3 (the gap between the first and the second term), and add 2 to the gap each time. Write the first five terms of this sequence. Find a general rule and a formula for the sequence.

c) Find a stepwise rule and a formula for this sequence: 3, 6, 11, 18, 27. Which one is easier to find? **(HINT: Use the sequence from part b.)**

d) Write the first five terms of the sequence given by this formula:

<table>
<thead>
<tr>
<th>Term Number</th>
<th>Term Number \times Term Number \times Term Number + 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>32</td>
</tr>
<tr>
<td>4</td>
<td>69</td>
</tr>
<tr>
<td>5</td>
<td>130</td>
</tr>
</tbody>
</table>

**ANSWERS:**

a) 1, 1, 2, 3, 5, 8, 13, 21

Formula: Term Number \times Term Number = (Term Number)^2
General rule: Multiply the Term Number by itself OR Square the Term Number

c) Stepwise rule: Start at 3 and add the gap each time. To find the gap, start at 3 (the gap between the first and the second term), and add 2 to the gap each time. Formula: Term Number \times Term Number + 2.

d) Sequence: 6, 13, 32, 69, 130.
Stepwise rule: To find the gaps in the gaps, start at 12 and add 6 each time. To find the gaps, start at 7 (gap between terms 1 and 2) and add the gaps in the gaps. To find the sequence itself, start at 6 and add the gaps you found.
Goals
Students will produce formulas for linear relations given in numerical and geometric form.

PRIOR KNOWLEDGE REQUIRED
Can create and extend a T-table for a pattern
Can find the gaps in a sequence
Is familiar with variables
Can identify a sequence that varies directly with the term number
Can produce a formula for a sequence that varies directly with the term number
Can identify increasing and decreasing sequences

MATERIALS
dice of two colours

The gap between the terms of a sequence is the coefficient in the formula of the sequence. Give each student a pair of dice of different colours. Ask students to roll the dice and write a sequence according to this rule: multiply the term number by the result on the red die and add the result on the blue die. Students need to write a formula for the rule, and produce the first three terms of the sequence. They can record the sequence in a T-table.

Ask students to find the difference between the terms of their sequences (the gap) each time. After they have created several sequences, ask students to identify where the gap appears in each formula. Review with students what they learned in the last lesson: the coefficient in the formula is the gap in the sequence. In this case, the gap will be the number rolled on the red die.

The gap in a geometric pattern. Draw or make the following sequence:

```
Figure Number | Number of Blocks
-------------|------------------
   1          |     1            
   2          |     3            
   3          |     5            
```

Ask students to describe what part of the pattern changes (the shaded part, the vertical stacks) and what part stays the same (the unshaded part, the bottom row). Draw a T-table for the number of blocks in each figure of the sequence as shown.

Ask students to predict the gap between the terms in the output column (Number of Blocks) before you fill in the column. How do they know? (The “gap” between terms in the T-table is the number of new blocks added to the pattern at each stage.) **ASK:** How can you find the number of shaded blocks?
in each figure, using the figure number? (multiply the figure number by the “gap”) Have students solve Question 2 on Workbook page 119 for practice.

Formulas for sequences that do not vary directly with the term number. Present several sequences (EXAMPLES: 7, 10, 13, 16; 5, 9, 13, 17; 12, 19, 26, 33) and have students fill in the first and the third columns in a chart with three columns:

<table>
<thead>
<tr>
<th>Term Number</th>
<th>Term Number × Gap</th>
<th>Term</th>
</tr>
</thead>
</table>

Then ask students to find the gap between the terms and to fill in the middle column of the table. Ask students to compare the numbers in the second and the third columns. **ASK:** How can you obtain the numbers in the third column from the numbers in the second column? (by adding the same number) Have students write both a general rule for the sequence (Multiply the Term Number by ____ and add ____), and a formula (Term Number × ____ + ____).

Compare one of these tables to the table for block patterns made from shaded and unshaded blocks above. Which column in this table would the number of shaded blocks correspond to? (Term Number × Gap) What does the gap in this table correspond to? (the number of blocks added each time, the number of vertical stacks) What does the number of unshaded blocks correspond to? (the number added to the second column to get the third column)

Next, present several patterns where the adjustment factor (the number used to get from the second column to the third column) should be subtracted from the product of the term number and the gap. **EXAMPLES:** 1, 4, 7, 10; 3, 7, 11, 15; 4, 11, 18, 25. Again, have students make tables, compare the columns, determine the adjustment factor, and find the general rule and the formula. Ask students also to check by substitution that their formula works for all of the rows of the T-table.

Let students practise finding rules for various T-tables, following the steps tin the box on Workbook page 120. They can use the following sequences to create T-tables:

a) 2, 7, 12, 17  
b) 21, 33, 45, 57  
c) 2, 23, 44, 65

**ANSWERS:**
a) Multiply Term Number by 5 and subtract 3, or 5 × Term Number – 3  
b) Term Number × 12 + 9  
c) Term Number × 21 – 19

Draw the following pattern and ask students to make a T-table for it, and then to find a formula for the pattern.

**Figure 1**  
**Figure 2**  
**Figure 3**

**ANSWER:** Figure Number × 3 – 2.

Ask students if they can shade the number of blocks equal to the Figure Number × gap in each pattern. (no) Why not? (there are not
enough blocks) How does the formula show that there will not be enough blocks to shade? (The total number of blocks is Figure Number × 3 − 2, and 3 is the gap. Since we subtract 2 from Figure Number × gap to get the total number of blocks, there will be not enough blocks to shade.) Point out that the method using T-tables to find a pattern rule works in any case, even when the shading blocks method does not.

Applications of pattern rules. Solve the following problem as a class, then have students practise solving similar questions (see Extra practice below).

Rita builds towers by placing cubes with side length 7 cm one on top of the other.

a) Find the formula for the surface area of her tower.

b) What are the height and the surface area of a tower that is 30 cubes tall?

c) How many cubes are in the first tower that has surface area greater than 1 m²?

d) How tall is the tower from part c) in metres? Can Rita build it?

SOLUTION:
a) Tower number \( n \) has \( n \) cubes in it. Each cube face has area 49 cm².

\( \text{(NOTE: Each time you add a cube to the tower, you are adding 4 cube faces to the total surface area. The area of the top and bottom of the tower are counted only once, at the beginning.)} \)

<table>
<thead>
<tr>
<th>Tower number (( n ))</th>
<th>( n \times \text{gap} )</th>
<th>Surface area (cm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>196</td>
<td>6 × 49 = 294</td>
</tr>
<tr>
<td>2</td>
<td>392</td>
<td>10 × 49 = 490</td>
</tr>
<tr>
<td>3</td>
<td>588</td>
<td>14 × 49 = 686</td>
</tr>
</tbody>
</table>

Formula: \( 196n + 98 \) cm².

b) A tower 30 cubes tall has \( n = 30 \). Its height is \( 7 \times 30 \) cm = 2.1 m, and its surface area is \( 196 \times 30 + 98 = 5976 \) cm².

c) Solving \( 196n + 98 = 10 000 \) for \( n \) gives \( n = \frac{50}{98} \). So if we take \( n = 51 \), the tower will have surface area greater than 1 m², whereas a tower with \( n = 50 \) will have a surface area less than that.

d) \( 51 \times 7 \) cm = 357 cm = 3.57 m. Rita will need help to build a tower that tall!

EXTRA PRACTICE:
1. a) Find the formulas for the perimeter and the area of the following figures. Use your formulas to predict the perimeter and the area of Figure 15.

![Figure 15](image)

b) Find a formula for the number of inner line segments in the figure.
How many inner line segments does the 20th figure in this pattern have?

c) Is there a figure in the pattern that has perimeter 1000? If yes, which one? If not, which one has a perimeter closest to 1000? What is its area? What is its number of inner line segments?

**ANSWERS:**

a) Perimeter = 4n + 8. Area = 4n + 3. For n = 15, perimeter is 68, area is 63.

b) Number of inner line segments = 6n + 2. For n = 20, it is 122.

c) When 4n + 8 = 1000, n = 248. Area = 4(248) + 3 = 995. Number of inner line segments = 6(248) + 2 = 1490.

2. The pattern at left is made from cubes with sides 2 cm.

   a) Find the formulas for the volume and surface area of the pattern.

   b) Find the volume and the surface area of the term with 100 cubes.

**ANSWERS:**

a) Volume: 8 cm³ × (3 × Term Number − 2) = 24 × Term Number − 16 cm³.

   Surface area: 48 × Term Number + 24 cm²

b) Term number: 3 × Term Number − 2 = 100, so Term Number = 34.

   Volume: 24 × 34 − 16 = 800 cm³ (this makes sense for a figure made from 100 cubes, each with a volume of 8 cm³).

   Surface area: 48 × 34 + 24 = 1656 cm².

**Extensions**

1. Find the rules for these T-tables. How do the rules relate to each other?

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
</tr>
</tbody>
</table>

**ANSWER:** The rule for the first table is Multiply Input by 3 and add 2. The rule for the second table is Subtract 2 from the Input and divide the result by 3. In the second table we are just undoing the operations in the first table since we get back to where we started (Output in one table = Input in the other table); thus, we have to do the “opposite” operations in the reverse order. Alternatively, some students might notice that another way to formulate the rule for the second table is to divide by 3 and then subtract 2/3. Indeed, by the distributive property, \((x − 2) ÷ 3 = x ÷ 3 − 2 ÷ 3\).
2. **Using patterns to find sum of terms of a linear sequence.** Tell students you want to add all the numbers to 100: \(1 + 2 + 3 + 4 + \ldots + 100\). **ASK:** How would you approach the problem? Would you just start adding and hope to finish quickly? Would you look for a pattern? Let students try the method they chose. Then explain that the mathematician K. F. Gauss came up with a clever answer that avoided both methods. He noticed that you could write this sum as follows:

\[
1 + 2 + 3 + \ldots + 48 + 49 + 50
+ 100 + 99 + 98 + \ldots + 53 + 52 + 51
+ 101 + 101 + 101 + \ldots + 101 + 101 + 101
\]

**ASK:** How many 101s there are in the sum? How do you know?

**ANSWER:** There are 100 numbers grouped in twos, so there are 50 pairs. Each pair adds to 101, so the sum is \(101 \times 50 = \frac{101 \times 100}{2}\).

Ask students to use this method to find the sum of the numbers from 1 to 10, from 1 to 20, and from 1 to 70.

Work together to find the sum from 1 to 25. **ASK:** What makes this problem different? (the numbers do not pair up) Ask students to think about how to write the sum a different way, to get around this problem. Make sure the following approaches are discussed:

**Method 1**
Find the sum from 1 to 24 and then add 25

\[
1 + 2 + 3 + \ldots + 12 + 24 + 23 + 22 + \ldots + 13
\]

\[
25 + 25 + 25 + \ldots + 25 = 25 \times 12
\]

\[
25 \times 12 + 25 = 25 \times 13 = 325
\]

**Method 2**
Find the sum from 1 to 25 directly.

\[
1 + 2 + 3 + \ldots + 12 + 13
+ 25 + 24 + 23 + \ldots + 14
\]

\[
26 + 26 + 26 + \ldots + 26 + 13 = 26 \times 12 + 13
\]

\[
= 312 + 13 = 325
\]

Point out that in any case the answer is always \(\frac{n(n+1)}{2}\) where \(n\) is the number of terms. It doesn’t matter whether \(n\) is even or odd.

**Where do pattern rules come in?** Ask students to add all the even numbers from 2 to 20—2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20—without actually doing the additions. There are several possible ways to rewrite the sum:

\[
(1 + 2 + 3 + \ldots + 10) \times 2
\]
OR $2 + 4 + 6 + 8 + 10$
$+ 20 + 18 + 16 + 14 + 12$

$22 + 22 + 22 + 22 + 22 = 22 \times 5$

OR $(1 + 3 + 5 + 7 + \ldots + 19) + 10$ and then try to find $1 + 3 + 5 + \ldots + 19$. (This is actually an easier pattern to notice if students are familiar with perfect squares.)

Note that all of these methods require noticing that the number of terms is 10. To find the number of terms you can look at the last term in the sum and find its number, using the formula for the pattern. For example, the formula for this pattern is $2n$, and the last term is 20, so $n = 10$.

**EXAMPLE:** Find $19 + 21 + 23 + \ldots + 83$.

**Step 1:** Make a T-table.

<table>
<thead>
<tr>
<th>Term Number</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19</td>
</tr>
<tr>
<td>2</td>
<td>21</td>
</tr>
<tr>
<td>3</td>
<td>23</td>
</tr>
</tbody>
</table>

**Step 2:** Write a formula for the table.

$2 \times \text{Term Number} + 17$ or $2n + 17$

**Step 3:** Find the number of the last term using your formula.

The last term is 83, and $2n + 17 = 83$, so $n = 33$

**Step 4:** There are 33 terms in the sum. So there are 16 pairs plus one in the middle. The middle (or $17^{th}$) term is: $2(17) + 17 = 51$

$19 + 21 + 23 + \ldots + 49 + 51$
$83 + 81 + 79 + \ldots + 53$
$102 + 102 + 102 + \ldots + 102 + 51$

The pairs all add to 102. So the sum is $16 \times 102 + 51 = 33 \times 51 = 1683$.

Use the steps above to add:

a) $7 + 8 + 9 + \ldots + 59$

b) $44 + 46 + 48 + \ldots + 96$

c) $35 + 37 + 39 + \ldots + 105$

d) $123 + 127 + 131 + \ldots + 203$

**ANSWERS:** a) 1749  b) 1890  c) 2520  d) 5423
**Review coordinates.** Draw a coordinate grid and show both (2, 4) and (4, 2). Remind students how the points are placed in different locations. The order we write the numbers in matters: The first coordinate tells us how far to go right of 0, and the second coordinate tells us how far to go up from 0. This is a convention used by mathematicians everywhere, the way > is used to mean more than and < is used to mean less than. Draw several grids on the board, add points, and have students write the coordinates of the points.

```
<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
```

Then make a chart with headings Ordered Pair, First Number, and Second Number (see Workbook page 122 Questions 1 and 2), and have students fill in the charts for the grids above. Students can also join the points with a straight line, find more grid points on these lines, and add them to the chart.

**Formulas for graphs.** Review writing formulas for T-tables. Students can write the formulas for the charts they created above, showing how to get the Second Number from the First Number. Point out that both First Number and Second Number are qualities that change, so they are both variables. Then present several graphs of straight lines with axes labelled First Number, Second Number, Input, Output; or \(x, y\). Have students:

a) draw grid points on the graphs (as shown in i)

b) make T-tables for the grid points on each line (there will be seven charts, one for each line shown)

c) write formulas showing how to get the value on the vertical axis from the value on the horizontal axis.
EXAMPLES:

i) 

ii) 

iii) 

Sample answer for ii):

a) Points (1, 0), (2, 2), (3, 4), (4, 6) should be marked on line D.

b) T-table for line D:

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

c) Formula for line D: \( y = 2x - 2 \).

EXTRA PRACTICE:

The graph shows the height of a fighter plane during takeoff.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Height (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

a) Fill in the T-table.
b) Write a formula for the T-table.
c) How high will the plane be after 10 seconds?
d) At this rate, when will the plane reach the height of 10 kilometres?

ANSWERS: b) \( h = 250t \)  
          c) 2.5 km  
          d) 40 seconds

Graphing T-tables. Remind students that they can think of a pair of variables (such as Input and Output or Term Number and Term) as ordered pairs. The input is the first number and the output is the second number. Have students change given T-tables of inputs and outputs first to ordered pairs and then to points on a graph. See Workbook page 124 Question 6.

Graphing formulas. Tell students that their next task will be slightly harder: they will produce the T-table themselves, using the formula that relates the variables. Give students formulas and have them produce the T-table and a set of ordered pairs, such as (Term Number, Term) or \((x, y)\), and then graph the pairs. EXAMPLES:

a) \( \text{Term} = \text{Term Number} \times 2 + 3 \)  
b) \( \text{Term} = 13 - \text{Term Number} \times 2 \)

c) \( y = 2x - 2 \)  
d) \( y = 17 - 3x \)  
e) \( \text{Output} = \text{Input} \times 4 - 2 \)
PROCESS ASSESSMENT
8m7, [R]
Workbook Question 4e)

SAMPLE ANSWERS:

a) 

\[
\text{Output} = \text{Input} \times \frac{1}{2} + 4
\]

<table>
<thead>
<tr>
<th>Term Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
</tbody>
</table>


b) 

\[
(2, 5), (4, 6), (6, 7), (8, 8)
\]


d) 

\[
(1, 14), (2, 11), (3, 8), (4, 5)
\]


Bonus

\[
(2, 5), (4, 6), (6, 7), (8, 8)
\]
Review relevant prior knowledge and vocabulary. Review key vocabulary related to sequences: term, term number, repeating, increasing, decreasing. Remind students how they used to find the difference between the terms to identify whether a sequence repeats, increases, or decreases, by the same amount or not. Have them classify the sequences below as repeating, increasing, and decreasing, and say whether each sequence increases/decreases by the same amounts or not. Keep the sequences and their descriptions on the board for later use.

**EXAMPLES:**

- a) 4, 6, 8, 10, 12
- b) 1, 4, 7, 10, 13
- c) 22, 17, 12, 7, 2
- d) 2, 5, 8, 2, 5, 8
- e) 0, 1, 4, 9, 16
- f) 15, 14, 12, 9, 5
- g) 12, 10, 8, 6, 12
- h) 15, 10, 5, 0, −5

Graphing sequences. To graph a sequence, change the sequence to a set of ordered pairs (term number, term) and then plot the ordered pairs on a graph. Have students convert the sequences in the examples above to a list of ordered pairs, then have them plot the pairs. Keep the plots for future use.

Graphs of increasing and decreasing sequences. Have students make a conjecture about how the graphs of increasing sequences are different from the graphs of decreasing sequences. Then have students verify their conjecture using the graphs they made. Students might find it helpful to circle the increasing sequences with one colour and the decreasing sequences with a different colour. Ask students to write in their notebooks a sentence about how the graphs of increasing and decreasing sequences are different, and then to compare their sentence with that of a partner. Students should try to improve both sentences by making a new sentence. Repeat with groups of four. Students can use their combined answers when answering Question 3d) on Workbook page 125.
**Linear graphs and sequences.** Tell students that a sequence is called *linear* if all the points on its graph can be joined by a straight line. Have students do parts A and B of Investigation 1 on Workbook page 127. Students can formulate their answer to C first individually, then with a partner, and finally with a group of four, to continually improve their answer before writing it. Have students check their answer by using it to solve Question 4 on the same page.

**EXTRA PRACTICE:** Return to the sequences and descriptions from the beginning of the lesson. Ask students to decide whether the points in each sequence can be joined by a straight line or not, and to check their conjecture by joining the points they plotted (the points can be joined by a straight line in a, b, c, and h).

**Bonus**

Is this sequence linear: 1, 3, 5, 7, then repeat?

**ANSWER:** No, but it looks like it for the first 4 terms.

**PROCESS EXPECTATION**

Making and investigating conjectures

**PROCESS ASSESSMENT**

8m3, 8m7, [R, C]

Workbook p 127 Question 4

**PROCESS EXPECTATION**

Representing, Visualizing

Emphasize that students can now determine the properties of a graph by looking at the corresponding sequence of points. Have students summarize what they can say about the graph if the sequence is increasing with the same gap between terms. (The points all lie on the same line and go from bottom left to top right.) Repeat with increasing sequences where the gaps are not all the same, (the points are not on the same line, but they all go from bottom left to top right), decreasing sequences where the gaps are all the same, and decreasing sequences where the gaps are not all the same.

**ASK:** Where have you seen pictures that look like the graphs in this lesson? (in line graphs) Pretend the graphs in Question 1 on Workbook page 126 are line graphs, with the horizontal axis being time, and the vertical axis being distance travelled. How would you describe the trends for each graph? (The distance increases with time for increasing sequences and decreases for decreasing sequences. When the gaps are the same we would say that the distance increases or decreases at a constant rate—the object or person whose position the graph describes moves with a constant speed.) Linear sequences increase or decrease at a constant rate. Have students suggest other relationships that graphs of linear sequences could represent. Students can also describe the trends for a relationship of their choice. **EXAMPLES:** recycled material collected, money earned, temperature, average precipitation.

**Formulas of linear sequences.** Remind students how to produce a sequence given a formula. Have students work through Investigation 2 on Workbook page 127. One possible conclusion could be that sequences are linear if Term Number appears only once in the formula. A more precise conclusion is that sequences are linear when Term Number is multiplied only by a constant (or divided by a constant, which can be either positive or negative) and has a constant added or subtracted. If the formula involves multiplying the Term Number by itself, the sequence will not be linear.
Patterns and Algebra 8-28, 29, 30

PA8-28 Graphing Formulas
PA8-29 Investigating Patterns
PA8-30 Problems and Puzzles

Pages 128–132

CURRICULUM EXPECTATIONS
Ontario: 7m60, 7m61, 7m64, 7m68; 8m1, 8m2, 8m5, 8m7, 8m57, 8m58, 8m59, 8m60
WNCP: 7PR1, 7PR2; 8PR1, 8PR2, [C, CN, R, PS, V]

VOCABULARY
formula
expression
substitution
variable
graph
term
T-table
ordered pair
term number
sequence
gap
coefficient

Goals
Students will graph linear patterns given by formulas and find further terms of patterns using a variety of tools.

PRIOR KNOWLEDGE REQUIRED
Can create and extend a T-table for a pattern
Is familiar with variables
Can draw points and identify coordinates of points in the first quadrant of the Cartesian plane
Can identify increasing and decreasing sequences
Can find the gaps in a sequence given numerically, by a formula or a table of values
Can graph a sequence

MATERIALS
dice

Review finding values of terms from a graph of a sequence.

Finding the term number given the term value. Have students fill in the table for the graph at left.

<table>
<thead>
<tr>
<th>Term number</th>
<th>1</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term value</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

ANSWERS: 4, 10
ANSWERS: 1, 3

If necessary, point out the strategy of starting at the term value on the vertical axis and going right until you reach the graph, and then descending to the horizontal axis to find the term number. Have students extend the line to find out what the next whole-number term value would be. (13th term, 5)

Have students graph these sequences: a) 4, 10, 16, 22; b) 8, 12, 16, 20; c) 4, 7, 10, 13. Alert students to the fact that they will need large grids, because they will need to extend the graphs. Then ask students to use their graphs to find the value of the 10th term of each sequence. To check their answers, students should find formulas for the sequences and find the value of the 10th term using algebra. Which method do students like better? Why?

Next, have students use the graphs to find which term of each sequence equals 40. Then have them substitute each term number they found into the formula to check their answers. (ANSWERS: a) 7th, b) 9th, c) 13th) As a challenge, have students check whether there is a term in each sequence that equals 36. If there is, which term is it? (ANSWERS: a) no, b) 8th, c) no)
Graphing expressions. Remind students that we can make a set of ordered pairs from a T-table, and we can make a T-table from an expression, so we can make a set of ordered pairs from an expression. Once we have a set of ordered pairs, we can plot the points on a graph, so if we are given an expression, we can draw a graph. Model drawing a graph from an expression, such as $3n - 1$:

**Step 1:** Make a T-table by substituting $n = 1$, $n = 2$, $n = 3$, $n = 4$, and $n = 5$ into the expression.

**Step 2:** Make a set of ordered pairs from the T-table with the first number being the value for $n$ and the second number being the value for the expression after substituting the first number for $n$.

**Step 3:** Graph the set of ordered pairs.

Have students do the exercises on Workbook page 128.

**EXTRA PRACTICE:**
a) $3n + 1$  
b) $n + 7$  
c) $2n + 5$  
d) $4n - 2$

**How are the gaps in the sequence reflected in the formula?** Remind students that when they substitute consecutive numbers into a formula (such as $3n + 4$) they produce a sequence of numbers, with the value of $n$ they substituted being the term number. For example, substituting $n = 1$, 2, 3, 4, 5 into $3n + 4$, produces sequence 7, 10, 13, 16, 19. What are the gaps in this sequence? (+3) Ask students to look at the sequences they produced in Question 1 on Workbook page 128. How is the gap reflected in the formula? (it is the coefficient of the variable) Remind students, that when they produced formulas for T-tables, they used the gap between the numbers. **ASK:** What did the gap become? (the coefficient of the variable)

**The gap in the sequence on the graph.** Have students find the gaps between the terms in the sequences they graphed in Questions 2 and 3 on Workbook page 128. **ASK:** Where would you look on the graph to see the difference in the term numbers? (the horizontal axis) The difference in the term values? (vertical axis) Where would you look to see the gap in the sequence? (on the vertical axis)

Graph several linear increasing sequences and have students identify the gap in each sequence without writing out the sequence itself. Then have them write out the sequence and check their answers. Present several formulas with different coefficients and ask students to tell which of these formulas could match the graph at left. **EXAMPLES:**

$$4n + 2 \quad 3n + 3 \quad 5n + 1 \quad 2n + 4$$

**ASK:** Why aren’t the other formulas able to match this graph? (The gap in the sequence is 3, so the coefficient of $n$ must be 3.)

**Using the first term of the sequence to identify the graph.** Ask students to think about how they could decide which of the following formulas with coefficient 3 could be the formula for the sequence in the graph:

$$3n + 1 \quad 3n + 2 \quad 3n + 3 \quad 3n - 2$$
If the idea of looking at the first term does not arise, ask students to think about what the stepwise rule for this sequence could be. Where would they start? What would the stepwise rule for the sequence given by each formula be? Does this information help them decide which formula matches the sequence in the graph?

Finally, present several graphs and their formulas (out of order) and have students match the graphs to the formulas:

a) \(3n - 1\)  
b) \(3n + 1\)  
c) \(2n + 2\)

---

**ANSWERS:**  
a) C  
b) B  
c) A

**Workbook page 132 (PA8-30 Problems and Puzzles)** can be used to review the material studied in this unit.

**ACTIVITY**

Students will need grid paper and two dice each. Students should roll the dice five times and record the pairs of numbers they get. For each pair of numbers, let \(l\) be the larger number and \(s\) be the smaller number. For each pair of numbers, have students write one of two formulas, \(l \times n + s\) or \(l \times n - s\), on separate cards. All formulas should be different, so if the same pair of numbers appears a second time, students should use \(l \times n - s\) if they used \(l \times n + s\) before, and vice versa. If the same pair of numbers appears a third time, students need to roll again. Students should graph the sequences given by these formulas on graph paper, but should not record the formula with the graph. Next they should swap their formulas and graphs with a partner and match the formulas produced by their partners to the graphs of these formulas.

**Extensions**

1. a) Use the formula for the sequence, divisibility rules, and factors to explain why 216 cannot be a member of the sequence 4, 10, 16, 22.

   b) Use a formula for the sequence and make an equation to check whether 216 can be a member of these sequences:

   i) 8, 12, 16, 20  

   ii) 4, 7, 10, 13.
ANSWER:

a) The formula for the sequence is $6n - 4$. $6n$ is always divisible by 6, but if we subtract 4, the number is never divisible by 6. Since 216 is divisible by 6, it cannot be a term in the sequence.

b) i) The formula for the sequence is $4n + 4$. If 216 is a member of this sequence for some $n$, $216 = 4n + 4$. Solving for $n$ we get $n = 53$, so 216 is the 53$^{rd}$ term in the sequence. Another way to look at it is to say $4n + 4 = 4(n + 1)$, so if 216 is divisible by 4, it is some term in the sequence. Indeed, $216 = 200 + 16$, both divisible by 4, so 216 is divisible by 4 and is a term in the sequence.

ii) The formula for the sequence is $3n + 1$. If 216 is a member of this sequence for some $n$, $216 = 3n + 1$. Solving for $n$ we get $n = \frac{71}{3}$, so 216 is not a term in the sequence. Another way to look at it is to say that if 216 is a term in the sequence, then 215 should be divisible by 3. However, $2 + 1 + 5 = 8$ is not divisible by 3, so 215 is not divisible by 3 either. This means 216 is not a term in the sequence.

2. Each arrow in the picture is 1 m long.

a) Find the areas of the rings, starting with the first ring (inner radius 1 m, outer radius 2 m).

b) Find a formula for the area of the rings. What is the area of the 100$^{th}$ ring?

SOLUTION:

a) Ring 1: $4\pi - \pi = 3\pi$ m$^2$
Ring 2: $9\pi - 4\pi = 5\pi$ m$^2$
Ring 3: $16\pi - 9\pi = 7\pi$ m$^2$

b) There can be two formulas for the area of the rings:
- using the subtraction of areas: $(n + 1)^2\pi - n^2\pi$
- using the gaps in the pattern: the gaps are $2\pi$, so the formula is $2\pi n + \pi$.

Using both formulas, we get the area of the 100$^{th}$ ring:

$101^2\pi - 100^2\pi = 10 \times 201\pi - 10 \times 000\pi = 201\pi$ m$^2 \approx 631.4$ m$^2$;
or $2\pi \times 100 + \pi = 201\pi$ m$^2$ as well.

3. This extension goes with Question 4 on Workbook page 132.

a) Find the difference between two consecutive perfect squares and complete the chart at left.

b) Look at the sequence in the right column. Find a formula (different from $N^2 - (N - 1)^2$) to calculate the values in this column in terms of $N$. 

---

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N^2$</th>
<th>Difference $N^2 - (N - 1)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$1^2 - 0^2 = 1$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$2^2 - 1^2 = 3$</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>$3^2 - 2^2 = 5$</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>49</td>
<td></td>
</tr>
</tbody>
</table>
c) Explain how $16 - 9 = 7$ can be modeled by the picture at left.

d) Without calculating it, predict the difference between $12^2$ and $13^2$ and draw a model to show the difference.

e) Write each of the following numbers as a difference of two consecutive perfect squares:

i) 71  
ii) 101  
iii) 137  
iv) 12 321

**ANSWERS:**

i) Solve $71 = 2n - 1$, so $n = 36$ and $71 = 36^2 - 35^2$  
ii) $51^2 - 50^2$  
iii) $69^2 - 68^2$  
iv) $6161^2 - 6160^2$

f) Note that 9 is both odd and a perfect square. Write 9 as a difference of two consecutive perfect squares. What Pythagorean triple does this produce? **NOTE:** Positive integers $a$, $b$, and $c$ form a Pythagorean triple if $a^2 + b^2 = c^2$.

**ANSWER:** 3, 4, 5.

g) Write the first 20 perfect squares. Which ones are odd? (the squares of the odd numbers)

h) Show how to obtain a Pythagorean triple starting from any odd number bigger than 1. **HINT:** Square the odd number to get a number that is both odd and a perfect square and write it as a difference of two consecutive perfect squares. How does this give you a Pythagorean triple?

i) Question h) asked you to start with an odd number bigger than 1. What would happen if you started with 1?

j) Use your list from question g) to write 10 Pythagorean triples, each time starting with an odd perfect square.

k) Conjecture: *The smallest number in all Pythagorean triples is odd*. Use your list of 20 perfect squares to check the conjecture or to find a counter-example. (One counter-example could be 6, 8, 10.)
Variables as changing quantities. Show the equations from the top of Workbook page 133. Emphasize that the variable \(a\) is not representing an unknown number, but instead a changing number. The equation \(2 \times a = a + a\) is always true, no matter what we substitute for \(a\), as long as we substitute the same number for all three \(a\)’s. This is very different from an equation like \(3a + 1 = 13\), where \(a\) represents an unknown number and we have to find the number that makes the equation true.

More equations in one variable. Show the following equations:

\[
\begin{align*}
8 + 1 - 1 &= 8 \\
8 + 2 - 2 &= 8 \\
8 + 3 - 3 &= 8 \\
8 + 17 - 17 &= 8 \\
8 + 134 - 134 &= 8
\end{align*}
\]

**ASK:** What numbers are changing? (the numbers we add and subtract)

Have volunteers replace these numbers with a variable, such as \(a\):

\(8 + a - a = 8\). Explain that it doesn’t matter what we add and subtract—as long as we add and subtract the same number, we will always end up with 8.

Have students try to come up with more examples of equations in one variable that are always true, no matter what you substitute for the variable. Use Scribe, Stand, Share to assess as many students as possible.

**EXAMPLES:**

\[
\begin{align*}
a + 4 - 4 &= a \\
3 \times a &= a + a + a \\
a \times 1 &= a \\
a \times 0 &= 0 \\
a + 0 &= a
\end{align*}
\]

Equations that are almost always true. Write the following equations on the board:

\[
\begin{align*}
1 \div 1 &= 1 \\
2 \div 2 &= 1 \\
3 \div 3 &= 1 \\
4 \div 4 &= 1 \\
5 \div 5 &= 1
\end{align*}
\]
Have a volunteer replace the changing number with a variable: \( a \div a = 1 \).

**ASK:** Can we substitute any value for \( a \) and make the equation true?

**PROMPT:** Is there a number that you are not allowed to divide by? (0)

Explain that \( 0 \div 0 \) has no answer, so the equation \( a \div a = 1 \) is true as long as \( a \) doesn’t equal 0.

**Equations in two variables.** Remind students that \( a + 4 - 4 = a \) is true for any value of \( a \), as long as we substitute the same number for both \( a \)'s.

Now, write these equations:

\[
\begin{align*}
a + 1 - 1 &= a \\
a + 2 - 2 &= a \\
a + 3 - 3 &= a \\
a + 4 - 4 &= a \\
a + 5 - 5 &= a
\end{align*}
\]

Have students create more equations that follow the same pattern. **ASK:** What number is changing in these equations? Have a volunteer come and replace the changing number with a variable. Then write on the board:

\[
a + a - a = a
\]

**ASK:** Does this equation show the pattern here? Why not? Emphasize that the equation \( a + a - a = a \) tells you what happens when you substitute the same number for all four variables; for example, \( 5 + 5 - 5 = 5 \) and \( 6 + 6 - 6 = 6 \). But for our pattern, we need two different variables, so we need the more general \( a + b - b = a \), which tells you that as long as both occurrences of \( a \) are replaced with the same number and both occurrences of \( b \) are replaced by the same number, the equation is true. The first equation, \( a + a - a = a \), is a special case of the second equation (the case where \( a = b \)).

The statements “\( a \) and \( b \) are both not zero” and “\( a \) and \( b \) are not both zero”. Explain that the first statement means that \( a \) is not zero AND \( b \) is not zero. What does the second statement mean? (\( a \) is not zero OR \( b \) is not zero) Assign different values, including 0, to \( a \) and \( b \) and have students decide when these statements are true. **EXAMPLES:**

\[
\begin{align*}
a = 3 \text{ and } b = 0 & \quad a = 0 \text{ and } b = 0 \quad a = 5 \text{ and } b = 2 \quad a = 0 \text{ and } b = 7
\end{align*}
\]

The statement “\( a \) and \( b \) are both not zero” is only true for \( a = 5 \) and \( b = 2 \); “\( a \) and \( b \) are not both zero” is true for the first, third, and fourth examples.

Have students finish this sentence: The equation \( (a \div b) \times (b \div a) = 1 \) is true provided that __________. (Answer: \( a \) and \( b \) are both not zero.)

**Substituting for the variables.** Have students substitute \( a = 3 \) and \( b = 5 \) into various equations.

\[
\begin{align*}
a + b - b &= a \quad (3 + 5 - 5 = 3) \\
a + b &= b + a \quad (3 + 5 = 5 + 3) \\
a \times b &= b \times a \quad (3 \times 5 = 5 \times 3) \\
2(b - a) &= 2b - 2a \quad (2(5 - 3) = 2(5) - 2(3)) \\
5(a + b) &= 5a + 5b \quad (5(3 + 5) = 5(3) + 5(5))
\end{align*}
\]
EXTRA PRACTICE: Substitute \( a = -3 \) and \( b = -6 \) into each expression above.

Verifying equations. Review with students how to verify that an equation is true for given values of the variable: calculate both sides and make sure they both equal the same number. See Question 6 on Workbook page 134.

EXTRA PRACTICE for Question 6:
Verify that each equation is true for \( a = -4 \) and \( b = 7 \).

a) \( a \times b = b \times a \)   b) \( 3(a + b) = 3a + 3b \)   c) \( (a ÷ 2) \times (b \times 2) = a \times b \)
d) \( a + 3 + b - 3 = a + b \)   e) \( (b + 3) - (a + 3) = b - a \)

Equations in two variables that are sometimes true. Tell students that two of the following equations are true for all values of \( a \) and \( b \).

1) \( a \times b + b = (a + 1) \times b \)
2) \( a \times b + b = a \times (b + 1) \)
3) \( a \times b + a = a \times (b + 1) \)

Have students predict which two equations are true for all \( a \) and \( b \). Then have students substitute values to determine which equations are true for

i) \( a = -3 \) and \( b = 4 \)   ii) \( a = 2 \) and \( b = -6 \)   iii) \( a = 4 \) and \( b = 4 \)

Have students use their answers to decide which two equations are always true. (Equations 1 and 3) Which equation is only sometimes true? (Equation 2)

Substituting variables for variables. Show students another way to check if the equation is true when \( a = b \): Substitute \( a \) for \( b \) into both sides. Model the process on Equation 1, and have students substitute \( a = b \) into Equations 2 and 3. Discuss what happens with Equation 2: \( a \times b + b \) becomes \( a \times a + a \), and \( a \times (b + 1) \) becomes \( a \times (a + 1) \), so the equation becomes \( a \times a + a = a \times (a + 1) \), an equation in one variable that is true for any value of \( a \). EXAMPLE: \( 7 \times 7 + 7 = 7 \times 8 \).

Equations in three variables. Show students the following equations in two variables.

\[ a + b + 2 = 2 + a + b \]
\[ a + b + 3 = 3 + a + b \]
\[ a + b + 4 = 4 + a + b \]

Explain that all these equations are true, no matter what you substitute for \( a \) and \( b \). Have students replace the changing number with a different variable (i.e., neither \( a \) nor \( b \)). (EXAMPLE: \( a + b + c = c + a + b \))
Explain that this is an equation in three variables. Have students verify these equations in three variables for $a = 2$, $b = -5$, and $c = -4$:

a) $3(a + b + c) = 3a + 3b + 3c$

b) $3(a + b - c) = 3a + 3b - 3c$

c) $a \times (b + c) = a \times b + a \times c$

d) $a \times (b - c) = a \times b - a \times c$

e) $(a + b) \times c = a \times c + b \times c$

f) $(c - a) \times b = c \times b - a \times b$

**Extension**

If $4 \Diamond 3 = (4 \times 3) + (4 + 3)$ and $2 \Diamond 5 = (2 \times 5) + (2 + 5)$, calculate $7 \Diamond 9$.

**ANSWER:** $7 \Diamond 9 = (7 \times 9) + (7 + 9) = 63 + 16 = 79$
**Goals**

Students will simplify and solve equations of the form $ax + b = cx + d$ and solve problems requiring equations of this form.

**PRIOR KNOWLEDGE REQUIRED**

- Can solve equations of the form $ax + b = c$
- Understands preservation of equality
- Can substitute for the variable
- Understands that $ax = x + x + \ldots + x$ (a times)
- Can perform operations with integers

**Review relevant prior knowledge.** Review the fact that adding and subtracting the same number does not change the result, so $a + b - b = a$ for all $a$ and $b$. Have students write this equation for different values of $a$ and $b$, including negative numbers. As well, review the fact that when a variable is multiplied by a coefficient, this is the same as adding the variable the number of times equal to the coefficient, so $3a = a + a + a$. From this, we know that $5a + 3a = (a + a + a + a + a) + (a + a + a) = 8a$. Have students simplify expressions such as: $5a - 3a$, $6x + 4x$, $y + 3y$, $5v + (-7)v$.

**Adding and subtracting the same variable term.** Write on the board: $5a + 3a - 3a = ____$. Have students predict what this expression will be equal to. Then ask students to check their prediction using several methods: substituting several different numbers for $a$, simplifying $(5a + 3a - 3a = 8a - 3a = 5a)$, and using the fact that we add and subtract the same number, $3a$.

Have students write the operation and the number that will "undo" the addition or the subtraction in an equation. **EXAMPLE:** $3 + 4x - 4x = 3$

a) $5a - 3a + _____ = 5a$  b) $6 + 4x _____ = 6$

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>b)</td>
</tr>
<tr>
<td>$5a - 3a + _____ = 5a$</td>
<td>$6 + 4x _____ = 6$</td>
</tr>
<tr>
<td>$c) y + 3y _____ = y$</td>
<td>$d) 5 + (-7)v _____ = 5$</td>
</tr>
</tbody>
</table>

**Moving terms from one side of an equation to the other changes the sign.** Review preservation of equality. Remind students that to solve equations of the type $5a + 3 = 18$, they would subtract 3 from both sides. Point out that this was a useful procedure, because subtracting 3 from both sides would turn the equation into an equation that is easy to solve: $5a = 15$. Point out that this equation is easy to solve because it has terms with a variable (we call them variable terms) on one side and numbers, or constant terms, on the other side. Explain that a common method of solving equations is to bring all the constant terms to one side, and all the variable terms to the other side.
Ask students to look at what happens to the signs of the terms when they add or subtract the same number from both sides when solving an equation. Use these **EXAMPLES**:

- a) $6 + 4x = 14$
- b) $3x - 6 = 24$
- c) $6x + 14 = 74$

$$
\begin{align*}
4x &= 14 - 6 \\
3x &= 24 + 6 \\
6x &= 74 - 14 \\
4x &= 8 \\
3x &= 30 \\
6x &= 60
\end{align*}
$$

The sign of the term that is moved to the other side changes.

**Moving variable terms from one side to the other.** Have students add or subtract the underlined term from both sides, and check that the sign of the variable term is indeed reversed:

- a) $5 + 4x = 2x$
- b) $16 - 4x = 4x$
- c) $6x + 14 = 8x$

$$
\begin{align*}
5 + 4x - 4x &= 14x - 4x \\
7 - 3x + 3x &= 4x + 3x \\
6x + 14 - 6x &= 8x - 6x
\end{align*}
$$

Have students use this observation to move variable terms (underline them first) to the other side. **EXAMPLE:**

$$
15 + 8x = 21x \rightarrow 15 = 21x - 8x
$$

**ANSWERS:**

- a) $5 = 2x - x$
- b) $16 = 4x + 4x$
- c) $14 = 8x - 6x$

Repeat without underlining the variable terms, and have students decide for themselves which term should move.

- a) $25 + 7x = 2x$
- b) $36 - 2x = 4x$
- c) $63x - 66 = 96x$

**ANSWERS:**

- a) $25 = 2x - 7x$
- b) $36 = 4x + 2x$
- c) $-66 = 96x - 63x$

Show students how to solve the equation from a) by grouping the like terms first:

$$
\begin{align*}
25 + 7x &= 2x \\
25 &= 2x - 7x \\
25 &= -5x \\
-5 &= x
\end{align*}
$$

Demonstrate checking the solution by substituting $x = -5$ into the initial equation. Have students solve b) and c) the same way and check their solutions by substitution.

Have students solve more equations of the same type by first moving the variable terms to the same side. Have students use substitution to check their answers. Remind students that when checking the answer it is important to use the same equation they started with. **EXAMPLES:**

- a) $27 + 5x = 2x$
- b) $56 - 2x = 6x$
- c) $63x - 66 = 96x$
- d) $35 + 9x = 2x$
- e) $35 - 2x = 5x$
- f) $10x + 66 = 21x$

**Bonus**

- g) $24 + 7x - 4x = 2x$
- h) $33 - 2x = 4x + 5x$
- i) $-6x - 120 = 6x$

**ANSWERS:**

- a) $27 = 2x - 5x$, so $27 = -3x$, $x = 27 \div (-3) = -9$
- b) $7$
- c) $-2$
- d) $-5$
- e) $5$
- f) $6$
Moving variable and constant terms between the sides. Tell students that now their task will be harder: they will need to move both constant and variable terms between the sides of equations. Remind students that the goal is to get all variable terms on one side and all constant terms on the other side, but it is up to them to decide which side will have the constant terms and which side will have the variable terms. Do the first example below together. **EXAMPLES:**

- \(23 + 5x = 2x - 4\)
- \(65 - x = 6x - 2\)
- \(3x - 6 = 19 - 6x\)
- \(21 + 9x = 2x - 7\)
- \(-35 + 2x = 5x - 3\)
- \(-11x + 62 = 21x - 2\)

**Bonus**

- \(24 = 2x - 7x + 4x, 24 = -x, \text{ so } x = -24\)
- \(24 = -x\)
- \(x = -24\)
- \(h) \ 3\)
- \(i) \ -10\)

### Word problems producing equations of type \(ax + b = cx + d\).

Work through the examples in the next set of questions together, and have students complete the questions after each example individually.

1. A box contains some red and yellow apples. Let \(x\) represent the number of red apples in the box. Write the number of yellow apples in terms of \(x\).
   
   **EXAMPLE:** There are 10 more yellow apples than red apples.
   
   **SOLUTION:** yellow apples = \(x + 10\)
   
   a) 3 fewer yellow apples than red apples (\(x - 3\))
   b) 8 more red apples than yellow apples (\(x + 8\))
   c) seven times as many yellow apples as red apples (\(x ÷ 7\))
   d) 9 fewer red apples than yellow apples (\(x + 9\))

2. A box contains 35 red and yellow apples. Find the number of red and yellow apples.
   
   **EXAMPLE:** There are 3 more red apples than yellow apples in the box.
   
   **SOLUTION:** red apples: \(x\), yellow apples: \(x - 3\).
   
   \[
   x + x - 3 = 35 \\
   2x = 35 - 3 \\
   2x = 32 \\
   x = 16
   \]
   
   a) There are 7 more red apples than yellow apples. (red: \(x\), yellow: \(x - 7, x + x - 7 = 35, 2x - 7 = 35, 2x = 42, x = 21\), so 21 red and 14 yellow apples)
   b) There are 11 more yellow apples than red apples. (red: \(x\), yellow: \(x + 11, 2x + 11 = 35, x = 12\), so 12 red and 23 yellow apples)
   c) There are 4 times as many yellow apples as red apples. (red: \(x\), yellow: \(4x, 5x = 35, x = 7\), so 7 red and 28 yellow apples)
   d) There are 6 times as many red apples as yellow apples. Let the smaller unknown be \(x\). (yellow: \(x\), red: \(6x, 7x = 35, x = 5\), so 30 red and 5 yellow apples)
3. Write expressions for the number of red, green, and yellow apples in a box. \textbf{HINT:} Underline the colour that appears in both sentences. Let \( x \) represent that colour.

\textbf{EXAMPLE:} There are 4 more yellow apples than \textcolor{green}{green} apples. There are 5 times as many red apples as \textcolor{green}{green} apples.

\textbf{SOLUTION:} \textcolor{green}{green}: \( x \); red: \( 5x \); yellow: \( x + 4 \).

a) There are 3 fewer yellow apples than red apples. There are 8 times as many green apples as yellow apples. (yellow: \( x \), red: \( x + 3 \), green: \( 8x \))

b) There are 6 fewer red apples than green apples. There are 3 more yellow apples than red apples. (red: \( x \), green: \( x + 6 \), yellow: \( x + 3 \))

Have students solve several word problems resulting in equations of the form \( ax + b = cx + d \). \textbf{EXAMPLES:}

1. There are 6 fewer red apples than green apples in a box. There are 4 times as many yellow apples as red apples. The number of red and green apples is equal to the number of yellow apples. How many apples are in the box?

\textbf{SOLUTION:} red: \( x \), green: \( x + 6 \), yellow: \( 4x \).

\begin{align*}
  x + x + 6 &= 4x \\
  6 &= 2x \\
  3 &= x
\end{align*}

\textbf{ANSWER:} 3 red, 9 green, 12 yellow

2. Parking lot A charges a flat fee of $5 and $2 for every hour. Parking lot B charges $3 for every hour, without a flat fee. Jeremy parked at lot A and Marissa parked at lot B for the same time. They paid the same amount. How long did they park for? \textbf{(ANSWER:} Let \( x \) represent the number of hours. Solve \( 5 + 2x = 3x \) to get \( x = 5 \), so they parked for 5 hours.)

3. The triangle and rectangle at left have the same perimeter. What are the side lengths of both shapes?

\textbf{(ANSWER:} Solve \( 5x - 2 = 4x + 6 \), so \( x = 8 \); sides of triangle: 8, 14, 16; sides of rectangle: 8 and 11)\)

\textbf{Bonus} Is the triangle a right, acute, or obtuse triangle? \((8^2 + 14^2 = 260 > 256 = 16^2). The triangle is an acute triangle, though very close to a right triangle.\)

\textbf{Extension}

Sometimes it is awkward to use \( x \) to represent the colour that appears in both sentences. \textbf{EXAMPLE:} There are 8 times as many red apples as green apples. There are 3 more yellow than red apples.
If we let \( x \) be the number of red apples, then the number of green apples is \( \frac{x}{8} \), and the number of yellow apples is \( x + 3 \). In total we get \( x + x + 3 + \frac{x}{8} \) apples. This is not very convenient.

However, if we let \( x \) be the number of green apples, then we have \( 8x \) red apples, and \( 8x + 3 \) yellow apples, so the total number of apples will be \( x + 8x + 8x + 3 \).

a) Decide which colour to represent using \( x \). Then write the expressions for the other colours.

i) There are 5 more yellow than red apples. There are 7 times as many yellow apples as green apples. (green: \( x \), yellow: \( 7x \), red: \( 7x - 5 \))

ii) There are 5 times as many red apples as green apples. There are twice as many green as yellow apples. (yellow: \( x \), green: \( 2x \), red: \( 5 \times 2x = 10x \))

b) Solve these problems:

i) There are 8 more yellow apples than red apples. There are 5 times as many yellow apples as green apples. There are 80 apples altogether. How many apples of each colour are there? (Green: \( x \), yellow: \( 5x \), red: \( 5x - 8 \). Total: \( x + 5x + 5x - 8 = 80 \) and \( 9x = 72 \), so \( x = 8 \). There are 8 green, 40 yellow, and 32 red apples.)

ii) A box contains red, green, blue, and yellow beads. The total number of beads is a perfect square. There are twice as many green beads as red beads, three times more blue beads than green beads, four times more yellow beads than blue beads. What could the number of the beads of each colour be?

**SOLUTION:** Let \( r \) represent the number of red beads. Then there are \( 2r \) green beads, \( 3 \times 2r \) blue beads, and \( 4 \times 3 \times 2r \) yellow beads. The total number of beads is \( r + 2r + 6r + 24r = 33r \). We know that \( 33r \) should be a perfect square, so if we take \( r = 33 \), we will get \( 33r = 33 \times 33 = 33^2 \). (Any multiple of 33 with a perfect square will work as \( r \) too.) In this case there are 33 red, 66 green, 198 blue, and 792 yellow beads in the box.
PS8-5 Using a Picture I

Teach this lesson after: 8.2 Unit 4

Goals:
Students will use pictures to help visualize a problem and to prove algebraic results including the Pythagorean Theorem.

Prior Knowledge Required:
Can apply the Pythagorean Theorem
Can draw scale drawings
Knows that the sum of the angles in a triangle is 180°
Knows that a straight line has 180°
Knows that two triangles with proportionate side lengths have corresponding angles equal
Can write formulas for patterns (for Problem Banks 1, 2, 4)
Can apply the SAS rule for similar triangles (for Problem Bank 10)
Knows that when corresponding angles on a transversal are equal, the lines are parallel (for Problem Bank 10)

Vocabulary: Pythagorean Theorem, scale factor

Materials:
grid paper or BLM 1 cm Grid Paper (p. P-65)

The importance of seeing given information visually. Tell students that you want them to think of a word that has the letters c, l, r, t (sample answers: cartwheel, clarity, clutter). After some students tell you an answer, ASK: Did anyone write the letters down so that you didn’t have to remember them? SAY: There are many ways to make a problem easier. One of them is to write things down so you don’t have to keep everything in your head.

Drawing a picture to solve a problem. Write on the board:

John decided to go for a walk in his neighbourhood.
He started by going 1 block east.
Then he turned left and went 2 more blocks.
Then he turned left again and went 3 more blocks.
He kept turning left and going one more block than after the previous turn.
After he had walked a total of 45 blocks, he got to his school.
He took a shortcut and walked home in a straight line.
To the nearest whole number, how many blocks did he walk in total?
Give students time to read the problem, then ASK: What makes this problem hard? (there are a lot of words, it’s hard to picture what is happening) SAY: Close your eyes and picture what is happening as I read the problem aloud. Read the problem aloud, and then have a volunteer draw a map of the first few turns of John’s walk. (see example below)

If “Home” is not already labelled, have a volunteer mark where it is. SAY: When you draw a picture, you don’t have to keep everything in your head. That means you can focus on solving the problem. ASK: What is the next step to solving the problem? (figure out where the school is) SAY: We know how many blocks John walked to get to school, and we know where his home is, so we have to use this information to figure out where the school is. ASK: How can you tell if he got to the school yet? (count the total number of blocks he’s gone so far and see if that is equal to 45) How can you get the number of blocks he’s gone so far? (add 1 + 2 + 3 + ... + 7 = 28) SAY: He’s gone 28 blocks so far, so he needs to keep going. Have a volunteer draw the next turn. ASK: Now how far has he gone? (36 blocks) Repeat for the next turn. (45 blocks) SAY: So now we know where the school is. Label the school, as shown below:

ASK: Is the school east or west of John’s home? (east) Is the school north or south of John’s home? (south) Draw on the board:
ASK: How many blocks east of John's home is the school? (5) How do you know? (allow some students to explain their reasoning) Have students draw the picture of John's trip on grid paper, or BLM 1 cm Grid Paper, as shown below:

![Grid Paper Diagram](image)

ASK: Does the grid make it easier to tell how many blocks east of John's home the school is? (yes) How does it make it easier? (I can just count the blocks instead of figuring out a calculation) How many blocks south of his home is the school? (4 blocks) How do you know? (I counted the blocks on grid paper) SAY: If you are ever taking a test and you don’t have grid paper, you can draw a grid yourself. Show students a rough drawing of a grid on the board. SAY: Now that you know how far south and east of John’s home the school is, you can add those numbers to the right triangle diagram. Label the side lengths of the right triangle on the board, as shown below:

![Right Triangle Diagram](image)

Remind students that the problem says there is a direct shortcut from school to home. ASK: How can we find the length of the shortcut? (use the Pythagorean Theorem) Have students find the distance of the shortcut. (distance\(^2 = 4^2 + 5^2\), so distance = \(\sqrt{16 + 25} = \sqrt{41} \approx 6.4\)) ASK: To the nearest whole number, what is the total distance that John walked? (45 + 6 = 51 blocks)

**Exercises:** Draw a picture to solve the problem.

a) Yu goes for a walk. She starts at home. She walks 6 blocks south, then 3 blocks east, then 3 blocks north, and then 1 more block east. Then she walks home in a straight line. How many blocks did Yu walk in total?

b) Yu is showing a friend around her neighbourhood. She starts at home. Then she walks 2 blocks north, 3 blocks east, 4 blocks south, 9 blocks west, 8 blocks north, 3 blocks east, 1 block south, and 15 blocks east. They end up at Yu’s school. Yu knows a shortcut home that goes in a straight line. How far do Yu and her friend need to walk from the school to home, measured in blocks?
c) Yu walks 1 block east, then turns right and walks 2 blocks, then turns right and walks 3 blocks, and then turns right again and walks 4 blocks. She then turns left and walks 4 blocks, turns left again and walks 3 blocks, turns left again and walks 2 blocks, and then turns left again and walks 1 block. How far does she end up from home, and in which direction?

**Bonus:** Yu follows the same pattern as in part c), but changes the direction of her turns from right to left after $4n$ blocks instead of after 4 blocks. How far does she end up from home and in what direction?

**Selected solution:** b) Draw a picture on grid paper to make an easier problem:

From the picture, the horizontal distance is 12 and the vertical distance is 5. The total distance is $\sqrt{12^2 + 5^2} = \sqrt{169} = 13$ blocks.

**Answers:** a) 18 blocks, c) 4 blocks west, Bonus: $4n$ blocks west

**Geometric proofs.** SAY: Pictures are often useful for proving statements true, especially geometric statements. Remember that you used pictures to help you prove the Pythagorean Theorem. I want to show you another proof of the Pythagorean Theorem that also uses pictures. Let’s start with a right triangle with side lengths 3, 4, and 5 units. Then we'll do the same thing with any side lengths. Draw on the board:

SAY: Now we will use three different scale factors to scale the original triangle. Draw on the board:

SAY: The first triangle is scaled by a factor of 3, the second one by a factor of 4, and the third one by a factor of 5. I used the same numbers for scale factors as the side lengths of the triangle, and that was on purpose. Have students draw the three triangles on grid paper or BLM 1 cm Grid Paper and cut them out. Have students rearrange the three triangles to make a rectangle. Ask them to think about why the triangles fit the way they did, in terms of both the side lengths and the angles.
Exercises:

a) Draw a diagram to show how the triangles fit together. Label all the side lengths.
b) All the triangles are similar to the original triangle with side lengths 3, 4, and 5. In your picture, mark the angles in each triangle: use one mark for the angle opposite the shorter leg and two marks for the angle opposite the longer leg, as shown below:

\[ \begin{array}{c}
3 \\
4
\end{array} \]

Answers: a–b)

Have a volunteer draw the answer on the board. SAY: The three angles at the top look like they make a straight line. ASK: How can you know this for sure? (they are the same as the three angles in the original triangle with side lengths 3, 4, and 5) How does marking the angles in the triangles help you know that? (I can see they are the three different angles in the triangle) Point to the two bottom angles in the rectangle, and ASK: These look like right angles, but how do you know for sure? (they are the two acute angles in the original right triangle, so they add to 90°) SAY: All the angles are right angles, so this is really a rectangle.

SAY: Remember, the goal is to prove the Pythagorean Theorem for any right triangle with side lengths \( a-b-c \), not just one with side lengths 3-4-5. ASK: Which numbers occur in two triangles? (12, 15, and 20) Where do they occur in the rectangle? (the 12s are two opposite sides of the rectangle and the 15 and 20 are common sides of two triangles) SAY: Having the 15 and the 20 sides as common sides ensures that the third triangle fits the gap left by the other two triangles. If we keep track of how we got the numbers in the first place, it might help us see what is going on. Draw on the board:

Point to the side marked 5 × 4 and 4 × 5 and ASK: How do you know that these side lengths are the same? (5 × 4 = 4 × 5) Do you think this will be true with any pair of numbers, or does it just work for 4 and 5? (it will be true for any numbers) How do you know? (multiplication is commutative; the order you multiply numbers in doesn’t matter) What is the length of the top side of the rectangle? (3 × 3 + 4 × 4 = 25) What is the length of the bottom side of the rectangle? (5 × 5 = 25) How does this prove the Pythagorean Theorem for the triangle with side lengths 3-4-5? (the top and bottom sides of the rectangle are equal)
**Exercises:** Draw three triangles similar to a right triangle with sides $a$, $b$, and $c$, where $a \leq b \leq c$:
- one triangle scaled by a factor of $a$
- one triangle scaled by a factor of $b$
- one triangle scaled by a factor of $c$

Then prove that the three triangles can be rearranged to make a rectangle. Explain why:
- the three angles at the top make a straight line
- the four corners of the shape are right angles
- the sides of the triangles that touch in your picture are identical

Then find equal opposite sides that prove the Pythagorean Theorem.

**Solution:**

The three angles at the top make a straight line because they are equal to the three angles in the original triangle. The bottom corners make right angles because they are made from the two acute angles in the original right triangle, which together add to 90°. The two top corners are 90° because they correspond to the right angles in the original triangle. The sides of the triangles fit exactly together because $a \times c = c \times a$ and $b \times c = c \times b$ (i.e., the commutative property). So the shape is a rectangle. Then the top side length equals the bottom side length, so $b \times b + a \times a = c \times c$, which is exactly what the Pythagorean Theorem says.

**Problem Bank**

1. Use the pictures to complete the formula.

   ![Pictures of square grids]

   $n^2 = (n - 1)^2 + \underline{\quad}$

   **Answer:** $2n - 1$

2. a) How many squares are on the border of an $n \times n$ square?

   ![Pictures of square grids]

   b) Complete the formula: $n^2 = (n - 2)^2 + \underline{\quad}$

   c) Do part b) another way. Complete the formulas.

   $n^2 = (n - 1)^2 + \underline{\quad}$ and $(n - 1)^2 = (n - 2)^2 + \underline{\quad}$

   So $n^2 = (n - 2)^2 + \underline{\quad}$

   **Answers:** a) $4n - 4$; b) $4n - 4$; c) $2n - 1$, $2n - 3$, $4n - 4$
3. The points on grid paper that are at the corners of the squares are called lattice points. How many lattice points are on the boundary of each square?

a) \[ \square \]  b) \[ \square \]  c) \[ \square \]  d) \[ \square \]  e) \[ \square \]

**Answers:** a) 4, b) 8, c) 12, d) 16, e) 20

4. How many lattice points are on the boundary of an \( n \times n \) square?

**Answer:** \( 4n \)

5. How many lattice points are completely inside an \( n \times n \) square?

**Answer:** \( (n - 1)^2 \)

6. A mathematical theorem says that the area, in square centimetres, of a polygon drawn on 1 cm grid paper is given by the formula:

\[
\text{Area} = (\text{number of inside lattice points}) + \frac{1}{2}(\text{number of boundary lattice points}) - 1
\]

Prove that this formula is correct for squares whose four corners are lattice points and whose edges lie along grid lines. Use your answer to Problem Bank 1.

**Answer:**
\[
(\text{number of inside lattice points}) + \frac{1}{2}(\text{number of boundary lattice points}) - 1
= (n - 1)^2 + 2n - 1 = n^2.
\]

7. Check by example that the statement from Problem Bank 6 is true for some squares with lattice point corners with edges that are not along grid lines:
8. a) Shade the part of the picture that has area $b^2 - a^2$.

b) Rearrange the shaded pieces from part a) to make a rectangle. What is the length of your rectangle? What is the width of your rectangle?

c) The areas from parts a) and b) are equal. Finish the equation to show this.

$$b^2 - a^2 = (b + a) \times (b - a)$$

**Answers:**
a) ___, b) ___, c) $b^2 - a^2 = (b + a) \times (b - a)$

9. a) Calculate the area of each shaded semicircle in terms of $\pi$.

i) ___, ii) ___, iii) ___

b) Is the sum of the areas of the semicircles on the legs of a right triangle always equal to the area of the semicircle drawn on the hypotenuse? How do you know? Did leaving the areas in parts i) and ii) of part a) in terms of $\pi$ help you notice the pattern?

**Answers:** a) i) $9\pi/2$, $16\pi/2$, $25\pi/2$; ii) $25\pi/2$, $144\pi/2$, $169\pi/2$; iii) $\pi/2 \times (a/2)^2 = \pi a^2/8$, $\pi/2 \times (b/2)^2 = \pi b^2/8$, $\pi/2 \times (c/2)^2 = \pi c^2/8$; b) yes, because $\pi a^2/8 + \pi b^2/8 = \pi c^2/8$

10. In the triangle, $AD = BD$ and $AE = EC$.

a) Prove that $DE$ and $BC$ are parallel.

b) $\angle ABC = 75^\circ$ and $\angle AED = 35^\circ$. Find $\angle BAC$. 
Answers:
a) \( \triangle DAE \) and \( \triangle BAC \) are similar by the rule SAS for similar triangles. That means \( \angle ADE \) equals \( \angle ABC \), so lines \( DE \) and \( BC \) are parallel.
b) \( \angle ABC = 75^\circ \) and \( \angle BCA = \angle DEA = 35^\circ \), so \( \angle BAC = 180 - 75 - 35 = 70^\circ \)

11. You need to fit a thin rectangular piece of wood through a window. The wood measures 90 cm by 110 cm. The window is 60 cm by 80 cm. How can you do it?
Answer: Place the 90 cm side diagonally across the window. This works because the diagonal of the window is 100 cm long, by the Pythagorean Theorem.

12. a) Construct a trapezoid from a right triangle by joining the midpoints of the two shorter sides of the right triangle. Label the sides as shown.

Use the picture to show that \( 3^2 + 5^2 + 4^2 = \frac{1}{2}(10^2) \).

b) Construct a trapezoid from a right triangle as shown:

Use the picture to write \( 4^2 + 10^2 + 3^2 \) as a fraction of \( 15^2 \).

Bonus:

b) Construct a trapezoid from a right triangle as shown:

Show that \( a^2 + b^2 + c^2 = \frac{1}{2}d^2 \).
d) Construct a trapezoid from a right triangle as shown.

Write \( a^2 + b^2 + c^2 \) as a fraction of \( d^2 \).

**Answers:**

a) \( 3^2 + 5^2 + 4^2 = 3^2 + 3^2 + 4^2 + 4^2 = 2 \times 3^2 + 2 \times 4^2 = 2 \times 5^2 \), but \( 10^2 = (2 \times 5)^2 = 2^2 \times 5^2 \), which is twice \( 2 \times 5^2 \)

b) \( 4^2 + 10^2 + 3^2 = 4^2 + 8^2 + 6^2 + 3^2 = 4^2 + (2 \times 4)^2 + (2 \times 3)^2 + 3^2 = 4^2 + 4 \times 4^2 + 4 \times 3^2 + 3^2 = 5 \times 4^2 + 5 \times 3^2 \) and \( 15^2 = (3 \times 3)^2 + (3 \times 4)^2 = 9 \times 3^2 + 9 \times 4^2 \), so the fraction is \( 5/9 \)

Bonus: c) \( b^2 = a^2 + c^2 \) by the Pythagorean Theorem, so \( a^2 + b^2 + c^2 = a^2 + a^2 + c^2 + c^2 = 2a^2 + 2c^2 \), and \( d^2 = (2a)^2 + (2c)^2 = 4a^2 + 4c^2 \), so \( a^2 + b^2 + c^2 = 1/2 \ d^2 \)

\[ a^2 + b^2 + c^2 = a^2 + (2a)^2 + (2c)^2 + c^2 = 5a^2 + 5c^2 , \text{ and } d^2 = (3a)^2 + (3c)^2 = 9a^2 + 9c^2 , \text{ so } a^2 + b^2 + c^2 = 5/9 \ d^2 \]
PS8-6 Using a Picture II

Teach this lesson after: 8.2 Unit 4

Goals:
Students will add information to a given picture in order to solve a problem.

Prior Knowledge Required:
Can apply the Pythagorean Theorem
Can apply the congruence rules for triangles: SSS, SAS, ASA
Knows that when lines are parallel, alternate angles through a transversal are equal
Knows that when lines are parallel, corresponding angles through a transversal are equal
Can draw a picture to solve a problem
Knows that when alternate angles on a transversal are equal, the lines are parallel
Can estimate square roots to the nearest whole number (for Extended Problem)
Can add and multiply decimals (for Extended Problem)
Can convert between cm and m (for Extended Problem)

Vocabulary: alternate angles, auxiliary line, congruence rule, corresponding angles (in congruent triangles), corresponding angles (with transversals), corresponding sides, hypotenuse, parallel, parallelogram, perpendicular, rhombus

Materials:
BLM Star Angles (pp. P-78–79, see Problem Bank 14)
BLM Fire Department Ladder Problems (pp. P-82–85, see Extended Problem)

NOTE: Students should complete Lesson PS8-5 before starting this lesson.

Proofs using auxiliary lines. SAY: Sometimes you are given a picture of a situation, but you need to add something to it to help you solve the problem. This type of situation comes up in angle problems with parallel lines. ASK: What types of angles are equal when you have parallel lines and a transversal? (corresponding angles and alternate angles) Have volunteers draw examples on the board. (see below)

Corresponding angles

Alternate angles

Draw on the board:
SAY: I want to figure out what $\angle B$ is. ASK: What line can I add to the picture so that I will have alternate angles? Have a volunteer draw it as a dashed line on the board, as shown below:

![Diagram showing line AB and line BC with angles 50° and 20° labeled]

ASK: Can you describe the dashed line you drew? (I drew it parallel to the other two lines and through B) To emphasize these features, draw lines that are obviously incorrect one at a time and then erase them. First, draw a dashed line that is parallel to the other lines but not through B, and then draw a dashed line that goes through B but is not parallel to the other lines. Mark the correct dashed line as parallel. SAY: By adding the dashed line, you have changed the problem into an easier one. If the line had already been there, it would have been an easier problem to begin with. Have volunteers solve the problem on the board. As they work, ASK: How do you know that angle is 50°? (it is alternate to the top 50° angle) How do you know that the other angle is 20°? (it is alternate to the 20° angle) The final picture should look like this:

![Diagram showing line AB and line BC with angles 50° and 20° labeled]

ASK: So what is $\angle B$? (70°) Tell students that when they add a line to a picture to help them solve a problem, they are drawing an auxiliary line. Draw on the board:

![Diagram showing auxiliary lines]

Conjecture: $\angle A = \angle B$

SAY: When both lines in two angles are parallel, I think that the angles will be equal, but I want to prove it for sure. Sometimes there are different ways to add lines to a picture, and you can still prove the result you want whichever way you use.
**Exercises:** Use three different ways of adding lines to the picture to show that $\angle A = \angle B$.

![Diagram A](image1)

![Diagram B](image2)

![Diagram C](image3)

**Answers:**

a) $\angle A$ and $\angle B$ are both corresponding to the same angle, so they are equal;
b) $\angle A$ and $\angle B$ are both alternate to the same angle, so they are equal;
c) the angles to the left of $\angle A$ and $\angle B$ are corresponding and the angles to the right of $\angle A$ and $\angle B$ are corresponding, so $\angle A$ and $\angle B$ are both $180^\circ$ minus the same two angles and so are equal

**Review congruence rules for triangles.** SAY: Another situation where it is useful to draw a picture is when you are using the congruence rules. It would be hard to keep track in your head which sides and angles are equal without seeing a picture. Write on the board:

- SSS (side-side-side)
- SAS (side-angle-side)
- ASA (angle-side-angle)

SAY: Remember that for side-angle-side to work, you need to know that two sides and the angle *between* them are equal in both triangles. For angle-side-angle to work, you need to know that two angles and the side *between* them are equal.

**Exercises:** Can you prove the triangles are congruent? If so, which congruence rule can you use?

![Diagram D](image4)

![Diagram E](image5)

![Diagram F](image6)

**Bonus:**

**Answers:**

a) SAS; b) no; c) SSS; d) ASA; e) no; Bonus: yes, the third angle is equal if the other two are, and the equal sides are in between corresponding angles, so ASA
SAY: Sometimes the triangles you are trying to prove congruent have a side in common. That side is always an equal side. Draw on the board:

\[ \triangle ABC \]

SAY: There are two triangles in the picture. Have a volunteer name them. (\( \triangle ABC \) and \( \triangle ACD \))

ASK: Are they congruent? (yes) What congruence rule did you use? (SSS) Write on the board:

\[ AB = AC = BC = \]

Have volunteers tell you what the corresponding equal sides are in \( \triangle ACD \). Write them in as volunteers tell you how to complete the equations. (\( AD, AC, CD \))

**Exercises:** Can you prove the triangles are congruent with the given information? If so, which congruence rule can you use? If not, draw a counter-example.

a) [Diagram]

b) [Diagram]

c) [Diagram]

**Answers:** a) yes, SSS; b) yes, SAS; c) no, a counter-example would be an isosceles trapezoid:

**Solving congruence problems using auxiliary lines.** Draw on the board:

\[ AB = BC \text{ and } AD = DC. \] Prove that \( \angle A = \angle C. \)

Have a volunteer mark the equal sides in the picture. ASK: What line could you draw so that you could use a congruence rule? (\( BD \)) Show this on the board:
ASK: What congruence rule can you use? (SSS) Do angles A and C correspond to each other? (yes) Remind students that the phrase “corresponding angles” has two different meanings, both of which you are using in this lesson. In this case, corresponding angles in congruent triangles are equal.

**Exercises:**
1. Show that $\angle B = \angle D$.

   a) ![Diagram](image1.png)
   
   b) ![Diagram](image2.png)

   **Bonus:** $AB = DE$ and $AD = BE$

**Answers:**
a) join $AC$, then $\triangle ADC \cong \triangle ABC$ by SSS, so $\angle B = \angle D$; b) join $AC$, then $\triangle ABC \cong \triangle CDA$ by SSS, so $\angle B = \angle D$; Bonus: join $AE$, then $\triangle ABE \cong \triangle EDA$ by SSS, so $\angle B = \angle D$

2. A parallelogram is a quadrilateral with opposite sides parallel. Show that the opposite sides of a parallelogram are equal by following the steps below.
   a) Draw a parallelogram and a diagonal to split it into two triangles.
   b) Show that the two triangles are congruent. Hint: What side do the triangles have in common?
   c) What are the corresponding sides of the triangles?

**Solutions:**
   a) Draw and label a parallelogram as shown, and join $AC$:

   ![Diagram](image3.png)

   b) $\angle BAC = \angle ACD$ by alternate angles, $\angle BCA = \angle CAD$ by alternate angles, and the side between the equal angles, $AC$, is common to both $\triangle ABC$ and $\triangle CDA$, so the triangles are congruent by ASA; c) $AB = CD$ and $BC = AD$ by corresponding sides of congruent triangles

**Problem Bank**
1. How can you show that the diagonals of a rectangle are equal without using a congruence rule?

   **Answer:** Use the Pythagorean Theorem.

2. Remember, a parallelogram has opposite sides equal and parallel. Follow the steps to show that the diagonals of the parallelogram bisect each other; i.e.,

   ![Diagram](image4.png)

   a) Label the vertices of the parallelogram $ABCD$.
   b) Draw diagonals $AC$ and $BD$. Label the point of intersection $E$.
   c) Show that $\triangle ABE \cong \triangle CDE$
   d) Show that the diagonals bisect each other.
Solutions:

a–b)

\( \angle EAB = \angle ECD \), because they are alternate angles; \( \angle EBA = \angle EDC \), because they are alternate angles; and \( AB = DC \), because they are opposite sides of the parallelogram. So \( \triangle ABE \cong \triangle CDE \); d) since \( \triangle ABE \cong \triangle CDE \), the corresponding sides \( DE \) and \( BE \) are equal and the corresponding sides \( AE \) and \( EC \) are equal, so the diagonals bisect each other.

3. A rhombus is a quadrilateral with all sides equal. Show that a rhombus is a parallelogram using the steps below.
   a) Draw a rhombus and a diagonal to divide it into two triangles.
   b) Prove that the two triangles are congruent.
   c) Label all the equal angles.
   d) Are alternate angles equal? Are opposite sides parallel?

Selected solutions:

a) \( \triangle ABD \cong \triangle CDB \) by SSS, since \( AB = CD \), \( BD = DB \), and \( DA = BC \)

b) \( \angle ABD \cong \angle CDB \) are corresponding angles in congruent triangles and hence equal.

d) They are alternate angles, so \( AB \) and \( CD \) are parallel. Also, \( \angle ADB = \angle CBD \), since they are corresponding angles in congruent triangles, and they are alternate angles, so \( AD \) and \( BC \) are parallel. So the rhombus is a parallelogram.

4. Prove that the angles opposite the equal sides in an isosceles triangle are equal by following the steps below.
   a) Sketch an isosceles triangle and mark the equal sides.
   b) Join the midpoint of the unequal side to the vertex opposite it.
   c) Your picture now has two smaller triangles. Prove that those triangles are congruent.
   d) Do the angles opposite the equal sides correspond to each other in the congruent triangles?

Answers:

a–b)

\( \triangle ABD \cong \triangle CBD \); d) yes, \( \angle BAD = \angle BCD \)
5. Reproduce all the steps for these proofs on your own, without looking at your notes.
   a) A quadrilateral has opposite sides parallel. Show that opposite sides are equal.
   b) In a parallelogram, the diagonals bisect each other.
   c) A quadrilateral has all sides equal. Show that opposite sides are parallel.

6. Show that the diagonals of a rhombus are perpendicular.
   **Solution:** Label the rhombus $ABCD$. Draw the diagonals $AC$ and $BD$.

   $\triangle ABC \cong \triangle ADC$ by SSS. So, $\angle BAC = \angle DAC$ and $\angle BCA = \angle DCA$. Also, $\triangle DAB \cong \triangle DCB$ by SSS. So, $\angle ABD = \angle CBD$ and $\angle ADB = \angle CDB$. Now look at all four triangles in the diagram. They have two angles the same, so the third one is the same, too. That means all four angles around the intersection point of $AC$ and $BD$ are the same and they add to 360°, so they are all 90°.

7. Show that, in an isosceles triangle, the line joining the midpoint of the unequal side to the opposite vertex is perpendicular to that side.
   **Answer:** The two triangles are congruent in the picture below, by SSS.

   $\angle BDA = \angle BDC$. But together, they make a straight line, so $\angle BDA = \angle BDC = 90^\circ$. So, $BD$ is perpendicular to $AC$.

8. Does the congruence rule for triangles hold for parallelograms, too? Either prove that it works or show a counter-example.
   a) SSS  b) SAS  c) ASA
   **Answers:**
   a) No, sample counter-example:
   
   b) Yes. Once you know two sides and the angle between them, you can determine all the angles, and you can determine all the sides in order around the parallelogram.
   c) No, sample counter-example:
9. A square is drawn on the hypotenuse of an isosceles right triangle.

\[ \triangle DAB \cong \triangle ADC \text{ by SAS, so } \angle ABD = \angle DCA \text{ and } \angle DCA = \angle BAC \text{ because they are alternate angles. In fact, all eight angles around the square are equal and they add to 360°, so they are each 45° and so } \triangle ABF \cong \triangle ADE \text{ by ASA. This is true of all four triangles, so the area of each triangle is one fourth the area of the square.} \]

b) Using \( x \) for the length of the equal sides of the isosceles triangle, the area of the triangle is \( x^2 + 2 \), and the side length of the square is by the Pythagorean Theorem, \( \sqrt{x^2 + x^2} = \sqrt{2x^2} = \sqrt{2}x \). So, the area of the square is \( (\sqrt{2}x)^2 = 2x^2 \), the ratio of the areas is \( 2x^2/(x^2 + 2) = 4 \).

10. Find the area and perimeter of the trapezoid. All measurements are in centimetres.

Solution: Draw a perpendicular from \( B \) to \( DC \), and label the point on \( DC \) as \( E \). Then \( EC = (x + 7) - (x + 1) = 6 \). Then, by the Pythagorean Theorem, the height of the trapezoid is \( \sqrt{10^2 - 6^2} = \sqrt{100 - 36} = \sqrt{64} = 8 \). Then the area of the trapezoid is \( 8(x + 4) \text{ cm}^2 \) and the perimeter is \( 2x + 26 \text{ cm} \).
11. Find the area. Hint: Draw the common hypotenuse to both right triangles.

![Triangle Diagram]

**Answer:** 18 cm²

12. Two sides of a shape are called adjacent if they have a common vertex.
   a) If you connect the midpoints of adjacent sides of a rectangle, what shape do you get?
   b) If you connect the midpoints of adjacent sides of the shape from part a), what shape do you get?

**Answers:**
   a) a rhombus, because all four sides are the hypotenuse of congruent right triangles
   b) a rectangle, because the line joining the midpoints are parallel to the diagonals by SAS for similar triangles, but the diagonals of a rhombus are perpendicular because all four angles around the centre are equal and so must be 90°

13. Here is another way to show that isosceles triangles have base angles equal. Make two copies of the same triangle, as shown below:

![Isosceles Triangle Diagram]

You can say the following are corresponding sides: \(AB = EF\), \(BC = ED\), and \(AC = DF\).

a) What are the corresponding angles? How does this show that the base angles of an isosceles triangle are equal?

b) Use the same method to show that all angles in an equilateral triangle are equal.

**Answers:**
   a) the corresponding angles are \(\angle A = \angle F\) and \(\angle C = \angle D\), but \(\angle D\) is a copy of \(\angle A\), and \(\angle F\) is a copy of \(\angle C\), so \(\angle A = \angle C\)
   b) you can pick any sides in any order as corresponding, so all angles can be picked as corresponding to any other, so all angles are equal

14. Have students complete **BLM Star Angles**. Students will discover that the sum of the angles in two different stars is 180° and explain why their reasoning will work for any five-sided star.
Answers:

1.

2. a)

b) Sample explanation: I think they will always add to 180° because I can make a straight angle parallel to any side of the star, and I used exactly the same reasoning for the non-regular star as I did for the regular one—even the angles are equal in exactly the same order along the straight line, e, c, a, d, b in both stars—so the same reasoning should work for any five-sided star.

15. Teach students a pencil trick for verifying that the sum of the angles in a triangle is 180°. Place the triangle so that one edge is horizontal and place the pencil so that the tip is facing to the right, as shown below:
Rotate the pencil CCW through the bottom left angle of the triangle, then continue in the CCW direction through the angle of the top vertex, and then again in the CCW direction through the angle of the bottom right vertex, as shown below:

After being rotated in the same direction through all the angles of the triangle, the pencil now faces the opposite direction. That means that the total angle it was rotated through was 180°.

a) Use this method to verify that the sum of the angles in a five-sided star is also 180°.

b) Verify that the sum of the angles in a quadrilateral is 360° by showing that the pencil goes through a complete circle.
Star Angles (1)

1. Use the parallel lines to show that the sum of the angles $a$, $b$, $c$, $d$, and $e$ in the star below is equal to $180^\circ$. 
2. a) Draw parallel lines to show that the sum of the angles $a$, $b$, $c$, $d$, and $e$ in this star is also $180^\circ$.

b) Do you think the sum of the angles in any five-sided star will be $180^\circ$? Explain how you know.
Extended Problem: Fire Department Ladder Problems

Materials:
BLM Fire Department Ladder Problems (pp. P-82–85)
calculators

Extended Problem: Fire Department Ladder Problems. Give students BLM Fire Department Ladder Problems. Students will solve extended problems that require them to choose what length of ladder to use and how to place the ladder safely. They will need to apply the Pythagorean Theorem and find the slope of a line.

Answers:
1. a) \( \sqrt{24} \approx \sqrt{25} = 5 \), too high; b) \( \sqrt{50} \approx \sqrt{49} = 7 \), too low; c) \( \sqrt{38} \approx \sqrt{36} = 6 \), too low
2. a) 3.6 m, b) 1.7 m
3. a) about 26.7 m high, b) about 31.9 m high
4. a) the truck with the 25 m ladder, b) the truck with the 25 m ladder, c) the truck with the 30 m ladder
5. a) Option A, b) Option A: 5, Option B: 2.5; c) the ladder in Option A is steeper, and you can see that by the steepness since 5 > 2.5; d) B only; e) no

Bonus: about 50.3 m long
You may use a calculator to complete this task.

1. Mentally estimate the square root to the nearest whole number by using a perfect square. Say whether your estimate is too high or too low.
   
a) \( \sqrt{24} \)

b) \( \sqrt{50} \)

c) \( \sqrt{38} \)

2. a) The base of a 3.9 m ladder is placed 1.5 m from a building. How high can the ladder reach up the side of the building?

b) The base of a 2 m ladder is placed 1 m from a building. How high, to the nearest tenth of a metre, can the ladder reach up the side of the building?
Fire Department Ladder Problems (2)

3. A fire department has ladders of different lengths mounted on different trucks. The ladder needs to reach the top of the building. The base of the ladder is 3 m above the ground and cannot go closer than 8 m to the building.
   a) How high up the side of a building can a 25 m ladder reach, to the nearest tenth of a metre?
   
   b) How high up the side of a building can a 30 m ladder reach, to the nearest tenth of a metre?

4. The fire department wants to send the truck with the shortest ladder that will reach the top of the building, in case there is a taller building that needs emergency help while the truck is away.

   Which truck from Question 3 should the fire department send if …
   a) the building is 26 m tall?
   
   b) the building has nine floors and each floor is 2.8 m high?
   
   c) the building has eight floors and each floor is 3.9 m high?
Fire Department Ladder Problems (3)

5. Two firefighters need to reach the top of a building. The building is 7.5 m high and has a stone wall around it. The stone wall is:

- 3 m high,
- 1.5 m out from the building, and
- 0.3 m thick.

The firefighters need to bring a ladder (not one mounted on the truck). The firefighters have two options.

Option A

Option B

a) Label each picture with the distances given.

b) The steepness of a ladder can be measured as the unit ratio:

\[
\text{steepness} = \frac{\text{vertical distance}}{\text{horizontal distance}}
\]

What is the steepness of the ladder in each option?

Option A: __________  Option B: __________

c) Which ladder is steeper? How does your answer to part b) show this?
**Fire Department Ladder Problems (4)**

**d)** To be safe to climb, the steepness of the ladder has to be less than 4. Which options are safe?

- Option A only
- Option B only
- Options A and B
- No option is safe

**e)** Two firefighters, with all their equipment, weigh a total of 230 kg. They estimate that they need a ladder with steepness 3 or greater in order to hold their weight. Using a safe option, can both firefighters climb the ladder at the same time?

**Bonus** Three square buildings, each as tall as they are wide, are laid out as shown and a firefighter places a ladder along all three of them. The ladder touches all three buildings as shown below, with its base on the ground.

The shortest building is 10 m tall and 10 m wide, and the middle building is 15 m tall and 15 m wide. How long does the ladder need to be to reach the top of the tallest building?
Unit 5  Number Sense

In this unit, students will study exponents, including powers of negative integers, and the order of operations with exponents and integers. Students will also study negative fractions and decimals.

Meeting Your Curriculum
Ontario students should cover all topics in this unit. For WNCP students, this unit is optional.
NS8-104 Introduction to Powers

Pages 137–138

CURRICULUM EXPECTATIONS
Ontario: 8m1, 8m2, 8m4, 8m7, 8m11
WNCP: optional, [R, T, C]; 9N1, 9N2

GOALS
Students will evaluate powers by using repeated multiplication and investigate properties of powers.

PRIOR KNOWLEDGE REQUIRED
Can multiply a sequence of more than two whole numbers (e.g., 2 × 3 × 5)
Can use a calculator to multiply numbers
Can add, subtract, multiply, and divide two whole numbers
Knows that multiplication commutes (e.g., 2 × 4 = 4 × 2)

INTRODUCE POWERS AS REPEATED MULTIPLICATION. First, remind students that multiplication is a short form for repeated addition. Then point out that just as we can add the same number repeatedly, we can multiply the same number repeatedly. Introduce the concepts power, base, and exponent as on Workbook page 137. Emphasize that in math, just as in English, a base is the bottom part of something. Assign questions according to the progression on Workbook page 137 Questions 1–5.

INVESTIGATE THE COMMUTATIVITY OF POWERS. Remind students that multiplication is commutative—the order we write the numbers in doesn’t matter. For example, 2 × 5 = 5 × 2. ASK: Do you think the same is true for powers—will it matter which number is written as the base and which number is written as the exponent? Have students explain the basis for their prediction. (Sample answers: I think it won’t matter because of the example I found in Question 5; I think it won’t matter because it doesn’t matter for both addition or multiplication and just as multiplication is repeated addition, powers are repeated multiplication; I can see in a simple example like 2³ and 3² that order does matter—2³ is even and 3² is odd, or 2³ = 8 and 3² = 9; I think it will matter because an odd base will give an odd result and an even base will give an even result, so interchanging an even number and an odd number will make a difference.) Then have students do the Investigation on Workbook page 137 and discuss the results. Emphasize that sometimes predictions turn out not to be true even when we have good reasons for making the predictions.


ADD, SUBTRACT, MULTIPLY, AND DIVIDE POWERS. Remind students of the order of operations. ASK: Which operation do you perform first, addition or multiplication? (multiplication) Why? (because multiplication is a short form for repeated addition and 3 + 4 × 5 actually means 3 + 5 + 5 + 5 + 5, so you have to add the 5 four times—that is, multiply 4 × 5—and then add the result to the 3) Emphasize that powers are repeated multiplication. When you see a power such as 2³, you are really multiplying 2 three times, so
$5 \times 2^3$ really means $5 \times 2 \times 2 \times 2 = 5 \times 8 = 40$. This means that you have to evaluate the powers before multiplying. Tell students to evaluate powers before doing multiplication or division, just as they do multiplication or division before addition or subtraction. So the new order of operations is:

1. Evaluate powers.
2. Do all multiplication and division, in order from left to right.
3. Do all addition and subtraction, in order from left to right.

(\textbf{\textit{NOTE:}} Do not introduce brackets in the same expression as powers at this point.) Have students do Workbook page 138 Question 12.

\section*{ACTIVITY}

The cards on BLM Magic Cards (pp Q-24–Q-25) are an application of the fact that every number is a sum of powers of 2 or one more than a sum of powers of 2. To create their own “magic cards” (BLM Question 10), students write 1, 2, 4, 8, and so on (1 and then the powers of 2) in sequence in the top left of each card, extend the table in BLM Question 4, then use the table to fill in the rest of the numbers on the cards. To extend the table in BLM Question 4, students will need to write the numbers from 32 to 63 as sums of powers of 2 (or one more than a sum of powers of 2). For example, $41 = 32 + 8 + 1$, so students will know to put 41 on cards 1 (for the 1 in the sum), 4 (for the 8 in the sum), and 6 (for the 32 in the sum).

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Card \#1 & Card \#2 & Card \#3 & Card \#4 & Card \#5 & Card \#6 \\
1 & 2 & 4 & 8 & 16 & 32 \\
\hline
\end{tabular}
\end{center}

\section*{Extensions}

1. The greatest common factor of $5^6$ and $5^7$ is \underline{\hspace{2cm}}. ($5^6$)
2. The lowest common multiple of $5^6$ and $5^7$ is \underline{\hspace{2cm}}. ($5^7$)
3. \textbf{Connecting algebra to geometry.} This extension is good preparation for NS8-105–106 Extension 3. Have students copy the following picture of two squares onto grid paper (not in their notebooks, as students will need to cut the squares out).
Have students find the area of the shaded part in two ways:

a) Subtract the area of the small square from the area of the big square. \((17^2 − 12^2)\)

b) Cut out the shaded region, and cut and rearrange it into a rectangle. What is the length of the rectangle in terms of 12 and 17? \((17 + 12)\) What is the width of the rectangle in terms of 12 and 17? \((17 − 12)\)

c) Have students write an equation using only the numbers 12 and 17 and the exponent 2, based on the two ways of finding the area of the shaded region. **ANSWER:** \(17^2 − 12^2 = (17 + 12)(17 − 12)\).

d) Have students make their own picture to prove that \(19^2 − 16^2 = (19 + 16)(19 − 16)\).

e) Tell students that the dimensions of the squares can be 12 and 17, or 16 and 19, or any two numbers. Because the numbers can be any numbers, we can use variables to represent them. Show students how to change the equation in c) to an equation with variables: \(a^2 − b^2 = (a + b)(a − b)\). Challenge students to substitute various numbers into this equation to see if it is true, and to fill in the table below (students should make up their own examples for the last two rows):

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>(a + b)</th>
<th>(a − b)</th>
<th>((a + b)(a − b))</th>
<th>(a^2)</th>
<th>(b^2)</th>
<th>(a^2 − b^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>8</td>
<td>2</td>
<td>16</td>
<td>25</td>
<td>9</td>
<td>16</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>11</td>
<td>3</td>
<td>33</td>
<td>49</td>
<td>16</td>
<td>33</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>11</td>
<td>7</td>
<td>77</td>
<td>81</td>
<td>4</td>
<td>77</td>
</tr>
<tr>
<td>10</td>
<td>9</td>
<td>19</td>
<td>1</td>
<td>19</td>
<td>100</td>
<td>81</td>
<td>19</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>11</td>
<td>5</td>
<td>55</td>
<td>64</td>
<td>9</td>
<td>55</td>
</tr>
</tbody>
</table>

Some students might notice that even when \(a − b\) is negative, the equation still holds. This is interesting, because the picture doesn’t apply to that case, but it was the picture that helped us guess the general formula in the first place.

4. Recall that taking a number to the exponent 2 is called squaring the number. Explain that taking a number to the exponent 3 is called cubing the number; and just as \(7^2\) can be read “seven squared,” \(7^3\) can be read “seven cubed.” Why is that? (because the volume of a cube of side length 7 is \(7 \times 7 \times 7 = 7^3\))

5. **Comparing powers without computing them directly.**

a) Write the smallest power of 2 that will make the statement true.

\[5 = 5^1 < 2 \boxed{2} \quad 5^2 < 2 \boxed{4} \quad 5^3 < 2 \boxed{8}\]
ANSWERS: 3, 5, and 7.

b) Continue the pattern above. Which power is bigger now?

\[
5^4 \quad \underline{2} \quad \text{ANSWER: } 5^4 > 2^3
\]

Challenge students to explain why the pattern doesn’t continue to hold. (Explanation: You are continually multiplying the powers of 2 by \(2^2 = 4\) and the powers of 5 by 5, so eventually the power of 5 will become larger.)

c) Determine how many digits the number \(2^{12}\) has without calculating it, as follows.

i) Use \(2^4 > 10\) to explain why \(2^{12} > 10^3\).

ii) Use \(2^3 < 10\) to explain why \(2^{12} < 10^4\).

iii) Use your answers to i) and ii) to determine how many digits \(2^{12}\) has.

ANSWERS:

i) \(2^{12} = 2^4 \times 2^4 \times 2^4 > 10 \times 10 \times 10 = 10^3\)

ii) \(2^{12} = 2^3 \times 2^3 \times 2^3 \times 2^3 < 10 \times 10 \times 10 \times 10 = 10^4\)

iii) Any number between \(10^3 = 1000\) and \(10^4 = 10000\) has 4 digits, so \(2^{12}\) has 4 digits.

d) i) Calculate the powers to determine the correct sign (\(>\) or \(<\)).

\[
5^3 \quad \underline{10^2} \quad 5^4 \quad \underline{10^3}
\]

ii) Without calculating it, determine how many digits the number \(5^{12}\) has.

ANSWERS:

i) \(5^3 = 125 > 100 = 10^2\) and \(5^4 = 625 < 1000 = 10^3\)

ii) \(5^{12} = 5^3 \times 5^3 \times 5^3 \times 5^3 > 10^2 \times 10^2 \times 10^2 = 10^6\) and \(5^{12} = 5^4 \times 5^4 \times 5^4 < 10^3 \times 10^3 \times 10^3 = 10^9\).

So \(5^{12}\) is between \(10^6\) and \(10^9\). This means that it has 9 digits. Encourage students who don’t see this immediately to use a T-table to see that if a number is between \(10^6\) and \(10^9\) (as \(5^{12}\) is), then it has 9 digits (some students might think it has 8 digits). The T-table could look like this:

<table>
<thead>
<tr>
<th>number is between...</th>
<th>number has ___ digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 and (10^2)</td>
<td>2</td>
</tr>
<tr>
<td>(10^2) and (10^3)</td>
<td>3</td>
</tr>
<tr>
<td>(10^3) and (10^4)</td>
<td>4</td>
</tr>
</tbody>
</table>
From the same pattern, a number between $10^8$ and $10^9$ will have 9 digits, since the number of digits is equal to the higher power of 10.

**PROCESS ASSESSMENT**

7m1, [R]

6. a) **A rule for the sum of powers of 2.** Find a rule for the sum of the first $n$ powers of 2 by completing the following chart:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$2^n$</th>
<th>sum of $n$ powers of 2</th>
<th>$2^n + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$2 + 4 = 6$</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>$2 + 4 + 8 = 14$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Write a rule for the sum of the first $n$ powers of 2. Predict the sum of the first 10 powers of 2.

**ANSWER:** $2^{n+1} - 2$, so the sum of the first 10 powers of 2 is $2^{10} - 2 = 2048 - 2 = 2046$.

b) **How big are powers of 2?** You have a choice between 2 prizes:

- Prize A: monthly payments of $1000
- Prize B: monthly payments of powers of 2 ($2, $4, $8, $16, and so on)

If you won the prize for 10 months, which prize would you prefer? (Prize A: $1000 \times 10 = $10 000 but $2^{10} - 2 = $2046) Which would you prefer if you won for 20 months? (Prize B: $1000 \times 20 = $20 000 but $2^{20} - 2 = $2 097 150) If you start receiving these prizes in January, what would be the first month where the total payout from the Prize B would be more than the total payout from Prize A? (the 13th month, which is January of the 2nd year)

7. a) Computer codes are written as sequences of 0s and 1s. **SAY:** There are two possible sequences of length 1 and four possible sequences of length 2. Show them on the board:

0 1 00 10 01 11

Have students write all the 0-1 sequences of length 3. **ASK:** How many are there? (8) Write them on the board to summarize. **ASK:** How many sequences of length 4 do you think there will be? Show students how you can quickly change the sequences of length 3 to get all the sequences of length 4, and how doing so demonstrates that the number of sequences of length 4 is double the number of length 3: add a 0 to the end of all the sequences of length 3 and then add a 1 to the end instead.
Write on the board:

- 2 sequences of length 1
- 4 sequences of length 2
- 8 sequences of length 3
- 16 sequences of length 4

Have students continue the pattern to determine how many 0-1 sequences there are of lengths 5, 6, 7, 8, 9, and 10. **ASK:** Do these numbers remind you of anything? (they are powers of 2) Where have you seen any of these numbers in relation to computers? (**EXAMPLE:** 256 Megabytes)

b) **ASK:** Which power of 2 is closest to 1000? (the 10th) Tell students that a kilobyte is really 1024 bytes, not actually 1000 bytes.

c) A megabyte is about 1 000 000 bytes. Exactly how many bytes are in a megabyte? **HINT:** 1 000 000 = 1000 × 1000. **ANSWER:**

\[
1024 \times 1024 = 1 \, 048 \, 576.
\]

**PROCESS ASSESSMENT**

8. Students will need to be very familiar with writing numbers as sums of powers of 2, or one more than a sum of powers of 2. Doing **BLM Magic Cards** will provide this familiarity. **EXAMPLE:**

\[
217 = 128 + 64 + 16 + 8 + 1 = 2^7 + 2^6 + 2^4 + 2^3 + 1.
\]

The ancient Egyptians used this observation to multiply any number by any other number! For example, to calculate 37 × 217, start with 37 and double it continually:

\[
\begin{align*}
37 \times 1 &= 37 \\
37 \times 2 &= 74 \\
37 \times 4 &= 148 \\
37 \times 8 &= 296 \\
37 \times 16 &= 592 \\
37 \times 32 &= 1184 \\
37 \times 64 &= 2368 \\
37 \times 128 &= 4736 \\
\end{align*}
\]

Since \(217 = 128 + 64 + 16 + 8 + 1\),

\[
\begin{align*}
37 \times 217 &= 37 \times (128 + 64 + 16 + 8 + 1) \\
&= 37 \times 128 + 37 \times 64 + 37 \times 16 + 37 \times 8 + 37 \times 1 \\
&= 4736 + 2368 + 592 + 296 + 37 \\
&= 8029
\end{align*}
\]

a) Have students check this answer using the standard algorithm for multiplication.

b) Use the ancient method of multiplying to do the following questions and check your answer by using the standard algorithm:

\[
\begin{align*}
15 \times 19 &= 285 \\
13 \times 27 &= 341 \\
37 \times 29 &= 1073
\end{align*}
\]
c) When you do these questions, you have a choice in which number to expand. For example, to multiply $15 \times 19$, you could either write $15 = 1 + 2 + 4 + 8$ and find $19 \times 1 + 19 \times 2 + 19 \times 4 + 19 \times 8$ or you could write $19 = 1 + 2 + 16$ and find $15 \times 1 + 15 \times 2 + 15 \times 16$. Do each of the above questions using the other way and decide which way you like better for each question and why.

d) State one thing you like about the ancient Egyptian method of multiplying. (Sample answer: As long as I can add, I only have to multiply by 2 or double numbers; I don’t have to remember the times tables.)

e) State one thing you don’t like about the ancient Egyptian method of multiplying. (SAMPLE ANSWER: It’s a lot more work than using the standard algorithm!)
**Goals**

Students will evaluate, compare, and order simple powers and investigate properties of powers. Students will solve equations involving powers. Students will write numbers in expanded form using power notation.

**PRIOR KNOWLEDGE REQUIRED**

Can evaluate a power by using repeated multiplication
Can write a number in expanded form

**Review power, base, and exponent.** Have students identify the base and exponent in several powers. Students can signal their answers by holding up the correct number of fingers.

**Patterns in powers of 2.** See Workbook page 139 Question 1. Point out that if you know a power of 2, you can find the next power of 2 by multiplying by 2, i.e., doubling the number.

**Patterns in powers of 10.** Have students make a chart like the one on Workbook page 139, Question 2a), for powers of 10 instead of powers of 2. Then have students do these questions:

\[
\begin{align*}
a) \ 10^3 \times 10^2 &= \underline{10^\square} \\
&= 10^{\square} \\
b) \ 10^4 \times 10^4 &= \underline{10^\square} \\
&= 10^{\square}
\end{align*}
\]

Continue with these problems:

\[
\begin{align*}
c) \ 10^4 \times 10^3 &= 10^{\square} \\
d) \ 10^5 \times 10^1 &= 10^{\square} \\
e) \ 10^3 \times 10^3 &= 10^{\square} \\
f) \ 10^6 \times 10^2 &= 10^{\square}
\end{align*}
\]

Draw students’ attention to the exponents. **ASK:** How can you get the exponent in the answer from the other two exponents? (add the two exponents) **PROMPT:** What operation can you use? **ASK:** Do you think this will be true for any base, or just base 10? Demonstrate why the same property holds for any base by using the teaching box on Workbook page 139. Then have students do Workbook page 139 Questions 2–3.

**Bonus**

\[
\begin{align*}
3^2 \times 3^3 \times 3^4 &= 3^{\square} \\
7 \times 7^2 \times 7^3 \times 7^4 \times 7^5 \times 7^6 \times 7^7 \times 7^8 &= 7^{\square}
\end{align*}
\]

**PROCESS EXPECTATION**

Looking for a pattern

**PROCESS EXPECTATION**

Looking for a pattern, Using logical reasoning

**Relating powers of 9 to powers of 3.** Have students do Workbook page 140 Question 4. When students finish, challenge students to explain why the pattern holds. Emphasize that because each 9 in the product is being replaced by a product of two 3s, there are twice as many 3s in the product as there were 9s.
After students finish Workbook page 140 Question 4, provide the following bonus problem.

**Bonus**
Write each product as a power of 3:

a) \(3^4 \times 9^2 = 3^4 \times 3^6 = 3^{10}\)  
b) \(3^3 \times 9^2\)  
c) \(9^2 \times 2^7\)  
d) \(9^4 \times 3^4\)

**Comparing and ordering powers with the same base or exponent.**

Have students recall how easy it was to compare and order perfect squares: \(43^2 < 47^2\) because 43 < 47; there is no need to compute the squares to check which one is larger. Have students articulate why this was the case. (Sample answer: multiplying two smaller numbers together will get a smaller result than multiplying two larger numbers together.) Then write on the board: \(43^3 < 47^3\). Ask students if they think this is true and why. To help them, rewrite the powers as repeated multiplication: \(43 \times 43 \times 43 < 47 \times 47 \times 47\). Emphasize that multiplying three smaller numbers together will get a smaller result than multiplying three larger numbers together.

**ASK:** Which is larger: \(43^{19}\) or \(47^{19}\)? (because 47 is larger than 43 and multiplying 19 larger numbers together will result in a larger number than multiplying 19 smaller numbers together) Emphasize that as long as the exponent is the same, the power with the larger base is larger. Then repeat the line of questioning for powers with the same base, but different exponents. Emphasize that, for example, multiplying more 5s together will get a larger result than multiplying fewer 5s together, so when the base is the same, the power with the greater exponent is greater.

After students finish Workbook page 140 Question 6, provide the following bonus problem.

**Bonus**

\(8^6 = 32\)  
(ANSWER: 3, because both sides of the equation are equivalent to \(2^{15}\))

**Solving equations with powers.** Show students how to use an organized list to solve for the variable. By trying each possibility for \(x\)—when it is either the base or the exponent—and evaluating the power, students can determine which value for \(x\) solves the equation.

**NOTE:** Workbook page 140 Question 9 is preparation for finding the bases in Question 10. After students finish Workbook Question 10, provide the following bonus problem.

**Bonus**
Write each expression as a power of a prime number.

a) \(2^6 \times 8^2\)  
b) \(343 \times 7^2\)  
c) \(3125 \times 5^3\)  
d) \(4^3 \times 8\)  
e) \(25 \times 5^5\)  
f) \(4^3 \times 2^7\)  
g) \(243^2\)

**ANSWERS:** a) \(2^{11}\)  
b) \(7^5\)  
c) \(5^8\)  
d) \(2^9\)  
e) \(5^7\)  
f) \(2^{13}\)  
g) \(3^{10}\)

**Powers and expanded form.** Review writing numbers in expanded form and then have students write numbers in expanded form using powers of 10; see Workbook page 141 Questions 1–4. Then have students write numbers written in expanded form, using power notation, as whole numbers; see Workbook page 141 Question 5. Finally, have students compare numbers written in expanded form as in Workbook page 141.
Question 6. Encourage students who struggle with Question 6 to circle the part in the two expressions that is different. Alternatively, students could write both expressions as whole numbers in standard form.

**Extensions**

1. How many 8s must I add together to get a sum equal to $8^3$?
   (64 because $8^3 = 8^2 \times 8 = 64 \times 8$)

2. How many $8 \times 8$ squares do I need to have in order to have a total area of $8^3$? (8 because $8^3 = 8 \times 8^2$)

3. Have students complete the chart by filling in the answers and continuing the patterns in the last two rows:

<table>
<thead>
<tr>
<th>1 + 2 + 3 + 4 + 5 + 6 = _____</th>
<th>$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = _____$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 + 2 + 3 + 4 + 5 = _____$</td>
<td>$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = _____$</td>
</tr>
<tr>
<td>$1 + 2 + 3 + 4 = _____$</td>
<td>$1^3 + 2^3 + 3^3 + 4^3 = _____$</td>
</tr>
<tr>
<td>$1 + 2 + 3 = 6$</td>
<td>$1^3 + 2^3 + 3^3 = _____$</td>
</tr>
<tr>
<td>$1 + 2 = 3$</td>
<td>$1^3 + 2^3 = ____$</td>
</tr>
<tr>
<td>1 = 1</td>
<td>1 = 1</td>
</tr>
</tbody>
</table>

Have students think about and describe how you can get the numbers in the second column from the numbers in the first column. (The numbers in the second column are the square of the sums in the first column. For example: $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1 + 2 + 3 + 4 + 5)^2$.)

Show students how to visualize the equations in the second column for small numbers:

Since $1^3 + 2^3 = 1 \times 1^2 + 2 \times 2^2$, draw one $1 \times 1$ square and two $2 \times 2$ squares. Then try to fit these three squares into a $3 \times 3$ square, since $3 = 1 + 2$, and the equation says $1^3 + 2^3 = (1 + 2)^2$.

![Visualizing](image)

This is almost a $3 \times 3$ square—just the bottom right corner is missing. But the two striped squares overlap in the middle, and if we take the overlapping part of one of the striped squares and move it to the bottom right corner, we get exactly a $3 \times 3$ square (see next page).
So $1^3 + 2^3 = (1 + 2)^2$.

The next case is $1^3 + 2^3 + 3^3 = 1 \times 1^2 + 2 \times 2^2 + 3 \times 3^2$, so draw one $1 \times 1$ square, two $2 \times 2$ squares, and three $3 \times 3$ squares, and try to fit them into a $6 \times 6$ square, since $6 = 1 + 2 + 3$ and the equation says $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$.

So $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$.

Challenge students to show the next case: $1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$. Give students 1-cm grid paper (see page U-25) and scissors. Students could colour the squares of different sizes different colours, and make a poster to show their work.
**Powers of Negative Numbers**

Exponents, Integers, and Order of Operations

Pages 142–143

**Goals**

Students will evaluate powers of negative integers. Students will evaluate expressions that involve integers, and that include brackets and powers, using order of operations.

**PRIOR KNOWLEDGE REQUIRED**

- Can add, subtract, multiply, and divide integers
- Can use the order of operations without powers
- Can write repeated multiplication using power notation

**Review multiplying integers.** Remind students that multiplying two positive or two negative integers results in a positive integer, whereas multiplying a positive integer and a negative integer results in a negative integer. Display the chart shown in the margin. Have students practise:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) (3 \times (-4))</td>
<td>-12</td>
</tr>
<tr>
<td>b) ((-2) \times (-5))</td>
<td>10</td>
</tr>
<tr>
<td>c) ((-3) \times 2)</td>
<td>-6</td>
</tr>
<tr>
<td>d) ((+2) \times (-7))</td>
<td>-14</td>
</tr>
<tr>
<td>e) ((-4) \times (-5))</td>
<td>20</td>
</tr>
</tbody>
</table>

**ANSWERS:** a) -12   b) 10   c) -6   d) -14   e) 20

Have students do examples that require multiplying many integers. Students can multiply two or more terms in each step.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) ((-2) \times (-3) \times (4))</td>
<td>24</td>
</tr>
<tr>
<td>b) ((-3) \times (5) \times (-2) \times (-4))</td>
<td>-120</td>
</tr>
<tr>
<td>c) ((-10) \times (-2) \times (-3) \times (-5) \times (-1) \times (-7))</td>
<td>2100</td>
</tr>
</tbody>
</table>

**ANSWERS:** a) 24   b) -120   c) 2100

Work through Workbook page 142 Question 1a) together as a class, then have students complete parts b)–e) on their own. When students are finished, discuss the pattern: which powers are positive and which are negative, and why. Emphasize that each time we multiply by a negative number, we change the sign of the product, so having an even number of negative terms makes the product positive, while an odd number of negative terms makes the product negative.

Then have students complete Workbook page 142.

**Review the correct order of operations with brackets but no powers.**

1. Do operations in brackets.
2. Do multiplication and division, from left to right.
3. Do addition and subtraction, from left to right.

Emphasize, in particular, that multiplication stands for repeated addition, so it makes sense to do multiplication before addition. In the expression
2 + 3 \times 4, the 3 isn’t being added to the 2, it is telling how many 4s to add together: 2 + 4 + 4 + 4. That’s why we do multiplication before addition. Furthermore, multiplication and division are “opposite” operations that undo each other, so it makes sense to do them at the same time, and not to do one before the other. The same goes for addition and subtraction.

**EXTRA PRACTICE:** Evaluate:

a) 3 − (7 + 2)  
b) 3 − 7 + 2  
c) 8 \times 3 − 5  
d) 8 \times (3 − 5)  
e) 8 − (5 + 1) + 3 \times (−2) − 8 ÷ (−2)  
f) 15 ÷ (−3) \times (−4) − 3 \times (2 − 4)

Solution:

e) Do operations in brackets: 8 − 6 + 3 \times (−2) − 8 ÷ (−2). Then do multiplication and division, from left to right: 8 − 6 + (−6) − (−4).

Then do addition and subtraction, from left to right:

\[
2 + (−6) − (−4) = −4 − (−4) \\
= 0
\]

f) Do operations in brackets: 15 ÷ (−3) \times (−4) − 3 \times (−2). Then do multiplication and division, from left to right:

\[
(−5) \times (−4) − 3 \times (−2) = 20 − (−6) \\
= 26
\]

**PROCESS ASSESSMENT**

8m2, [R]  
Workbook p 142 Question 4

Review the correct order of operations with powers but no brackets.

1. Evaluate powers.
2. Do all multiplication and division, in order from left to right.
3. Do all addition and subtraction, in order from left to right.

Emphasize, in particular, that a power stands for repeated multiplication, so it makes sense to evaluate the powers before doing any multiplication.

For example, look at 2 \times 3^4. This expression really means 2 \times 3 \times 3 \times 3 \times 3, because 3^4 means 3 \times 3 \times 3 \times 3. Also, point out that the expressions (−2)^4 and −2^4 mean different things:

\[
(−2)^4 = (−2) \times (−2) \times (−2) \times (−2) \\
= (−2)^2 \times (−2)^2 \\
= (+4) \times (−2) \\
= −8
\]

\[
−2^4 = −(2) \times (2) \times (2) \times (2) \\
= −2 \times 2 \times 2 \times 2 \\
= −16
\]

So −2^4 is the opposite of 2^4.

**EXTRA PRACTICE:** Evaluate:

a) 3 \times 5^2  
b) 1000 ÷ 5^3  
c) 3^2 − (−2)^3  
d) −4^3 ÷ (−2)^2  
e) 3 + 4^2 ÷ (−2)  
f) 1000 ÷ (−2)^3  
g) 7 − 3^2 \times (−2)  
h) 12 ÷ (−4) \times 5

**ANSWERS:** a) 75  
b) 8  
c) 17  
d) −16  
e) −5  
f) −125  
g) 25  
h) −15

The order of operations with brackets and powers. Emphasize that expressions inside brackets are evaluated first, even before powers.
EXTRA PRACTICE: Evaluate:

a) \((7 - 5)^3\)  
b) \(7^3 - 5^3\)  
c) \((5 - 8)^2\)
d) \(5^2 - 8^2\)  
e) \(5 + 3^4 - (2 \times 5)\)  
f) \(3375 \div (5^2 \times (-3))\)
g) \(3375 \div 5^2 \times (-3)\)  
h) \((7 - 3)^2 \times (-2)\)

ANSWERS: a) 8  
b) 218  
c) 9  
d) -39  
e) 76  
f) -45  
g) -405  
h) -32

Include expressions in exponents. Tell students that sometimes there is an expression in an exponent that needs to be calculated before you can evaluate the power. All the numbers in the same exponent should be treated as if they were in brackets and so evaluated first. EXAMPLE: \(3^2 + 4 = 3^6\).

Have students practise:

a) \(3^8 - 6\)  
b) \((-2)^{6 + 2}\)  
c) \(5^8 - 3 \times 2\)  
d) \((7 - 10)^3 - 2 + 1\)
e) \((-2)(4 + 2)\div 2\)  
f) \((-2)^4 + 2 \div 2\)  
g) \((−8)^2 \div (−2)^4 \times (3 + 2)^20 \div 5\)

ANSWERS:

a) 9  
b) -8  
c) 25  
d) 9  
e) -8  
f) -32  
g) 2500

Summarize the order of operations.

1. Do operations in brackets.
2. Calculate exponents and evaluate powers.
3. Do multiplication and division, from left to right.
4. Do addition and subtraction, from left to right.

Have students do Workbook page 143 Question 3 for practice.

Adding brackets changes the value of an expression. Write on the board and evaluate: \(25 - 3^2 + 2 \times 2^2 (= 24)\).

Show students different ways of adding brackets to this expression and ask them to evaluate the new expressions using the correct order of operations:

a) \(25 - 3^2 + (2 \times 2)^2\)  
b) \((25 - 3)^2 + 2 \times 2^2\)  
c) \((25 - 3)^2 + (2 \times 2)\)
d) \((25 - 3^2 + 2 \times 2^2)\)  
e) \(25 - (3^2 + 2) \times 2^2\)  
f) \(25 - (3^2 + 2) \times 2^2\)
g) \((25 - 3^2 + 2 \times 2)^2\)

ANSWERS: a) 32  
b) 492  
c) 500  
d) 8  
e) 0  
f) -19  
g) -144

Then point out that some ways of adding brackets don’t make sense. For example, it wouldn’t make sense to add an opening bracket between a base and an exponent in a power, e.g., \(25 - 3(2 + 2) \times 2^2\). Write this expression on the board and ASK: Why doesn’t this make sense? (the exponent doesn’t mean anything without the base) Point out, however, that a closing bracket can be put between a base and its exponent (as in parts a, b, c, e, and g above) because the expression in brackets becomes the base for the exponent.

Have students add brackets to these expressions to get as many different answers as they can.

a) \(16 + 8 \div 2 + 6\)  
b) \((-5)^6 + 3 \times 2\)  
c) \((-2)^8 + 2 \div (-5)^2 + 5\)

ANSWERS: a) 26, 3, 18, 17  
b) 625 or -5  
c) -22, -1500, 372.8
Patterns in ones digits of powers. Have students complete this chart:

<table>
<thead>
<tr>
<th>$2^n$</th>
<th>$2^1$</th>
<th>$2^2$</th>
<th>$2^3$</th>
<th>$2^4$</th>
<th>$2^5$</th>
<th>$2^6$</th>
<th>$2^7$</th>
<th>$2^8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ones digit</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

After students complete the chart, have students predict the ones digit of $2^9$, $2^{10}$, $2^{11}$, and $2^{12}$, and explain the basis for their predictions. Ask students to predict the ones digit for $2^{22}$. (4, because every fourth term in the sequence has the same ones digit, so $2^{22}$ has the same ones digit as $2^{18}$, $2^{14}$, $2^6$, and $2^2$)

ASK: What is the ones digit of $2^{7051}$? Start by listing the powers of 2, in backwards order, that have the same ones digit: $2^{7047}$, $2^{7043}$, $2^{7039}$. ASK: Is this a good method to use? Why not? Why did it work well for $2^{22}$? Tell students that although the method we used for $2^{22}$ doesn’t work as well for $2^{7051}$, we can still look at the easier problem for ideas. Write on the board:

$2^2$, $2^6$, $2^{10}$, $2^{14}$, $2^{18}$, $2^{22}$

Then write the sequence of exponents on the board: 2, 6, 10, 14, 18, 22. ASK: How can we describe these numbers? (sample answers: they are all of the form $4n + 2$; they are all 4 more than the previous term) Then ask students to think in terms of remainders: How can you describe these numbers in terms of remainders? (They all have remainder 2 when dividing by 4.) Emphasize that all these powers of 2 have the same ones digit as $2^0$ (which is the first exponent in the sequence that has remainder 2 when dividing by 4).

ASK: How does this help us with the harder problem of finding the ones digit of $2^{7051}$? PROMPT: What do we have to know about 7051 to help us answer this question? (the remainder when dividing by 4) Have students use long division to find this remainder, and then use this information to determine the ones digit of $2^{7051}$. (The remainder when dividing 7051 by 4 is 3, so $2^{7051}$ has the same ones digit as 2 to the exponent 3. Since $2^3$ is 8, $2^{7051}$ has ones digit 8.)

Bonus: What is the ones digit of $2^{151}$, $2^{251}$, $2^{351}$, $2^{451}$, $2^{551}$, $2^{651}$, $2^{751}$, and $2^{800651}$? (these all have remainder 3 when dividing by 4; in
fact the last 2 digits determine the remainder when dividing by 4; students using the WNCP curriculum might remember this fact from Grade 7)

When students finish, point out that it is quite surprising that we can find the ones digit of $2^{7051}$ when we know so little about the number itself. We don’t know any of its other digits, or even how many digits there are!

**Extensions**

1. a) Write each power of 10 in standard form.
   \[
   \begin{align*}
   10^4 &= \underline{\quad} \\
   10^3 &= \underline{\quad} \\
   10^2 &= \underline{\quad} \\
   10^1 &= \underline{\quad}
   \end{align*}
   \]

   b) How do we get each number on the right side of the equations in part a) from the number above it? (divide by 10) How do we get each exponent in the left side of the equation from the exponent above it? (subtract 1)

   c) Continue the pattern in the exponents and in the numbers:
   \[
   10 = \underline{\quad}
   \]

   d) Repeat parts a) and b) for powers of 2, powers of 3, and powers of 5.

   e) What is the zeroth power of any number? Explain. (Each power is obtained from the previous one by dividing by the base. But the first power of any base is the base itself, so the zeroth power is the base divided by the base, which is always 1.)

   f) Introduce negative powers. Have students continue the pattern:
   \[
   \begin{align*}
   10^2 &= 100 \\
   10^1 &= 10 \\
   10^0 &= 1 \\
   \quad^{-1} &= \underline{\quad} \\
   10^{-2} &= \underline{\quad}
   \end{align*}
   \]

   **ANSWERS:** 1/10 or 0.1, and 1/100 or 0.01, because you divide by 10 to get the next term in the pattern.

2. Remind students that to find two or three consecutive numbers that add to a number, we can use equations (e.g., $x + x + 1 = 37$).
   a) Use an equation to find two consecutive numbers that add to:
   \[
   \begin{align*}
   \text{i)} & \quad 37 \quad \text{i)} & \quad 79 \quad \text{iii)} & \quad 63 \quad \text{iv)} & \quad 99
   \end{align*}
   \]
b) Find half of:
   
i) 37  ii) 79  iii) 63  iv) 99

c) How are your answers to parts a) and b) related?

   Explain to students that the two consecutive numbers are very close together—they are only one apart. If they were exactly the same and added to 37, they would be exactly half of 37, but because they are only close to each other and not exactly the same, they are only close to half of 37.

d) Use half of 85 to find two consecutive numbers that add to 85.

   Tell students that you want to find three consecutive numbers that add to 60. **SAY:** The three numbers are very close together—they are only one apart. If the same number was added 3 times to get 60, what number would that be? (20) Tell students that they should look for numbers close to 20 to find three consecutive numbers that add to 60. (19 + 20 + 21 = 60)

e) Use 1/3 of 72 to find three consecutive numbers that add to 72.

   (1/3 of 72 = 24, so 23 + 24 + 25 = 72)

   Tell students that you want to find two consecutive numbers that **multiply** to 156. **SAY:** These two numbers are close together—they are only one apart. If a number was multiplied by itself to get 156, what would the number be? (the square root of 156) What two consecutive numbers are closest to the square root of 156? (12 and 13 because 12² = 144 and 13² = 169) Tell students that if numbers that are close together multiply to 156, they should be close to the square root of 156, so a good guess for the two numbers is 12 and 13. Indeed, 12 × 13 = 156.

f) Which whole number, when squared, is close to 182? Calculate the square root of 182 on a calculator and use the result to find two consecutive numbers that multiply to 182. (13 × 14 = 182)

g) Guess and check to find the whole number which, when cubed, is closest to 1320. (10³ = 1000 and 11³ = 1331, so 11)

h) The product of three consecutive numbers is 1320. Find these numbers. (10 × 11 × 12 = 1320)

i) The product of three consecutive numbers is 7980. Find these numbers. (19 × 20 × 21 = 7980)

3. **Prime factorizations, powers, squares, and square roots.**

a) Write each square of a power as a power itself. **EXAMPLE:**

   \[(2^3)^2 = 2^3 \cdot 2^3 = (2 \times 2 \times 2) \times (2 \times 2 \times 2) = 2^6\]

   i) \((3^2)^2\)  ii) \((2^3)^2\)  iii) \((5^3)^2\)  iv) \((7^4)^2\)  v) \((8^5)^2\)

   **Bonus** \((2^{324})^2\)
b) Teach students to write prime factorizations using power notation. For example, $24 = 2 \times 2 \times 2 \times 3$ can be written as $24 = 2^3 \times 3$. Have students write prime factorizations for these numbers using power notation:

i) 12  
ii) 75  
iii) 96  
iv) 72

c) Why is $5^3$ a prime factorization, but $8^2$ is not a prime factorization? (Because $5$ is prime, but $8$ is composite and can be factorized further) PROMPT: Draw the factor tree for each number—$5^3 = 125$ and $8^2 = 64$.

d) If the prime factorization of a number is $2^3 \times 7^5 \times 11^8$, what is the prime factorization of its square? ($2^6 \times 7^{10} \times 11^{16}$)

e) How can you tell immediately from the exponents in a prime factorization whether the number is a perfect square? (If all exponents are even, the number is a perfect square; if any exponent is odd, the number is not a perfect square.)

f) Write each square root of a power as a power itself. EXAMPLE:

\[
\sqrt{2^6} = \sqrt{2 \times 2 \times 2 \times 2 \times 2 \times 2} = \sqrt{(2 \times 2) \times (2 \times 2) \times 2} = 2 \times 2 \times 2 = 2^3
\]

i) $\sqrt{2^4}$  
ii) $\sqrt{3^6}$  
iii) $\sqrt{3^{10}}$  
iv) $\sqrt{5^{16}}$

ANSWERS: i) $2^2$  
ii) $3^3$  
iii) $3^5$  
iv) $5^8$

4. a) Mathematicians have proven that if $a$ and $b$ have GCF = 1, and $a$ is a prime number, then $a$ is a factor of $b^{a-1} - 1$. This is called Fermat’s Little Theorem.

Check this for:

\[
a = 2 \text{ and } b = 3 \\
a = 2 \text{ and } b = 5 \\
a = 3 \text{ and } b = 2 \\
a = 3 \text{ and } b = 4 \\
a = 3 \text{ and } b = 5 \\
a = 3 \text{ and } b = 10 \\
a = 5 \text{ and } b = 2 \\
a = 5 \text{ and } b = 3 \\
a = 5 \text{ and } b = 4 \\
a = 7 \text{ and } b = 10 \\
\]

your own example: $a =$ _____ and $b =$ _____

(make sure $a$ is prime and GCF of $a$ and $b$ is 1)

b) For $a = 4$ and $b = 7$, check whether $a$ is a factor of $b^{a-1} - 1$. Is this a counter-example to the statement in part a)? Explain.

c) For $a = 5$ and $b = 10$, check whether $a$ is a factor of $b^{a-1} - 1$. Is this a counter-example to the statement in part a)? Explain.
5. Read the book *One Grain of Rice* by Demi, and then work through these questions individually, stopping after each one to check answers and discuss.

a) If 1 grain of rice weighs 25 mg, how much rice (by mass) would Rani get in a week? In 2 weeks? In 3 weeks? After the whole month (30 days)? **NOTE:** Review the rule for finding the sum of \( n \) powers from NS8-104 Extension 6: The sum of the first \( n \) powers of 2 is \( 2^n + 1 – 2 \). For example, the sum of the first 6 powers of 2 is \( 2 + 4 + 8 + 16 + 32 + 64 = 2^6 - 2 = 126 \).

**ANSWERS:**

1 week: Number of grains = 1 + 2 + 4 + 8 + 16 + 32 + 64 = 127. The mass of 127 grains is 127 × 25 mg = 3175 mg = 3.175 g.

2 weeks: The number of grains after 1 week is \( 2^7 - 1 \). Using the same pattern, the number of grains after 2 weeks is \( 2^{14} - 1 = 16383 \) grains. The mass is 16 383 × 25 mg = 409 575 mg = 409.575 g.

3 weeks: \( 2^{21} - 1 = 2097151 \) grains, for a total mass of \( 2097151 \times 25 \text{ mg} = 52428775 \text{ mg} = 52428.775 \text{ g} \).

30 days: There are \( 2^{30} - 1 = 1073741823 \) grains, for a total mass of

\[
1073741823 \times 25 \text{ mg} = 26843545575 \text{ mg} = 26843545575 \text{ g} = 26843545575 \text{ kg} = 26843.545575 \text{ tonnes}.
\]

So Rani would get almost 27 tonnes of rice after 30 days.

b) If the volume of a grain of rice is 33.3 mm\(^3\), what would Rani need in order to carry the rice home on the 7\(^{th}\) day? The 14\(^{th}\) day? The 21\(^{st}\) day? After the whole month?

**ANSWERS:** Notice that on day 1, she needs to carry 1 grain; on day 2, 2 grains; on day 3, 2\(^2\) grains; on day 4, 2\(^3\) grains; and on day \( n \), 2\(^n-1\) grains.

7\(^{th}\) day: 2\(^6\) grains = 64 grains for a volume of 64 × 33.3 mm\(^3\) = 2131.2 mm\(^3\) = 2.1312 cm\(^3\). Rani could carry this much rice home in the palm of her hand.

14\(^{th}\) day: 2\(^{13}\) grains = 8192 grains for a volume of 8192 × 33.3 mm\(^3\) = 272793.6 mm\(^3\) = 272.7936 cm\(^3\). Rani could carry the rice in a bowl.

21\(^{st}\) day: 2\(^{20}\) grains = 1 048 576 grains for a volume of 1 048 576 × 33.3 mm\(^3\) = 34 917 580.8 mm\(^3\) = 34 917.580 8 cm\(^3\) = 34.917 580 8 dm\(^3\). Rani needs a capacity of almost 35 L. You can demonstrate this capacity if your school has 35 thousands cubes. Four big pails should do.
A whole month: \(2^{20}\) grains = 536,870,912 grains for a volume
of 536,870,912 \(\times 33.3\) mm\(^3\) = 17,877,801,369.6 mm\(^3\) =
17,877,801.369 cm\(^3\) = 17,877,801.369 dm\(^3\) = 17.877,801.369 m\(^3\).
This is almost 18 \(m^3\). Rani would need many sacks... and other
villagers to help her carry them!

c) After a whole month, would all the grains they have to store fit into
the classroom?

ANSWER: There would be \(2^{30} - 1\) grains = 1,073,741,823 grains
for a volume of

\[
1,073,741,823 \times 33.3 \text{ mm}^3 = 35,755,602,705.9 \text{ mm}^3
\approx 35.8 \text{ m}^3
\]

(This is likely to fit in most classrooms. Calculate the volume of your
classroom, roughly, by measuring the length and width of the floor
and the height of the walls.)

d) The residents of the village decide they won’t start eating the rice
until there is enough for everyone in the village to have 3 bowls of
rice a day. If a bowl of rice has 2000 grains, and the village has
250 people, on what day can the villagers start eating the rice?

ANSWER: They need to have 1,500,000 grains before they can start
eating. From part a), the answer is between 2 weeks and 3 weeks.
Check each day in turn:

- 15 days: \(2^{15} - 1 = 32,767\) grains
- 16 days: \(2^{16} - 1 = 65,535\) grains
- 17 days: \(2^{17} - 1 = 131,071\) grains
- 18 days: \(2^{18} - 1 = 262,143\) grains
- 19 days: \(2^{19} - 1 = 524,287\) grains
- 20 days: \(2^{20} - 1 = 1,048,575\) grains
- 21 days: \(2^{21} - 1 = 2,097,152\) grains

So on day 21, the villagers can each have 3 bowls of rice a day.
Notice that on day 22, they would get \(2^{21} = 2,097,152\) grains of rice,
so they have enough to eat 3 bowls each on that day too, and so on.

e) Assuming the villagers eat 3 bowls of rice a day as described in
part d), what is the volume of rice the villagers have to store at the
end of the month? How long can they eat from this supply of rice?

ANSWER: The villagers eat for 10 days (day 21 to day 30). So we
need to subtract \(1,500,000 \times 10 = 15,000,000\) grains from the total
to be stored (see part c):

\[
1,073,741,823 - 15,000,000 = 1,058,741,823 \text{ grains of rice.}
\]

To determine how many more days the villagers can eat for, divide
this total by the 1,500,000 grains they eat each day: \(1,058,741,823 \div 1,500,000 = 705.827\) days.
So the villagers can eat for 705 more
days, or almost 2 years.
NS8-110  Negative Fractions and Decimals

Page 145

CURRICULUM EXPECTATIONS
Ontario: 8m1, 8m6, 8m7, 8m13
WNCP: optional, [V, R, C], 9N3

VOCABULARY
integer
opposite integer
positive
negative
fraction
decimal
> and < symbols

Goals
Students will represent, compare, and order rational numbers (i.e., positive and negative fractions and decimals to thousandths).

PRIOR KNOWLEDGE REQUIRED
Can compare and order fractions
Can compare and order decimals
Can compare and order integers
Can use long division to write fractions as decimals

Review comparing and ordering positive fractions and decimals.
Progress as follows:

1. Compare tenths, hundredths, and thousandths. EXAMPLE: Compare 0.132 and 0.105

2. Compare fractions with denominator 2, 4, 5, 8, or 20 to decimals by changing the fraction to a fraction with denominator 10, 100, or 1000, and then to a decimal. EXAMPLES: Compare
   i) \( \frac{1}{4} \) and 0.23
   ii) \( \frac{7}{20} \) and 0.431
   iii) \( \frac{8}{5} \) and 1.584
   iv) \( \frac{3}{8} \) and 0.36
   SAMPLE ANSWER: iv) \( \frac{3}{8} = \frac{3 \times 125}{8 \times 125} = \frac{375}{1000} = 0.375 > 0.36. \)

3. Compare any fraction to a decimal by changing the decimal to a fraction and comparing the fractions. EXAMPLE: Compare \( \frac{2}{3} \) to 0.65 by comparing 200/300 to 65/100 = 195/300.

4. Compare any fraction to a decimal by changing the fraction to a decimal by long division. EXAMPLE:
   Compare \( \frac{2}{3} \) to 0.65 by writing \( \frac{2}{3} \) as the repeating decimal 0.\overline{6}, which is greater than 0.65. (NOTE: If students are not comfortable converting fractions to repeating decimals, students will need to use the method of changing the decimal to a fraction instead of changing the fraction to a decimal.)

5. Order lists of fractions and decimals. EXAMPLE:
   Order the list: 5 \( \frac{2}{3} \), 5 \( \frac{3}{5} \), 5.712, 5.615, 5.67.

Review comparing and ordering integers. First, recall that all negative numbers are smaller than all positive numbers. Second, remind students that an “opposite integer” is the number that is the same distance from 0 on a number line, but in the other direction; a number and its opposite add to 0; to obtain the opposite number, keep the number part the same, but change the sign (from + to – or from – to +).
To order negative numbers, order the opposite positive numbers and put the negative numbers in reverse order to their positive opposites. For example, since \(3 < 4 < 5\), we know that \(-5 < -4 < -3\). The fact that 4 is less than 5 tells us that 5 is further from 0, but this means that -5 is further from 0 than -4, which means it is more to the left, so \(-5\) is less than \(-4\).

Placing negative numbers on number lines. Have students mark the positive integers 3, 7, and 8 with an X on a number line from -10 to 10 that has only three points labelled: -10, 0, and 10. Then have students mark their opposites, -3, -7, and -8, on the same number line. **ASK:** Do you see any symmetry? (a vertical line through 0 is a mirror line)

Review placing positive fractions and decimals on number lines, and then show students how to place negative fractions and decimals on number lines by using the mirror image of the opposite positive numbers.

**Compare and order positive and negative fractions and decimals.**
Combine the concepts above. **EXAMPLE:**

Order these numbers: \(2, \, 1.3, \, -1.4, \, -2\frac{1}{4}, \, \frac{5}{3}, \, -1\frac{1}{2}\).

**Step 1:** Order the positive numbers: \(2, \, 1.3, \, \frac{1}{4}, \, \frac{5}{3}\).
So, \(\frac{1}{4} < 1.3 < \frac{5}{3} < 2\).

**Step 2:** Order the opposite of the negative numbers: \(1.4, \, 2, \, \frac{1}{2}\).
So, \(1.4 < \frac{1}{2} < 2\).

**Step 3:** Order the negative numbers in reverse order from Step 2:
\(-2 < -1\frac{1}{2} < -1.4\)

**Step 4:** Combine the lists from Steps 1 and 3, with all negative numbers less than all positive numbers:
\(-2 < -1\frac{1}{2} < -1.4 < \frac{1}{4} < 1.3 < \frac{5}{3} < 2\)
Magic Cards

Look at these “magic cards”:

<table>
<thead>
<tr>
<th>Card #1</th>
<th>Card #2</th>
<th>Card #3</th>
<th>Card #4</th>
<th>Card #5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 3 5 7</td>
<td>2 3 6 7</td>
<td>4 5 6 7</td>
<td>8 9 10 11</td>
<td>16 17 18 19</td>
</tr>
<tr>
<td>9 11 13 15</td>
<td>10 11 14 15</td>
<td>12 13 14 15</td>
<td>12 13 14 15</td>
<td>20 21 22 23</td>
</tr>
<tr>
<td>17 19 21 23</td>
<td>18 19 22 23</td>
<td>20 21 22 23</td>
<td>24 25 26 27</td>
<td>24 25 26 27</td>
</tr>
<tr>
<td>25 27 29 31</td>
<td>26 27 30 31</td>
<td>28 29 30 31</td>
<td>28 29 30 31</td>
<td>28 29 30 31</td>
</tr>
</tbody>
</table>

Here’s the magic trick: You pick a number between 1 and 31. I ask you if it’s on card 1 and you say yes or no. Then I ask if it’s on card 2, then card 3, then card 4, and finally card 5. In each case, you answer yes or no. I can then tell you, without even looking at the cards or having them memorized, what your number is.

**EXAMPLE:** If you tell me that your number is on cards 1, 2, and 4 but not on 3 and 5, then I know that your number is 11.

How do I do it? To find out, answer the following questions.

1. Which numbers are only on one card?
   - _____ is only on card # _____.
   - _____ is only on card # _____.
   - _____ is only on card # _____.
   - _____ is only on card # _____.
   - _____ is only on card # _____.

2. Except for 1, all the numbers you found are powers of the same base. Write these numbers as powers with the same base: _____, _____, _____, and ____.
   - What pattern do you see in the numbers that are only on one card?

3. Look at the numbers that occur only on one card. Where do they occur on that card? Are they easy to find?
4. For each number that is on more than one card, write in the chart below the numbers you found in Question 1 that are on the same card.

<table>
<thead>
<tr>
<th>Number on more than one card</th>
<th>Numbers from Question 1 on the same card</th>
<th>Number on more than one card</th>
<th>Numbers from Question 1 on the same card</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1, 2</td>
<td>19</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1, 4</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2, 4</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1, 2, 4</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1, 8</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>26</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td>27</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
<td>28</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>29</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td>31</td>
<td></td>
</tr>
</tbody>
</table>

5. What operation can you use to get the first number in the first column from the numbers in the second column? _______________________

6. What number appears in the top left corner of card 1? _____ of card 3? _____ of card 4? _____

Without looking at the cards or your chart, which number is on cards 1, 3, and 4 but not on cards 2 and 5? ______

What operation did you use to find the answer? _______________________

Check your answer by looking at the cards.

7. I am thinking of a number that is on cards 1, 3, and 5 but not on cards 2 and 4. What number am I thinking of? ______

8. I am thinking of a number that is only on cards 4 and 5. What number am I thinking of? ______

9. Without looking at the cards or your chart, which cards is 19 on? _______________________

10. Make up 6 cards with numbers from 1 to 63 by using the same trick.
PS8-7 Searching Systematically II

**Teach this lesson after:** 8.2 Unit 5

**Goals:**
Students will use a number’s prime factorization to count the number of factors the number has.

**Prior Knowledge Required:**
Can write prime factorizations of numbers within 100
Can list all the factor pairs of numbers within 100
Can use exponent notation for repeated multiplication
Can search systematically when two or three related quantities are changing
Can identify prime numbers within 100

**Vocabulary:** exponent, factor, factor tree, power, prime, prime factorization

**Review prime factorizations.** Write on the board:

```
    20
   /\  \
  4   5
```

SAY: Remember that you can start to make a factor tree for a number by putting that number on the top and then writing the number as a product of two smaller numbers. Have a volunteer add numbers to the second tree using different numbers (either 2 and 10 or 10 and 2) SAY: You keep going until the bottom numbers cannot be written as a product of two smaller numbers.

ASK: What are numbers called that cannot be written as a product of two smaller numbers? (prime numbers) SAY: Once all the bottom numbers are prime, you can stop. Extend the trees as shown below:

```
    20    20
   /\  /\  \
  4   5 10  2
 /\  /\  /\  \
 2 2  2  5
```

ASK: Can any numbers be factored further? (no) SAY: All the numbers without a branch below them are prime, so they have all been factored as much as possible. This shows two ways of writing 20 as a product of prime factors. Write on the board:

\[ 20 = 2 \times 2 \times 5 \quad 20 = 2 \times 5 \times 2 \]
SAY: These are both prime factorizations of 20. ASK: What is the same about both prime factorizations? (they both have two 2s and one 5) What is different about them? (the numbers are not in the same order) SAY: You can multiply numbers in any order and get the same answer.

**Exercises:**

a) Complete the factor trees to write the prime factorization of 36 in three ways.

\[
\begin{array}{c}
36 \\
\quad / \ \\
\quad \quad 4 \\
\quad / \\
\quad 2 \\
\end{array}
\begin{array}{c}
36 \\
\quad / \\
\quad 9 \\
\quad / \\
\quad 3 \\
\end{array}
\begin{array}{c}
36 \\
\quad / \\
\quad 6 \\
\quad / \\
\quad 2 \\
\end{array}
\]

b) What is the same about all three ways? What is different?

**Sample answers:**

\[
\begin{array}{ccc}
36 & 36 & 36 \\
4 & 9 & 6 \\
2 & 3 & 3 \\
2 \times 2 \times 3 \times 3 & 2 \times 3 \times 2 \times 3 & 3 \times 3 \times 2 \times 2 \\
\end{array}
\]

b) All three ways have two 2s and two 3s. The prime factors are in a different order.

SAY: All prime factorizations of a number result in the same product, just the order of the prime factors might be different. We usually write the prime factorization in order from least prime factor to greatest prime factor, using exponents, like this. Write on the board:

\[36 = 2^2 \times 3^2\]

**Exercises:** Using a factor tree, find the prime factorization of the number. Use exponent notation.

a) 30 \hspace{1cm} b) 27 \hspace{1cm} c) 28 \hspace{1cm} d) 75

**Answers:** a) \(2 \times 3 \times 5\), b) \(3^3\), c) \(2^2 \times 7\), d) \(3 \times 5^2\)

**Review listing all the factors of a number.** Write on the board:

<table>
<thead>
<tr>
<th>Factor Pairs of 24</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
</tbody>
</table>

SAY: You can stop once you repeat a number because if 4 and 6 multiply to 24, so do 6 and 4, and all the factors of 24 that are greater than 6 are already multiplied by something smaller than 4 in the table, so you don’t need to look for them anymore.
**Exercises:** Use a table to list all the factors of the number.

a) 24  b) 32  c) 80  d) 81  e) 36

**Answers:**

a) 1, 2, 3, 4, 6, 8, 12, 24; b) 1, 2, 4, 8, 16, 32; c) 1, 2, 4, 5, 8, 10, 16, 20, 40, 80; d) 1, 3, 9, 27, 81; e) 1, 2, 3, 4, 6, 9, 12, 18, 36

**Finding all the factors of a number using prime factorization.** Tell students that there is another way to list all the factors of a number and that this way uses the prime factorization. Ask a volunteer to draw the factor tree of 24 and write the prime factorization of 24. (see below)

```
24
/  \
2   12
  /  \
3   4
 / \
2 2
```

Prime factorization: $24 = 2^3 \times 3$

**Exercises:** Find the prime factorization of each factor of 24.

<table>
<thead>
<tr>
<th>Factor of 4</th>
<th>Prime Factorization</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 1</td>
<td>N/A</td>
</tr>
<tr>
<td>b) 2</td>
<td>2</td>
</tr>
<tr>
<td>c) 3</td>
<td></td>
</tr>
<tr>
<td>d) 4</td>
<td></td>
</tr>
<tr>
<td>e) 6</td>
<td></td>
</tr>
<tr>
<td>f) 8</td>
<td></td>
</tr>
<tr>
<td>g) 12</td>
<td></td>
</tr>
<tr>
<td>h) 24</td>
<td></td>
</tr>
</tbody>
</table>

**Answers:** c) 3, d) $2 \times 2$, e) $2 \times 3$, f) $2 \times 2 \times 2$, g) $2 \times 2 \times 3$, h) $2 \times 2 \times 2 \times 3$

ASK: What are the prime factors of 24? (2 and 3) Point to the prime factorization and ASK: What is the exponent of 3? (1) SAY: That means a factor of 24 can have at most one 3. It might have no 3s and it might have one 3. ASK: Which factors of 24 have no 3s? (1, 2, 4, 8) Which factors have one 3? (3, 6, 12, 24) What is the exponent of 2 in the prime factorization of 24? (3) SAY: That means that a factor of 24 can have at most three 2s. ASK: Which factors of 24 have no 2s? (1, 3) Which factors have one 2? (2, 6) Which factors have two 2s? (4, 12) Which factors have three 2s? (8 and 24) Explain to students that the factors of 24 can include no 2, one 2, two 2s, three 2s, no 3, or one 3.

SAY: Let’s use this idea to list the factors of 18. Have a volunteer tell you the prime factorization of 18. ($2 \times 3^2$) Draw on the board:

<table>
<thead>
<tr>
<th>Number of 2s</th>
<th>Number of 3s</th>
<th>Factor of 18</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Explain to students that if there is one 2 and two 3s in a factor of 18, then that factor is the product of one 2 and two 3s. Write “$2 \times 3 \times 3 = 18$” on the board and write “18” in the “Factor of 18” column. ASK: What if there are no 2 and two 3s? (3 × 3 = 9) PROMPT: When there are two 3s, the factor would be the product of two 3s. Complete the second row of the table, as shown below:

<table>
<thead>
<tr>
<th>Number of 2s</th>
<th>Number of 3s</th>
<th>Factor of 18</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>$2 \times 3 \times 3 = 18$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>$3 \times 3 = 9$</td>
</tr>
</tbody>
</table>

ASK: What about one 2 and one 3? What factor of 18 do you get with one 2 and one 3? (2 × 3 = 6) Complete the third row of the table with 1, 1, and 6. Point out to students that continuing in this way might miss some possibilities. SAY: Searching in an organized way can help make sure you do not miss any possibilities. Draw on the board:

<table>
<thead>
<tr>
<th>Number of 2s</th>
<th>Number of 3s</th>
<th>Factor of 18</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

SAY: When there are no 2s and no 3s in the factor, then the factor is 1 because 1 is a factor of any number. Write “1” in the first row of the “Factor of 18” column. ASK: What factor of 18 has no 2 and one 3? (3) Write “3” in the second row of the “Factor of 18” column. ASK: What factor of 18 has no 2 and two 3s? (9) Write “$3 \times 3 = 9$” in the next row. Then ask a volunteer to complete the table. (see below)

<table>
<thead>
<tr>
<th>Number of 2s</th>
<th>Number of 3s</th>
<th>Factor of 18</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>$3 \times 3 = 9$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$2 \times 3 = 6$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$2 \times 3 \times 3 = 18$</td>
</tr>
</tbody>
</table>

Write all factors of 18 below the table. (1, 2, 3, 6, 9, 18)

**Exercise:** Use $40 = 2^3 \times 5$ to find all factors of 40.

**Bonus:** List the factors of $7^2 \times 5^3 = 6125$. 

---

Q-30  
Teacher's Guide for Grade 8 — Problem Solving Lessons
### Answers:

<table>
<thead>
<tr>
<th>Number of 2s</th>
<th>Number of 5s</th>
<th>Factor of 40</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2 × 5 = 10</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2 × 2 = 4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2 × 2 × 5 = 20</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2 × 2 × 2 = 8</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2 × 2 × 2 × 5 = 40</td>
</tr>
</tbody>
</table>

Factors of 40: 1, 2, 4, 5, 8, 10, 20, 40
Bonus: 1, 5, 25, 125, 7, 35, 175, 875, 49, 245, 1225, 6125

### Counting factors

Say: Once you know the prime factorization of a number, you can list all the factors, so you can determine how many factors it has, too. But I wonder if there is a way to count the factors without listing them all. Let’s start with numbers that have only one prime factor. For example, \(2^7 = 128\) has only 2 as a prime factor. Write on the board:

<table>
<thead>
<tr>
<th>Number of 2s</th>
<th>Factor of 128</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2 × 2 = 4</td>
</tr>
<tr>
<td>3</td>
<td>2 × 2 × 2 = 8</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

Say: When there is only one prime factor, the factors are really easy to list. The first column tells you how many 2s to multiply to write the factor in the second column. Ask: Do I need to finish the second column to know how many factors there are? (No) How can I tell from the table how many factors there are? (There are 8 rows so there are 8 factors) How can you tell from the exponent how many rows there will be? (The rows go from 0 to the exponent, so add 1 to the exponent to get the number of rows) Say: When the exponent of a prime number is 7, there are 8 factors.

### Exercises

How many factors does each prime power have?

- a) \(2^{10}\)
- b) \(3^8\)
- c) \(2^{99}\)
- d) \(7^{15}\)
- e) \(11^8\)

**Answers:** a) 11, b) 9, c) 100, d) 16, e) 9
SAY: When there are two prime factors, it’s a bit harder, but you can still do it. Write on the board:

\[ 3 \times 2^7 = 3 \times 128 = 384 \]

<table>
<thead>
<tr>
<th>Number of 2s</th>
<th>Number of 3s</th>
<th>Factor of 384</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

SAY: If we just want to know how many factors there are and not what they are, we don’t need the last column. We can just count the rows. Erase the last column, then ASK: How many factors have no 2s? (2) How many factors have one 2? (2) How many have two 2s? (2) How many have three 2s? (2) How many have seven 2s? (2) Write on the board:

\[ 2 + 2 + 2 + \ldots + 2 = \ldots \]

ASK: How many 2s are being added? (8) How do you know? (there are 8 numbers from 0 to 7) Write on the board:

There are \[8 \times 2 = 16\] factors.

**Exercises:** How many factors does the number have?
- a) \[3^4\]
- b) \[2 \times 3^4\]
- c) \[2^2 \times 3^4\]
- d) \[2^3 \times 3^4\]
**Bonus:** \[2^m \times 3^n\]

**Answers:**
- a) 5
- b) 10
- c) 15
- d) 20
- Bonus: \[(m + 1) \times (n + 1)\]

SAY: A number can have three prime factors, too. Write on the board:

\[ 45 = 3^2 \times 5 \text{ and } 90 = 2 \times 3^2 \times 5 \]

Have a volunteer list the factors of 45. (1, 3, 5, 9, 15, 45) Have another volunteer list the factors of 90. (1, 2, 3, 5, 6, 9, 10, 15, 18, 20, 45, 90) ASK: Are all the factors of 45 also factors of 90?
(yes) SAY: I’m going to write down the factors of 90 that are factors of 45 and then I’m going to write down the rest underneath. Write on the board:

\[
\begin{array}{cccccccc}
1 & 3 & 5 & 9 & 15 & 45 \\
2 & 6 & 10 & 18 & 30 & 90
\end{array}
\]

ASK: How can you get the factors of 90 from the factors of 45? (double the factors of 45 to get the rest of the factors of 90) SAY: All the factors of 90 have either no 2s or one 2. All the factors of 45 are the factors of 90 that have no 2s. All the other factors are the ones that have one 2. You can get those by multiplying by 2 each of the factors of 45.

**Exercises:**

a) List the factors of 28 = 2\(^2\) × 7.
b) Use your list from part a) to list the factors of 140 = 2\(^2\) × 5 × 7.

**Answers:**
a) 1, 2, 4, 7, 14, 28; b) 1, 2, 4, 7, 14, 28, 5, 10, 20, 35, 70, 140

**Problem Bank**

1. a) Write the prime factorization of 39 000.
b) What is the largest prime factor of 39 000?

**Answers:**
a) 39 000 = 3 × 13 × 2 × 2 × 2 × 5 × 5 × 5, b) 13

2. Find the prime factorization of the numerator and denominator, then simplify the fraction by cancelling.
a) \(\frac{60}{72}\)
b) \(\frac{6615}{24 696}\)

**Answers:**
a) \(\frac{(2 \times 2 \times 3 \times 5)}{(2 \times 2 \times 2 \times 3 \times 3)} = \frac{5}{2 \times 3} = \frac{5}{6}\)
b) \(\frac{(3 \times 3 \times 3 \times 5 \times 7 \times 7)}{(2 \times 2 \times 2 \times 3 \times 3 \times 7 \times 7 \times 7)} = \frac{(3 \times 5)}{(2 \times 2 \times 2 \times 7)} = \frac{15}{56}\)

3. Jay says that 6\(^3\) has exactly four factors because the exponent is 3.
a) Explain why his reasoning is wrong.
b) How many factors does 6\(^3\) actually have?

**Answers:**
a) Jay’s reasoning only works for prime numbers, but 6 is not prime; b) 16

4. a) List the factors of 3 × 5\(^3\). How many two-digit factors does it have?
b) The factors of 2\(^2\) × 3 × 5\(^3\) have either no 2s, one 2, or two 2s. List the factors of each type: no 2s, one 2, and two 2s.
c) How many two-digit factors does 2\(^2\) × 3 × 5\(^3\) have?

**Answers:**
a) 1, 5, 25, 125, 3, 15, 75, 375, it has 3 two-digit factors; b) no 2s: 1, 5, 25, 125, 3, 15, 75, 375; one 2: 2, 10, 50, 250, 6, 30, 150, 750; two 2s: 4, 20, 100, 500, 12, 60, 300, 1500; c) 9

5. How many factors does each number have?
a) 2\(^2\) × 7 \hspace{1cm} b) 2 × 5 × 11 \hspace{1cm} c) 3\(^7\)

**Answers:**
a) 4 × 2 = 8, b) 2 × 2 × 2 = 8, c) 8
6. a) Evaluate $4^2 \times 5$ and list its factors.
b) Amir says that $4^2 \times 5$ has $3 \times 2 = 6$ factors. Is he correct? Explain why his reasoning is correct or why it is incorrect.

**Answers:**
a) $4^2 \times 5 = 80$ and its factors are: 1, 2, 4, 5, 8, 10, 16, 20, 40, 80; b) his reasoning would be correct if 4 was prime, but in fact $4^2 = 2^4$, so the correct number of factors is $5 \times 2$, not $3 \times 2$

7. What is the smallest number that has exactly eight factors?

**Solution:** You need to compare the smallest number with each type of prime factorization that gives exactly eight factors: $2^3 \times 3 = 24$, $2 \times 3 \times 5 = 30$, and $2^7 = 128$, so the smallest number is 24.

8. The numbers in Problem Bank 5 all have eight factors, and they all have different formats for the prime factorization. How many different formats for the prime factorization can a number have with the given number of factors?

- a) 6 factors
- b) 7 factors
- c) 9 factors
- d) 10 factors
- e) 12 factors
- f) 15 factors

**Answers:**
a) 2 formats: $2^5$ and $2 \times 3^2$; b) 1 format, such as $2^6$; c) 2 formats, because 9 can be obtained using $3 \times 3$ or using $1 \times 9$; d) 2 formats; e) 4 formats, namely $2^3 \times 3^2$, $2 \times 3^5$, $2^{11}$, and $2 \times 3 \times 5^2$; f) 2 formats

9. What is the smallest number with exactly 12 factors?

**Solution:** Choose from among $2^3 \times 3^2 = 72$, $2^5 \times 3 = 96$, $2^2 \times 3 \times 5 = 60$, and $2^{11} = 2048$, so the smallest number with 12 factors is 60.

10. What is the smallest number with exactly 15 factors?

**Answer:** $2^4 \times 3^2 = 144$

11. What is the product of all the factors of $2 \times 3^2$?

**Answer:** $2^3 \times 3^6 = 5832$

12. What number am I?

- a) I have 8 factors. The product of my factors is 2 560 000.
- b) I have 9 factors. The product of my factors is 69.
- c) I have 10 factors. The product of my odd factors is 59 049 = $3^{10}$. I am even.

**Answers:**
a) 40, b) 36, c) 162

13. Raj has 12 roses and 15 lilies. He wants to make bouquets so that:

- he uses all his flowers,
- each bouquet has the same number of roses, and
- each bouquet has the same number of lilies.

What is the greatest number of bouquets he can make?

**Solution:** The number of bouquets $\times$ the number of roses in each bouquet is 12, so the number of bouquets is a factor of 12. By the same logic, it is also a factor of 15, so the answer is the GCF of 12 and 15, that is, 3.
14. Liz is organizing a Canada Day celebration. She bought 9873 red flowers and 5485 white flowers. She wants to make bouquets so that:
   • she uses all her flowers,
   • each bouquet has the same number of red flowers, and
   • each bouquet has the same number of white flowers.
What is the greatest number of bouquets she can make?
**Solution:** \(9873 = 9 \times 1097\) and \(5485 = 5 \times 1097\), so the GCF of 9873 and 5485 is 1097. The greatest number of bouquets that can be made is 1097.

15. Clara organized a large food drive to collect food for a food bank. She received 244 675 cans of corn, 704 664 cans of peas, and 880 830 cans of peaches. She decides to make packages of food so that:
   • she uses all the donated food,
   • each package has the same number of cans of corn,
   • each package has the same number of cans of peas, and
   • each package has the same number of cans of peaches.
What is the greatest number of packages that can be made?
**Solution:** By trying 2, 3, and 5 as factors successively:
\[704 \text{,}664 = 2^3 \times 3^2 \times 9787\]
\[880 \text{,}830 = 2 \times 3^2 \times 5 \times 9787\]
\[244 \text{,}675 = 5^2 \times 9787\]
So the GCF of all three numbers is 9787, and that is the greatest number of packages that can be made.
PS8-8 Using Structure I

Teach this lesson after: 8.2 Unit 5

Goals:
Students will represent algebraically situations that require two variables.
Students will use two variables to do algebraic proofs.

Prior Knowledge Required:
Can collect like terms
Can represent algebraically situations that are linear in one variable
Can multiply integers

Vocabulary: collect like terms, constant term, distributive property, like terms, multiple, remainder, variable, variable terms

Review collecting like terms. Write on the board:

\[ 3x + 4x + 5 - 2 - x + 6 \]

ASK: How could you write this more simply? (group all the variable terms together and group all the constant terms together) Write on the board:

\[ 3x + 4x - x + 5 - 2 + 6 \]

SAY: You can calculate the constant terms: \( 5 - 2 + 6 \) is \( 3 + 6 \), which is 9. Write the simplified expression on the board:

\[ 3x + 4x - x + 9 \]

ASK: How can you simplify the variable terms? (find \( 3 + 4 - 1 \) to get the coefficient) SAY: You are adding three \( x \)'s to four \( x \)'s and then taking away one \( x \), so to get the number of \( x \)'s you end up with, you are just calculating \( 3 + 4 - 1 = 6 \). Write on the board:

\[ 6x + 9 \]

Exercises: Simplify.

a) \( 4x + 3 + x \)  
b) \( 5x + 3 - 2 \)  
c) \( 7 + 4x - x + 2 \)  
d) \( 5x - x + 4 - 2x \)  
e) \( 3 + 3x + 4 - x - 2 \)  
f) \( 5x - 7 - 4x + 5 + 3x \)

Answers: a) \( 5x + 3 \), b) \( 5x + 1 \), c) \( 3x + 9 \), d) \( 2x + 4 \), e) \( 2x + 5 \), f) \( 4x - 2 \)

Relate collecting like terms to the distributive property. Write on the board:

\[ 3x + 4x = 7x \]
SAY: No matter what number you put for $x$, this is always true. Write on the board:

\[
\begin{align*}
3 \times 1 + 4 \times 1 &= 7 \times 1 \\
3 \times 2 + 4 \times 2 &= 7 \times 2 \\
3 \times 18 + 4 \times 18 &= 7 \times 18
\end{align*}
\]

ASK: What property is this? (the distributive property) SAY: So, collecting like terms is just one application of the distributive property.

**Introduce problems with two variables.** Write on the board:

<table>
<thead>
<tr>
<th>Words</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Add 3 to a number.</td>
<td>$x + 3$</td>
</tr>
<tr>
<td>Multiply a number by 4.</td>
<td></td>
</tr>
<tr>
<td>Add a number to itself.</td>
<td></td>
</tr>
<tr>
<td>Multiply a number by itself.</td>
<td></td>
</tr>
<tr>
<td>Add a number to another number.</td>
<td></td>
</tr>
<tr>
<td>Multiply a number by another number.</td>
<td></td>
</tr>
</tbody>
</table>

Have volunteers write expressions for the next three phrases on the board as you read them aloud. (4$x$, $x + x$ or 2$x$, $x \times x$ or $x^2$) Then read the next phrase aloud and ASK: How is this one different from adding a number to itself? (you are adding a number to another number) SAY: Because you don't know what the other number is, it might be the same as the first number or it might not be; you have to use a different letter to represent it. Write “$x + y$” for that phrase, then have a volunteer write the expression for the last phrase ($x \times y$ or $xy$).

**Exercises:** Write an expression for the phrase. Use two variables when you need to.

a) Add 4 to a number.
b) Subtract a number from 10.
c) Subtract a number from another number.
d) Double a number.
e) Double a number, then add another number.
f) Double a number, then subtract another number.
g) Add a number to another number, then double the result.
h) Multiply a number by 3, then subtract another number.

**Sample answers:** a) $x + 4$, b) $10 - x$, c) $x - y$, d) $2x$, e) $2x + y$, f) $2x - y$, g) $2(x + y)$, h) $3x - y$

**Collecting like terms in two variables.** Write on the board:

\[
2x + 3 + 4y + 3x - y + 7 = \underline{\hspace{5cm}}
\]

Circle the $2x$ and the $3x$. SAY: When you are adding 2 and 3 of the same thing, you get 5 of them, so you can combine the $x$'s. Begin the expression by writing “5$x$” in the blank. SAY: But
you can’t combine the x’s and the y’s because you are adding different things. You have to combine like terms, the x’s with the x’s and the y’s with the y’s. ASK: How many y’s do you have altogether? (3) How did you get that? (I had 4 and subtracted 1) Continue writing the expression, as shown below:

5x + 3y

ASK: What is the total constant term? (10) Finish writing the expression, as shown below:

5x + 3y + 10

Exercises: Simplify.

a) 2x + 5y – x – 3y               b) 3x + 4 + 4y – 1 + 7x – 3y 

3x – 4y + 2x + 7y – x – 5y       d) 2x + y – x + 4 + 3y + 4x – 5 

Answers: a) x + 2y, b) 10x + y + 3, c) 4x – 2y, d) 5x + 4y – 1

The sum of two multiples of a number is still a multiple of that number. SAY: I want to show that the sum of two multiples of a number is still a multiple of that number. Let’s start with multiples of 2. Ask a volunteer to tell you any multiple of 2 less than 100 (for example, 56).

SAY: Let’s add 56 to various numbers. Write on the board:

56 + 12 =

56 + 300 =

Have volunteers do the additions. (68, 356) ASK: Is 12 even? (yes) Is 56 + 12 even? (yes) Is 300 even? (yes) Is 56 + 300 even? (yes) Write on the board:

56 + 12 = 2 × 28 + 2 × 6 

= 2 × ______

Have a volunteer fill in the blank. (34) ASK: How did you get that? (added 28 + 6) What property of multiplication allows you to do that? (multiplication distributes over addition) SAY: Let’s prove it algebraically for any two even numbers. Both the even numbers are two times something because that’s what even numbers are, but they might be two times different things. Write on the board:

first number = 2x
second number = ______

ASK: How can we show the second number algebraically? (sample answers: 2y, 2a, 2b) Write “2y” in the blank, then ASK: Why can’t we use 2x? (the first number is 2x and the second number might not equal the first number) Write on the board:

2x + 2y = 2(______)
SAY: Let’s say the first number is two times $x$ and the second number is two times $y$. ASK: The number they add to is two times what? $(x + y)$ Continue writing on the board:

$$2x + 2y = 2(x + y)$$

ASK: Is the result of adding two even numbers still even? (yes) SAY: This is something you probably knew for even numbers from lower grades; but now you can prove it algebraically, and that is very powerful because you can prove it for much more than just even numbers.

**Exercises:**

a) Show that the sum of two multiples of 3 is also a multiple of 3.
b) Show that the sum of two multiples of 7 is also a multiple of 7.
c) Show that the difference of two multiples of 3 is also a multiple of 3.

**Answers:**

a) Let the two numbers be $3x$ and $3y$. Then $3x + 3y = 3(x + y)$ is also a multiple of 3.
b) Let the two numbers be $7x$ and $7y$. Then $7x + 7y = 7(x + y)$ is also a multiple of 7.
c) Let the two numbers be $3x$ and $3y$. Then $3x - 3y = 3(x - y)$ is also a multiple of 3.

**Following a procedure to get an unexpected result.** Provide students with the following exercise.

**Exercise:** Pick any two-digit number. Then follow the procedure below. Do this for several two-digit numbers. What do you notice about how the result compares with the number you started with?

Step 1: Add the digits.
Step 2: Multiply the result by 11.
Step 3: Subtract the number you started with.

**Sample answers:** starting with 26, get 8, then 88, then 62; starting with any two-digit number, the result will always be the number with the digits reversed.

Tell students that the procedure seems like it gets the result by magic, but you want to prove algebraically that it will always work. Tell students that you want to represent the situation algebraically. ASK: What numbers are you adding in Step 1? (the digits) SAY: Because it is the digits that I want to operate on, I need a way to represent the digits algebraically. ASK: How many digits are there in the number? (2) So how many variables will I need? (2) SAY: Let’s use $x$ and $y$ for the digits. Write on the board:

- tens digit = $x$
- ones digit = $y$

**Exercises:** What are the values of $x$ and $y$ for the given two-digit number?

a) 36  b) 55  c) 42  d) 20  e) 11

**Answers:** a) $x = 3, y = 6$; b) $x = 5, y = 5$; c) $x = 4, y = 2$; d) $x = 2, y = 0$; e) $x = 1, y = 1$
When students have completed the exercises, point out that $x$ and $y$ are two different variables because they can represent different numbers. They might represent the same number as in parts b) and e), but they don’t have to. Then have a volunteer write on the board the expression for the result of Step 1 (add the digits):

$$x + y$$

Have another volunteer write the expression for Step 2 (multiply the result by 11):

$$11(x + y)$$

Now refer students to Step 3 and tell them that you are going to subtract the two-digit number that you started with, so now you need a way to represent that algebraically. Write on the board:

$$36 = 10 \times 3 + 6$$

SAY: Remember that you can write a number in expanded form by writing 10 times the tens digit and adding the ones digit. The number 36 really means ten times three plus six.

**Exercises:** Write the number in expanded form.

a) 55  
 b) 42  
 c) 20  
 d) 11  
 e) a number with tens digit $x$ and ones digit $y$

**Answers:** a) $10 \times 5 + 5$, b) $10 \times 4 + 2$, c) $10 \times 2$ or $10 \times 2 + 0$, d) $10 \times 1 + 1$, e) $10x + y$

Write on the board:

Any two-digit number can be written as $10x + y$.

Have a volunteer write the expression for Step 3 (subtract the number you started with):

$$11(x + y) - (10x + y)$$

SAY: This is the number I will always end up with, but I want to write it in a simpler form so that I can see why it is the reverse. I want to be able to collect like terms, but I can only do that if there are no brackets. Point to the first set of brackets, in $11(x + y)$, and ASK: How can I get rid of these brackets? (change it to $11x + 11y$) Write on the board:

$$11(x + y) = 11x + 11y$$

Point to the second set of brackets, in $(10x + y)$, and ASK: How can I get rid of these brackets? (subtract both terms inside the brackets) SAY: When I subtract the number I started with, $10x + y$, I am subtracting both $10x$ and $y$. Write on the board:

$$11(x + y) - (10x + y)$$

$$= 11x + 11y - 10x - y$$
ASK: What makes this easier to work with than when there were brackets? (now we can collect like terms) SAY: Now we can collect the x’s together and the y’s together. ASK: How many x’s are there? (1) SAY: 11x – 10x is 1x. ASK: How many y’s are there? (10) SAY: 11 y’s minus 1 y is 10 y’s. Continue writing on the board:

\[= 11x - 10x + 11y - y\]
\[= x + 10y\]

SAY: We started with 10x + y and we ended, after the procedure, with x + 10y. Now it’s easy to see why it’s the reverse number. Write on the board:

36 has \(x = 3\) and \(y = 6\). Then after the procedure, you get \(3 + 10 \times 6\), which is 63.

**Exercises:** Here is another procedure.
Step 1: Pick a two-digit number and subtract its tens digit from its ones digit—this might be a negative number.
Step 2. Multiply the result by 9.
Step 3: Add the result to the original number.

a) Show for three examples of your choice that this gets the reverse of the starting number.
b) Prove algebraically that this procedure always gets the reverse of the starting number.

**Answers:**
b) Let the number be \(10x + y\), so that \(x\) is the tens digit and \(y\) is the ones digit. Then Step 1 gets \(y - x\), Step 2 gets \(9(y - x)\), Step 3 gets \(10x + y + 9(y - x) = 10x + y + 9y - 9x = x + 10y\), which is the reverse of the number: it has tens digit \(y\) and ones digit \(x\).

**Problem Bank**
1. Show that the sum of three multiples of 7 is still a multiple of 7.

2. Which of these statements is true? Find a counter-example for the other statement.
   A. If two numbers are both multiples of 3, so is their sum.
   B. If the sum of two numbers is a multiple of 3, so are both numbers.
   **Answer:** Statement A is true. A counter-example for Statement B is 4 + 5 = 9, since 9 is a multiple of 3, but 4 and 5 are not multiples of 3.

3. The sum 1 + 2 + 3 = 6 is a multiple of 3.
   a) Is the sum 1 + 2 + 3 + 4 a multiple of 4?
   b) Is the sum 1 + 2 + 3 + 4 + 5 a multiple of 5?
   c) Is the sum 1 + 2 + 3 + 4 + 5 + 6 a multiple of 6?
   d) Use the grid to explain why \(1 + 2 + 3 + \ldots + 6 = 6 \times 7 \div 2\)
   e) Write an expression for \(1 + 2 + 3 + \ldots + n\) that is similar to the expression in part d).
   f) Is \(1 + 2 + 3 + \ldots + n\) a multiple of \(n\)?
Answers: a) the sum is 10, so no; b) the sum is 15, so yes; c) the sum is 21, so no; d) the area of the rectangle is \(2 \times (1 + 2 + 3 + \ldots + 6)\) and the area is also equal to \(6 \times 7\), so \(1 + 2 + 3 + \ldots + 6 = 6 \times 7 + 2\); e) \(1 + 2 + 3 + \ldots + n = n \times (n + 1)/2\); f) When \(n\) is odd, \(n + 1\) is even, so \((n + 1)/2\) is a whole number. Then \(n \times (n + 1)/2\) is a multiple of \(n\). When \(n\) is even, then \(n + 1\) is odd and \(n \times (n + 1)/2\) is halfway between two consecutive multiples of \(n\). For example, when \(n = 6\), then 21 is halfway between 18 and 24.

4. a) Show that the sum of three consecutive numbers is always a multiple of 3.
b) Is the sum of four consecutive numbers always, sometimes, or never a multiple of 4?
c) Is the sum of five consecutive numbers always, sometimes, or never a multiple of 5?
d) Is the sum of six consecutive numbers always, sometimes, or never a multiple of 6?
e) Investigate when is the sum of \(n\) consecutive numbers always a multiple of \(n\), when is it sometimes a multiple of \(n\), and when is it never a multiple of \(n\).

Answers: a) If the smallest number is the multiple of 3, then the sum is \(3x + 3x + 1 + 3x + 2 = 9x + 3\), which is a multiple of 3. If the smallest number has remainder 1, then the sum is \(3x + 1 + 3x + 2 + 3x + 3\), which is \(9x + 6\), which is a multiple of 3. If the smallest number has remainder 2, then the sum is \(3x + 2 + 3x + 3 + 3x + 4 = 9x + 9\), which is again a multiple of 3; b) never, because the sum of four consecutive numbers is either \(4x + 6\), \(4x + 10\), \(4x + 14\), or \(4x + 18\), which is never a multiple of 4; c) always; d) never; e) when \(n\) is odd, the sum is always a multiple of \(n\), and when \(n\) is even, the sum is never a multiple of \(n\).

5. Create a procedure to reverse a three-digit number. Prove algebraically that your procedure works.

Answer:
Step 1: Subtract the hundreds digit from the ones digit.
Step 2: Multiply by 99.
Step 3: Add to the original number.
This works because \(100x + 10y + z + 99(z - x) = 100x + 10y + z + 99z - 99x = x + 10y + 100z\), which is the reverse number.

6. Create a procedure to reverse a four-digit number. Prove algebraically that your procedure works.

Answer:
Step 1: Subtract the thousands digit from the ones digit. Multiply by 999.
Step 2: Subtract the hundreds digit from the tens digit. Multiply by 90.
Step 3: Add the results of Step 1 and Step 2 to the original number.
This works because \(1000x + 100y + 10z + w + 999(w - x) + 90(z - y) = 1000x + 100y + 10z + w + 999w - 999x + 90z - 90y = x + 10y + 100z + 1000w\), which is the reverse number.

7. Create a procedure to reverse a five-digit number. Prove algebraically that your procedure works.

Answer:
Step 1: Subtract the ten thousands digit from the ones digit. Multiply by 9999.
Step 2: Subtract the thousands digit from the tens digit. Multiply by 990.
Step 3: Add the results of Steps 1 and 2 to the original number.
This works because \(10 000a + 1000b + 100c + 10d + e + 9999(e - a) + 990(d - b) = 10 000a + 1000b + 100c + 10d + e + 9999e - 9999a + 990d - 990b = a + 10b + 100c + 1000d + 10 000e\), which is the reverse number.
8. To find the digital root of a number, add its digits, then add the digits of the resulting number until you reach a single digit. Example: 395 \rightarrow 17 \rightarrow 8.

a) Find the digital root of …
   i) 25  ii) 384  iii) 670  iv) 36  v) 198

b) Use a pattern to predict what can be the digital root of …
   i) a multiple of 3
   ii) a multiple of 2
   iii) a multiple of 9
   iv) a number in the sequence 1, 4, 9 16, 25, 36, …
   v) a number in the sequence 1, 2, 4, 8, 16, 32, …

c) Can zero ever be the digital root of a positive number? Explain how you know.

Answers:
   a) i) 7, ii) 6, iii) 4, iv) 9, v) 9; b) i) 3, 6, or 9; ii) any digit; iii) only 9; iv) 1, 4, 7, or 9; v) 1, 2, 4, 5, 7, or 8; c) no, because the sum of a number’s digits can never be zero

9. a) Find the remainder when dividing by 9.
   i) 25  ii) 384  iii) 670  iv) 36  v) 198

b) Look at your answers to part a). Predict when the digital root is equal to the remainder when dividing by 9 and when it is not.

c) Explain why the number of shaded squares is a multiple of 9.

i) 25

ii) 384


d) In part c), how does the picture show that you can subtract a multiple of 9 from the number to get the sum of its digits?

e) Draw a picture to show that you can subtract a multiple of 9 from each number to get the sum of its digits.

i) 670  ii) 36  iii) 198

f) Explain why a number and its digital sum always have the same remainder when dividing by 9.

Selected answers:
   a) i) 7, ii) 6, iii) 4, iv) 0, v) 0; b) when the number is not a multiple of 9, the digital root is equal to the remainder when the number is divided by 9; when the number is a multiple of 9, the digital root is 9 and the remainder is zero; c) i) each tens block has nine shaded squares, so there are 2 \times 9 = 18 shaded squares, ii) each hundreds block has 99 shaded squares, and each tens block has 9 shaded squares, so the total number of shaded squares is also a multiple of 9, as it is: 99 + 99 + 99 + 9 + 9 + 9 + 9 + 9 + 9 + 9 + 9 = 9 \times (11 + 11 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1); d) taking away the shaded squares leaves the unshaded squares, and the number of these is the sum of the digits; f) because subtracting a multiple of 9 doesn’t change the remainder when dividing by 9
Goals:
Students will prove algebraically the divisibility rules for 3, 4, 7, 8, 9, 11, and 13.

Prior Knowledge Required:
Can collect like terms in linear expressions involving two variables
Can represent algebraically linear situations that involve two variables
Can prove algebraically that the sum of two multiples of a number is again a multiple of that number

Vocabulary: divisibility rule

NOTE: Students should complete Lesson PS8-8 before starting this lesson.

Introduce divisibility rules. SAY: We are going to create divisibility rules. That means you will be able to decide easily whether a number is divisible by another number or not. You already know some divisibility rules. ASK: What is the rule for determining if a whole number is divisible by 2? (if the ones digit is divisible by 2, so is the number) What other numbers are easy to make divisibility rules for? (5 or 10) What are those divisibility rules? (a number is divisible by 5 if its ones digit is 0 or 5; a number is divisible by 10 if its ones digit is 0)

NOTE: Some students may already know and be able to apply the divisibility rules for 3, 9, and 11. If so, point out that by the end of the class everyone will be able to not only apply those rules but also to prove algebraically why they work.

Adding or subtracting a multiple of a number doesn’t change the remainder when divided by that number. Write on the board:

Adding or subtracting a multiple of 3 doesn’t change the remainder when dividing by 3.

Read the statement aloud. Have volunteers tell you examples of numbers that have remainder 1 when dividing by 3 (sample answers: 1, 4, 7, 10) Draw on the board a number line from 0 to 10 and have a volunteer mark the multiples of 3 with dots.

SAY: Because I started with one more than a multiple of 3, if I keep adding 3s, I will always get one more than a multiple of 3. That means the remainder stays at 1 no matter how many 3s I add. That means I can add any multiple of 3 and the remainder is still 1.
**Exercises:** Decide if the statement is true. Use a number line to explain your reasoning.

a) Subtracting a multiple of 3 doesn’t change the remainder when dividing by 3.
b) Adding a multiple of 4 doesn’t change the remainder when dividing by 4.
c) Subtracting a multiple of 4 doesn’t change the remainder when dividing by 4.

**Answers:** All statements are true. Sample explanation: a) When you start 1 more than a multiple of 3 and move back 3 spaces, you are still 1 more than a multiple of 3, and it doesn’t matter how many times you move back 3 spaces—you can subtract any multiple of 3 and the result still has remainder 1. If you start with remainder 2, that is, 2 more than a multiple of 3, and you move back 3 spaces any number of times, you end up still 2 more than a multiple of 3.

**Review the reason for the divisibility rules for 2, 5, and 10 for two-digit numbers.** Write on the board:

\[ 74 = 70 + 4 \quad 65 = 60 + 5 \quad 98 = 90 + 8 \]

SAY: I want to understand the divisibility rule for 2. Remember that any multiple of 10 is also a multiple of 2, so you can remove them. Show this on the board:

\[ 74 = \Box + 4 \quad 65 = \Box + 5 \quad 98 = \Box + 8 \]

SAY: It looks like you only need to check if the ones digit is a multiple of 2, but we’ve only shown it for three examples, 74, 65, and 98, and I want to show it for any number. You can use algebra to generalize each step. ASK: How did I split each number up? (into tens and ones) SAY: In every case, I multiplied the tens digit by 10 and added the ones digit. Let’s use variables for the digits. Let’s use \( a \) for the tens digit and \( b \) for the ones digit.

Write on the board:

\[ 74 = 10 \times 7 + 4 \quad 65 = 10 \times 6 + 5 \quad 98 = 10 \times 9 + 8 \]

Any two-digit number with tens digit \( a \) and ones digit \( b \) is \( 10a + b \).

SAY: We’ve just generalized the first step. ASK: What was the second step for all three numbers? (remove the tens) Write on the board:

\[ \Box a + b \]

ASK: What mathematical operation are you using when you remove \( 10a \)? (subtraction) Write on the board:

\[ 10a + b - 10a = b \]

SAY: You can subtract the tens and keep the remainder the same because \( 10a \) is always a multiple of 2, no matter what \( a \) is. So, \( 10a + b \) has the same remainder as \( b \) when dividing by 2. So, \( 10a + b \) is divisible by 2 if and only if \( b \) is divisible by 2.
Exercises:
a) Use algebra to explain why a two-digit number is divisible by 5 if and only if its ones digit is 0 or 5.
b) Use algebra to explain why a two-digit number is divisible by 10 if and only if its ones digit is 0.

Selected answer: a) If the tens digit is $a$, and the ones digit is $b$, the number is $10a + b$. Since $10a$ is always a multiple of 5, you can subtract it and the result (namely, $b$) will have the same remainder when dividing by 5 as the original number. This will be 0 if and only if $b$ is 0 or 5.

SAY: You can do the same thing with three-digit numbers. Write on the board:

$$375 = 300 + 70 + 5$$

ASK: How would you generalize this way of writing 375 to any number? (write it as 100 times the hundreds digit + 10 times the tens digit + the ones digit) Have a volunteer show how to write a three-digit number with hundreds digit $a$, tens digit $b$, and ones digit $c$:

$$100a + 10b + c$$

Then write on the board:

$$1000a + 100b + 10c + d$$

ASK: What does this represent? (a 4-digit number with thousands digit $a$, hundreds digit $b$, tens digit $c$, and ones digit $d$)

Exercises:
1. a) Use algebra to explain why a three-digit number is divisible by 2 (or 5 or 10) if and only if its ones digit is divisible by 2 (or 5 or 10). Repeat for four-digit numbers.
b) Use algebra to explain why a three-digit number is divisible by 4 if and only if its last two digits form a number divisible by 4. Repeat for four-digit numbers.

Selected answer: b) $1000a + 100b$ is always a multiple of 4, so $1000a + 100b + 10c + d$ has the same remainder when dividing by 4 as does $10c + d$. So, a number leaves no remainder when divided by 4 precisely when its last two digits form a number that leaves no remainder.

2. Decide if the number is a multiple of 4.
   a) 236    b) 8142    c) 35 216    d) 217 384 211
   Answers: a) yes, because 36 is; b) no, because 42 is not; c) yes, because 16 is; d) no, because 11 is not

Teach the trick for divisibility by 3. Write on the board:

$$471$$

$$4 + 7 + 1 = 12$$

SAY: To check if a number is divisible by 3, add the digits. If the result is divisible by 3, then the number is divisible by 3. In this case, 12 is a multiple of 3, so 471 is as well.
**Exercises:** Use the trick to decide if the number is a multiple of 3. Verify your answer by doing the division.

a) 815  
b) 927  
c) 532  
d) 6142  
**Bonus:** 78 326 184

**Answers:** The sum of the digits is: a) 14, so no; b) 18, so yes; c) 10, so no; d) 13, so no; 
**Bonus:** 39, so yes

**Using expanded form to understand the divisibility rule for 3.** Write on the board:

\[ 25 = 20 + 5 \]

Say: You can take away 9 from each tens block. Then you’re left with 2 ones from the tens blocks and 5 ones. Draw on the board:

\[
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\]

Write on the board:

\[ 25 \div 3 \text{ has the same remainder as } (2 + 5) \div 3. \]

Say: I could take more from the ones blocks, but the reason I don’t want to is that I’m trying to understand the general rule that you just add the digits to test for divisibility by 3.

**Exercises:** Draw a picture to show that …

a) 34 \div 3 \text{ has the same remainder as } (3 + 4) \div 3  
b) 86 \div 3 \text{ has the same remainder as } (8 + 6) \div 3  
c) 158 \div 3 \text{ has the same remainder as } (1 + 5 + 8) \div 3  
d) 231 \div 3 \text{ has the same remainder as } (2 + 3 + 1) \div 3  
**Bonus:** 2142 \div 3 \text{ has the same remainder as } (2 + 1 + 4 + 2) \div 3

**Using algebra to show the divisibility rule for dividing by 3.** Say: You can generalize this to any number, not just the specific examples. Write on the board:

\[ 10a + b \]
SAY: In this case, \(a\) is the number of tens blocks and \(b\) is the number of ones blocks. ASK: When you take nine from each tens block, what are you subtracting? \((9 \times a)\) Write on the board:

\[
10a + b - 9a =
\]

SAY: Remember, you can collect like terms to simplify the expression. Have a volunteer simplify the expression.

\[
10a + b - 9a = a + b
\]

SAY: When you start with \(10 \times a\)'s and take away \(9 \times a\)'s, you still have \(1 \times a\) left. ASK: Is \(9 \times a\) a multiple of 3? (yes) SAY: Any multiple of 9 is a multiple of 3, so you can subtract a multiple of 3 from any two-digit number and get the sum of its digits. Have a volunteer show how to write a three-digit number with hundreds digit \(a\), tens digit \(b\), and ones digit \(c\). (see below)

\[
100a + 10b + c
\]

Then write on the board:

\[
1000a + 100b + 10c + d
\]

ASK: What does this represent? (a four-digit number with thousands digit \(a\), hundreds digit \(b\), tens digit \(c\), and ones digit \(d\))

**Exercises:** Show how to subtract a multiple of 3 to get the sum of the digits.

a) \(100a + 10b + c\), where \(a\), \(b\), and \(c\) are the digits
b) \(1000a + 100b + 10c + d\), where \(a\), \(b\), \(c\), and \(d\) are the digits
c) any five-digit number

**Bonus:** any number

**Answers:** a) subtract \(99a + 9b\); b) subtract \(999a + 99b + 9c\); c) let \(a\), \(b\), \(c\), \(d\), and \(e\) be the digits starting with the ten thousands digit, then subtract \(9999a + 999b + 99c + 9d\); Bonus: For each digit other than the ones digit, subtract \((10^n - 1) \times \) that digit. This is always a multiple of 3 because the number \(10^n - 1\) always consists of 9s.

SAY: You can always subtract a multiple of 3 from a number to get the sum of its digits. So, if the sum of the digits is a multiple of 3, so is the number. If the sum of the digits is not a multiple of 3, neither is the number.

**Discovering and applying the divisibility rule for 11 for three-digit numbers.** SAY: There is also a divisibility rule for 11. ASK: What are the two-digit numbers that are multiples of 11? (11, 22, 33, 44, 55, 66, 77, 88, 99) What is a rule for deciding if a two-digit number is divisible by 11? (if the digits are the same, the number is divisible by 11; if not, the number is not divisible by 11) SAY: You can use two-digit multiples of 11 to make more three-digit multiples of 11. Write on the board:

\[
220 + 33 = 253
\]

ASK: Is this a multiple of 11? (yes) How do you know? (both 220 and 33 are, so adding them is another multiple of 11)
NOTE: Review the answers to Exercises 1 and 2 with students before providing Exercises 3 and 4.

**Exercises:**

1. Add to find more multiples of 11.
   a) 440 + 33  
   b) 550 + 11  
   c) 770 + 22  
   d) 220 + 66  
   **Answers:** a) 473, b) 561, c) 792, d) 286

2. Look for a pattern in your answers to Exercise 1. How can you get the tens digit from the other two digits?
   **Answer:** the tens digit is the sum of the other two digits

3. Add to find more multiples of 11.
   a) 440 + 88  
   b) 550 + 77  
   c) 770 + 88  
   d) 220 + 99  
   **Answers:** a) 528, b) 627, c) 858, d) 319

4. Look for a pattern in your answers to Exercise 3. How can you get the tens digit from the other two digits? What is different about the additions in Exercises 1 and 2?
   **Answers:** The tens digit is 11 less than the sum of the other two digits. In Exercise 1, no regrouping was required, but in Exercise 2, regrouping was required.

When students have completed Exercises 3 and 4, review the answers with them. Then tell students that you wonder if this type of addition gets all the three-digit multiples of 11. SAY: Let's list the additions in an organized way. Write on the board:

\[
\begin{array}{llll}
110 + 0 & 220 + 0 & 330 + 0 \\
110 + 11 & 220 + 11 & 330 + 11 \\
110 + 22 & 220 + 22 & 330 + 22 \\
\ldots & \ldots & \ldots \\
110 + 99 & 220 + 99 & 330 + 99 & \ldots
\end{array}
\]

ASK: How can we check if this is all the multiples of 11? (check if there are any multiples missed in between columns) SAY: The first column does start at the first multiple of 11 bigger than 99 and all the numbers go up by 11 in each column, so we just have to check whether or not there are any missed in between the columns. Write on the board:

\[
\begin{array}{llll}
110 & 220 & 330 & 440 \ldots
\end{array}
\]

ASK: How far apart are these numbers? (110 apart) SAY: You can get the next multiple of 11 after adding 99 by adding 110, so there aren't any missed. This means that all the multiples of 11 that have three digits follow the rule: either the tens digit is the sum of the other two digits or it is 11 less than the sum of the other two digits. Write on the board:

\[
\text{If } 100a + 10b + c \text{ is a multiple of 11, then either } a + c = b \text{ or } a + c = b + 11.
\]
You can write this by moving all the variables to one side. Write on the board:

\[ a + c - b = 0 \text{ or } a + c - b = 11 \]

**Exercises:** Use the divisibility rule for 11 to decide if the number is divisible by 11.

- a) 508  
- b) 528  
- c) 374  
- d) 384  
- e) 661  
- f) 616

**Answers:**

- a) 5 + 8 - 0 = 13, no;  
- b) 5 + 8 - 2 = 11, yes;  
- c) 3 + 4 - 7 = 0, yes;  
- d) 3 + 4 - 8 = -1, no;  
- e) 6 + 1 - 6 = 1, no;  
- f) 6 + 6 - 1 = 11, yes

**Proving algebraically the divisibility rule for 11.** Remind students about the divisibility rule for 3. SAY: I want to prove the divisibility rule for 11 the same way we did for 3. Let’s remember what worked for 3. Write on the board:

\[ 100a + 10b + c - 99a - 9b = \]

SAY: We used multiples of 3 that are close to 10 and 100 and subtracted from the original number. Have a volunteer collect like terms to simplify the equation. (see below)

\[ 100a + 10b + c - 99a - 9b = a + b + c \]

SAY: So, the number is divisible by 3 as long as the sum of the digits is divisible by 3.

ASK: What number close to 10 is divisible by 11? (11) What is the closest number to 100 that is divisible by 11? (99) Write on the board:

\[ 100a + 10b + c - 99a - 11b = \]

SAY: We’re using multiples of 11 that are close to 10 and 100 to subtract from the original number. Have a volunteer collect like terms to simplify the equation. (see below)

\[ 100a + 10b + c - 99a - 11b = a - b + c \]

SAY: The three-digit number is divisible by 11 as long as \( a - b + c \) is divisible by 11. That means \( a - b + c \) can be 0 or 11. ASK: Can \( a - b + c \) be 22? (no) Why not? (\( a + c \) is less than 20, so \( a - b + c \) is also less than 20) Can \( a - b + c \) be -11? (no) Why not? (you are subtracting at most 9 from a positive number)

SAY: Using algebra makes it easy to create a divisibility rule for dividing four-digit numbers by 11.

**Exercises:**

1. a) Use long division to show that 1001 is divisible by 11.  
   b) Subtract 1001a + 99b + 11c - (1000a + 100b + 10c + d).  
   c) Create a divisibility rule to decide if a four-digit number is divisible by 11.  
   d) Evaluate 783 \times 11. Check that your rule says that the resulting number is a multiple of 11.  
   e) Evaluate 680 \times 11. Check that your rule says that the resulting number is a multiple of 11.  
   f) Evaluate 831 \times 11 + 2. Check that your rule says that the resulting number is not a multiple of 11.
Answers: a) 1001 ÷ 11 = 91 R 0; b) \( a - b + c - d \); c) Add the ones and hundreds digits and add the tens and thousands digits. Then subtract one sum from the other. If the result is a multiple of 11 (−11, 0, or 11), then the original number is also a multiple of 11; d) \( 783 \times 11 = 8613 \), and \( 8 - 6 + 1 - 3 = 0 \), so 8613 is a multiple of 11; e) \( 680 \times 11 = 7480 \), and \( 7 + 8 - 4 = 11 \), so 7480 is a multiple of 11; f) \( 831 \times 11 + 2 = 9143 \) and \( 9 + 4 - 1 - 3 = 9 \), so 9143 is not a multiple of 11.

2. Is the number divisible by 3, 4, or 11?
   a) 275  b) 572  c) 561  d) 516  e) 1716
   Answers: a) 11; b) 4 and 11; c) 3 and 11; d) 3 and 4; e) 3, 4, and 11

Combining divisibility rules. SAY: I want to look at how prime factorizations can help us make divisibility rules for new numbers. Write on the board:

\[
396 = 2 \times 2 \times 3 \times 3 \times 11
\]

Have volunteers use the prime factorization to call out factors of 396. (possible answers: 1, 2, 3, 4, 6, 9, 11, 12, 18, 22, 33, 36, 44, 66, 99, 132, 198, 396) Have volunteers justify some of the answers. (sample answer: 12 is a factor because it is \( 2 \times 2 \times 3 \))

ASK: Is 6 a factor? (yes) How do you know? (\( 2 \times 3 \) is part of the factorization) SAY: A number is divisible by 6 if and only if its prime factorization has both 2 and 3, so you can make a divisibility rule for 6 from the rules for 2 and 3. Just check if it is divisible by 2 and by 3. Both answers need to be yes. ASK: How would you make a divisibility rule for 33? (check if it’s divisible by 3 and by 11)

Exercises:
1. Is the number divisible by 6?
   a) 216  b) 312  c) 231  d) 352  Bonus: 217 302 480
   Answers: a) yes, b) yes, c) no, d) no, Bonus: yes

2. Is the number divisible by 33?
   a) 264  b) 563  c) 1782  d) 85 380
   Answers: a) yes, b) no, c) yes, d) no

ASK: What two divisibility rules did you combine to make a divisibility rule for 6? (divisibility rules for 2 and for 3) What two divisibility rules can you combine to make a divisibility rule for 12? (the divisibility rules for 4 and 3) Would checking for divisibility by 2 and 6 work? (no) Write on the board:

\[
\begin{align*}
2 \text{ and } 2 \times 3 \text{ are factors} \\
2 \times 2 \text{ and } 3 \text{ are factors}
\end{align*}
\]

SAY: The 2 says that 2 is a factor, and the 6 says that \( 2 \times 3 \) is a factor, but it might be the same 2 as from the 2. But 4 says that \( 2 \times 2 \) is a factor and 3 says that 3 is a factor, so \( 2 \times 2 \times 3 \) is a factor because there is no overlap between \( 2 \times 2 \) and 3. Write on the board:

\[
30 = 2 \times 3 \times 5
\]
This number has 2 as a factor and it has 6 as a factor, but it doesn’t have 12 as a factor because you would need $2 \times 2 \times 3$, and so 30 is not a multiple of 12. If you want to combine divisibility rules, you have to make sure that the two numbers you are combining do not have any factors in common.

Now you have divisibility rules for 2, 3, 4, 5, 6, 10, and 11. Write those numbers on the board:

2 3 4 5 6 10 11

**Exercises:**
1. Which divisibility rules would you combine to make a divisibility rule for each number?
   a) 15  b) 20  c) 30  d) 44  Bonus: 60
   **Answers:** check if the number is divisible by: a) 3 and 5; b) 4 and 5; c) 5 and 6, or 2, 3, and 5; d) 4 and 11; Bonus: 3, 4, and 5

2. Is the number divisible by 15?
   a) 1215  b) 315  c) 321  d) 125  Bonus: 1 234 567 890
   **Answers:** a) yes, b) yes, c) no, d) no, Bonus: yes

**Problem Bank**
1. Find the missing digit so that $38\,12\,\square$ is divisible by both 2 and 3.
   **Answer:** To be divisible by 3, the missing digit must be 1, 4, or 7. Since the number is divisible by 2, the missing digit must be 4.

2. a) Write 561 561 as a multiple of 1001.
   b) 1001 = $7 \times 11 \times 13$. Decide which of the prime numbers 7, 11, and 13 the number 561 568 is divisible by. Explain how you know.
   **Answers:** a) 561 561 = $561 \times 1000 + 561 \times 1 = 561 \times 1001$; b) 561 568 is 7 more than a multiple of 7, so it is a multiple of 7. It is 7 more than a multiple of 11, so it is not a multiple of 11. It is 7 more than a multiple of 13, so it is not a multiple of 13.

3. a) Can you rearrange the digits so that 931 is a multiple of 3? Explain how you know.
   b) Can you rearrange the digits so that 931 is a multiple of 11? Explain how you know.
   **Answers:** a) no, because no matter how you rearrange the digits, the sum of the digits is still 13, which is not a multiple of 3; b) yes, 913 is a multiple of 11 because $9 - 1 + 3 = 11$

4. What three-digit number am I?
   a) My digits add to 11. I am a multiple of 11. My hundreds digit is 4.
   b) I am a multiple of 3, 4, and 11. My tens digit is one greater than my ones digit.
   **Answers:** a) 407, b) 132

5. a) Find two missing digits A and B so that 8AB3 is a multiple of both 3 and 11.
   b) Can you find two missing digits A and B so that A8B3 is a multiple of both 3 and 11? Explain.
   **Answers:** a) sample answer: 8613; b) A − 8 + B − 3 = A + B − 11 must be either 0 or 11, so that A + B is either 11 or 22. It can’t be 22 because A and B are both at most 9, so A + B is 11. But then the sum of the digits is A + 8 + B + 3 = 22, which is not a multiple of 3.
6. a) A number $3x + 1$ has remainder 1 when divided by 3. Another number, $3y$, is a multiple of 3. What is the remainder when you divide their sum by 3? 
b) A number $3x + 2$ has remainder 2 when divided by 3. When you add a multiple of 3, what is the remainder when you divide by 3? 
c) A number has remainder 4 when divided by 7. When you add a multiple of 7, what is the remainder when you divide by 7? 
**Bonus:** A number has remainder $m$ when divided by $n$. When you add a multiple of $n$, what is the remainder when you divide by $n$?

**Selected solution:**
a) $3x + 1 + 3y = 3x + 3y + 1 = 3(x + y) + 1$ is 1 more than a multiple of 3, so it has remainder 1 when you divide by 3 

**Answers:** b) 2, c) 4, Bonus: $m$

7. One number has remainder 1 when dividing by 3 and another number has remainder 2 when dividing by 3. What remainder does the sum leave when divided by 3? 
a) Try various pairs of numbers:  

<table>
<thead>
<tr>
<th>remainder 1</th>
<th>remainder 2</th>
<th>remainder?</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

b) Prove algebraically that the sum of two numbers with remainders 1 and 2 when divided by 3 is always a multiple of 3.

**Answers:** b) $3x + 1 + 3y + 2 = 3x + 3y + 3 = 3(x + y + 1)$, which is a multiple of 3

8. Fill in the “Remainder when dividing by 3” addition chart.

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Answer:**

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

9. Subtract a multiple of 9 from the number to get the sum of its digits. 
a) 718 
b) 738 
c) 352 
d) 495 
e) $10a + b$, where $a$ and $b$ are single digits 
f) $100a + 10b + c$, where $a$, $b$, and $c$ are single digits 
g) $1000a + 100b + 10c + d$, where $a$, $b$, $c$, and $d$ are single digits

**Answers:**
a) $718 = 700 + 10 + 8$, and if you subtract $7 \times 99 + 1 \times 9$, you get $7 + 1 + 8 = 16$ 
b) $738 = 700 + 30 + 8$, and if you subtract $7 \times 99 + 3 \times 9$, you get $7 + 3 + 8 = 18$ 
c) $352 = 300 + 50 + 2$, and if you subtract $3 \times 99 + 5 \times 9$, you get $3 + 5 + 2 = 10$ 
d) $495 = 400 + 90 + 5$, and if you subtract $4 \times 99 + 9 \times 9$, you get $4 + 9 + 5 = 18$ 
e) $10a + b - 9a = a + b$ 
f) $100a + 10b + c - (99a + 9b) = a + b + c$ 
g) $1000a + 100b + 10c + d - (999a + 99b + 9c) = a + b + c + d$
b) Use your rule to check if the number is a multiple of 9. Check by long division.

   i) 855       ii) 679       iii) 5432       iv) 8712  

   **Answers:** a) A number is divisible by 9 if the sum of its digits is divisible by 9. This works because you can subtract a multiple of 9 from any number to get the sum of its digits. For example, 718 = 700 + 10 + 8, and if you subtract 7 × 99 + 1 × 9, you get 7 + 1 + 8 = 16; 
b) i) yes, ii) no, iii) no, iv) yes; **Bonus:** yes

11. The number is divisible by 9. Find the missing digit.

   a) 8□2          b) 134□250  

   **Answers:** a) 8, b) 3

12. How many zeros does the number end with?

   a) 3 × 4 × 5 × 6 × 7  
   b) 30 × 40 × 50 × 60 × 70  

   **Answers:** a) 1, b) 6

13. a) Write 200 as a multiple of 8. 
b) 56 is a multiple of 8. 100 is not a multiple of 8.

   i) Is 156 a multiple of 8?  
   ii) Is 256 a multiple of 8?  
   iii) Is 356 a multiple of 8?  
   iv) Is 456 a multiple of 8?

c) 10b + c is a two-digit number with tens digit b and ones digit c. If 10b + c is a multiple of 8, 
when is the three-digit number 100a + 10b + c also a multiple of 8?  

   **Answers:** a) 200 = 25 × 8; b) i) no, ii) yes, iii) no, iv) yes; c) when a is even

14. 52 is a multiple of 4 but not a multiple of 8.

   a) What is the remainder of 52 ÷ 8?  
   b) What is the remainder of 100 ÷ 8?  
   c) What is the remainder of 152 ÷ 8?  
   d) Is the number divisible by 8?

   i) 152       ii) 252       iii) 352       iv) 452  

   e) 10b + c is a two-digit number with tens digit b and ones digit c. If 10b + c is a multiple of 4 but 
not 8, when is the three-digit number 100a + 10b + c a multiple of 8?  

   **Answers:** a) 4; b) 4; c) 0, because it has the same remainder as 4 + 4 = 8; d) i) yes, ii) no, 
   iii) yes, iv) no; e) when a is odd
In this unit, students will distinguish between right and skew prisms and construct nets of 3-D shapes. They will also determine the relationships between the height, the area of the base, and the volume of right prisms and cylinders and connect volume to capacity. They will develop and use the formulas for finding the volume of right prisms and cylinders. Students will also determine the surface area of right prisms and cylinders and solve problems involving volume and surface area of prisms and cylinders.

Materials
Students will benefit from seeing a variety of 3-D shapes, both prisms (right and skew) and not prisms. Such shapes are required in some lessons. If you do not have a commercial set of 3-D shapes, you can either make some from modelling clay (you will find tips for working with modelling clay on our website or create shapes using nets provided on BLM Nets of 3-D Shapes (pp R-48, U-1–U-24). You can also find examples of different 3-D shapes among boxes. For example, some chocolate boxes (e.g., Toblerone) are triangular or hexagonal prisms, and you can turn a standard milk carton into a pentagonal prism by cutting off the strip on the top and covering the dips in the bases with paper.

You will also need a variety of boxes and cylindrical cans of different dimensions. Ahead of time, ask students to bring from home various boxes and cans, such as empty medicine packages, soup cans, tea boxes, cereal boxes, and so on. Boxes in the shape of prisms that are not rectangular prisms will be very useful as well.

Capacity vs. Volume
Volume is the amount of space taken up by a three-dimensional object and capacity is defined as how much a container can hold. One way to distinguish volume from capacity (at least at this level) is to look at the units in which they are measured: volume is measured in linear units cubed—centimetres cubed (cm³), metres cubed (m³), kilometres cubed (km³)—while capacity is measured in millilitres or litres (mL or L).

Meeting Your Curriculum
Lessons ME8-9 through ME8-12 and ME8-16 deal with nets, volume, and the surface area of prisms, which students in Ontario learned in Grade 7. However, to be able to solve problems involving volume and surface area of both prisms and cylinders, as required by the Ontario curriculum, students need a good review of this material. Ontario teachers are encouraged to go through these lessons at least briefly.

For students following the WNCP curriculum, lesson ME8-9 contains essential review material, and the rest of the unit is core curriculum.
ME8-9  Right Prisms
Pages 146–147

CURRICULUM EXPECTATIONS
Ontario: 7m34, 7m49; 8m5, 8m7, review
WNCP: 6SS3; essential for 8SS2, [C, CN, V]

VOCABULARY
right prism
skew prism
face
edge
vertex
skeleton
volume
length
width
height

PROCESS ASSESSMENT
8m7, [C]
Workbook Question 3b)

Goals
Students will identify and sketch right prisms.

PRIOR KNOWLEDGE REQUIRED
Can identify right angles
Can identify right prisms
Can find the volume of a right rectangular prism
Can multiply or divide decimals
Is familiar with cubic units of measurement

MATERIALS
a variety of right and skew prisms (see Introduction)
BLM Nets of 3-D Shapes (pp R-48, U-1–U-24)
modelling clay and toothpicks (to make a skeleton)
grid paper

Review prisms. Prisms have two identical (congruent) polygonal faces called bases and side faces that are parallelograms. Students might be familiar only with right prisms, whose side faces are rectangles. Present several 3-D shapes (do not include skew prisms for now) one at a time and have students tell whether each shape is a prism or not. To assess students at a glance, you can ask them to answer “yes” and “no” in ASL (shake your fist up and down for “yes,” touch the thumb with the pointer and the middle finger together for “no”). Then ask volunteers to place all the prisms base down.

Right prisms and skew prisms. A skeleton of a prism is a model that has only edges and vertices, no faces. Show students how you can make a skeleton of a prism using modelling clay and toothpicks: make two copies of a base, add vertical edges to one of the bases, and attach the other base on top. Place two copies of a skeleton on the table, base down, and shift the top base of one of them so that the whole prism is tilted to the side. Explain that the prism you have created is called a skew prism. The original prism is called a right prism. Display several prisms (as below; you can find the nets for them on BLM Nets of 3-D Shapes) and have students sort them into right prisms and skew prisms as a class. Add several shapes that are not prisms and have students explain why these are not prisms. Note that answers will vary for shapes that are not prisms (side faces are not parallelograms, bases are not the same size, bases are congruent but are not translations of each other, etc.)
### Type of Shape Properties BLM Nets of 3-D Shapes

<table>
<thead>
<tr>
<th>Type of shape</th>
<th>Properties</th>
<th>BLM Nets of 3-D Shapes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Right prisms</td>
<td>The top face is directly above the bottom face. Side edges are vertical.</td>
<td>Shapes 1–11 (3, 4, 5 shown here)</td>
</tr>
<tr>
<td>Skew prisms</td>
<td>The top face is shifted from the bottom face. Side edges are not vertical. Side faces are parallelograms.</td>
<td>Shapes 12–15 (12 and 15 shown here)</td>
</tr>
<tr>
<td>Not prisms</td>
<td>The side faces are not parallelograms, bases are not the same size, bases are facing different ways, etc.</td>
<td>Shapes 16–24 (20, 21, 22 shown here)</td>
</tr>
</tbody>
</table>

**NOTE:** For prisms that have all faces in the shape of a parallelogram, any pair of faces can be taken as bases. If four of the faces are rectangles, you can take the remaining two faces to be bases and conclude that the prism is a right prism. The following shapes on BLM Nets of 3-D Shapes are composed only of parallelograms, should you want to focus on them and have students determine which are right prisms and which are not.

- 3—a right rectangular prism (any pair of opposite faces can be placed directly above each other)
- 8—a right prism with a parallelogram base (only the non-rectangular faces—the parallelograms—can be placed directly above each other)
- 12—a skew prism with three different pairs of identical parallelogram faces (no pair of faces can be placed directly above each other)
- 13—a skew rectangular prism (two faces are rectangles, but no pair of faces can be placed directly above each other)

**The angle between the base and the side faces.** Show students a book that is open partway. Explain that just as the space between two rays with the same endpoint forms the angle between two lines, the space between two faces joined at an edge forms the angle between the faces. This angle can be acute, right, or obtuse (show each with the pages of the book).

Sketch two lines, one vertical and the other horizontal. **ASK:** What is the angle between the lines? (90°, right angle) Place a tall box (such as a large cereal box) on your desk. **SAY:** The sides of the box are vertical and the desk is horizontal. What is the angle between the sides of the box and the desk? (90°, right angle) Hold up prisms one at a time and invite volunteers to stand them on their bases next to the cereal box. **ASK:** Can you place the prism so that it touches the side of the box with a whole side face? Can you do it for all the side faces? Have volunteers check all the side faces. As a class, sort the prisms into those that have at least one side face that doesn’t sit flush against the box and those for which any
side face can be placed flush against the box. If the prism has all faces that are parallelograms, you can try different faces as bases. Look for a pair of bases such that all side faces can be placed flush against the box (for example, the prism on BLM Nets of 3-D Shapes (8) has one pair of bases, the parallelogram faces, that allow all side faces to be placed flush against the box, so it should be placed in the first group). 

**ASK:** How would you describe the shapes in the two groups? (right prisms and skew prisms)

What is the angle between the side faces and the base for right prisms? (a right angle)

**Hidden lines.** Draw a picture of a cube using dashed lines for the hidden edges. Explain to students that the edges on the back of the shape are often drawn using dashed lines to indicate that we can’t see them. The dashed lines and the solid lines might intersect in the picture, but if the point of intersection is not a vertex, there is no real intersection between the edges there. The lines intersect in the picture because the picture is flat, but the shape itself is 3-dimensional.

**Sketching cubes.** Show your students how they can draw a picture of a cube.

**Step 1:** Draw a square that will become the front face.

**Step 2:** Draw another square of the same size, so that the centre of the first square is a vertex of the second square.

**Step 3:** Join the corresponding vertices with lines as shown.

**Step 4:** Erase parts of the lines that represent hidden edges, to make them dashed lines.

Struggling students will find it helpful to draw cubes on dot paper or grid paper.

**Sketching right rectangular prisms.** Sketch the two diagrams at left on the board and draw students’ attention to the difference between these shapes and the previous cube. Both shapes at left are not cubes—they are rectangular prisms of different lengths, with the front and back faces being squares. The side faces, the top, and the bottom look different because the prisms have different lengths. 

**ASK:** How was Step 2 performed differently in each drawing? (the bottom left vertex of the back face is not at the centre of the front face—it is closer to the bottom left vertex of the front face in the thinner shape and farther from that vertex in the longer shape)

**ASK:** What would you do in Step 2 to draw a very long rectangular prism? Point out that the corner of the back face should not sit on the diagonal, so that the edges do not overlap:
Have students draw a long rectangular prism.

Add dimensions to one of the prisms you sketched, as shown. **ASK:** Could these be the dimensions of this prism? What is wrong? Have a student rearrange the dimensions to better match the picture.

Remind students that to find the volume of a right rectangular prism, they multiply the length, the width, and the height. Have students find the volume of the prism above.

Ask students to sketch a rectangular prism, add some dimensions, and swap their sketch with a partner. Have students find the volume of their partner's prism. Then ask students to sketch a rectangular prism that has a volume of 300 cm$^3$ and compare their sketches with a partner. Did partners draw the same prism? Can they draw a different prism with the same volume?

**Sketching other right prisms using the same method.** Have students draw other prisms, such as triangular or pentagonal prisms, the same way they drew rectangular prisms. Point out that the rectangular faces in these sketches are distorted—they look like parallelograms. The bases (the front and back faces) are not distorted.

**Sketching a prism standing on a base rather than on one of the rectangular faces.** Point out to students that in this position, the front face is a rectangle and is not distorted, while the bases, which are now at the top and the bottom, are distorted. To see the distortion of the base in this position, suggest that students hold a pattern block or a paper polygon horizontally, slightly below eye level. They should see that the polygon in the base appears to be squashed vertically (shorter and wider) compared to the polygon viewed head on. To draw a prism standing on a base, draw the base shorter and wider than it is when viewed head on, then draw the second (top) base directly above the bottom base and join the vertices to produce the side faces.

**Extension**

**Sketch a skew prism.** Sketch the first (bottom) base as when drawing a right prism, but sketch the second (top) base so that it is not directly above the first base. Also, add a line perpendicular to the top base to show that there is an angle. **EXAMPLE:**
Identify 3-D shapes from their faces. On the board, draw several shapes (that together are the faces of a right prism) and ask your students which 3-D shape they make. If students have trouble identifying the shapes, ask them to circle the base(s) first, by looking for shapes that are not rectangles or parallelograms. **EXAMPLES:**

(pentagonal prism) (triangular prism)

**Introduce nets.** Hold up a net for a cube, e.g., **BLM Nets of 3-D Shapes (1)**. Ask students to identify the shapes the net is made of. (squares) **ASK:** What 3-D shape that you know has all its faces that are squares? (cube) Fold the net, to show that it indeed folds into a cube. Explain that a *net* of a 3-D shape shows all the faces of the shape attached together and can be folded into the 3-D shape.

**Making nets.** Hold up a triangular or pentagonal prism (you can make one using **BLM Nets of 3-D Shapes (2, 6)**). **ASK:** Which shapes are the faces of this prism? How many bases does it have? What is the shape of the side faces? How many side faces does it have? If you wanted to make a net for this prism, the easiest way would be draw the band of rectangles for side faces (created by rolling a prism and tracing the side faces in turn) and then add the bases. Illustrate this on the board.

Explain that another way to make a net for this prism is to start with a base (draw it on the board and write “base” on it), add a side face along each edge of the base (draw one side face and ask volunteers to draw the rest), and finish with the second base. Model this method on the board as well.
Different nets for the same prism. This would be a good time to do Activity 1, below.

A net or not a net? After students finish Activity 1, draw several incorrect examples of nets for a triangular prism on the board and ask volunteers to explain why these drawings cannot serve as nets for prisms:

- The bases are not the same
- The middle face is too short
- The base at the bottom is flipped
- A side face is missing

Have students work in pairs, with one student drawing a picture that will not work as a prism net and the other student explaining why the drawing cannot be a prism net. Have students switch roles several times. Then have each pair choose two of their favourite “nets” and swap them with another pair of students. Afterwards, each group of four can present one of their drawings to the class and explain why it cannot be a prism net. For a more challenging task, students can draw pictures that might or might not work as nets, and the class can predict if these are prism nets (students can use thumbs up and thumbs down to show their answer). After each vote, ask students to sketch the net on a sheet of paper. When the voting is done, have different students redraw the nets to scale, cut them out, and fold them, to check the prediction.

EXTRA PRACTICE:

1. How many of each type of face would you need to make this prism?

2. Have students complete BLM A Net or Not a Net? Students can use tape to join the faces. This exercise emphasizes the fact that it’s very hard to correctly identify nets for skew prisms just by looking at them. You have to cut them out and check! **ANSWERS:**

   - skew square prism
   - skew triangular prism
   - (no)
   - (no)

Nets and dimensions. Give each student a small rectangular box with different length, width, and height (e.g., small empty pill package). Have students number the faces of their prism. Ask students to measure the sides of the prism and to mark the dimensions of each side on a drawing of the side. Have them number the sides in their drawings as they numbered the sides on the box. How many sides should they have drawn? (6) Ask
students to sketch a net for the same prism, and mark the numbers on
the faces of the net as well. Then ask students to mark the dimensions on
each face of the net. Finally, ask them to place the prism on their desks and
to identify each face of the prism, and then each face of the net, as top,
bottom, right, left, front and back faces.

Have pairs swap boxes and construct nets for each other’s boxes. This time
they need to draw the sides of the boxes to scale. Students should mark
the dimensions on the nets. Do the nets they drew to scale look similar to
the sketches their partners made beforehand? Have students cut out their
nets, fold and tape them, and check that the folded net is the same size as
the prism they started with.

**ACTIVITIES 1–2**

1. Let students explore various ways of creating nets for the same solid
rather than memorizing a single net. They will need various prisms
with faces that are not regular polygons. You can ask students to
make such prisms using modelling clay (they can use plastic knives
to cut the faces flat) or nets from BLM Nets of 3-D Shapes (3, 5, 6,
7, 8, 9). Each student should work with at least two different prisms.

   Have students trace the faces of their prisms on a piece of paper in
order to create a net. **ASK:** them to cut out the nets they have drawn.
   Let them cut off faces of the net (one at a time) and re-attach the
   faces at different places. Will the new net fold into the same prism?
   Which edges are places where you would want to re-attach faces
   and which are not?

2. Give each student nets for three different shapes: a right prism,
a skew prism, and a non-prism. (You can use nets from BLM Nets
for 3-D Shapes.) Have students identify which net belongs to a
right prism and which belongs to a skew prism, and then guess
which shapes these are. Students can also try to describe what the
non-prism will look like. Students should then construct the shapes
from the nets to check their predictions. Pairs can compare the
shapes they produced and explain to each other why their non-
prisms are not prisms.

**Extension**

Give students BLM Is It a Net? Ask students to predict whether each
drawing is the net of a 3-D shape. How are the nets the same? (They have
the same overall shape, and both have two square faces.) How are they
different? (Net A has 4 parallelogram faces, net B has 8 triangular faces)
Have students cut out the nets, fold them, and check their predictions.
(Net A does not fold into a 3-D shape, but net B does—it folds into a shape
called an antiprism.) How does breaking the parallelograms into triangles
help the picture become a net? (When the net is folded, the triangles have
an angle between them, allowing the side faces to fit around both squares.)
Measurement 8-11: Volume of Rectangular Prisms

Goals
Students will find the volume of rectangular prisms.

Prior Knowledge Required
- Is familiar with area and perimeter
- Can find the area of a rectangle
- Knows the formula for the area of a rectangle
- Can multiply or divide decimals
- Is familiar with linear and square units of measurement

Units of length, area, and volume. Review with students the various units used to measure length and area. Point out that 1-dimensional objects, like strings and line segments, have only 1 dimension, length, which we measure in centimetres, metres, kilometres, and so on. Objects that have area are 2-dimensional; they have length and width, and we measure their area with square units, such as m² (where the raised 2 reminds us that they are 2-dimensional). Remind students how the square metres show up in the calculation of area: \(1 \text{ m} \times 1 \text{ m} = 1 \text{ m}^2\). Explain that objects that have length, width, and height are 3-dimensional, and we measure their volume in cubic units, such as cm³. Show students a centimetre cube and point out that its sides are all 1 cm long. Ask students what other measurement units for volume they know. How large are these units?

Remind students that the third dimension in 3-D figures is called height. Identify the length, width, and height in the prism at left. Students can review the formula for the volume of a rectangular prism using Questions 1 and 2 on Workbook page 151.

Use the terms length, width, and height to label the multiplication statement that gives the volume of the prism at left:

\[
3 \text{ cm} \times 2 \text{ cm} \times 4 \text{ cm} = 24 \text{ cm}^3
\]

**ASK:** What does the raised 3 mean? (three 1 cm sides were multiplied to get one cubic cm)

Draw several prisms on the board, mark the height, width, and length (you can use different units for different prisms), and ask students to find the volume. **EXAMPLES:**

a) \(10 \text{ cm} \times 6 \text{ cm} \times 2 \text{ cm}\)  
b) \(2 \text{ m} \times 3.4 \text{ m} \times 5 \text{ m}\)  
c) \(3 \text{ km} \times 4.6 \text{ km} \times 7.2 \text{ km}\)

**ANSWERS:** a) 120 cm³  
b) 34 m³  
c) 99.36 km³

Remind students to include the correct units in their answers.
Volume of a rectangular prism and area of faces. Draw a rectangular prism on the board, mark the dimensions (say, 2 cm, 3 cm, and 10 cm) and have students find the volume of the prism. **ASK:** What is the length of this prism? (10 cm) The width? (3 cm) The height? (2 cm) Write that information on the board. Invite a volunteer to write the volume of the prism in terms of length, width, and height.

Ask students to sketch the net for the prism and label the top, bottom, side, front, and back faces on the net. Then ask them to mark each edge of the prism as length, width, or height. Label the first several edges together, and have students label the rest of the edges individually. See the answer below.

Have students find the area of each face on the net and then write each area in terms of length, width, and height (**EXAMPLE:** top face = length × width).

**ASK:** What does the expression “length × width” represent in the formula of the volume of the prism? What do you find when you multiply length by width? (the area of the top face—or the bottom face—of the prism) Rewrite the formula as “area of top face of prism × height.”

Remind students that order does not matter in multiplication, so length × width × height can be rewritten as, say, length × height × width. What does the expression “length × height” represent in the formula? (the area of the left or right side) Ask students to rewrite the formula using the area of one of the side faces. Rewrite the formula for the volume as height × width × length. Have students identify the expression “height × width” as the area of the front face and rewrite the formula using the area of the front face.

Have students find the volume of the prism with length 4 cm, width 6 cm, and height 7 cm in three ways: using the area of the top face, the area of the front face, and the area of one of the side faces. Did they get the same answer all three ways?

**Finding dimensions of a prism given its volume.** Draw a prism on the board, and give students the area of the bottom face and the volume. **EXAMPLE:** volume 32 cm³, area of bottom face 16 cm². Mark the height as $h$, and ask students to write an equation for the volume of the prism.
Then have them solve the equation. Have students use this method to find the height of several more prisms. Proceed to problems where the missing dimension is not height, but length or width.

**EXTRA PRACTICE:**

1. a) Volume = 75 cm³  
   b) Volume = 12 m³  
   c) Volume = 105 cm³

![Diagram](image)

**ANSWERS:**  a) 3 cm  
   b) 2.4 m  
   c) 7.5 cm

2. The area of the base of a right prism is 8 cm² and its volume is 32 cm³. What is its height? (4 cm)

3. Find the volume of the prism in the margin.

**ANSWER:** The height of the prism is 6 cm² ÷ 2 cm = 3 cm, so the volume is 3 cm × 10 cm² = 30 cm³. Another answer: The length of the prism is 10 cm² ÷ 2 cm = 5 cm, so the volume is 6 cm³ × 5 cm = 30 cm³.

Proceed to problems where the volume and one linear dimension are given, and students need to find the area of the face perpendicular to that direction. **EXAMPLE:** volume 35 cm³, height 4 cm. What is the area of the bottom face? (35 cm³ ÷ 4 cm = 8.75 cm²)

**Word problems practice:**

Jon brought a cake to class to celebrate his birthday. The cake was a rectangular prism 28 cm by 30 cm by 7 cm.

a) What is the total volume of the cake? (5 880 cm³)

b) There are 40 students in the class. How much cake will each person get?

   _______ cm³ ÷ 40 = _______ (5 880 cm³ ÷ 40 = 147 cm³)

c) Use cm grid paper to draw the base of the cake. Cut the cake into 40 equal-sized pieces.

d) What are the dimensions of each piece? Use this to check your answer to part b). **(SAMPLE ANSWER:** 7 cm × 3 cm × 7 cm = 147 cm³)

**Extension**

The volume of a rectangular prism is 24 cm³ and its height is 2 cm. What can be the dimensions of the base of the prism? **SAMPLE ANSWER:** The base of the prism has area 24 ÷ 2 = 12 cm², so the dimensions of the base could be 1 cm × 12 cm, 2 cm × 6 cm, 3 cm × 4 cm, 2.4 cm × 5 cm, and so on.
Volume of rectangular prism \(= \text{area of base} \times \text{height}\). Draw a rectangular prism on the board, mark the dimensions (say, 2 cm, 3 cm, and 4 cm) and have students find the volume of the prism. Invite volunteers to write the volume of the prism in terms of the area of one of the faces and height, length, or width. Remind students that in the case of a rectangular prism, any pair of opposite faces can be bases and the dimension perpendicular to the bases is called \(\text{height}\). So if we take, say, the bottom face to be the base, we can rewrite the formula for the volume of the prism as “\(\text{height of prism} \times \text{area of base of prism}\).”

Volume of triangular prisms with a right triangle in the base. Ask students to think about how they could calculate the volume of such prisms. Ahead of time, photocopied \text{BLM Nets of 3-D Shapes (6)} onto two pages of different colours and make two copies of each prism, say, green and blue. The prisms on the BLM are both right rectangular, but they have different triangles as bases. Show students two identical prisms (one green, one blue) side by side, then show how you can put them together to make a rectangular prism. \text{ASK:} What fraction of the volume of the rectangular prism does each triangular prism make? (half)

Volume of triangular prisms with a scalene triangle in the base. Review finding the area of triangles by splitting them into two right triangles. Join the congruent faces (numbered 1) of the two green prisms together so that they make a single triangular prism with a scalene obtuse base and show that prism to students. Repeat with the blue prisms. Place the green and blue prisms side by side to emphasize that they are identical. Do they have the same volume? (yes) Show how you can make a rectangular prism
by separating the parts of the blue prism and attaching each smaller blue prism to the green one (attach the blue face 2 to the green face 2 and the blue face 3 to the green face 3 to make a single rectangular blue-and-green prism). What fraction of the rectangular prism is the green prism? (half)

Summarize: The volume of a triangular prism with any triangle in the base is half the volume of the rectangular prism with the same height and a base that is twice as large as the base of the triangular prism.

The triangular prisms of each colour can be joined into larger triangular prisms and then combined to form a rectangular prism.

**Volume of prisms with a parallelogram in the base.** Draw a parallelogram on the board and review with students how they can convert a parallelogram to a rectangle with the same area by cutting off a triangle and shifting it to the other side. Remind students that the area of the parallelogram is base × height. Point out that the word “base” has a different meaning here—the base of a parallelogram is the length of the side to which we draw a perpendicular (but the base of a prism is a face, which can itself be a parallelogram).

Display the two green and two blue prisms again. Join the prisms of each colour into larger prisms with a scalene triangle in the base, as above, but this time place the prisms so that they stand on their bases. Show how you can combine the prisms to make a larger prism with a parallelogram in the base (join face 3 of the combined blue prism to face 3 of the combined green prism as shown below). Ask students to identify the base and the height of the parallelogram in the base of the prism. (Again, emphasize that the word base refers to two different things here—the length of the side of a parallelogram and the parallelogram itself, which is the base of the prism.)

**ASK:** How could we convert this parallelogram-based prism into a rectangular prism? Take suggestions, then shift the smallest blue prism to the other side of the combined blue-and-green prism, so that face 2 is joined to face 2 to obtain a rectangular prism as before.
ASK: What is the length of the new prism? (the base of the parallelogram)
What is its width? (the same as the height of the parallelogram in the base)
What is its height? (the same as the height of the parallelogram-based prism)
What is the volume of this prism? (base of parallelogram \( \times \) height of parallelogram \( \times \) height of the prism)
What do the first two terms in the product make? (area of parallelogram)
Ask students to rewrite the formula for the volume of the parallelogram-based prism using the area of the base of the prism. (area of base \( \times \) height of prism)

Ask students to find the volume of a prism with height 7 cm and a parallelogram in the base that has base 5 cm and height 4 cm. Repeat with more prisms.

**Volume of triangular prisms** = area of base \( \times \) height of prism. Explain that now that you have a nice formula for the volume of a prism with a parallelogram in the base (area of base \( \times \) height of prism), you would like to go back to the volume of a triangular prism, to see whether a similar formula would work there. Draw a triangular prism inside a rectangular prism, as shown at left, and ask students what the volume of the triangular prism should be. (half the volume of the rectangular prism) The volume of the rectangular prism is area of base \( \times \) height of prism. Let’s choose the top face of the rectangular prism to be the base, so that it contains the base of the triangular prism and both prisms have the same height. So the volume of the triangular prism is:

\[
\frac{1}{2} \text{ volume of rectangular prism} = \frac{1}{2} \text{ area of base of rectangular prism} \times \text{ height of prism} = \frac{1}{2} \text{ area of rectangle} \times \text{ height of prism} = \text{ area of triangle} \times \text{ height of prism} = \text{ area of base of triangular prism} \times \text{ height of prism}
\]

This means that the same formula (area of base \( \times \) height of prism = volume of prism) works for triangular prisms as well.

**EXTRA PRACTICE:** Find the volume of these prisms.

**ANSWERS:** a) 17 cm\(^3\)  b) 18 cm\(^2\)  c) 18.27 m\(^3\)

**Volume of a prism with any polygon in the base.** Combine any three of the green and blue triangular prisms (all standing on a base) so that they
make a right prism with a complicated polygon in the base. Ask students how they could find the volume of this prism. Draw the shape of the base on the board and invite a volunteer to show how they would split the prism into triangular prisms by splitting the base into triangles. Point out that any polygon can be decomposed into triangles. Ask students to show how to do so for each of the following polygons.

Have students draw two different polygons (with at least 5 sides) and show how to cut them into triangles. Ask students to think about what measurements they will need to take to find the area of each polygon. Then have students take the measurements and find the area of the polygons they drew and the volume of prisms with those polygons as bases and height 20 cm.

**EXTRA PRACTICE:**

Valerie’s teacher says that a triangular prism has volume that is half the volume of a rectangular prism of the same height. Valerie looks at this picture and thinks that the volume of the grey triangular prism is half the volume of the rectangular prism, so the volume of the triangular prism is $5 \text{ cm} \times 2 \text{ cm} \times 10 \text{ cm} \div 2 = 50 \text{ cm}^3$. Is she correct? Explain.

**ANSWERS:** No, the triangular prism is not half the rectangular prism, it is smaller than that. In this triangular prism the front face is the base, so if we take the front face to be the base in the rectangular prism, we can see that the triangle is less than half the $5 \text{ cm} \times 10 \text{ cm}$ rectangle. In fact, the volume of the triangular prism is $(3 \text{ cm} \times 10 \text{ cm} \div 2) \times 2 \text{ cm} = 30 \text{ cm}^3$.

**ACTIVITY**

Give students a variety of right prisms (shapes made from BLM Nets of 3-D Shapes and boxes) and have them measure the prisms and find their volume.

**Extension**

A wealthy king had a treasure chest in the shape of a rectangular prism. He ordered his carpenters to create a larger chest for his treasure.

a) The first carpenter doubled the length of the box and left the width and the height the same. The second carpenter doubled the width of the box and left the length and the height the same. The third carpenter doubled the height of the box and left the length and the width the same. Who made the largest chest for the king’s treasure? (nobody—they all have the same volume)
b) The fourth carpenter doubled the length, the width, and the height of the king’s old treasure chest to create his new chest. How many times larger was the new chest than the old one? (8 times)

c) The fifth carpenter, being jealous of the money the fourth carpenter was paid, decided to make a chest that had the same volume as the chest of the fourth carpenter. He wanted his chest to have the same height as the king’s old chest, but he decided that the length of the new chest would be two times more than the length of the old chest. How many times wider than the king’s old chest did this carpenter make his chest? (4 times)
Review characteristics of circles. Review with students the formula for the area of a circle, the terms radius and diameter, and the meaning of $\pi$. Remind them that $\pi \approx 3.14$, and because this is an approximation and not an exact value, any calculation involving this value is only an approximation. Point out that because we use the value of pi rounded to two decimal places, it makes sense to round the answers to two decimal places as well.

Introduce cylinders. Remind students how regular polygons with many sides look almost like a circle. **ASK:** What will a right prism with a polygon with many sides in the base look like? Introduce the term cylinder, show students a cylinder, and explain that the circles are also called bases, just as the non-rectangular sides of prisms were called bases. To illustrate that a cylinder and a prism with a regular polygon with many sides in the base look very much alike, you can show two stacks of pennies, one made from round pennies, the other from 12-sided pennies.

Develop the formula for the volume of a cylinder. Review the formula for the volume of polygonal prisms: area of base $\times$ height of prism. Ask students to predict, based on what they know about the volume of right prisms, what the formula for the volume of cylinders should be. Draw students’ attention to the stacks of pennies again. The stacks are so much alike that the volume should be almost the same. Have students do Parts A to D of the Investigation on Workbook page 157 individually. If necessary, remind students that the notation $a \div b$ means $a \div b$, and that $a \times b$ can be written as $ab$. Continue through parts E to H of the Investigation as a class.
Practice finding the volume of cylinders. Use questions similar to Questions 2 and 3 on Workbook page 158.

EXTRA PRACTICE: Find the volume of each cylinder.

a) \[ \text{Volume} = \pi \times 4^2 \times 2.25 \approx 90 \text{ cm}^3 \]

b) \[ \text{Volume} = \pi \times 5^2 \times 6.47 \approx 323.5 \text{ cm}^3 \]

c) \[ \text{Volume} = \pi \times 7.5^2 \times 7 \approx 1153.95 \text{ cm}^3 \]

d) \[ \text{Volume} = \pi \times 11^2 \times 7.5 \approx 2127.66 \text{ cm}^3 \]

Word problems practice:

1. A glass of water is full to the brim.
   a) The glass of water measures 68 mm across and is 9 cm tall. How much space (in whole cm³) does the glass of water take up?
   b) The sides of the glass are 3 mm thick, and its bottom is 9 mm thick. 1 cm³ of water is 1 mL. How much water (in whole mL) does the glass hold?

   ANSWERS:
   a) Volume = \( \pi \times 3.42^2 \times 9 \approx 327 \text{ cm}^3 \)
   b) Inner volume = \( \pi \times 3.12^2 \times 8.1 \approx 244 \text{ cm}^3 \), so the capacity of the glass is 244 mL.

2. A railway car is a cylinder 326 cm in diameter. It is 17.4 m long. What is its volume (in m³, rounded to two decimal places)? (145.16 m³)

Extension

Lipstick A is 13 mm wide and 25 mm long, and it costs $5.95. Lipstick B is 15 mm wide and 17 mm long, and it costs $5.87. Which one is larger? How much do they cost per mm³? Which one is cheaper (per mm³)?

(Volume: A \( \approx 3\,316.63 \text{ mm}^3 \), B \( \approx 3\,002.63 \text{ mm}^3 \); cost: A \( \approx 0.18 \text{ cents/mm}^3 \), B \( \approx 0.20 \text{ cents/mm}^3 \). Lipstick A is larger and cheaper.)
Goals

Students will solve problems related to capacity of right prisms and cylinders.

PRIOR KNOWLEDGE REQUIRED

Can find the volume of a rectangular prism
Can identify right prisms as prisms
Can identify the base of a prism
Can multiply or divide decimals
Is familiar with cubic units of measurement
Can find area of a circle
Can find the volume of a cylinder

MATERIALS

small graduated cylinders
centicubes
cans and boxes of different shapes

Capacity. Explain that the capacity of a container is how much it can hold. Write the term on the board.

Remind students that capacity is measured in litres (L) and millilitres (mL).

**ASK:** Where have you seen the prefix “milli” before and what did it mean? (millimetre; one thousandth) How many millilitres are in 1 litre? (1 000) In 2 litres? (2 000) In 7 litres? (7 000) What do you do to change litres to millilitres? (multiply by 1 000)

Write on the board:

\[
\begin{align*}
1 \text{ metre} &= 1 000 \text{ millimetres} \\
1 \text{ litre} &= 1 000 \text{ millilitres}
\end{align*}
\]

Ask students to think of three quantities that are measured in litres and three that are measured in millilitres. (EXAMPLES: mL—cup of juice, small carton of milk, dosage of liquid medicine, bottle of perfume; L—big carton of juice, gas tank of a car, bag of potting soil)

**Connection between mL and cm³.** Hold up a graduated cylinder with some water in it. Drop a centicube (≈ 1 cm³) into the cylinder. **ASK:** your students if they can see how much liquid is displaced by the cube. (no, the amount is too small) What can you do to find how much water is displaced by one cube? (One answer: Drop in 10 cubes and divide the displacement by 10.) If possible, have all students drop centicubes into graduated cylinders and measure the displacement. (should be 10 mL) What is the capacity of 1 cm³? (1 mL)
Show a small rectangular box and ask students how they could measure its capacity. We know the capacity of 1 cm³. What is the capacity of 10 cm³? Of 20 cm³? Invite volunteers to measure the sides of the box and calculate its volume. What is the capacity of the box?

Draw a cube on the board and tell students that it has a capacity of 1 L. ASK: What are the dimensions of the cube? How many millilitres are in 1 L? How do you find the volume of the cube? (You multiply the side by itself 3 times.) Which number is multiplied by itself 3 times to get 1 000? So how long is the side of the cube? (10 cm) Mark the cube sides as 10 cm × 10 cm × 10 cm, and invite volunteers to write both the volume of the cube and its capacity (in mL and L) beside the cube.

Connection between L and dm³. Remind students that 10 centimetres make 1 decimetre. You can draw the conversion “stairs” at left on the board and add decimetres to the empty step. Draw a cube, mark its sides as 1 dm × 1 dm × 1 dm, and ask students to find its volume in dm³ and in cm³. (1 dm³ and 1 000 cm³) What is the capacity of this cube? (1 000 mL = 1 L)

Finding the capacity of prisms. Draw a box on the board and write its dimensions: 30 cm × 40 cm × 50 cm. ASK: What is the capacity of the box? Let students find the capacity in millilitres first, then ask them to convert it to litres.

Ask students if they can solve the problem another way. (HINT: a cube 1 dm × 1 dm × 1 dm has capacity 1 L) They can convert the dimensions to decimetres and get the result in litres: 3 dm × 4 dm × 5 dm = 60 dm³, so the capacity is 60 L.) Did students get the same answer?

Give students more problems of this kind. Include triangular prisms, polygonal prisms, and cylinders. EXAMPLES:

1. A cylindrical water barrel is used to collect rain water. It is 70 cm wide. After a heavy rain, the height of the water in the barrel is 7 cm. How much water is in the barrel?

   ANSWER: volume \(\approx 3.14 \times 35^2 \times 7 \approx 26.9 \text{ cm}^3 \approx 26.9 \text{ L} \)

2. A can of soup has diameter 6.5 cm and height 9.3 cm. The can is made of tin that is 1 mm thick. Find the capacity of the can, rounded to the nearest mL.

   ANSWER: Inner diameter = 6.3 cm (inner radius 3.15 cm), inner height = 9.1 cm, so capacity is 284 mL.

3. Find the volume and the capacity (in L, rounded to one decimal place) of a prism with height 12 cm and the base as shown.

   SOLUTION: Split the pentagon into a trapezoid (with bases 33.2 and 21.2 cm and height 20 cm) and a triangle (with base 33.2 cm and height 34.9 − 20 = 14.9 cm). The area of the trapezoid is 544 cm², and the area of the triangle is 247.34 cm². The area of the base of the prism is thus 791.34 cm², and the volume is 9 496.08 cm³. The capacity is about 9.5 L.
EXTRA PRACTICE:
1. Find the volume and the capacity of the rectangular prisms with these dimensions:
   a) $1 \text{ m} \times 1 \text{ km} \times 1 \text{ m}$  
   b) $5 \text{ cm} \times 0.3 \text{ m} \times 2 \text{ m}$  
   c) $1 \text{ mm} \times 1 \text{ m} \times 1 \text{ km}$

   ANSWERS:
   a) Volume: $1 \text{ m} \times 1 000 \text{ m} \times 1 \text{ m} = 1 000 \text{ m}^3$
      $= 1 000 \times (1 \text{ m} \times 1 \text{ m} \times 1 \text{ m})$
      $= 1 000 \times (10 \text{ dm} \times 10 \text{ dm} \times 10 \text{ dm})$
      $= 1 000 000 \text{ dm}^3$, capacity $1 000 000 \text{ L}$
   b) Volume: $0.05 \text{ m} \times 0.3 \text{ m} \times 2 \text{ m} = 0.03 \text{ m}^3 = 30 000 \text{ cm}^3$,
      capacity $30 000 \text{ mL} = 30 \text{ L}$
   c) Volume: $0.001 \text{ m} \times 1 \text{ m} \times 1 000 \text{ m} = 1 \text{ m}^3$, capacity $1 000 \text{ L}$

2. Daniela wants to find the volume of an apple. She puts the apple into a glass jar with 600 mL of water. The jar is a cylinder $12 \text{ cm}$ in diameter. The water reaches a height of $9.8 \text{ cm}$. What is the volume of the apple?
   (Volume of apple and water together $= \pi \times 6^2 \times 9.8 \text{ cm}^3 \approx 1 107.8 \text{ cm}^3$, so the volume of the apple is $\approx 407.8 \text{ cm}^3$.)

Word problems practice:
1. A recipe for pumpkin pie filling calls for $\frac{13}{4}$ cups of pumpkin puree,  
   1 cup of sugar, 2 beaten eggs, and $\frac{3}{4}$ cups of cream. A cup has capacity $240 \text{ mL}$, and a beaten egg is about $0.2 \text{ cups}$.

   a) Convert the measurements to decimals.
   b) What is the total capacity of the ingredients?
   c) Katie has a round pan $25 \text{ cm}$ in diameter and $4 \text{ cm}$ deep. She made the pie shell about $0.5 \text{ cm}$ thick, and it reaches to the top of the pan. If she uses the recipe above for the filling, will she have too little filling for her pie shell, too much filling, or just enough?
   d) About how far will the top of the filling be from the top of the shell (and the top of the pan)? Round your answer to the nearest millimetre (which is a tenth of a centimetre).

   ANSWERS:
   b) $1.75 + 1 + 0.4 + 0.75 = 3.9 \text{ cups} = 936 \text{ mL}$
   c) The inner radius of the shell is $12.5 - 0.5 = 12 \text{ cm}$ and the height is $3.5 \text{ cm}$, so the capacity of the pie shell is about $1 583 \text{ mL}$. The recipe will not produce enough filling to fill the shell to the top.
   d) The area of the base of the inside of the pie shell is $472.16 \text{ cm}^2$, so the filling will fill about $936 \text{ cm}^3 \div 472.16 \text{ cm}^2 \approx 2.0 \text{ cm}$, so there will be about $4.0 \text{ cm} - 0.5 \text{ cm} - 2.0 \text{ cm} = 1.5 \text{ cm}$ between the top of the pie shell and the top of the filling.
2. Ellie made a triangular prism from modelling clay, as shown. She rolled it into a cylinder with diameter 3 cm. What is the height of the cylinder she made?

**ANSWER:** Volume of prism = \(5 \text{ cm} \times 3 \text{ cm} \times 5 \text{ cm} \div 2 = 37.5 \text{ cm}^3\).

Height of cylinder \(\approx 37.5 \text{ cm}^3 \div (3.14 \times 1.5^2) \text{ cm}^2 \approx 5.3 \text{ cm}\).

**ACTIVITY**

Give students cylindrical cans and boxes in the shape of different prisms (such as chocolate boxes) and have them find the volume and the capacity by taking the necessary measurements.

**Extensions**

1. A cylindrical can of soup measures 97 mm from top to bottom and 67 mm across.

   a) Find the volume of the can.

   b) The label says that the can contains 284 mL of soup. Examine a similar can and explain why the volume on the label can be different from the volume you found in a). (ANSWER: The can measurements include the rim of the can and its thickness. The inner radius and height, which determine capacity, will be smaller.)

2. When baking brownies, it is important that you keep the height of the brownie mix in the pan very close to what the recipe calls for. How should you change the recipe if it asks you to use a 10" square pan and you only have an 8" round pan? (An 8" round pan has a diameter of 8" and a radius of 4").

   **SOLUTION:** A 10" square pan has area of base 100 in\(^2\). An 8" round pan has area of base about 50.24 in\(^2\), which is about half of the base of the square pan. To keep the height the same, we need to halve the volume, so we need to make only half of the recipe.
Make sure students know how to multiply and divide decimals by 10, 100, and 1 000 by shifting the decimal point. You can use the questions below as a diagnostic test. If necessary, review NS8-46 to NS8-51.

Fill in the blanks:

a) 2.7 \times 100 = \ 

b) 29 \div 100 = \ 

c) .45 \times 100 = \ 

d) 3.6 \div 100 = \ 

e) 2.32 \times 1 000 = \ 

f) 254 \div 1 000 = \ 

g) .36 \times 1 000 = \ 

h) 5.07 \div 1 000 = \ 

i) .043 \times 1 000 = \ 

j) .79 \div 1 000 = \ 

k) 4.3 \times 10 000 = \ 

l) 37 \div 10 000 = \ 

m) .18 \times 10 000 = \ 

n) 5.9 \div 10 000 = \ 

o) 6.253 \times 10 000 = \ 

p) 34.56 \div 10 000 = \ 

q) 41.31 \times 1 000 000 = \ 

r) 3 278 \div 1 000 000 = \ 

Review relationships. Review the relationships between units of length (metre, kilometre, millimetre), capacity (litre and millilitre), and mass (gram, kilogram, milligram). Discuss the meaning of the prefixes kilo and milli (both mean 1000, but kilo is used for larger units and milli is used for smaller units).

Do we multiply or divide? Draw the diagram in the margin on the board.

ASK: Which unit is the largest? Which unit is the smallest? Let’s look at a measurement in metres: 2 000 m. How many kilometres is that? (2 km) To convert the measurement in metres to the new unit, kilometres, did we need more or fewer of the new units? (fewer) How many times fewer? (1 000 times fewer) To get the measurement in kilometres from a measurement in metres, should we multiply by 1 000 or divide by 1 000?
(divide) Why? (The new unit is larger than the old unit, so we need fewer new units.) Add an arrow from m to km on the diagram and label the division by 1 000.

Discuss the conversion from kilometres to metres, then from metres to millimetres and vice versa, adding more arrows to the diagram. The finished diagram should look like the diagram at left.

Choose pairs of units and ask students if they need to multiply or divide by 1 000 to convert a measurement from one to the other. EXAMPLES: from m to km, from km to mm, from mm to m.

Convert between units by multiplying or dividing by 1 000. Fill in the blanks in the following two questions together, then have students practise converting units individually using the same thinking.

Convert 275 mm to m:

The new units are ______ times ______. (1 000 / bigger)
So I need ______ times ______ units. (1 000 / fewer)
I ___________ by _______. (divide / 1 000)
275 mm = ______ m (0.275)

Convert 27.5 km to m:

The new units are ______ times ______. (1 000 / smaller)
So I need ______ times ______ units. (1 000 / more)
I ___________ by _______. (multiply / 1 000)
27.5 km = ______ m (27 500 m)

EXAMPLES:
Change the units:

a) 245 m to km   b) 2.67 m to mm   c) 0.76 km to m
d) 345 mm to m   e) 36.9 m to km   f) 1 560 m to mm
g) 3.587 mm to m  h) 8 765 km to mm  i) 76.98 mm to m

ANSWERS:
a) 0.245 km   b) 2 670 mm   c) 760 m   d) .345 m   e) .036 9 km
f) 1 560 000 mm   g) .003 587 m   h) 8 765 000 000 mm   i) 0.076 98 m

Review the relationships between m, cm, dm, and mm. Draw the diagram at left on the board. Have students copy the diagram and add arrows and multiplication or division prompts. They should explain, orally and/or in writing, why they multiply or divide in each case and by how much. Review answers on the board.

EXAMPLE: Convert 47 cm to metres. To get from centimetres to metres, I go up 2 steps. This means the new unit is $10 \times 10 = 100$ times larger, so I need 100 times fewer units. So I divide by 100. To divide by 100, I shift the decimal point 2 places to the left: 47 cm = 0.47 m.

Ask students to tell whether they need to multiply or divide, and by how much, to convert measurements from:
a) m to cm  

b) cm to mm  
c) cm to m  
d) mm to cm  

**Bonus**  
e) mm to km  
f) km to cm  

**SAMPLE ANSWER:**  
a) We need more centimetres than metres and there are 2 steps from one to the other in the diagram, so we have to multiply by 100 to convert a measurement in metres to a measurement in centimetres.  

Now have students perform some conversions between these units.  

**EXTRA PRACTICE:**  
Change the units.  
a) 240 m = ____ cm     
b) 2.61 mm = ____ cm  
c) 0.78 cm = ____ m     
d) 38.5 dm = ____ m  

**Bonus**  
e) 36.9 km = ____ cm      
f) 23 568.9 dm = ____ km  
g) 1 234 567 890 mm = ____ km  

**ANSWERS:**  
a) 24 000 cm  
b) 0.261 cm  
c) 0.007 8 m  
d) 3.85 m  

**Bonus**  
e) 3 690 000 cm  
f) 23.456 89 km  
g) 1 234.567 89 km  

**Review the names of the units of area:** metres squared (m²) and centimetres squared (cm²).  

1 m² = 10 000 cm². Draw a very large square on the board and mark its sides as 1 m. **ASK:** What is the area of this square? (1 m²) Write the area below the square. Divide the square into a 10 × 10 grid, and **ASK:** How many squares are in the large square? (100) How do you know?  

**ASK:** How many centimetres are in a metre? Change the markings on the sides to show 1 m = 100 cm. Ask whether the smaller squares are 1 cm × 1 cm. (no) What is the side length of each smaller square? (10 cm) What do you have to do to get squares 1 cm × 1 cm? (divide the smaller squares into another 10 × 10 grid) Show the division on one of the squares. (As an alternative, you can photocopy BLM Square Metre on a transparency and project it on the board; cover the labels to start and uncover them as you go.) How many small squares are in the medium square? (100 × 100 = 10 000) How do you know? What is the area of the medium square? (100 cm²) How many small squares will fit into the largest square? (10 000) How do you know? (There are 100 medium squares in the large square, and 100 small squares can fit in each medium square, so in total there will be 100 × 100 cm² = 10 000 cm²). Write the equation for the area of the largest square on the board.  

**Find the area a different way and compare the two methods.**  
**ASK:** How many small squares fit along the side of the large square? (100) Why? (because 1 m = 100 cm) How do you find the area of a square? What would that give for the large square? (100 cm × 100 cm = 10 000 cm²) What is the area of the large square in metres? Ask students to write the equality for the area: 1 m² = 1 m × 1 m = 100 cm × 100 cm = 10 000 cm².
Now compare these equalities: \(100 \times 100 \text{ cm}^2 = 10000 \text{ cm}^2\) and \(100 \text{ cm} \times 100 \text{ cm} = 10000 \text{ cm}^2\). How are they the same? How are they different? In the first equality, the first number does not have units—it is the number of times the medium square appears—and the second number is the area of the medium square. In the second equality, both numbers are length measurements, so they are measured in centimetres and produce centimetres squared when multiplied.

**Converting between m\(^2\) and cm\(^2\).** Draw a rectangle on the board and mark the sides as 120 cm and 200 cm. Ask students to find the area of the rectangle in cm\(^2\). (24 000 cm\(^2\)) Then ask them to convert the lengths to metres and to find the area in m\(^2\). (1.2 \times 2 = 2.4 \text{ m}^2) Ask students to compare the answers. Which measurement has a larger numeral? Which measurement has a larger unit? Do you need more or fewer cm\(^2\) than m\(^2\)? To get the measurement in cm\(^2\), what do you have to do to the measurement in m\(^2\)? (multiply by 10 000) Why do you multiply? (because the new unit is smaller we need more units, so we have to multiply by 10 000) Repeat with a rectangle 60 cm \times 70 cm, and again with a rectangle 25 cm \times 40 cm.

Write several measurements on the board and ask students to convert the units between cm\(^2\) and m\(^2\).

**EXTRA PRACTICE:**

<table>
<thead>
<tr>
<th>Change the units:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 23 m(^2) =</td>
<td>b) 2.61 m(^2) =</td>
</tr>
<tr>
<td>__________ cm(^2)</td>
<td>__________ cm(^2)</td>
</tr>
<tr>
<td>c) 7 865 cm(^2) =</td>
<td>d) 38.5 cm(^2) =</td>
</tr>
<tr>
<td>__________ m(^2)</td>
<td>__________ m(^2)</td>
</tr>
<tr>
<td>e) 0.076 5 m(^2) =</td>
<td>f) 0.54 m(^2) =</td>
</tr>
<tr>
<td>__________ cm(^2)</td>
<td>__________ cm(^2)</td>
</tr>
<tr>
<td>g) 137 845 cm(^2) =</td>
<td>h) 0.5 cm(^2) =</td>
</tr>
<tr>
<td>__________ m(^2)</td>
<td>__________ m(^2)</td>
</tr>
<tr>
<td>i) 0.002 3 m(^2) =</td>
<td>j) 0.000 06 m(^2) =</td>
</tr>
<tr>
<td>__________ cm(^2)</td>
<td>__________ cm(^2)</td>
</tr>
<tr>
<td>k) 456 cm(^2) =</td>
<td>l) 4.72 cm(^2) =</td>
</tr>
<tr>
<td>__________ m(^2)</td>
<td>__________ m(^2)</td>
</tr>
</tbody>
</table>

**ANSWERS:**

| a) 230 000 cm\(^2\) | b) 26 100 cm\(^2\) | c) 0.786 5 m\(^2\) | d) .003 85 m\(^2\) |
| e) 765 cm\(^2\)     | f) 5 400 cm\(^2\)   | g) 13.784 5 m\(^2\) | h) 0.000 05 m\(^2\) |
| i) 23 cm\(^2\)      | j) 0.6 cm\(^2\)     | k) 0.045 6 m\(^2\)  | l) 0.000 472 m\(^2\) |

**Review the names of the units of volume:** metres cubed (m\(^3\)), millimetres cubed (mm\(^3\)) and centimetres cubed (cm\(^3\)).

1 cm\(^3\) = 1 000 mm\(^3\). Draw a cube on the board and mark its sides as 1 cm. **ASK:** What is the volume of this cube? (1 cm\(^3\)) Write the volume below the cube. Draw a copy of the cube and mark the sides as 10 mm. **ASK:** Are these cubes the same or different? Why? (same, 10 mm = 1 cm)

Ask students to find the volume of the cube in millimetres cubed. Draw a square and mark its sides 1 cm = 10 mm. Have students find the area in cm\(^2\) and in mm\(^2\). Then compare the units of area and volume: a square is 2-dimensional, so 1 cm\(^2\) = 10 mm \times 10 mm = 100 mm\(^2\).
(we multiply the linear factor 10 two times), and a cube is 3-dimensional, so 1 cm³ = 10 mm × 10 mm × 10 mm = 1 000 mm³ (we multiply the linear factor 10 three times).

Review that 1 m³ = 1 000 000 cm³. Draw a square and a cube again, and mark the sides of both as 1 m = 100 cm. Again, have students find the area and volume, in smaller units and in larger units, and again draw attention to the fact that the linear factor is multiplied by itself twice for area and three times for volume.

Connect to powers. ASK: How do mathematicians write an expression such as 100 × 100 in a short way? (100²) How do they write 100 × 100 × 100? (100³) Where do we see a similar notation? (in the units cm² and cm³) Point out that to know what number to multiply or to divide, we need to look at the units. For example, to convert 345 m² to km², we can think this way:

The old units are m² and the new units are km². We know 1 km = 1 000 m, and we are dealing with square units, so the new units are 1 000² times larger. This means we need 1 000 000 fewer units, and so we divide by 1 000 000: 345 m² = 0.000 345 km².

Have students say whether they need to multiply or to divide, and by how much, to convert:

m² to mm² (× 1 000²) cm² to m² (÷ 100²) km³ to m³ (× 1 000³) m³ to cm³ (× 100³)

Practise converting areas and volumes. Have students perform conversions such as:

a) 500 m² = ________ cm²  
b) .9 m³ = ________ cm³  
c) 3 cm² = ________ m²  
d) 1 950 cm³ = ________ m³  
e) 15.4 cm² = ________ m²  
f) 0.05 m³ = ________ cm³  
g) 4 200 m³ = ________ mm³  
h) .7 mm² = ________ cm²  
i) 2.3 km² = ________ m³  
j) 145.4 mm² = ________ m²  
k) 15.34 cm³ = ________ mm³  
l) 0.007 dm³ = ________ cm³

ANSWERS:

a) 5 000 000 cm²  
b) 900 000 cm³  
c) 0.000 3 m²  
d) .001 95 m³  
e) .001 54 m²  
f) 50 000 cm³  
g) 4 200 000 000 000 mm³  
h) .007 cm²  
i) 2 300 000 000 m³  
j) 0.000 145 4 m²  
k) 15 340 mm³  
l) 7 cm³

Measurements need to use the same units in calculation of volume. Remind students that when a prism or a cylinder has measurements given in different units, say metres and centimetres, you would need to convert some of the measurements before multiplying. Metres cannot be multiplied by centimetres! For example, if a prism has sides 2.2 m × 37 cm × 50 cm, we could convert all the measurements to centimetres and find the volume in cm³: 220 cm × 37 cm × 50 cm = 407 000 cm³. Alternatively, we could convert all the measurements to metres and find the volume in m³: 2.2 m × .37 m × .5 m = 0.407 m³. These answers agree, since there are 1 000 000 cm³ in 1 m³. Have students find the volume of prisms and
cylinders with mixed measurements by converting to each of the units used, and check that the answers agree.

a) $105 \text{ cm} \times 0.6 \text{ m} \times 24 \text{ cm}$  

b) $2 \text{ m} \times 3.4 \text{ m} \times 58 \text{ cm}$  

c) radius $3.1 \text{ m}$, height $46 \text{ cm}$

**ANSWERS:**

a) $151200 \text{ cm}^3 = 0.1512 \text{ m}^3$  

b) $3944000 \text{ cm}^3 = 3.944 \text{ m}^3$  

c) $\approx 13.880684 \text{ m}^3 = 13880684 \text{ cm}^3$

Point out to students that an easy way to make sure you do not mistakenly multiply metres by centimetres is to write all the measurements with the units at every stage of a calculation, as in the example above.

**Extensions**

1. Which has larger area? Which has larger perimeter?

   a) a rectangle $30 \text{ cm} \times 1.5 \text{ m}$ or a rectangle $920 \text{ mm} \times 45 \text{ cm}$

   b) a square $1 \text{ m} \times 1 \text{ m}$ or a rectangle $70 \text{ cm} \times 130 \text{ cm}$

   c) a square $1 \text{ m} \times 1 \text{ m}$ or a rectangle $90 \text{ cm} \times 110 \text{ cm}$

   d) a square $1 \text{ m} \times 1 \text{ m}$ or a rectangle $99 \text{ cm} \times 101 \text{ cm}$

   e) a square $1 \text{ m} \times 1 \text{ m}$ or a circle of radius $63.7 \text{ cm}$?

**ANSWERS:**

**Areas:**

a) $30 \text{ cm} \times 150 \text{ cm} = 4500 \text{ cm}^2$ and $92 \text{ cm} \times 45 \text{ cm} = 4140 \text{ cm}^2$, so the first rectangle is larger

b) the rectangle has area $.7 \times 1.3 = 0.91 \text{ m}^2$, so the square is larger

c) the rectangle has area $.9 \times 1.1 = 0.99 \text{ m}^2$, so the square is larger

d) the rectangle has area $.99 \times 1.01 = 0.9999 \text{ m}^2$, so the square is larger

Perimeters: In a), the first rectangle has a larger perimeter. All three rectangles in b), c), and d) have the same perimeter as the $1 \text{ m} \times 1 \text{ m}$ square. The circumference of the circle in d) is about the same as the perimeter of the square. Point out that the square has the largest area of all rectangles with the same perimeter, and the circle is the most economical of shapes—it will cover the largest area of all shapes with the same distance around.

2. Which insect travels faster: an insect moving $24 \text{ mm}$ per second on an insect moving $24 \text{ m}$ per hour?

**ANSWER:** $24 \text{ m/h} = 24000 \text{ mm/h} / 3600 \text{ s} = 24000 \div 3600 \text{ mm/s} = 6.67 \text{ mm/s}$, so the first insect is faster. Here is another way to see this: The first insect is moving at $24 \text{ mm/s} = 0.024 \text{ m/s}$. There are $3600$ seconds in an hour, so in an hour this insect moves $3600 \times 0.024 \text{ m}$, meaning its speed is $3600 \times 0.024 \text{ m/h} = 86.4 \text{ m/h}$, which is more than $24 \text{ m/h}$. Again, the first insect is faster.
3. Lisa has to convert units of density, from \( \text{g/cm}^3 \) to \( \text{kg/m}^3 \). Does she need to multiply or to divide, and by how much?

**Answer:** \( 1 \text{ g/cm}^3 \) is a rate: \( 1 \text{ g} : 1 \text{ cm}^3 \) (or \( \frac{1\text{ g}}{1\text{ cm}^3} \) in fractional notation).

Multiply both sides of the rate by 1000 to change grams to kilograms:

\[
1 \text{ g} : 1 \text{ cm}^3 = 1000 \text{ g} : 1000 \text{ cm}^3 = 1 \text{ kg} : 1000 \text{ cm}^3
\]

(in fractional notation: \( \frac{1\text{ g}}{1\text{ cm}^3} = \frac{1000\text{ g}}{1000\text{ cm}^3} = \frac{1\text{ kg}}{1000\text{ cm}^3} \))

Multiply by 1000 again to convert \( \text{cm}^3 \) to \( \text{m}^3 \):

\[
1 \text{ kg} : 1000 \text{ cm}^3 = 1000 \text{ kg} : 1000000 \text{ cm}^3 = 1000 \text{ kg} : 1 \text{ m}^3
\]

(in fractional notation: \( \frac{1\text{ kg}}{1000\text{ cm}^3} = \frac{1000\text{ kg}}{1000000\text{ cm}^3} = \frac{1000\text{ kg}}{1\text{ m}^3} \))

This means \( 1 \text{ g/cm}^3 = 1000 \text{ kg/m}^3 \). Since we need 1000 of the new units for 1 of the old units, the unit is smaller. Note that the new unit is smaller even though it uses kg and \( \text{m}^3 \), which are both larger than \( \text{g} \) and \( \text{cm}^3 \)! This happens because we need more units from the old denominator to get the unit in the denominator of the new ratio, than we do units from the old numerator to get the unit in the numerator of the new ratio. The first time we multiplied by 1000, we turned g to kg, but we had to multiply by 1000 a second time to turn \( \text{cm}^3 \) to \( \text{m}^3 \).

4. Can you convert 500 \( \text{cm}^3 \) to \( \text{dm}^3 \)? Why or why not? (No, because \( \text{cm}^3 \) are units of volume, and \( \text{dm}^3 \) are units of area.)
Goals

Students will find the surface area of right prisms.

Prior Knowledge Required

Can find area of polygons
Can perform basic operations with decimals
Is familiar with square units of measurement
Can draw a net for a right prism
Can convert units of area
Can find the volume of right prisms

Materials

empty cartons from medicine, soup, tea, etc

Introduce surface area. Review area for rectangles, parallelograms, and triangles. Tell students that the surface area of a prism is the sum of the areas of all faces of the shape. Ask students when they might need to know the surface area of a prism. (Example: to calculate the amount of paper needed to wrap a present—see Extension 1)

Find the surface area of rectangular prisms by adding the areas of all faces. Present a rectangular prism. Invite volunteers to measure the sides of the prism. Ask students to draw the faces of the prism and to mark the dimensions of each face. Have students check that they drew all the faces. How many should there be? (6) Ask students to write a multiplication statement for the area of each face and to add the results for all the faces. Point out to students that because they are measuring area, the measurement units are cm² and not cm³, even though this is a 3-D shape.

Identify identical faces and use multiplication to find the surface area. Draw a cube on the board and mark the faces (top, bottom, right, left, front, back). Ask students to name pairs of opposite faces. Ask: Which faces are the same as other faces? Which faces come in pairs? How can we use this to shorten the calculation of the surface area? Draw several rectangular prisms and mark dimensions on them. Have students find the areas of the top, front, and right side faces, and then double these areas to find the surface area of each prism. Examples:

3 cm × 4 cm × 5 cm  4.5 m × 2 m × 3 m  2.3 km × 1.2 km × 4 km

Students can also measure some empty prism-shaped cartons and find their surface area.

Bonus Find the surface area of the prism at left using as few calculations as you can.
EXTRA PRACTICE:
The surface areas of the front, top, and right faces of a prism add to 450 cm². How can you find the total surface area of the prism?

Sketching nets for prisms with given dimensions. Draw several right rectangular prisms on the board and write the dimensions beside them.

EXAMPLES: a) 3 cm × 3.4 cm × 6 cm    b) 12.2 cm × 4 cm × 20.2 cm

Ask students to sketch the nets for the prisms. Then ask students to mark on the nets which face is which (top, bottom, right, etc.). One by one, go through the faces on the net and have students identify their dimensions. Do the first example as a class and have students do the rest on their own, but share answers on the board. Leave the pictures on the board for future use.

Finding volume and surface area of prisms using nets. Review with students how to find the volume of prisms. Then have students find the surface area and the volume of the prisms they sketched nets for above. Point out that to find the volume students only need the dimensions of the prisms, and these can be read from the nets. ASK: How can we use nets when finding surface area? (the area of the net is the same as the surface area of the shape) Then present several more sketches of prisms with dimensions and have students find the volume and the surface area of the prisms. EXAMPLES:

Finding missing dimensions. Have students solve several problems in which they have to find the missing dimension in a 2-D shape given the area and the dimension. EXAMPLE: A rectangle has area 72 cm². Its width is 4 cm. What is its length? (72 ÷ 4 = 18 cm) Next, move to problems involving 3-D shapes, as in Question 12 on Workbook p. 163. Then ask students to find the surface area of the prisms in Question 12. Students who have trouble solving Question 13 on Workbook page 163 can try to organize their search by using a table with headings, a, b, c, and a × c. They can start by looking for pairs of numbers a and c that multiply to 18, and then find the number b that produces b × c = 6.

EXTRA PRACTICE:
1. Find the missing length, the volume, and the surface area of the prism at left.

   ANSWER:
   Missing length = 4 cm
   Volume = 8 cm² × 1.5 cm = 12 cm³
   Surface area = 2 × (2 cm × 4 cm + 1.5 cm × 4 cm + 1.5 cm × 2 cm)
   = 2 × (8 cm² + 6 cm² + 3 cm²) = 34 cm²
2. The prism in the margin has volume 90 cm³. Find the dimensions and
the surface area of the prism.

**ANSWER:**
length = 15 cm² ÷ 3 cm = 5 cm, width = 90 cm³ ÷ 15 cm² = 6 cm,
surface area =
2 × (3 cm × 5 cm + 3 cm × 6 cm + 5 cm × 6 cm) = 126 cm²

3. Sally’s teacher tells her that she can find the surface area of a prism
by adding the areas of three faces and then multiplying by 2. Which
of the following will give Sally the right answer? Explain.

   i) (area of top + area of bottom + area of right side) × 2
   ii) (area of top + area of left side + area of back) × 2
   iii) (area of top + area of right side + area of front) × 2
   iv) (area of bottom + area of right side + area of left side) × 2
   v) (area of bottom + area of left side + area of front) × 2
   vi) (area of bottom + area of front + area of top) × 2

**ANSWER:** ii), iii), and v) because we need to choose one from each
pair of sides: top and bottom, left and right, front and back.

**Prisms with mixed units.** Review the need to keep track of the units used
in calculations of volume and surface area. Review the formula for the
volume of prisms: base of prism × height of prism. Have students practise
finding the surface area and volume of prisms with measurements given
in mixed units. **EXAMPLES:**

a) ![Image](image1.png)
   **ANSWERS:**
   a) surface area = 5.610 2 m², volume = .386 835 m³
   b) surface area = 2.136 09 m², volume = 0.125 307 m³
   c) surface area = 6.412 6 m², volume = .630 832 m³

Tell students that you heard someone say that in a) the surface area is
larger than the volume. Is that correct? Have students explain the answer.
(no, you cannot compare m³ to m²)

**Bonus** Use the Pythagorean Theorem to find the surface area
of the prism:

![Image](image2.png)
ANSWER: Surface area $= 19392.8 \text{ cm}^2 = 1.939\,28 \text{ m}^2$

EXTRA PRACTICE:
Find the volume and the surface area of the prism in the margin.

ANSWER:
Missing length $= 3.5 \text{ m}$
Volume $= 14 \text{ m}^2 \times 2.1 \text{ m} = 29.4 \text{ m}^3$
Surface area $= 2 \times (2.1 \text{ m} \times 4 \text{ m} + 14 \text{ m}^2 + 3.5 \text{ m} \times 2.1 \text{ m})$
$= 2 \times (8.4 \text{ m}^2 + 14 \text{ m}^2 + 7.35 \text{ m}^2) = 59.5 \text{ m}^2$

Extensions

1. Project: Which way of wrapping a present uses the smallest amount of paper?
   a) Find a box with all three dimensions different, such as a shoe box. Use old newspapers as wrapping paper.
   b) Wrap the box so that at least 5 cm of paper overlap at the first wrapping. Unwrap and check how much paper you used.
   c) Turn the box 90° and repeat b) with a new sheet of paper. Turn the box again, in a different direction, and repeat.
   d) Which way uses the least amount of paper?
   e) Find the area of the overlap during the first step of wrapping in each case. Does the smallest initial overlap area correspond to the smallest amount of paper used?
   f) Find the surface area of the box.
   g) For each way of wrapping, find how much paper is used to cover the unwrapped sides. Is there a correspondence between the largest amount of paper used in total and the greatest amount of paper used to cover the sides?

Answers will vary for all a) to g), depending on the boxes students use!

2. a) A big block of butter has dimensions $15 \text{ cm} \times 7.5 \text{ cm} \times 7.5 \text{ cm}$.
The same amount of butter can be bought in four smaller sticks of the same length, but half the width and height of the larger pack. What are the dimensions of the smaller sticks? $(15 \text{ cm} \times 3.75 \text{ cm} \times 3.75 \text{ cm})$

Draw the nets for the large block of butter and the small stick. Find the surface area of both blocks. What is the additional surface area of the butter when it is sold in four sticks? $(450 \text{ cm}^2)$
b) A wrapper for the block of butter is a rectangle that covers three of
the larger sides of the block once, and has an overlap on the fourth
larger side. The overlap is 1/3 of the width of the side. What is the
length of the wrapper for the bigger block? for the smaller block?
(bigger block: \(4 \times 7.5 \text{ cm} + 2.5 \text{ cm} = 32.5 \text{ cm}\); smaller block:
\(4 \times 3.75 \text{ cm} + 1.25 \text{ cm} = 16.25 \text{ cm}\))

When the wrapper is folded to cover the two square faces of the
block of butter, it covers only 2/3 of the height of the square, on both
sides. What is the width of the wrapper for the bigger block? For the
smaller block? (bigger block: \(2 \times 5 \text{ cm} + 15 \text{ cm} = 25 \text{ cm}\), smaller
block: \(2 \times 2.5 \text{ cm} + 15 \text{ cm} = 20 \text{ cm}\))

How much more wrapping do four smaller sticks use?
\((1 \text{ 300 cm}^2 - 812.5 \text{ cm}^2 = 487.5 \text{ cm}^2)\)

c) Is the additional amount of wrapping the same as the additional
surface area of butter? (no, the additional amount of wrapping is
larger by 37.5 cm²) Explain why there is a difference. (The additional
amount of wrapping is larger because there is some overlap on
each of the four sticks and it is more than the overlap on one large
butter block.)

3. Research project:

a) One litre of paint covers 7 m². How much paint would someone
need to paint a wall with dimensions 6 m by 3 m? What if the wall
has a door that is 2 m high by 80 cm wide? What if the wall also has
a window that is a 1 m by 1 m square?

b) A room has a closet that is 1 m deep, 2 m wide and 2.5 m high. A
door that is 2 m high and 80 cm wide leads to it. I want to paint the
sides, back, and top of the closet, but neither the floor nor the door.
How much paint do I need?

c) Choose a room in your school or at home. Calculate the amount of
paint needed to repaint the room. Consider all surfaces and fixtures,
such as doors, windows, closets, electrical outlets, built-in shelves,
or ledges. What will you paint and what does not need to be
painted? Do you want to use more than one colour? Will you need
more than one coat of paint? (You will if you are using a dark colour,
or painting over a dark colour.)

4. A wealthy king has a treasure chest in the shape of a rectangular prism
30 cm wide, 40 cm long, and 25 cm high. He ordered his carpenters to
design a chest that can hold twice as much treasure.

a) The first carpenter doubled the length of the box and left the width
and the height the same. The second carpenter doubled the width
of the box and left the length and the height the same. The third
carpenter doubled the height of the box and left the length and
the width the same. The chest with the smallest surface area uses
the least amount of wood and so is the least expensive. Which carpenter made the least expensive chest?

**ANSWER:**
1st carpenter: \[2 \times (30 \times 80 + 30 \times 25 + 25 \times 80) = 10300 \text{ cm}^2\]
2nd carpenter: \[2 \times (60 \times 40 + 60 \times 25 + 25 \times 40) = 9800 \text{ cm}^2\]
3rd carpenter: \[2 \times (30 \times 40 + 30 \times 50 + 50 \times 40) = 9400 \text{ cm}^2\]

The 3rd carpenter made the least expensive chest.

b) None of the three carpenters got the job of making the chest for the king’s treasure. Instead, he ordered the chest from a fourth carpenter, who suggested a chest with a volume that was 101.4% of what the king wanted but that would cost less than the chests of the other carpenters. Her chest had a square at the base and was 40 cm high. What were the dimensions of this carpenter’s chest? What was its surface area? Compare the surface area of this chest to the chest with the smallest area in part a).

**ANSWER:**
Volume: 101.4% of \((30 \times 25 \times 40 \times 2) \text{ cm}^3 = 60840 \text{ cm}^3\)

The height is 40 cm, so the base should be \(60840 \div 40 = 1521 \text{ cm}^2\), and \(1521 = 39^2\), so the box is \(39 \text{ cm} \times 39 \text{ cm} \times 40 \text{ cm}\).

The surface area is \(2 \times (39 \times 39 + 39 \times 40 + 39 \times 40) = 9282 \text{ cm}^2\), 118 cm² smaller than the box made by the 3rd carpenter.

**NOTE:** The last box is the closest to a cube, which is the most economical of prisms. Of all prisms with the same volume, a cube will have the smallest surface area. Compare this to a square, which is the most economical of all rectangles. Of all rectangles with the same area, the square will have the smallest perimeter. See Extension 1 of ME8-15.
Develop the formula for the area of the side face of a cylinder. Give each student an empty toilet paper roll. Have students measure and record its height and diameter, and the circumference of the circle in the base. Then have students cut the roll vertically (as in Question 1 on Workbook page 164) and lay it flat. What shape do they get? (a rectangle) What is the width of this rectangle? What is its length? What measurements of the cylinder are these equal to? (length = circumference, width = height of cylinder)

Give students cans of different sizes and paper. Ask students to check whether a rectangle that has length equal to the circumference of a cylinder and width equal to the height of a cylinder will wrap around the can precisely. **ASK:** Did that work for all cans? So what formula could we use for finding the area of the side face of a cylinder? (circumference of base \times height of cylinder)

Draw a cylinder on the board and mark its diameter as 6 cm and its height as 2 cm. **ASK:** What will the area of the side face be? Can you find the circumference of this cylinder? Which formula will you use? Have students find the area of the side face. (circumference \times height = (\pi \times 6 \text{ cm}) \times 2 \text{ cm} \approx 37.68 \text{ cm}^2) Repeat with a cylinder with height 3 cm and radius 5 cm. (area of side face = (2 \times 5 \times \pi) \times 3 = 30\pi \approx 94.2 \text{ cm}^2) Then ask students to predict a formula for the area of the side face of a cylinder with radius \(r\) and height \(h\). (2\pi rh)

Remind students that since we use \(\pi \approx 3.14\), which is only exact to two decimal places, our answers would not be correct for larger number of decimal places. For the moment we multiply by \(\pi\) in a problem, we should round the answers to at most two decimal places and use the approximately equal sign.
Analogue of the formula circumference of base \times height for prisms.

Hold up a prism with a many-sided polygon in the base (e.g., BLM Nets of 3-D Shapes (11)) so that the bases are horizontal. **ASK:** How is this prism the same as a cylinder and how is it different? Look at the formula for the area of the side face of a cylinder: circumference of base \times height of cylinder. What would you replace “circumference of base” with to get a formula that will work for a prism? (perimeter of base) Have students use this formula to find the surface area of prisms given the height of the prism, and the area and circumference of the base. **EXAMPLES:**

![Prism diagram](image)

**ANSWERS:**

a) \(14 \times 2 + 13.6 \times 3 = 68.8 \text{ cm}^2\)  
b) \(79.04 \times 2 + 40.8 \times 3.1 = 284.56 \text{ cm}^2\)

**Surface area of a cylinder.** **ASK:** To find the surface area of the prism, what did you add to the surface area of the side faces? (the area of the bases) To find the surface area of a cylinder, what will you add to the area of the side face? (the area of the bases) What are the bases of a cylinder? (circles) What is the area of a circle? To summarize, the surface area of a cylinder will be the sum of three components: area of the bottom base, area of the top base, and area of the side face, which we find as area of a rectangle. Have students find the total surface area of several cylinders. Do the first one or two examples as a class, then have students work individually. Have students do at least half of the problems by hand. If they use a calculator for the rest of the problems, have them estimate the answer first, to avoid mistakes. **EXAMPLES:**

![Cylinder diagram](image)

**ANSWERS:**

b) \(105.12 \text{ cm}^2\)  
c) \(622.47 \text{ m}^2\)  
d) \(66.316 \text{ m}^2\)  
e) \(32.91 \text{ cm}^2\)
Nets of cylinders are similar to nets of prisms. Show students a hexagonal prism (see BLM Nets of 3-D Shapes (4)) and ask them to sketch a net for it. Encourage multiple solutions. Make sure that the two nets at left are shown. Then show a prism with a regular 12-sided polygon in the base (see BLM Nets of 3-D Shapes (11)) and ask students to sketch a net for it. If you need to construct an exact net for this prism, which net would be more convenient to draw? Why? (a net in which the side faces are joined side by side in a long rectangle, because you can draw all the rectangles together) Then ask students to think about what a net for a cylinder would look like. How would it be similar to the net of a prism with a 12-sided polygon in the base? What would you need to draw differently? (There is only one long rectangle for the side face, and circles instead of polygons.) What will the dimensions of the rectangle be? (circumference of base \times height of prism)

Other nets of cylinders. Give students another empty roll of toilet paper, and have them cut it diagonally, as in Question 1 b) on Workbook page 164, to lay the cut tube flat. What shape did they get? (a parallelogram) ASK: How do we find the area of a parallelogram? (base \times height) What is the height of this parallelogram? (the height of the cylinder) What is the base of this parallelogram? (the circumference of the base of the cylinder) Have students do the Activity below.

Matching nets to cylinders. Draw a cylinder on the board and mark its diameter as 10 cm and its height as 15 cm. Ask students what the dimensions of the rectangle that can be folded into the side face should be (about 31.4 cm \times 15 cm) Draw several rectangles as shown below and ask which of them could be the rectangles that fold into the side face of this cylinder. (B and C) How do you know? What is wrong with the others? (the rectangle in question is about twice as long as it is wide, and only these two rectangles have these proportions) Ask students to tell which side would be the height and which side would be the circumference of the circle. (The shorter side is the height.)

Draw the two nets at left. ASK: Which of these nets could be a net for a cylinder? What is wrong with the other picture? (Picture A will work as a net, but picture B will not because the circles are too small.) Approximately what should the ratio between the side that becomes the edge of the cylinder and the diameter of the circle be? Why? (The ratio between the side and the diameter is the ratio between the circumference of the circle and its diameter, which is \pi, so the first ratio should be about 3 : 1.)

Draw three cylinders below and have students decide which one the cylinder folded from net A will look like. (C)
Have students sketch the nets for the other two cylinders.

**ACTIVITY**

Have students construct a variety of quadrilaterals (students should try at least one of each special quadrilateral and one general quadrilateral), then cut them out and try to roll them into a cylinder. Which ones will roll into the side face of a cylinder? What properties do such quadrilaterals have? Students can also experiment with other polygons.

**ANSWER:** Quadrilaterals that can roll into the side face of a cylinder have parallel sides of the same length and adjacent angles that add to 180°, so they have to be parallelograms. Other polygons need pairs of parallel sides of the same length, with pairs of angles at the sides that will become the edges of the side faces adding to 180°, and other angles adding in pairs to 360°. See an example of a hexagon and an octagon that will roll into a cylinder below. The thick lines mark the sides that will be glued to circles. **EXAMPLES:**

\[
\begin{align*}
\angle A + \angle B &= 180° \\
\angle E + \angle F &= 180° \\
\angle C + \angle D &= 360°
\end{align*}
\]

**Extension**

Which of the pictures below works as a net for a cylinder? What is wrong with the others? Draw a similar picture, cut it out, and try to fold it into a cylinder to check.
**EXPLANATION:** B is the only net that works.

In A, both trapezoids would be glued to one of the circles using the longer base, and to the other with the shorter base. These sides do not add to the same length, and are either longer or shorter than the circumference of the circle.

In C, when you use the circle with the letter C as the bottom base and fold the trapezoids up to become the side faces, you see that each non-base side of one trapezoid will attach to the side of a different length in the other trapezoid, so the sides do not fit.

In D, the circles are attached to the adjacent sides of one of the trapezoids. If you try to wrap the trapezoid attached to both circles around them, the circles will meet. In a cylinder, the circular faces do not meet.
Goals
Students will find solve problems involving volume and surface area of prisms and cylinders.

Prior Knowledge Required
Can find the volume of a rectangular prism
Can find the volume of a cylinder
Can find the area and circumference of a circle
Can identify the base of a prism
Can multiply or divide decimals
Is familiar with cubic and square units of measurement
Can find the area of polygons
Can find the surface area of cylinders and prisms

Review Some Relevant Prior Knowledge. Review formulas for the areas of triangles, rectangles, parallelograms, and circles. Review finding missing dimensions of rectangles and triangles when given the area. Review finding the volume of right prisms (volume = area of the base × height) and the surface area of right prisms (add areas of all faces, nets can help keep track of faces). Remind students that writing the units at every step of the calculation helps both to avoid multiplying dimensions in different units and to ensure the right number of dimensions are multiplied (two for area, three for volume).

Finding Dimensions of Prisms with a Given Volume Using an Organized Search. Tell students that you want to find all possible whole-number dimensions of a rectangle with area 60 cm². Have students solve the problem, then discuss possible solutions. Solutions include: produce a factor rainbow (each arc in the factor rainbow matches a rectangle); use a T-table to search for factors in an organized way; find the prime factorization of 60 and list all possible factors, then pair them; draw rectangles in an organized way. Point out that it is preferable to perform your search in an organized way because of the large number of possible solutions. All of the solutions above have some sort of organized structure. Then present a harder problem: Find all right rectangular prisms with volume 60 cm³.

Ask: How are these problems the same and how are they different? (We are again looking for numbers that multiply to 60, but there are 3 numbers involved in the second problem, and only 2 in the first.) Will a factor rainbow help here? (no) Why not? (we need triples, not pairs of numbers) Will drawing prisms be a convenient strategy? (no) Why not? (We sketch prisms, we do not draw them precisely the way we do rectangles, and it will take too much time to draw all possible prisms)

Suggest that students use a chart with three columns (tell them to leave space for a fourth column to be added later) and headings height, width, and length. Point out that when we have three values that can change,
it makes sense to solve a simpler problem first. In this case, we can set the first value, look at what the other two values can be in an organized way, then modify the first value and repeat. So we will start with height 1 cm for the prism, and see what length and width the prism can have. What do we know about the area of the base of a prism with height 1 cm and volume 60 cm³? (area of the base = 60 cm²) Have students list all possible lengths and widths for a prism with height 1 cm and volume 60 cm³. How did they do that? Point out that this was a problem that they solved before—finding all rectangles that have area 60 cm².

**ASK:** Do all prisms with volume 60 cm³ have height 1? (No) Tell students that they will now look for all prisms of height 2 cm. What will the area of the base be? (30 cm²) **ASK:** Does it make sense to include prisms with width 1 cm? Why not? (because any prism with width 1 cm was already listed as a prism with height 1 cm) Have students find all other prisms with height 2 cm. Next, use height 3 cm (only one prism is possible) and height 4 cm. Ask students to look at the first prism with height 4 cm that they’ve written (it is either $4 \times 1 \times 15$ or $4 \times 3 \times 5$). What do they notice? (it is already in the list, row 4 or row 10) Does it make sense to continue the search? Why not? (Since we are only looking for prisms that have height 4, and width at least 4 (all smaller widths are already accounted for), the prisms will have height × width at least 16, so the length is definitely smaller than 4. We have already listed all prisms with these dimensions.)

**ASK:** Which of these prisms will have the smallest surface area? Have students add a column for surface area and find the surface area of all these prisms. The prism with the smallest surface area is the one closest to a cube—the last one in the table.

Ask students to estimate the volume of a cube with sides 3.92 cm. Will it be a lot more, a lot less, or about the same as the volume of the prisms they found? Have them explain their answer. Then have students check the volume using a calculator. **ASK:** Will the surface area of the cube with sides 3.92 cm be more, less, or about the same as the surface area of the last prism you found? Again, have students explain their thinking and check by using a calculator.

### Mat plans
Remind students that a mat plan shows what a shape made of connecting cubes or 1-cm cubes would look like from the top. The number of cubes stacked above each square in the mat plan is listed in the square. For example, show students an L-shape made of connecting cubes (or draw one on the board) and draw the mat plan for it (see sample in margin).

Ask students to find the volume of the L-shape. **ASK:** How can you do that from the mat plan? (add the numbers on the mat plan). Then ask students whether this shape is a prism. (Yes) What are its bases? (the L-shaped faces) Have students draw a mat plan of the shape as if it was standing on a base (see answer at left). What do they notice? (all numbers in the mat plan are the same) Does turning the shape on the side change its volume? (No) Remind students that the mat plan for any prism standing on its base...
will have the same numbers in all squares. Point out that this provides us with a self-checking mechanism for the mat plan after we turned the shape.

**ASK:** Which of the mat plans will be more useful for finding the surface area of the shape? (the second one) How will you find the surface area from it? (The shape has the same height everywhere, and the mat shows the base of the shape. Find the perimeter of the base and multiply by the height. Then add twice the area of the base.) Have students find the surface area of the shape. (perimeter of base = 10 cm, so area of side face is $2 \times 10 = 20$ cm$^2$, area of base is 4 cm$^2$, and total surface area $= 20 + 2 \times 4 = 28$ cm$^2$)

Have students find the volume and the surface area of the shape with mat plan in the margin. Students who have trouble visualizing the shape can use connecting cubes to construct it, identify it as a prism, turn it base down, and draw the second mat plan (see answer in margin). **ANSWER:** volume = 12 cm$^3$, surface area = 40 cm$^2$.

**Word problems involving volume or surface area.** Work through the following problems as a class.

1. Pam has a box with a hexagonal base, a lid, and rectangular sides. The box is 40 cm tall and its base (which is the same as the lid) is shown at left.
   a) Sketch the box.
   b) Sketch a net for the box and mark the dimensions on the faces.
   c) Find the surface area of the box.
   **ANSWER:** perimeter of the base $= 100$ cm, so side faces have total area $4000$ cm$^2$. Splitting the base into two trapezoids, with bases 20 cm and 38 cm and height 12 cm, the area of each base is $(38 + 20) \div 2 \times 12) = 696$ cm$^2$, so the surface area of the box is $5392$ cm$^2$.
   d) Pam needs 5 mL of paint for each 100 cm$^2$ of area. How much paint will she need to paint the outside of the box (including the lid)? ($53.92 \times 5 = 269.6$ mL of paint)

2. A rectangular box without a lid has volume 40 000 cm$^3$. It is 40 cm wide and 50 cm long.
   a) What is the height of the box?
   b) What are the length and the width of the lid (the missing face)?
   c) What is the surface area of the box? Remember, there is no lid.
   d) The material for the box costs $12.35 per m$^2$. How much will the material for the box cost?
   **ANSWERS:** a) 20 cm  b) 40 cm $\times$ 50 cm  c) $40 \times 50 cm + 2 \times (40 cm \times 20 cm + 50 cm \times 20 cm)$  
   $= 3800$ cm$^2 = 0.38$ m$^2$  
   d) $0.38 \times 12.35$ which rounds to $4.69$.  

**ANSWER:**

- 3 1 2
- 3 1 2

- 2
- 2 2
- 2 2 2

- 38 cm
- 20 cm
- 15 cm
- 24 cm
Finding the height of a cylinder from the volume and radius. Show students a can that has a rim on top. Explain that you know the capacity of this cylinder and thus you can find the volume. You can measure the diameter of the can, but you cannot measure the height because of the rim. Tell students that the diameter of the can is 5 cm and its capacity is 156 mL. What is the inside height of the can? Solve the problem as a class. (PROMPTS: What is the volume of the can? Which units, mm or cm, will be the most convenient for this problem? What is the radius of the base of the can? What is the area of the base of the can? The area of the base of the can is $\pi \times 2.5^2 \approx 19.63$ cm$^2$. The inside volume of the can is 156 cm$^3$, so the height is $156 \div 19.63 \approx 7.95$ cm.)

Have students practise finding the heights of cylinders given their radius (or diameter) and volume (or capacity). EXAMPLES:

1. A cylinder has volume 125 cm$^3$. It has radius 42 mm. What is the height of the cylinder? ($\approx 2.26$ cm)

2. A can has capacity 376 mL. It has inner diameter 6.8 cm. What is the inner height of the can (rounded to the nearest mm)? (10.4 cm)

Review finding the radius of a circle from the circumference or area. Review the formulas for the area and circumference of a circle, and how to work backwards when you know the circumference or area to find the radius. Solve several problems like the following:

1. A circle has area 10 m$^2$. What is its radius, rounded to the nearest centimetre? (ANSWER: $10 \text{ m}^2 = \pi \times r^2$, so $r^2 \approx 10 \div 3.14 \approx 3.19 \text{ m}^2$, so $r \approx 1.79$ m)

2. A circle has area 15.6 cm$^2$. What is its diameter (rounded to the nearest millimetre)? (ANSWER: $15.6 \text{ cm}^2 = \pi \times r^2$, so $r^2 \approx 15.6 \div 3.14 \approx 4.97 \text{ cm}^2$, so $r \approx 2.28$ cm, and $d \approx 4.6$ cm)

3. A can has circumference 34 cm. What is its diameter in millimetres, rounded to one decimal place? (ANSWER: $34 \text{ cm} = \pi \times d$, so $d \approx 34 \div 3.14 \approx 10.83 \text{ cm} = 108.3$ mm)

Finding radius or diameter from the volume and height of cylinders. Present the following problem:

A round can has capacity 1.145 L. It is 15.5 cm tall. How many such cans will fit in a box 30 cm $\times$ 40 cm $\times$ 32 cm?

ASK: To know how many cans will fit into the box, what do we need to know? (the width of the can) The can is round. What shape it is? (a cylinder) What do we know about this cylinder? (capacity and height) What does “width” mean in terms of a cylinder? (diameter) Do we know a formula for a cylinder that involves capacity, diameter, and height? (no) Is there a formula that we know that could help us? What can we find easily from diameter? From capacity? Which formula that we know involves radius and height? (surface area and volume both use radius and height) Have students find the volume of the can. (1 145 cm$^3$) Again, suggest that
students work backwards from the formula for the volume to find the radius of the can from the volume and the height. (1 145 cm³ = π × r² × 15.5 cm, so r² ≈ 1145 ÷ 15.5 ÷ 3.14 ≈ 23.525 8 cm², r ≈ 4.85 cm, and d ≈ 9.7 cm)

How many cans will fit at the bottom of the box? (3 × 4 = 12 cans) How many cans will fit in the box in total? (24 cans)

**SAY:** Suppose we need a label that will cover the side face of the can and have an overlap of 1 cm. What shape should the label be? (rectangle) What should the dimensions of the label be? What should the height of the label be? (the same as the height of the can) What should the length of the label be? (the circumference of the base + 1 cm) Have students find the circumference of the base and the length of the label. (C = π × 9.7 cm ≈ 30.5 cm, so the length of the label should be about 31.5 cm.)

Have students practise finding the diameter or the radius of cylinders given the volume, capacity, or surface area. Include some problems with prisms.

**EXAMPLES:**

1. A round can is 10.2 cm tall. Its capacity is 398 mL. What is the radius of the can (rounded to the nearest millimetre)? What is the surface area of the can (in whole cm²)? (radius ≈ 3.5 cm, surface area ≈ 302 cm²)

2. A round can has circumference 257.4 mm. It is 3 cm tall. What is its capacity in whole mL? **(ANSWER:** 257.4 = 2πr, so r ≈ 257.4 ÷ 6.28 ≈ 40.99 mm, so volume = πr²h ≈ π × 4.1² × 3 ≈ 158 cm³, and capacity ≈ 158 mL)

3. a) A rectangular prism has width = length = 30 cm. Its surface area is 1 m². What is its height?

   b) A cylinder has width 30 cm and surface area 1 m². Without calculating, predict whether it is taller than the prism in a). Explain your prediction.

   c) Find the height of the cylinder in b) and check your prediction.

   d) Calculate the volumes of both the prism and the cylinder to see which is larger.

**ANSWERS:**

   a) Let h be the height of the prism. Then its surface area is

   \[ 2 \times (30 \times 30 + 30 \times h + 30 \times h) \text{ cm}^2 = 1 \ 800 \text{ cm}^2 + 120 \times h \text{ cm}^2 = 10 \ 000 \text{ cm}^2, \ h \approx 68.3 \text{ cm}. \]

   b) The base of the cylinder is a circle with diameter 30 cm, which would fit inside the square that is the base of the prism in a), and the circumference of the circle is smaller than the perimeter of the square. The surface area of the prism and the cylinder are the same, so to compensate for the smaller bases and smaller perimeter, the height of the cylinder has to be larger than the height of the prism. The cylinder will be taller.
c) The bases of the cylinder each have area $\pi \times 152 \text{ cm}^2 \approx 706.5 \text{ cm}^2$, so the side face should have area about $10000 - 2 \times 706.5 = 8587 \text{ cm}^2$. Height of cylinder $\approx 8587 \div (3.14 \times 30) \approx 91.2 \text{ cm}$.

d) Volume of prism $\approx 30 \times 30 \times 68.3 = 61470 \text{ cm}^3$.
Volume of cylinder $\approx 706.5 \times 91.2 = 64432.8 \text{ cm}^3$.
The cylinder has larger volume.

**Bonus**

A cylindrical thermos has circumference 26 cm and inner diameter 6.5 cm. Its inner height is 8 cm.

a) What is the capacity of the thermos?

b) What is the thickness of the thermos walls?

c) If the thickness of the bottom is twice the thickness of the walls, what is the outer height of the thermos without the lid?

d) What is the volume of the thermos when it is full (not including the lid)?

**ANSWERS:**
a) capacity $= \pi \times 3.25^2 \times 8 \approx 265.33 \text{ mL}$

b) outer radius $= 26 \div (2\pi) \approx 4.14 \text{ cm}$, so the thickness of the walls is about $4.14 - 3.25 = 0.89 \text{ cm} = 8.9 \text{ mm}$

c) The bottom is 1.78 cm thick, and the outer height is 9.78 cm.

d) The volume is about $\pi \times 4.14^2 \times 9.78 \approx 526.38 \text{ cm}^3$.

**Extensions**

1. a) Find the surface area and volume of each right rectangular prism

   i) [Diagram]

   ii) [Diagram]

   iii) [Diagram]

   **ANSWERS:**

   Surface area: 62 cm$^2$   Surface area: 248 cm$^2$   Surface area: 558 cm$^2$
   Volume: 30 cm$^3$   Volume: 240 cm$^3$   Volume: 810 cm$^3$

   b) How does the surface area of a right rectangular prism change when each side length is multiplied by the same amount? (it is multiplied by the square of the scale factor) How does the volume of a right rectangular prism change when each side length is multiplied by the same amount? (it is multiplied by the third power of the scale factor)
c) If each dimension is enlarged by 50% (so the new dimensions are 150% of the old dimensions), what percentage do the surface area and volume increase by? HINT: a 50% increase is the same as multiplying by what factor? (each dimension is multiplied by 1.5, so surface area is multiplied by $1.5^2 = 2.25$ and volume is multiplied by $1.5^3 = 3.375$)

2. Find the surface area and volume of the tissue box shown below. Remember to leave out the opening on the top.

![Diagram of a tissue box with dimensions 12 cm x 22 cm x 7 cm]

ANSWERS:
Volume: $12 \text{ cm} \times 22 \text{ cm} \times 7 \text{ cm} = 1\,848 \text{ cm}^3$

Surface area: $2 \times (12 \text{ cm} \times 22 \text{ cm} + 12 \text{ cm} \times 7 \text{ cm} + 22 \text{ cm} \times 7 \text{ cm}) - 12 \text{ cm} \times 5 \text{ cm} = 1\,004 \text{ cm}^2 - 60 \text{ cm}^2 = 944 \text{ cm}^2$

3. You are designing a cereal box for a cereal company. The box needs to have a volume of 2 000 cm$^3$. There are many possible boxes you could make with this volume.

a) Verify that the three sets of measurements (in cm) below produce boxes with volume 2 000 cm$^3$.

$1 \times 1 \times 2\,000 \quad 2 \times 25 \times 40 \quad 5 \times 25 \times 16$

b) Calculate the surface area of each box above. (8 002 cm$^2$, 2 260 cm$^2$, 1 210 cm$^2$)

c) If it costs 25¢ for each cm$^2$ for the material to make the box, which box would you recommend? (5 $\times$ 25 $\times$ 16)

d) Find three more boxes with the same volume and calculate the surface area of each. Now which box would you recommend? (Answers will vary.)

e) The cereal company wants the front of the box to be at least 20 cm wide and 20 cm high and the depth of the box to be at least 4 cm. Find two boxes satisfying these conditions that each have a volume of 2 000 cm$^3$. Which box would you recommend? (A: $20 \times 20 \times 5$ and B: $20 \times 25 \times 4$, Surface area of A = 1 200 cm$^2$, Surface area of B = 1 360 cm$^2$, so box A is better.)
Nets of 3-D Shapes — List of 3-D Shapes

Right prisms
1. Cube
2. Pentagonal prism
3. Rectangular prism
4. Hexagonal prism
5. Triangular prism with an obtuse scalene base
6. Triangular prisms with right scalene bases
7. Right prism with a trapezoid base
8. Right prism with a parallelogram base
9. Right prism with an irregular hexagonal base
10. Right octagonal prism
11. Right dodecagonal prism

Skew prisms
12. Skew prism with three different pairs of parallelogram faces
13. Skew rectangular prism
14. Skew square prism
15. Skew triangular prism

Other shapes
16. Skew rectangular pyramid
17. Right pentagonal pyramid
18. Right hexagonal pyramid
19. Tetrahedron
20. Square antiprism
21. Truncated square pyramid
22. Dodecahedron
23. Icosahedron
24. Octahedron
A Net or Not a Net?

Does this picture make a net of a 3-D shape? If yes, what shape?
Predict, then cut out the net and fold it to check your prediction.
Is It a Net?

A

B
Square Metre

$1 \text{ cm}^2 = 1 \text{ cm} \times 1 \text{ cm}$

$1 \text{ m} = 100 \text{ cm}$

$1 \text{ m}^2 = 1 \text{ m} \times 1 \text{ m} = 100 \text{ cm} \times 100 \text{ cm} = 10000 \text{ cm}^2$
PS8-10 Using Logical Reasoning III

Teach this lesson after: 8.2 Unit 6

Goals:
Students will solve problems involving workers who work at different, but constant, rates.

Prior Knowledge Required:
Can add fractions
Can divide whole numbers by a fraction
Can solve worker problems where everyone works at the same rate
Can calculate a percentage of a fraction (for Problem Bank 5)
Can multiply fractions (for Problem Bank 5)
Can convert among percentages, fractions, and decimals (for Problem Banks 5, 6)
Can multiply decimals (for Problem Bank 6)
Can subtract fractions (for Extended Problem)
Can calculate the area of a circle (for Extended Problem)
Can calculate the volume of a cylinder (for Extended Problem)
Can convert between m$^3$ and cm$^3$ (for Extended Problem)

Vocabulary: rate

Materials:
BLM Pool Maintenance (pp. R-62–65, see Extended Problem)

NOTE: Students should complete Lesson PS8-4 before starting this lesson.

Determining the time to finish a job given the fraction completed in a unit time period.
Write on the board:

Nora painted a wall. After one hour, she was done $\frac{1}{3}$ of the wall. How long would it take her to paint the whole wall?

Read the problem aloud and have a volunteer tell you the answer. (3 hours) Repeat by erasing 1/3 and replacing with 1/2 (2 hours), 1/4 (4 hours), and 1/10 (10 hours). Next, replace the fraction with 2/3 and SAY: This problem is a bit harder, but I think you can still do it. To guide students, write on the board:

After one hour, Nora was done $\frac{1}{3}$ of the wall.

After one hour, Amy was done $\frac{2}{3}$ of the wall.

ASK: Who is working faster, Nora or Amy? (Amy) How much faster? (Amy is working twice as fast) SAY: If Amy can get twice as much done in one hour as Nora can, it would take her half as long to finish the wall. ASK: How long does it take Nora to finish the wall? (3 hours) How long does it take Amy? (half as long as 3 hours, so 3/2 hours or 1 1/2 hours)
**Exercises:** How many hours would it take the person to paint the whole wall, working alone?

a) Lewis can paint $\frac{1}{5}$ of the wall in one hour.

b) Avril can paint $\frac{2}{5}$ of the wall in one hour.

c) Marko can paint $\frac{3}{5}$ of the wall in one hour.

**Answers:** a) 5 hours, b) 5/2 hours or 2 1/2 hours, c) 5/3 hours or 1 2/3 hours

SAY: You can answer the same type of question given how much is done in any unit of time.

**Exercises:** How long would it take to complete the task?

a) Jin walks $\frac{1}{9}$ of the distance from home to school in one minute.

b) Sally walks $\frac{2}{9}$ of the distance from home to school in one minute.

c) Jin runs $\frac{1}{15}$ of the race in one minute.

d) Sally runs $\frac{4}{15}$ of the race in one minute.

**Answers:** a) 9 minutes, b) 9/2 minutes or 4 1/2 minutes, c) 15 minutes, d) 15/4 minutes or 3 3/4 minutes

To summarize some of these answers, write on the board:

- $\frac{1}{5}$ of the wall in one hour, so five hours to finish the wall
- $\frac{2}{5}$ of the wall in one hour, so $\frac{5}{2}$ hours to finish the wall
- $\frac{4}{15}$ of the race in one minute, so $\frac{15}{4}$ minutes to finish the race

ASK: Do you see a shortcut to find the amount of time required to finish the job? (look at the fraction of the job completed in one hour or one minute and turn the fraction upside down)

**Exercises:** How long would it take to complete the task?

a) Randi completes $\frac{3}{8}$ of the multiplication questions in one minute.

b) Edmond completes $\frac{4}{7}$ of his homework in one hour.

c) Randi watched $\frac{5}{9}$ of the movie in one hour.

d) Edmond walked $\frac{2}{11}$ of the distance in one minute.

**Answers:** a) 8/3 minutes or 2 2/3 minutes, b) 7/4 hours or 1 3/4 hours, c) 9/5 hours or 1 4/5 hours, d) 11/2 minutes or 5 1/2 minutes
ASK: When you turn the fraction upside down, what operation does that remind you of? (division, or, more precisely, dividing 1 by the fraction) SAY: To get the amount of time you need to paint the whole wall, you are really dividing 1 by the fraction of the wall that is painted in one hour. Let’s see why this works. Remember that the fraction of the wall painted and the amount of time taken are proportional. Suppose that you can paint half of the wall in one hour. Write on the board:

\[
\frac{1}{2} \text{ of wall painted : 1 hour} \\
1 \text{ whole wall painted : ?}
\]

SAY: Whatever you multiply 1/2 by to get 1, you have to multiply 1 by to get how much time it takes to paint the whole wall. Show this on the board:

\[
\frac{1}{2} \text{ of wall painted : 1 hour} \\
1 \text{ whole wall painted : ??} \\
\frac{1}{2} \times ?? = 1 \text{ so } 1 \div \frac{1}{2} = ?
\]

SAY: All the question marks in the picture have to stand for the same thing, so you have to divide 1 by 1/2 to get how long it takes to paint 1 whole wall. In this case, 1 divided by 1/2 is 2, so you can paint the whole wall in two hours.

**Problems where workers do not work at the same rate.** Write on the board:

- Tess can paint a wall in three hours, working alone.
- Ethan can paint the same wall in six hours, working alone.
- If they work together, how long will it take them to paint the wall?

SAY: Tess and Ethan don’t work at the same rate. ASK: Will two people take more time or less time than one person working alone? (less time) Draw on the board:

While pointing to their names, SAY: Tess is going to start on this side and Ethan is going to start on that side. ASK: How much of the wall will Tess have done after one hour? (1/3) How do you know? (she takes three hours to do the whole wall, and in one hour, she can do 1/3 as much) How much of the wall will Ethan have done after one hour? (1/6) How do you know? (he takes six hours to do the whole wall, and in one hour, he can do 1/6 as much)
Exercises: How much of the wall does each person paint in one hour?

a) Billy can paint the wall in four hours
b) Lynn can paint the wall in seven hours

Answers: a) $\frac{1}{4}$, b) $\frac{1}{7}$

SAY: Once you know how much each person paints in one hour, you can find out what fraction of the wall they can paint together in one hour. Colour on the board how much each person will have done after one hour, as shown below:

ASK: What fraction of the wall is painted altogether? (1/2) How did you get that? (added $\frac{1}{3}$ and $\frac{1}{6}$) SAY: To get the total amount painted, add the amounts they each painted. Write on the board:

\[
\begin{align*}
\frac{2}{3} \times \frac{1}{6} &+ \frac{1}{6} \\
\frac{2}{6} &+ \frac{1}{6} \\
\frac{3}{6} &+ \frac{1}{6} \\
\frac{1}{2} &
\end{align*}
\]

Exercises: Find the total fraction of the wall painted in one hour.

a) Billy painted $\frac{3}{10}$ of the wall and Lynn painted $\frac{1}{5}$ of the wall.

b) Billy painted $\frac{1}{8}$ of the wall and Lynn painted $\frac{1}{4}$ of the wall.

c) Billy painted $\frac{1}{5}$ of the wall and Lynn painted $\frac{2}{15}$ of the wall.

Answers: a) $\frac{1}{2}$, b) $\frac{3}{8}$, c) $\frac{1}{3}$

SAY: Remember, you can figure out the amount someone can paint in one hour if you know how long it would take them to paint the whole wall by themselves.
Exercises: Find the total fraction of the wall painted in one hour.
a) Cam can paint the wall in two hours, working alone.
Emma can paint the wall in three hours, working alone.
b) Cam can paint the wall in four hours, working alone.
Emma can paint the wall in three hours, working alone.
c) Cam can paint the wall in five hours, working alone.
Emma can paint the wall in two hours, working alone.
Answers: a) 5/6, b) 7/12, c) 7/10

SAY: Once you know the total fraction painted in one hour, you can decide how much time they would need to finish the job. ASK: If two people paint 1/2 of the wall in one hour, how long does it take them to paint the whole wall? (2 hours) What if they painted 1/3 of the wall in one hour, then how long would it take them to paint the whole wall? (3 hours) What if they painted 2/3 of the wall in one hour, then how long would it take them to paint the whole wall? (3/2 hours or 1 1/2 hours) How did you get that? (it takes half as long to paint the wall as if they painted 1/3 of the wall in 1 hour)

Exercises:
1. Tess can paint a wall in three hours, working alone. Ethan can paint the same wall in five hours, working alone.
a) How much of the wall will Tess have done after one hour?
b) How much of the wall will Ethan have done after one hour?
c) How much of the wall will be done altogether after one hour?
d) How long will it take them together to paint the wall?
Answers: a) 1/3, b) 1/5, c) 8/15, d) 15/8 hours or 1 7/8 hours

2. Sharon can paint a wall in three hours, working alone, and so can Ren. How long will it take them working together?
Answer: They can each paint 1/3 of the wall in one hour, so working together they can paint 2/3 of the wall in one hour. So, it takes 3/2, or 1 1/2, hours to paint the wall.

3. Look at your answers to Exercises 1 and 2. Does it take Sharon and Ren more or less time than Tess and Ethan? How could you have predicted that?
Answer: less, because Ren is faster than Ethan

SAY: It is always a good idea to go back and check that your answers make sense. ASK: Does it make sense that it takes just under two hours for Tess and Ethan to work together? (yes) Why? (you would expect it to take less than three hours because that is how long it takes Tess alone, but more than 1 1/2 hours because that’s how long it would take if Tess and Ethan both worked at Tess’s pace, but Ethan is slower, so it should take longer)

Exercise: Ella can paint a wall in two hours. Rob can paint the same wall in one hour. How long will it take them, working together, to paint the wall? Check that your answer makes sense.
Answer: In one hour, Ella can paint half the wall and Rob can paint the whole wall by himself, so in one hour, they can do 1 1/2 of the job, or 3/2 of the job. So, they can do the job in 2/3 of an hour. This is less time than it takes Rob by himself, but more time than it would take both of them together, working at Rob’s speed. That makes sense because Ella is slower than Rob.
Problem Bank
1. An engineer can write a program in six days. Another engineer can write the same program in three days. How long would it take if they worked together to write the program? 
**Solution:** The first engineer can write the program in six days. That means he can write 1/6 of the program per day. The second engineer can write the program in three days. That means she can write 1/3 of the program per day. Together they can write 1/6 + 1/3 = 1/2 of the program per day, so it would take two days to complete the program.

2. Mandy can paint a fence by herself in four hours. Kyle can paint the same fence in six hours. They work together for two hours and then Mandy leaves.
   a) How much of the fence will be done after one hour?
   b) How much of the fence will be done after two hours?
   c) How much of the fence will be left for Kyle to do on his own?
   d) How long will it take Kyle to paint the amount of the fence from part c) on his own?
**Answers:** a) in one hour, Mandy paints 1/4 of the fence and Kyle paints 1/6 of the fence, together they paint 1/4 + 1/6 = 5/12 of the fence; b) in two hours, they paint 10/12, or 5/6, of the fence; c) so when Mandy leaves, Kyle has 1/6 of the fence left to paint; d) it will take him another hour.

3. Armand and Jessica work at a factory. Armand is training Jessica to pack bottles. Armand can pack 20 000 bottles in five hours. Jessica can pack 3000 bottles in two hours. How long will it take them to pack 44 000 bottles?
**Answer:** In one hour, Armand packs 4000 bottles and Jessica packs 1500 bottles, so together they pack 5500 bottles in one hour. So, it will take them 44 000 ÷ 5500 = 8 hours.

4. Tristan can mow the lawn by himself in three hours. Working together, Tristan and Lily mow the lawn in two hours. How long would it take Lily to mow the lawn by herself?
**Answer:** Tristan can mow 1/3 of the lawn in one hour. In two hours, he completes 2/3 of the lawn. Lily, then, must have done 1/3 of the lawn in two hours. So, she would take six hours to mow the lawn by herself.

5. Jasmin works 20% faster than Arsham. That means that she can do 20% more work than Arsham can do in the same amount of time. Arsham can paint 1/4 of a wall in one hour.
   a) What fraction of the wall is 20% of 1/4 of the wall?
   b) What fraction of the wall can Jasmin paint in one hour?
   c) How long will Arsham and Jasmin take to paint the wall if they work together?
**Answers:** a) 1/20, b) 3/10, c) 20/11 hours or 1 9/11 hours

6. Zack and Tina are fixing a bike together. Tina works 30% faster than Zack. If they work together, they can do the job in five hours.
   a) How long would it take Zack if he worked alone?
   b) Is your answer more than 10 hours or less than 10 hours? Why does that make sense?
Answer: a) If Zack does \( x \) of the job, Tina does \( 1.3x \), and together they do \( 2.3x = 1 \) whole job. So, in five hours, Zack does \( 1/2.3 = 10/23 \) of the job. In one hour, he does \( 2/23 \) of the job. So, it will take him \( 23/2 \), or \( 11 \frac{1}{2} \), hours to do the job on his own; b) More than 10 hours, which makes sense because if Zack and Tina worked at the same pace and did the job in five hours, it would take each of them 10 hours working alone. But Zack is slower than Tina, so it should take him longer than 10 hours.

NOTE: In Problem Bank 7, parts a) to e) are guiding questions to help students solve the problem. Some advanced students may be able to solve the problem without guidance.

7. Jayden can paint a fence in 10 hours. Megan can paint the fence in 12 hours. They work for 3 hours and then Alice comes to help. They finish the job 2 hours later. How long would it take Alice to do the job on her own?
   a) How much of the job did Megan and Jayden finish in 1 hour?
   b) How much of the job did Megan and Jayden finish in 5 hours?
   c) How much of the job did Alice do in 2 hours?
   d) How much of the job did Alice do in 1 hour?
   e) How long would it take Alice to complete the job on her own?

Selected solutions: a) \( 1/10 + 1/12 = 11/60 \), b) \( 55/60 = 11/12 \)

Answers: c) \( 1/12 \), d) \( 1/24 \), e) 24 hours
Extended Problem: Pool Maintenance

Materials:
BLM Pool Maintenance (pp. R-62–65)

Extended Problem: Pool Maintenance. This extended problem allows students the opportunity to work with areas of circles and volume of cylinders, converting between units of volume (cm$^3$ and m$^3$), and multiplying decimals. Provide students with BLM Pool Maintenance.

Answers:
1. a) $320, b) $80, c) $62.80, d) $111.64
2. a) 17.584 m$^3$; b) 100, 1 000 000; c) 17 584 000 cm$^3$; d) 17 584 L; e) 53.28 days
3. a) 15.7 m$^2$, b) 1.57 m$^3$, c) $177.41$

Bonus: a) 12 hours, b) 1/4 full, c) 2 1/4 hours or 2 hours 15 minutes
Pool Maintenance (1)

You may use a calculator to complete this task. Use 3.14 for π.

1. A store sells pool covers in various sizes and shapes. It decides to make the cost of the cover proportional to the area of the cover. It sells a 4 m by 9 m rectangular cover for $320. How much will the store sell the following covers for? Write your answers to the nearest dollar.

a) a 6 m by 6 m square cover

b) a 3 m by 3 m square cover

c) a circular cover with diameter 3 m

d) a circular cover with diameter 4 m
Pool Maintenance (2)

2. Simon's circular pool has a diameter of 4 m and is 1.5 m deep. He fills the pool until the water is 10 cm from the top.
   
   a) How much water does Simon use in m³?

   b) 1 m = ________ cm, so 1 m³ = ______________ cm³.

   c) How much water does Simon use in cm³?

   d) Remember, 1 cm³ = 1 mL and 1 L = 1000 mL. How much water does Simon use in L?

   e) One estimate suggests that Canadians use an average 330 L of water per day per person. How long would it take for an average Canadian to use as much water as Simon used to fill his pool?
Pool Maintenance (3)

3. Simon wants to put a 1 m wide sidewalk around his circular pool, which has diameter 4 m.
   a) What is the area of the sidewalk in m²?

   b) The sidewalk will be made from concrete 10 cm deep. How much concrete will be needed to make the sidewalk in m³?

   c) Concrete costs $100 per m³, plus a 13% tax. How much will the concrete for the sidewalk cost?
Pool Maintenance (4)

**Bonus** ► One hose can be used to fill a pool in three hours. Another hose can be used to drain the full pool in four hours.

a) If the hose used to drain the pool is accidentally left on while the owner is trying to fill the pool, how long will it take to fill the pool?

b) After three hours, the owner checks on the pool, expecting it to be full. How full is the pool at the three-hour mark?

c) The owner of the pool turns off the drain hose at the three-hour mark. How much longer will the pool take to fill completely?
Unit 7  Geometry

In this unit, students will investigate the relationships between the number of faces, edges, and vertices of 3-D shapes. They will create and describe tessellations with regular and irregular polygons and identify criteria for polygons to tessellate. Students will also draw and interpret side, front, and top views of 3-D shapes made from prisms.

Materials
In many lessons you will need either isometric or regular dot paper. Such paper is provided on BLM Dot Paper (p S-36) and BLM Isometric Dot Paper (p S-37). You can also photocopy the BLMs onto a transparency and project them on the board when you need pre-drawn dots.

In lessons G8-42 to G8-46 students will need connecting cubes during course of the lesson and for some of the exercises in the workbook.

Meeting Your Curriculum
Lesson G8-37 addresses Ontario curriculum expectation 8m51, which does not appear on the WNCP curriculum for grades K–9, so this lesson is optional for those who follow the WNCP curriculum.

Lessons G8-38 to G8-46 address WNCP core curriculum expectations 8SS5 and 8SS6 (tessellations and views of 3-D structures). The material covered in these lessons was studied using a slightly different approach in grades 6 and 7 in Ontario. Therefore, these lessons are optional for Ontario students.
Relationships between the numbers of faces, edges, and vertices of prisms. Divide your students into groups of four and give each group the five prisms used in Question 1 on Workbook page 168. (You can make them using the nets on BLM Nets of 3-D Shapes (2, 3, 4, 6, 10).) Students will share the shapes within their group. Have students count the edges, faces, and vertices of the shapes to complete Question 1 individually. Point out that the number of vertices in each base (you can count them in the pictures in the first row) is the same as the number of sides (or edges) in the base. Check the formulas obtained in the last column and ask students to pick one formula and to explain it geometrically. (SAMPLE EXPLANATION: A prism with \( n \) edges in the base also has \( n \) vertices in the base. There are two copies of the base in any prism. Any vertex belongs to one of the bases, therefore there are \( 2 \times n \) vertices altogether in a prism with \( n \) base edges.) Students can pair up with other students who explained the same formula to try to come up with a better explanation together. Then they can share their answers with the class.

Now give each group four different pyramids (e.g., from BLM Nets of 3-D Shapes (16–19)). Have each student count the faces, edges, and vertices of one pyramid, as well as the number of sides in its base. Then ask students to substitute the number of sides in the base into the formulas for the pyramids. Did they get the right number of faces, edges, and vertices for the pyramids? (no) Do the formulas they obtained for the prisms work for the pyramids? (no)

Have students find the formulas that will work for the pyramids. They can either use geometrical reasoning, as they did when explaining the formulas for the prisms, or they can combine their data with the data of other students in their group and look for a pattern.
Ask students to compare the formulas for pyramids and prisms using a chart like the one below. **ASK:** Which values make sense for \( n \), which is the number of sides in the base? (**ANSWER:** whole numbers greater than 2. **PROMPTS:** Can \( n \) be 2? Can it be 5.5?) Can the formulas produce the same number of vertices, edges, or faces for any relevant value of \( n \)? (No. For example, \( 2n \) is never equal to \( n + 1 \), because if \( 2n = n + 1 \), then \( n = 1 \), and we know \( n \) must be at least 3.)

<table>
<thead>
<tr>
<th></th>
<th>Prism</th>
<th>Pyramid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of sides in the base</td>
<td>( n )</td>
<td>( n )</td>
</tr>
<tr>
<td>Number of vertices</td>
<td>( 2n )</td>
<td>( n + 1 )</td>
</tr>
<tr>
<td>Number of edges</td>
<td>( 3n )</td>
<td>( 2n )</td>
</tr>
<tr>
<td>Number of faces</td>
<td>( n + 2 )</td>
<td>( n + 1 )</td>
</tr>
</tbody>
</table>

**Relationship between the number of edges, faces, and vertices.** Have ready six large cardboard squares of the same size, each with one of the following words written clearly on it: top, bottom, front, back, left, right. (Use thick cardboard, e.g., from a moving box.) Loosely tape the squares into a cube (the words should be on the outside) so that you can easily take it apart. Present the cube to students and show how you can take it apart. Then work through Question 3 together as a class: Ask the questions in the workbook and have students signal the answer for each one whenever possible. Give the answer and explanations after each question. Point out that the answers to both c) and f) count the number of vertices on all faces when the faces are separated.

Explain that you want to check whether the formula from Question 3—\( F \times \) (number of vertices on each face) = \( V \times \) (number of faces that meet at each vertex)—works for a different shape. Show students a triangular pyramid (e.g., from BLM Nets of 3-D Shapes (19)). **ASK:** How many faces does it have? (4) Draw four triangles on the board to represent the faces and **ASK:** How many vertices are on each face? (3) How many vertices are in total on all faces? (3 \( \times \) 4 = 12) Write on the board:

\[
F \times \text{(the number of vertices on each face)} = 3 \times 4 = 12.
\]

How many vertices does the triangular pyramid have? (4) How many faces meet at each vertex? (3; students can use an actual pyramid to find the answer or to check it) Have students substitute the values they found into the expression \( V \times \) (number of faces that meet at each vertex) to verify the formula.

**ASK:** Do you think the formula will work for any shape? Will it work for, say, a triangular prism? What is the problem? (The number of vertices is not the same on each face.) **NOTE:** See Extensions 1 and 2.

Let students work individually on the Investigation on Workbook page 169. The investigation is similar to the previous question, but works on edges instead of vertices.
Introduce Platonic solids. Have students each pick one of the prisms used at the beginning of the lesson and check how many faces meet at each vertex. Summarize the results on the board. Is the number of faces that meet at a vertex the same for all vertices of all prisms? (yes) Repeat with pyramids. Are there pyramids where the same number of faces meet at all vertices? (yes, triangular pyramids) Explain that Platonic solids are 3-D shapes with faces that are congruent regular polygons and the same number of faces meeting at each vertex. Are there any prisms that are Platonic solids? (cube) Are there any pyramids that are Platonic solids? (a triangular pyramid is a Platonic solid when its faces are equilateral triangles) If possible, show students some other shapes, such as a truncated square pyramid (a pyramid with the apex cut off), a dodecahedron, an icosahedron, an octahedron, and a square antiprism—see nets for these shapes on BLM Nets of 3-D Shapes (20–24)—and ask students to check whether these shapes are Platonic solids.

For Bonus Question c) on Workbook page 169, have students make the shape from clay or join two triangular pyramids together. Do not use the picture in the Workbook, since it might be incorrect in your copies.

Explain that there are only five Platonic solids. Provide nets for these solids from BLM Nets of 3-D Shapes (1, 19, 22, 23, 24), and have students construct the shapes from the nets.

Euler’s formula. Have students do Question 6 on Workbook page 170, then assign different polyhedra to different students (assign tetrahedrons and octahedrons together, as they need less work) and have students verify the data in their tables by counting the vertices and edges in actual shapes. Then have students complete the Workbook pages for this lesson.

NOTE: Students can use a soccer ball to verify the number of faces, vertices, and edges they find in Question 9 on Workbook page 171.

Extensions

1. In the formula $F \times (\text{number of vertices on each face}) = V \times (\text{number of faces that meet at each vertex})$, the total number of vertices is counted in two different ways. Check this for triangular prisms:
   
   a) Draw the faces of a triangular prism separately. Add the number of vertices on each face. (You have just written out the left side of the formula.)
   
   b) Multiply the number of vertices by the number of faces that meet at a vertex.
   
   c) Compare your answers in a) and b). Are your answers to a) and b) the same? If they are, explain how this happens. (HINT: How many times did you count each vertex in a)? If they aren’t, go back and check your answers.
ANSWERS:

a)  

The total number of vertices is $3 \times 4 + 2 \times 3 = 18$

b) Since three faces meet at each vertex,

$V \times (\text{number of faces meeting at each vertex}) = 6 \times 3 = 18$

c) The answers are the same because when we count the vertices in a), we count each vertex three times, because three faces meet at each vertex.

2. The formula $F \times (\text{number of vertices on each face}) = V \times (\text{number of faces that meet at each vertex})$ does not work for shapes that have a different number of vertices on each face (e.g., some triangles and some rectangles) or a different number of faces that meet at each vertex. However, you can use the average number of vertices per face and the average number of faces that meet at a vertex. Which measure of central tendency would you use in this case? Calculate the mean, the median, and the mode of the number of vertices of each face of a pentagonal prism. Substitute each into the formula above. Does the formula hold for any measure of central tendency? Repeat with a pentagonal pyramid, which has both a different number of vertices on each face and a different number of faces that meet at each vertex. Use the mean, median and mode of both in the formula.

Write a formula that works for any prism or pyramid.

**ANSWER:** $F \times (\text{mean number of vertices on each face}) = V \times (\text{mean number of faces that meet at each vertex})$

3. Students can verify Euler’s formula for a cube ($V = 8$, $E = 12$, $F = 6$) and then remove a corner of a cube at the midpoint of the 3 edges adjacent to a single vertex and verify that the resulting 3-D shape still satisfies the formula ($V = 10$, $E = 15$, $F = 7$).

They can then repeat the process for every vertex, one at a time, until they obtain a cuboctahedron (see margin).

4. Use the formulas for the number of faces, edges, and vertices in prisms and pyramids with an $n$-sided polygon in the base obtained in the beginning of the lesson to identify the shapes below as either prisms or pyramids. Explain how you arrived at your answers.

a) What shape has 15 vertices? (14-gonal pyramid. **EXPLANATION:**

A prism has an even number of vertices, so the shape is a pyramid.
The number of vertices of a pyramid is one more than the number of vertices in the base, so the base has 14 sides.)

b) What shape has 111 edges? (37-gonal prism. **EXPLANATION:** 111 is a multiple of 3 but not a multiple of 2, so the shape is a prism. The number of edges for a prism is $3n$, so given $3n = 111$, we have $n = 37$, where $n$ is the number of edges in the base.)

c) What shape has 120 edges and 80 vertices? (40-gonal prism)

d) What shape has 120 edges and 61 faces? (60-gonal pyramid)

Solution for c) and d): 120 is a multiple of both 3 and 2, so the shape could be either a prism or a pyramid. If it is a prism, the number of edges/vertices in the base is $120 \div 3 = 40$. Then the shape has 80 vertices, and 42 faces, so it is the answer in c). If the shape is a pyramid, it has $120 \div 2 = 60$ edges/vertices in the base, and the number of faces (and vertices) is $60 + 1 = 61$. This is the shape from d).

5. Give students **BLM A Doughnut** (p S-32) and ask them to construct a shape by following these instructions:

   a) Cut out the nets. Fold the nets along the edges.
   
   b) Tape sides A and B together to make a square prism without bases.
   
   c) Tape side C to side D and side E to side F to make two rings joined at a common edge. Join the rings along the wider sides.
   
   d) Place the prism without bases from b) inside the other piece, and tape the top edges of one piece to the top edges of the other. Repeat with the bottom edges. The result should look like a doughnut.
   
   e) Does Euler’s formula hold for the “doughnut”? The answer is No! The shape has 20 faces, 16 vertices, and 36 edges. The shape has a hole in the middle—like the hole in a doughnut—and Euler’s formula works only for shapes without such holes.
Goals
Students will extend existing tessellations and describe tessellations in terms of transformations.

PRIOR KNOWLEDGE REQUIRED
Can perform, identify, and describe rotations, reflections, and translations and their combinations

MATERIALS
BLM Cards for Transformations (p S-35)
rulers
protractors

Tessellations. Explain to students that a tessellation is a pattern made up of one or more shapes that completely covers a surface, without any gaps or overlaps. One example of a tessellation is the pattern made by floor tiles (polygons or other shapes). If you can tile the floor with a shape, this shape is said to tessellate. ASK: Do you know any shape that tessellates? Your students will almost certainly say “square,” but they might also name other shapes, such as rectangles (ask them to draw a picture: there is more than one way to tessellate with rectangles; to prompt students suggest that they think about brick houses or paving), diamonds, other parallelograms, triangles or hexagons. Show a picture of a honeycomb. Is it a tessellation? If there are any interesting tessellation patterns around the school, mention them as well. Have students do part a) of the Activity on pages S-10–S-11.

Review transformations. Show students the shapes at left. Discuss with students what transformation would take one shape onto the other. (reflection) Review how to describe translations on a grid by marking a point and its image, and checking how much the point moved and in which direction. Review the fact that reflections, rotations, and translations produce congruent shapes. Point out that since tessellations are made up of copies of the same shape, we can use transformations to describe the tessellations.

Tessellating by translating a group of shapes that make a square. Show the pattern at left on the board or on an overhead. ASK: Can you use these L-shapes to tessellate? Ask students to identify the single transformation that takes each shape into the others. (rotations of 90°, 180°, or 270°, clockwise or counter-clockwise) What can you do with all four shapes together to tessellate the whole surface? (translate the square) How many units do you need to translate the square? In which direction? Lead students to the idea of making a row of squares by translating the group of four shapes, 4 units left (or right) repeatedly, then translating the whole
row 4 units up or down repeatedly. Another common solution is to translate
the group 2 units up and 4 units right to create a “diagonal” row, and then
to translate the diagonal row 4 units up or down repeatedly.

Ask students to look at the tessellations they produced during part a) of
the Activity and to describe them in terms of transformations.

EXTRA PRACTICE:

a) Which of the pictures below shows a tessellation using the same
shape? Explain.

i)  

ii)  

b) Which transformations take the dark shape into the other shapes
of the tessellation?

Describing transformations in a tessellation on the Cartesian plane.
Draw the partial tessellation shown at left on the board or on an overhead.
ASK: Which transformation takes shape A onto shape B? Draw a translation
arrow between shapes A and B. Ask students to describe the translation.
Mark a vertex on shape A and ask students to identify the vertex of B which
is the image of the marked vertex under the translation. Ask them to draw
the translation arrow between the new vertices. ASk: What do you notice?
(the second translation arrow is the same length and points in the same
direction as the first)

Ask students to find a pair of shapes that can be transformed onto one
another by a reflection (C and D), and ask a volunteer to identify the mirror
line (y-axis). Draw a vertical mirror line through the left-hand side of C and
ASK: What should I do to the image of C (after the reflection) to move it
onto shape D? (translate it 8 units right)

Have students find a pair of shapes that can be transformed onto one
another by a rotation, and identify the amount of rotation needed and the
centre of rotation. SAMPLE ANSWERS: shape D to E, 90° CCW rotation
around (2, 0); shape E to F, 180° rotation around (0, 1), shape G to A,
90° CW rotation around (2, 4).

Ask students to identify a pair of transformations that would take shape C
onto shape E. Encourage students to find multiple answers to this question.
SAMPLE ANSWERS: reflect C through the y-axis to obtain D, and rotate the
image (D) 90° counter-clockwise; reflect C through its right side (vertical
line through (−2,0)) and rotate around (0, 2); rotate C 90° clockwise around
(−3, 1), then reflect through the right side of the shape.

Changes in orientation of a shape under repeated transformation.
Ask students to describe shape C without using the word “L-shape.” For
example: A rectangle with base 1 and height 3, with an additional 1 × 1
square at the bottom of the right side. Then ask them to describe shape D.
SAY: I reflect shape C through a vertical line. Where is the additional square attached now? (at the bottom of the left side) I reflect shape C through a horizontal line. Where is the additional square attached? (top of the right side) Invite volunteers to check the predictions.

ASK: Can I take shape C onto shape D using only reflections through the sides? How many reflections will I need? (3) Can you tell where the additional square will be attached after each reflection without actually making the reflections? (bottom left, then bottom right, then bottom left, then bottom right, and so on) I reflected shape C thirty-seven times through vertical lines. Where is the square—on the left or on the right? On the top or on the bottom? (bottom left) I slid shape C in some direction. Where is the square now? On which side—right or left? Top or bottom? (right, bottom)

Have students draw a copy of shape C on grid paper and try to rotate it 90° clockwise around different points. Does the additional square point to the same direction every time? (yes, bottom left) Which way will the additional square point if you turn shape C 90° counter-clockwise? (top right)

**Bonus**

I slid shape C, turned it 180°, and reflected it through a vertical line. Where is the additional square after all those transformations? (top right)

**EXTRA PRACTICE:**

a) Describe a series of transformations that could be used to get shape A onto shape C. Give at least two answers.

b) Which transformations were used to move shape B onto shape D? Give at least two answers. (There are infinitely many possible!)

**Bonus**

Find a sequence of two transformations that will take shape A onto shape D.

**Describing designs.** Use the triangular cards from BLM Cards for Transformations. Ask students to arrange the triangular cards in a hexagon, such that each card is a reflection of each adjacent card through their common side, as shown below. Ask students to tell which transformation takes each card onto each other card.
Repeat with the square cards from BLM Cards for Transformations, arranging them first into a square and then into a $1 \times 4$ rectangle, as shown below).

**What is the result of two reflections?** In arranging the cards above, the only transformation performed was a reflection. Discuss the results of two reflections in each case, as follows.

**SAY:** In the hexagon, when you start with the bottom-most shape and reflect it twice, you get the shape at the top right. You can also get from the bottom-most shape to the top right shape by a $120^\circ$ counter-clockwise rotation. In this case, two reflections produce a $120^\circ$ rotation. What is the angle between the mirror lines? ($60^\circ$) Now look at any two opposite shapes. What transformation takes one of them onto the other? (a reflection)

Look at two opposite shapes in the square. **ASK:** Which transformation takes one shape onto the other? ($180^\circ$ rotation) What is the angle between the mirror lines in this case? ($90^\circ$) So the result of two reflections in the axes that are perpendicular to each other is a $180^\circ$ rotation.

Now look at the rectangle. **ASK:** What transformation takes the first shape onto the third shape? (translation) What is the angle between the mirror lines? ($0^\circ$, the lines are parallel)

One can also notice that reflections over intersecting lines produce alternating reflections and rotations. If you number the shapes as if you were going around the hexagon or the square, you can see a pattern in the transformations needed to get each next shape from the first shape. The pattern is reflection, rotation, then repeat. If the reflections are made in parallel lines (as they are in the rectangle), the pattern is reflection, translation, then repeat.
Have students create and describe tessellations using part b) of the Activity.

**ACTIVITY**

a) Divide your students into two groups. Students in each group will individually construct a pair of shapes using a ruler and a protractor, and cut out at least 6 copies of each shape.

Group A: An isosceles trapezoid with sides and smaller base 5 cm and angles of 135°. A square with sides 5 cm.

Group B: A parallelogram with base 5 cm and height 5 cm, with an angle of 45°. A square with sides 5 cm.

Each student should create at least three different tessellations: one using one of the two shapes, a second using the other shape, and a third using both shapes together. Students will sketch the tessellations in their notebooks. Students with the same pair of shapes will pair up and share their tessellations. They will try to come up with another tessellation using one or two of their shapes. Then students will form groups of four and share their tessellations again. Can they come up with a new tessellation? The goal is to produce as many different tessellations as possible.

b) Have students describe their tessellations in terms of transformations. To make the task easier, ask students to draw a very simple asymmetric design on each shape (e.g., a tilted flower, the number 2, the letter P), so that the shapes are exactly the same. Ask them to turn the shapes over and trace the design on the back, so that when a shape is reflected, the design is reflected as well. Have students sketch one of their tessellations (that involves at least two different transformations) in their notebooks and describe it in terms of transformations. Students can again share tessellations in groups of two and four, and have their partners describe the tessellations they produced.

**Extension**

**Working backwards.** To get from shape A to shape B, translate A 3 units left and 4 units up, then rotate it 90° CCW, and then reflect it in the x-axis. How can you get from shape B to shape A? Check your prediction. (reflect it in the x-axis, rotate 90° CW, and translate A 3 units right and 4 units down)
Construction worker problem:
Joshua says that he can tile the floor of a bathroom using only regular octagons. Is this correct?

Josef says that he can tile the floor using only regular pentagons. Is this correct?

Do Joshua and Josef encounter the same problem?

Let your students use regular pentagons, hexagons, and octagons (e.g., from BLM Polygons for Tessellating) to check whether these shapes tessellate. Discuss the results. ASK: How many shapes of each kind can you place together so that they share a vertex and do not overlap? (3 pentagons, 3 hexagons, 2 octagons) Are there any gaps left?

Answer the construction worker problem: Joshua cannot place more than 2 octagons together—the gap is not large enough for a third. Joseph can place 3 pentagons together but there is a gap that is not enough for a fourth pentagon. Neither worker can tile the floor using only his shape because of the gaps that are left.

Ask volunteers to sketch tessellations with triangles on the board. Repeat with squares and hexagons.

**Sum of the angles in a quadrilateral.** Remind students how to measure angles with a protractor. Ask students to each draw a quadrilateral with four different interior angles (i.e., not a special quadrilateral). Then have students
measure the angles in their quadrilateral and add them. What is the sum of the angles in the quadrilateral? Have students fold their quadrilateral along a diagonal and measure the angles in the resulting triangle. What is the sum of the angles in the triangle? (180°) Does this sum fit with the sum of the angles in the quadrilaterals? Did all students get the same result? Why could that be? As a class, work through the proof of the sum of the angles in a quadrilateral in the box on Workbook page 174.

ASK: What is the degree measure of a straight angle? (180°) Draw a straight angle and ask what the degree measure around the vertex is. There are two straight angles, so there are 360° around the point. Draw a picture of three line segments in the shape of the letter Y and ask students what the sum of the angles around the vertex should be. (360°) If the three angles in the Y are the same, what is the measure of each angle? (120°)

Have students work individually through Questions 1 to 5 on Workbook page 174.

Sum of the angles in any polygon. Have students work through Question 6 on Workbook page 175 shape by shape. Invite students to share their answers and check them as a class, making any corrections necessary.

To prompt students to develop a formula for the sum of the angles in any polygon, ask them to think how they get a number in each column from the corresponding number in the previous column.

A different way to find the sum of the angles in a polygon. Draw an irregular heptagon on the board, pick a point inside the polygon, and connect it to the vertices of the polygon, thus splitting it into seven triangles. ASK: What is the sum of the angles in each triangle? (180°) Write 180° in each triangle on the diagram. What is the total angle measure in the triangles? (180° × 7 = 1260°) Then ask students to mark all the angles that this count covers. Does this count include all the interior angles of the heptagon? (yes) Are there any angles in this count that are not interior angles in the heptagon? (yes) Have students mark these angles with a different colour (shown as a double line on the diagram at left). What is the total measure of the superfluous angles? (360°) What is the sum of the angles in the heptagon? (1260° − 360° = 900°)

Ask students whether the answer is reasonable. Point out that 360° = 180° × 2, so you could calculate the answer using the distributive law. Write on the board: 180° × 7 − 180° × 2 = 180° × (______) ASK: What goes in the brackets? (7 – 2) Have students calculate the answer this way. (180° × 5 = 900°) Did they get the same answer?

Ask students to check whether this answer fits with the expression for the sum of the interior angles, 180° × (n − 2), which they developed in Question 6 on Workbook page 175. (yes) ASK: Will this method of breaking the polygon into triangles using a point inside the polygon work for any polygon? (yes) If a polygon has n vertices, how many triangles will be there? (n) What will the sum of the angles in all the triangles be? (180° × n) What will the superfluous amount be? (always 360°) Have students write the formula for the sum of the angles in a polygon with n vertices.
(180°n – 360°) Then ask students to show that both formulas produce the same number. (180° × (n – 2) = 180°n – 360° by using the distributive law)

**Which regular polygons tessellate?** Review with students what regular polygons are: they have equal sides and all their angles are the same. What is a regular quadrilateral? Is a rhombus a regular quadrilateral? Why not? (angles are not equal) Is a rectangle a regular quadrilateral? Why not? (sides are not equal) Can there be a triangle that has equal sides but is not regular? (no, equal sides means equilateral, and equilateral triangles have equal angles) Continue to the Investigation on page 176. Discuss the patterns students notice in the tables and the answers to Parts E and F in particular. When discussing the answers to E, point out that the expression 360° ÷ x° shows how many polygons meet at a vertex of a tessellation. The answer is a whole number for triangles, squares, and hexagons. Indeed, 6 triangles, 4 squares, or 3 hexagons meet at a vertex when these shapes are used to tessellate. To guide the students to the answer in F, ask them to identify whether the sequence of the measures of interior angles is increasing or decreasing. (increasing) This means the more angles a polygon has, the larger these angles are. So for any regular polygon with more than 6 vertices, the measure of each interior angle is more than 120°.

**ASK:** What is the smallest number of polygons that can meet at a vertex of a tessellation? (3) What does this say about the largest possible angle of a tessellating regular polygon? (It is at most 120°) This means any polygon with more than 6 vertices has angles that are too large to allow three shapes to meet at a vertex, and so cannot tessellate.

**Extensions**

1. a) The formula for the sum of the angles in a polygon with n sides is 180° × (n – 2). In a regular polygon all angles are equal. What is the measure of each angle in a regular polygon with n sides? (180° × (n – 2) ÷ n)

   Verify the formula for equilateral triangles, squares, regular pentagons, hexagons, heptagons, and octagons using your answers in part A of the Investigation on Workbook page 176.

   b) Write the following expressions without the brackets:

   i) 18 ÷ (6 ÷ 3)           ii) 24 ÷ (3 × 8)  
   iii) 24 ÷ (3 × 8 ÷ 4)     iv) 24 ÷ (8 ÷ 4 × 3) 

   **ANSWERS:** i) 18 ÷ 6 × 3    ii) 24 ÷ 3 ÷ 8  iii) 24 ÷ 3 ÷ 8 × 4  
   iv) 24 ÷ 8 × 4 ÷ 3

   c) Use your answer from a) and the distributive law to rewrite the expression for 360° ÷ (measure of one interior angle) so that you end up with one pair of brackets. Then show that the expression is equal to 2n ÷ (n – 2).
SOLUTION: 360° ÷ (measure of one interior angle)
= 360° ÷ (180° × (n − 2) ÷ n)
= 360° ÷ 180° ÷ (n − 2) × n
= 2 ÷ (n − 2) × n
= 2 × n ÷ (n − 2)
= 2n ÷ (n − 2)

NOTE: Remember that multiplication and division are interchangeable. This is why we can change the order in which elements are multiplied and divided in the fourth line of the solution.

<table>
<thead>
<tr>
<th>n</th>
<th>2n ÷ (n − 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

d) Fill in the table at left for n = 3 to n = 10. For which values of n is 2n ÷ (n − 2) a whole number?
Use this answer to explain which three types of regular polygons tessellate.

ANSWER: The expression 360° ÷ (measure of one interior angle) tells how many polygons meet at a vertex of a tessellation. For a shape to tessellate, the answer must be a whole number. For the answer to be a whole number, (n − 2) should divide into 2n. This can happen in the following cases:

\[ n = 3, n - 2 = 1, 2n ÷ (n - 2) = 6. \text{ Triangles tessellate,} \]
6 triangles meet at a vertex of the tessellation.

\[ n = 4, n - 2 = 2, 2n ÷ (n - 2) = 4. \text{ Squares tessellate,} \]
4 squares meet at a vertex of the tessellation.

\[ n = 6, n - 2 = 4, 2n ÷ (n - 2) = 3. \text{ Hexagons tessellate,} \]
3 hexagons meet at a vertex.

e) Look at the pattern of numbers in the right column of the table in d). Is it increasing or decreasing? (decreasing)
If n > 6, the number in the right column is less than 3. However, it never becomes 2. To see that, look at the following quotients:

\[ 3 ÷ 5 \quad 4 ÷ 7 \quad 8 ÷ 6 \quad 9 ÷ 4 \quad 6 ÷ 7 \quad 7 ÷ 6 \]
How can you tell if a quotient is larger than 1?
Is \( n ÷ (n - 2) \) larger or smaller than 1? (larger)
If you double a number that is larger than 1, what can you say about it? (it is larger than 2)

EXPLAIN why \( 2n ÷ (n - 2) \) is always larger than 2.
f) Explain why no other regular polygon tessellates.

SOLUTION: The number 360° ÷ (measure of one interior angle)
= 2n ÷ (n − 2) is the number of polygons meeting at a vertex.
From e), we know that 2n ÷ (n − 2) is always larger than 2, so it is between 2 and 3 for regular polygons with number of sides greater than 6. This means that no regular polygon other than the triangle,
square, or hexagon will have a whole number of shapes meeting at a vertex of the tessellation, so no other regular polygons tessellate.

2. Look at the pattern in the measure of one interior angle in regular polygons (see the Investigation on Workbook page 176). Does the pattern increase or decrease? (increase)

Now find the gaps between the rows (or the terms of the pattern). Do the gaps increase or decrease? (decrease) Do you think the gaps might at some point become negative? If the gaps become negative, we will be adding a negative number, and the pattern in the angles will decrease. So maybe there are polygons with really large numbers of sides that tessellate! Let’s investigate what happens at really high numbers.

Find the size of interior angles for polygons with 100 and 101 sides, then find the gap between them. Is the gap still positive? (Yes. For \( n = 100 \) the formula for the sum of interior angles \( 180° \times (n - 2) \), which gives \( 180° \times 98 = 17640 \), so each angle is about 176.4°. For \( n = 101 \) we get each angle \( 180° \times 99 \div 101 \approx 176.44° \))

Repeat for polygons with 1 000 and 1 001 sides. (\textbf{ANSWER}: for 1 000: 179.64°, for 1 001: about 179.64036°)

As a matter of fact, the gap becomes smaller and smaller, but never reaches zero. At the same time, the size of each interior angle always increases, approaching 180° but never reaching it. From a geometric point of view, when the number of sides increases, a regular polygon looks more and more like a circle, though it never becomes a circle. Circles do not tessellate, and polygons with more than 6 sides cannot tessellate either.
Review the sum of the angles in polygons. Students will need to find the angles in regular and irregular pentagons, hexagons, and octagons. Emphasise that student don’t need to remember the formula for the sum of the angles, it is enough to remember how to find the formula by breaking a polygon into triangles and using the fact that the sum of the angles in a triangle is 180°. Point out that there are at least two ways to do this: you can have triangles that share a vertex that is a vertex of the polygon itself, as in Questions 1 and 6 on Workbook pages 174 and 175, or in the middle of the shape, as in Question 4 on Workbook page 174. The sum of the angles will not depend on it.

Review finding missing angles in a polygon using the sum of the angles and variables. For example, present the hexagon below.

50° 135° 135°

Have students identify the shape. Mark one of the missing angles as x and ask what the measure of the second missing angle will be. (x) Then ask students to write an equation for the sum of the angles in the hexagon 

\[2x + 135° + 135° + 50° + 90° = 720°\]

and solve it to find the measure of the missing angles (x = 155°).
Students should do the Activities below before completing the Workbook pages individually. Note that the measure of angle C in Workbook page 178 Question 7 is 50°, and the unnamed angle in Question 8 is 120°. Ask students to add this data if it is missing. Also, when students are asked to make copies of shapes from the book, they should not trace them. Instead, they should draw the shapes themselves using a ruler and protractor, to make sure the angles are exact. (In some copies of the student book, the shapes are not drawn accurately.)

**ACTIVITIES 1–2**

1. Fold a sheet of paper three times, so that there are eight layers. Draw a quadrilateral that does not have a line of symmetry (and is not a parallelogram) and cut it out, cutting through all eight layers. Number your quadrilaterals. Try to arrange the eight quadrilaterals so that they do not have gaps and do not overlap. Does your quadrilateral tessellate? Name the transformations used to obtain quadrilaterals 2 through 8 from quadrilateral 1.

2. Repeat Activity 1 with a triangle. **HINT:** Two triangles make a quadrilateral.
Give students blank paper, grid paper, scissors, and tape and let them work through the problems on Workbook page 180. Students should cut out multiple copies of their tessellating shapes to help them create the tessellations.

**Bonus** for Workbook page 180, Question 1:

This shape was created by repeating the procedure in a) twice.

a) Does it tessellate?
b) Which transformation is used in the tessellation?
c) Can you use a reflection when tessellating with this shape? Explain.

**Bonus** for Workbook page 180, Question 2:

Create a tessellation with the same shape using a reflection and a 180° rotation.

If students have trouble seeing that Ahmad’s shape in Question 4 does not tessellate, suggest that they place the pieces together so that the curved edges of one shape match the curved edges of another shape. Six shapes together will fit into a ring with an empty hexagon in the middle, which Ahmad’s shape can’t fill. The same method will show that shape C in Question 5 does not tessellate.

**Extensions**

1. Create a shape that will tessellate using a rotation. Start by choosing a regular tessellating polygon, e.g., a square. Find the centre of the shape, then draw a design such that if you rotate the polygon around
EXAMPLE

If the square is rotated 90° around the centre, both the square and the design do not change.

If the square is rotated 90° around the centre and it turns onto itself, the design does not change. (See example in margin.) Cut along the lines of the design. The resulting identical shapes tessellate. You can tessellate the square formed by four copies of the shape using any transformations you want, in addition to rotations. For example, the tessellation below can be described using all three transformations: rotations, reflections, and translations.

2. Project idea: Tessellations in the work of M.C. Escher.

Give students BLM Tessellating Monsters (p S-34).

a) Identify a point on the picture where more than two shapes meet. Label it P. How many shapes meet at P?

b) Label the shapes around P (use A, B, C, D). Which transformation or combination of transformations will take shape A onto each of the other shapes?

c) Copy one shape from the design (say, shape A) onto tracing paper. Go along the perimeter of the shape. Can you find another point on the picture where more than two shapes meet? How many points like that can you find along the perimeter of shape A? Mark all these points on your tracing.

d) Check each of the points you marked on the tracing. How many shapes meet at each of these points? Write the answer beside each point.
e) Join the points you marked on the tracing of A in the order they appear along the perimeter of the shape. Name the polygon you obtained.

f) Using the same method as in e), draw the polygon on each of the shapes on the design. Did you get a tessellation? Copy the tessellation with polygons to a clean sheet of paper and describe the transformations that could be used to create this tessellation.

g) Look at the tessellation with polygons you created in f). Can you describe it using a different set of transformations than the original design?

h) Find several designs by M. C. Escher that show tessellations. Which of the designs are made from many copies of one shape? Which are made from more than one different shape? Sort the designs.

i) Pick a design made using multiple copies of the same shape and repeat a) through h).

j) For each tessellation below, find a design by M. C. Escher that would produce this tessellation by the method of parts f) and g).

SAMPLE ANSWERS for a) to g):

a) Four shapes meet at P.

b) B: 90° counter-clockwise rotation around P.
   C: 180° clockwise or counter-clockwise rotation around P.
   D: 90° clockwise rotation around P.

c) There are four points where more than two shapes meet.

d) Four shapes meet at each point.

e) You get a square.

f) The tessellation is the same as in j) part iii). Multiple descriptions are possible, e.g., translate each shape one unit up repeatedly, then translate the whole column one unit right repeatedly.

g) Multiple answers are possible, e.g., reflect the square through the upper side repeatedly, then reflect the whole column through the right side repeatedly.
Introduce isometric dot paper. Give each student a sheet of isometric dot paper and write the term *isometric* on the board. **ASK:** Which other words that start with “iso” do you know? (isosceles) What does “iso” mean in that word? (the same) What does “metric” remind you of? (metre) Explain that “metric” means length (or distance). Ask students to join several closest dots with line segments. **ASK:** How is this dot paper different from regular dot paper? (it makes triangles, not squares) Which type of triangles? Have students measure the sides to check. (the triangles are equilateral) Explain that this paper is called isometric dot paper because the distances between adjacent dots are all equal. Project a sheet of regular dot paper on the board, and show how distances between adjacent dots on regular dot paper are not equal (the distance along a diagonal in a square is larger than the distance along a side of the same square). Regular dot paper is not isometric.

On isometric dot paper all distances are equal

On regular dot paper the distances are not equal

Explain that these two kinds of dot paper are used to produce two different types of views of 3-D shapes. Show the pictures of a cube at left and ask students to compare them. How are they the same? (both show a cube, both show three faces of a cube) How are they different? (A does not distort one face but distorts the other two so that they look like parallelograms. B distorts all three faces the same way—they all look like rhombuses.) Point the edges of cube B are all the same length, whereas A the edges perpendicular to the front face look shorter. Which cube is easier to produce on regular dot paper? (A) on isometric dot paper? (B)

**Drawing cubes on isometric dot paper.** Project a sheet of isometric dot paper onto the board. Show students how to draw a cube using the dots. Start with the top face, then draw the vertical edges (no hidden edges!), and then draw the visible bottom edges. (See the box on Workbook page 181.)

---

**CURRICULUM EXPECTATIONS**
Ontario: 6m50, 6m51; 8m3, 8m6, optional
WNCP: 8SS5, [V]

**VOCABULARY**
top view
right side view
left side view
front view
back view
bottom view
isometric

**On isometric dot paper all distances are equal**

**On regular dot paper the distances are not equal**

**Goals**
Students will draw top, front, and side views of 3-D structures constructed from cubes.

**PRIOR KNOWLEDGE REQUIRED**
Can identify top, bottom, front, back, left, and right sides of 3-D structures

**MATERIALS**
BLM Isometric Dot Paper (p S-37)
BLM Dot Paper (p S-36)
connecting cubes
Drawing 3-D shapes on isometric dot paper. Explain that to create an isometric drawing, it helps to start from the top. Look at the topmost layer and draw the top face or faces first. Then draw the vertical edges that are part of the topmost layer as you did with the single cube.

Hold up the shape made with three connecting cubes at left. Invite a volunteer to draw the top layer, a single cube (see below). What does the next layer look like? It consists of two cubes. Take two cubes locked together and compare this shape to the original shape made with three cubes: Which edges of the new shape are hidden by the top cube in the original shape? We do not need to draw them. Which visible edges of the new shape are already drawn (because they are the bottom edges of the cube on top)? Ask a volunteer to draw the remaining visible edges of the second layer.

Have students do Questions 1 and 2 on Workbook page 181. Students may find it easier to copy a shape onto isometric dot paper if they start by shading the top layer of the shape. They will need connecting cubes for Question 2.

Views of a structure. Remind students what the views from different sides are called, as in the box on Workbook page 182 (left side view, front view, and so on). You can draw the picture at left on the board and keep it there for reference for the next few lessons.

Explain that when we draw shapes from a different angle, as we do when we draw them on isometric dot paper, we have to decide which of the two vertical sides to make the front face. Depending on our choice, the other side will be either the left side or the right side.

One 3-D picture might be not enough to build the structure. Show students the picture at left. ASK: Was it drawn on isometric dot paper? (no) Ask students to build it from connecting cubes. ASK: How many cubes did you use? (7 or 8) Ask if anybody used a different number of cubes to make the structure. Have students present solutions with 7 and 8 cubes. Why are two solutions possible? (because the eighth cube is not visible in the picture; the picture can’t tell us if it’s there or not) How could we make clear what the structure looks like? (possible answers: show a second
PROCESS EXPECTATION
Reflecting on other ways to solve the problem

picture from a different angle, tell how many cubes were used) Explain that engineers and workers often use pictures of several views of a shape to give all the information what the shape looks like. Have all students construct the shape pictured with 7 cubes (a $2 \times 2 \times 2$ cube with one cube missing on the back), then ask them to hold the shape so that they see only the front face. Have them draw the front face.

**Thick lines show change of level.** Ask students to turn the shape so that they see only the right side. What is its shape? (square) Now ask them to turn the shape so that they only see the left side. What shape do they see? (square) How is the left side different from the right side? (there is a cube missing) What is the shape of the left side? (an L-shape) Have students draw the square that they see. Point out that if you draw only a square as the left side view, there will be no indication for the viewer that a cube is missing. Ask students to shade the place where the cube is missing on the left side view. Explain that in such cases people often add thicker lines to show that there is a change of depth on the picture. Have students draw thicker lines to separate the shaded square from the rest of the picture (see margin for sample).

Have students work through all the questions on Workbook pages 182–185 except Question 6. Students who have trouble drawing side views from pictures might find it useful to build the actual shapes from connecting cubes and to turn the shapes when drawing the views. Students can also check their work by building the shapes and looking at them from different sides. Another way to help struggling students: ask them to shade the sides that face in the same direction as the view they’re drawing. For example, when drawing the left side view, shade the sides that face.

**Drawing three views together.** Explain to the students that when we refer to several views of a shape, we often say “side views” for short, but we actually mean the top view, the front view, and at least one side view (right or left).

Before doing Question 6 on Workbook page 185, show students the structure at left. Have students draw the front view and the right side view of the structure on regular dot paper. **ASK:** What is the height of the front view? (3) What is the height of the right side view? (3) Are these the same? (yes) Will this happen for any structure? (yes) Why? (the height of both views is the height of the structure) Explain that we emphasize that the views have the same height by drawing the right side view and the front view side by side, aligning the top and the bottom of the views. We also draw the right side view to the right of the front view, so that the front side of the structure is shown on the left of the right side view, closest to the front view. This provides a self-checking mechanism. If you see that the front view and the side view are not the same height, you know right away that there is a mistake. Repeat with the top view, which is drawn directly above the front view with the front side at the bottom, closest to the front view. The top view has the same width as the front view, because the width of both views is the width of the structure. When students finish, draw the three views aligned as shown.
ASK: Height and width are two of the dimensions of the structure, and we used them for self-checking. The third dimension of the structure, depth, is perpendicular to the front view. In which two of the three views do we see how deep the structure is? (right side view and top view) Which dimensions of these views are the same as the depth of the structure? (the “height” of the top view and the “width” of the right side view) This provides us another opportunity for self-checking.
Side views of rectangular prisms. Show students a box with length, height, and width all different and labelled. (The box should be large enough that all students can see it across the room. A rectangular tissue box would be a good choice.) Invite volunteers to measure the box, and write down the measurements on the board. Then decide as a class which side will be the front view, and have students draw the front, top, and both side views, so that the right side view is to the right of the front view, and the left side view is to the left of the front view. Remind students that when they are asked to draw the “side views”, they need to draw the front view, the top view, and at least one of the left or right side views.

Run your finger along one of the edges, say the edge joining the front and the top sides of the box. **ASK:** Which faces meet at this edge? Have students identify this edge on both faces. **ASK:** Do you see this edge on the right side view? What does it look like? (a point, the top left vertex of the right side view rectangle) Where is this edge on the left side view? (top right vertex) Repeat with other edges. Students can trace the edges in different colours on their pictures.

Return to the edge between the top and the front faces. Ask students how long it is. Then have students write the length of this edge and all edges equal to it on all the side views. Repeat with other edges, so that all the side views have all sides marked with appropriate measurements.

**Side views of shapes made from rectangular prisms.** If available, show students a box that is not a rectangular prism but looks like two rectangular prisms combined. (See the sample in the margin. You can glue two boxes with one equal dimension together, and cover the sides with paper to hide the common edges.) Invite volunteers to take the measurements of the shape. Discuss which measurements to take and how to write them down. (One way to take the measurements is to take the total length, width, and height, and then measure just the length and width of the additional part.)
It might also help to write the measurements on a sketch of the shape. Then tell students that you would like to draw the side views of the shape. Invite volunteers to draw the right side view (the L-shape) and to mark the dimensions on it. Then discuss how to draw the top view and the front view. Lead students to the idea of drawing a rectangle with a line showing the change in depth. Point out that students do not need to use a thicker line because there are no extra lines in these side views as there were in side views of shapes made from connecting cubes. (Draw an L-shape made with connecting cubes to remind students how they needed to show the depth change with a thick line because of the lines for separate cubes.) As well, explain that it is good to have the lines showing the change in depth at the same level on different views (see picture in margin). This provides us with a self-checking mechanism. For example, if I know that the top part of the structure is half of its total height, and I see that the line showing the break between the top and the bottom part is at different heights in the front view and the side view, I know that there is a mistake somewhere. This means it is best to start by drawing the view that does not have a change in levels, if there is one.

You can add lines showing the change in depth to other views later.

Have students draw the side views of the box and mark the dimensions. Then have students practice doing the same with other prisms. Give students pairs of boxes to tape together themselves, to create the prisms. Repeat with prisms made from three boxes. Finally, have students draw side views of various objects in the classroom, such as desks, cupboards, shelves, and so on.

**ACTIVITY**

Give each student a small box and have them draw the side views. Students should mark the dimensions of the box on the side views. Have students get into groups of four so that students in each group have four different boxes. Have them mix the boxes and the pictures of side views. Then have groups of four swap their pictures and boxes. Each student then takes one picture of the side views and has to match it to a box.

**Extension**

Use the top, right, and front views of a prism to find the surface area of a prism. **EXAMPLE:**

<table>
<thead>
<tr>
<th>top view</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 m</td>
</tr>
<tr>
<td>2 m</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>front view</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2 m</td>
</tr>
</tbody>
</table>

| right side view |
Relative position of faces on side views. Present the structure at left and have students draw the front, top, and both side views on grid or dot paper.

ANSWER:

front view

right side view

left side view

top view

bottom view

Ask students to look at the top view. **ASK:** Where is the left side on this picture? (on the left) Write “left” on the left side of the view. Where is the right side? (on the right) Mark that on the picture. Where is the front on the top view? (at the bottom; mark it) Repeat with other sides and other views. The labelled side views will look like this:
Identifying structures from side views. Present the side views and the structures below and have students signal which of the structures has these side views.

**a)**  
- Top view
- Front view
- Right side view

**ANSWER:** Structure 2

**b)**  
- Top view
- Front view
- Right side view

**ANSWER:** Structure 3

Building structures from side views. Have students do Questions 1 through 3 on Workbook pages 188–189.

Before doing Question 4, divide students into two groups. Explain that you want to do an experiment to find out which method of constructing shapes works better. One group of students will build the structure starting from the largest view without thick lines—in this case, the front view. The other group will start from the smallest view without thick lines—the left side view. Have students suggest ways of deciding which method is faster. **ASK:** Should we wait till everybody does the work, and then take average? Should we take the average after discarding the best and the worst times in each group? Should we time the groups and see how many students built the structure correctly. Discuss as a class, then perform the experiment using the views at left. Which group was faster on average? (The group that starts with the larger view is likely to finish more quickly because although the initial shape is more complicated, it is still very simple, and they have fewer cubes to add to it afterwards than the other group has to add to their structure.)

Have students complete Workbook page 189.
Horizontal and vertical rotation. Explain to the students that in a plane there are only two ways to rotate a shape: clockwise and counter-clockwise. In three dimensions, there are more than two ways to do it. **ASK:** Where do you see rotations in real life? (**EXAMPLES:** wheels of a car, cogs in a motor, merry-go-round, fan) Explain that the rotation in a Ferris wheel is called **vertical rotation**, because the wheel moves in a vertical plane, and the rotation in a merry-go-round is called **horizontal rotation**, because a merry-go-round stays in the same horizontal plane. Have students sort the examples they named into horizontal and vertical rotation.

Give each student a die and ask them to place it so that 1 is at the front and 2 is at the top. Have students make a table showing which number appears on which face. Then ask them to perform a vertical clockwise rotation of 90° (show what you mean; leave the front face at the front) and to list which numbers appear on which faces again. **ASK:** Which faces stayed the same? Which faces changed? How did they change? Point out that you can perform a vertical rotation in different ways: you can leave the front side as the front side, or you can leave the left side and the right side as they are) Have students identify the changes in the positions of the faces under other rotations, horizontal and vertical. Students might find the die (and the charts they made) useful for completing Questions 1 and 2 on Workbook page 190.

When a shape rotates horizontally, the top view rotates as well. Give your students connecting cubes and have them perform Investigation 1 on Workbook page 190, first with the shape in the workbook, then with a more complicated shape, such as the top shape in the margin. Then ask students to construct the second shape in the margin from connecting cubes, and draw the top view. Ask them to predict the top views after various rotations, such as 90° clockwise, 180° counter-clockwise, 270° clockwise, and so on. After drawing the predicted shapes, have students check their predictions.
**ASK:** What will happen if you rotate the second shape 90° clockwise vertically? (Actually rotate an object this way as you ask the question, or use your finger to show the direction and size of the rotation—a quarter of a circle—on a picture or in the air.) Will the top side stay the stop side? (no) Which sides will stay the same side? (front side and back side) Will the front view stay the same? (no) How will it change? (it will turn 90° clockwise)

Have students draw the front view of the last shape to predict how it will change in a 90° counter-clockwise vertical rotation that leaves the front side the same. Have them verify their predictions. Repeat with a different vertical rotation, such as 180°.

**When a shape rotates horizontally, the side, front, and back views switch. ASK:** When you rotate a shape horizontally, the top face stays the top face, but it turns. Which other face stays in the same place? (bottom) What happens to the other faces? Have students investigate this question by filling out the chart used in Investigation 2 on Workbook page 191 for the second shape in the margin above. (Students will notice that the right side view and the left side view, if you disregard the thick lines, are reflections of each other. So are the front and the back views. This happens because you are looking at the same structure from opposite sides. When students compare the views of the same structure in different rows, they will notice that the views shift (including the thick lines). For example, a front view becomes a) a left side view after a 90° clockwise rotation, b) a back view after a 180° rotation, and c) a right side view after a 270° clockwise (or 90° counter-clockwise) rotation. So when you move from the second row to the third or from the third to the fourth, all the views are “shifted” one column to the left. When the structure is rotated counter-clockwise, each 90° rotation corresponds to a shift one column to the right."

**PROCESS ASSESSMENT**

8m6, 8m7, [C, V]

After doing the Investigation on Workbook p. 191, students can predict their answers for a different shape, such as the shape below.
A Doughnut
Polygons for Tessellating

[Diagram of polygon arrangements for tessellation]
Tessellating Monsters
Cards for Transformations
PS8-11 Choosing Strategies

Teach this lesson after: 8.2 Unit 7

Goals:
Students will choose between the strategies from previous lessons to solve problems.

Prior Knowledge Required:
Can apply the problem-solving strategies learned so far
Can apply the divisibility rules for 2, 3, 5 (for Problem Bank 1)
Knows that the angles in a regular \( n \)-gon are all equal (for Problem Banks 7–9)
Knows that the angles in any \( n \)-gon add to \( 180° \times (n - 2) \) (for Problem Banks 7–9)
Can apply the Pythagorean Theorem (for Problem Banks 13–17, 20, 21)
Can compute the area of a triangle (for Problem Bank 15)
Can apply the congruence rules for triangles (for Problem Banks 15, 19)
Can interpret percentage increase (for Problem Bank 17)
Can calculate the surface area of a cube (for Problem Bank 17)
Can identify a tessellation of the plane (for Problem Banks 22–27)

Materials:
BLM 1 cm Grid Paper (p. S-49, see Problem Bank 22)
BLM Isometric Grid Paper (p. S-50, see Problem Bank 22)

NOTE: The following Problem Bank questions reflect a selection of the problem-solving strategies used in the problem-solving lessons for Grade 8. Students will need to choose among all the strategies they have learned this year to solve these problems.

Problem Bank
1. The number 4333 is a product of a one-digit number and a three-digit number. What are the two numbers?
   Answer: Eliminate 2, 3, and 5 as prime factors using the divisibility rules. Then 7 must be a factor because it is the only one-digit prime number left. Divide to find the other number: \( 4333 \div 7 = 619 \). The two numbers are 7 and 619.

2. The number 37 is prime. Can you add a prime number to 37 to get another prime number? Explain how you know.
   Answer: No, because if you add 2, you get 39, which is \( 3 \times 13 \), and if you add any other prime number, you will get an even number as the sum.

3. The ratio of the three angles in a triangle is 6 : 7 : 7. What are the angles? Hint: Remember that the angles in a triangle add to 180°.
   Answer: 54°, 63°, 63°
4. Ava wrote \(1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times 50\) as a product of prime numbers.
   a) How many times did she write 2?
   b) How many times did she write 5?
   c) How many zeros does the number end with?

**Solutions:**

a) There are 25 multiples of 2 from 1 to 50: they are: 2, 4, 6, ..., 50. Take out one 2 for each of them: \(2 \times 2 \times \ldots \times 2 \times (1 \times 2 \times 3 \times \ldots \times 25)\). Now we have reduced the problem to finding the answer for 1 to 25 instead of 1 to 50. There are 12 multiples of 2 from 1 to 25, namely: 2, 4, 6, ..., 24. Again take out one 2 for each of them: \(2 \times 2 \times \ldots \times 2 \times (1 \times 2 \times 3 \times \ldots \times 12)\).

There are six multiples of 2 from 1 to 12, namely: 2, 4, 6, 8, 10, 12. Take out one 2 for each of them: \(2 \times 2 \times 2 \times 2 \times 2 \times 2 \times (1 \times 2 \times 3 \times 4 \times 5 \times 6)\). There are three multiples of 2 from 1 to 6, namely: 2, 4, and 6. Take out one for each of them: \(2 \times 2 \times 2 \times (1 \times 2 \times 3)\). There is one multiple of 2 from 1 to 3, namely 2. Adding together all the 2s we put in the prime factorization, we get 25 + 12 + 6 + 3 + 1 = 47.

b) The multiples of 5 from 1 to 50 are 5, 10, 15, 20, 25, 30, 35, 40, 45, 50. Each multiple of 5 has one 5, except 25 and 50, which have two 5s, so the total number of 5s is 1 + 1 + 1 + 2 + 1 + 1 + 1 + 2 = 12.

c) 12

5. On an analog clock, the hour hand is 10 cm long and the minute hand is 14 cm long. How many times faster is the tip of the minute hand moving than the tip of the hour hand?

**Hint:** How much distance does each cover in one hour?

**Solution:** The tip of the minute hand travels \(\frac{7}{5}\) of the distance in \(\frac{1}{60}\)th of the time as the tip of the hour hand does, so it is \(1.4 \times 60 = 84\) times faster.

6. Three years ago, Cathy's age was nine times as much as her nephew's age, but today Cathy's age is five times as much as her nephew's age. How old is Cathy?

**Answer:** 30

7. How many sides does a regular polygon have if each angle is …
   a) 179°   b) 178°   c) 177°   d) 176°   e) 175°   f) 174°

**Selected solution:**

c) \(180 \times (n - 2)/n = 177\)
\[180(n - 2) = 177n\]
\[180n - 360 = 177n\]
\[3n = 360\]
\[n = 120\]

**Answers:** a) 360, b) 180, d) 90, e) 72, f) 60

8. Can a regular polygon have angles equal to 173°? Explain how you know.

**Answer:** No, because if it did, you would have \(180 \times (n - 2)/n = 173\), which simplifies to \(7n = 360\), and \(n\) is not a whole number. But any polygon has to have a whole number of sides.
9. a) Define concave and convex polygons by using the examples in the following table. 
Hint: Look at the angles.

<table>
<thead>
<tr>
<th>Concave</th>
<th>Convex</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Concave polygon examples" /></td>
<td><img src="image2.png" alt="Convex polygon examples" /></td>
</tr>
</tbody>
</table>

b) In a convex polygon, create an exterior angle for each angle the same way as you would for a triangle. What is the sum of these external angles?
c) The exterior angles in a regular polygon are 10°. How many sides does the polygon have?

**Answers:**
a) convex polygons have all angles less than 180°, concave polygons do not;
b) 360°; c) 36

10. An isosceles triangle has side lengths 5 cm, 5 cm, and 6 cm. Another isosceles triangle has side lengths 5 cm, 5 cm, and 8 cm. Predict which triangle has greater area, then check your prediction.

**Answer:** Students are likely to predict that the triangle with sides 5, 5, and 8 is larger, but they both have the same area. Draw the height and use the Pythagorean Theorem. The height of the triangle with base 8 cm is 3 cm and the height of the triangle with base 6 cm is 4 cm. Both triangles have area 12 cm².

11. Which diagonal in the parallelogram is longer? Explain how you know.

**Answer:** Draw perpendiculares from the top vertices to the base and label the points as shown.

A E is the hypotenuse of the right triangle ADE, and B C is the hypotenuse of the right triangle BCF. Since C F is longer than D E and A D = B F, then B C is longer than A E.
12. In both pictures, the circle has radius 1 cm. Is a greater fraction of the circle grey or is a greater fraction of the square grey?

Answer: The fraction of the circle that is shaded is \( \frac{2}{\pi} \). The fraction of the square that is shaded is \( \frac{\pi}{4} \). The question is asking which is greater, \( \frac{2}{\pi} \) or \( \frac{\pi}{4} \)? Multiplying both numbers by \( 4\pi \), this is like asking which is greater, 8 or \( \pi^2 \), but 8 < \( \pi^2 \) because 3 < \( \pi \). So, in fact, 9 < \( \pi^2 \), so \( \frac{2}{\pi} < \frac{\pi}{4} \), which tells us that a greater fraction of the square is grey than of the circle.

13. The size of a TV screen is described in terms of the distance across its diagonal.
   a) A traditional TV screen has width to height ratio equal to 4 : 3. What is the width and height of a traditional TV screen with a diagonal measurement of 1.25 m?
   b) A wide-screen TV has width to height ratio equal to 16 to 9. What is the width and height of a wide-screen TV screen with a diagonal measurement of 1.25 m?
   c) The ideal viewing distance is four times the height of the screen. How far away would you sit for each TV in parts a) and b)?

Selected solution:

b)

By the Pythagorean Theorem, \((125)^2 = (16x)^2 + (9x)^2 = 256x^2 + 81x^2 = 337x^2\), so \(x^2 = \frac{15625}{337}\), and \(x \approx 6.81 \text{ cm}\). So the height is about 61 cm and the width is about 109 cm.

Answers: a) 1 m wide and 75 cm high, c) 3 m for the traditional TV screen and 2.44 m for the wide-screen TV

14. An overhang has its top edge 2 m above the ground and it extends 2.4 m from the side of the building. A ladder needs to reach the top of the building 6.8 m high. How long does the ladder have to be?

Solution: The bottom of the ladder is 3.4 m from the building, so the ladder needs to be \( \sqrt{(3.4)^2 + (6.8)^2} = \sqrt{57.8} \approx 7.60 \text{ m} \) long.
15. Squares are drawn on the sides of a right triangle. Show that all four triangles in this picture have the same area.

Solution: \( B \) and \( C \) are congruent, so they have the same area. Now, rotate \( B \) inside the square twice, as shown, and call the resulting triangles \( E \) and \( F \):

Triangles \( A \) and \( E \) have equal base and the same height, so they have equal areas. Triangles \( D \) and \( F \) have equal base and the same height, so they have equal areas.

16. a) Prove that the shape shown is a square.

b) How many squares could you draw on each size of dot paper?
   i) 3 by 3 dot paper  
   ii) 4 by 4 dot paper  
   iii) 5 by 5 dot paper

Hint: How many different sizes of squares can you draw? How many can you draw altogether?

Answers:
a) Draw four surrounding right triangles.
They are all congruent and the two acute angles add to 90°, so the angles in the shape are right angles \((180 - 90 = 90)\). The sides in the shape are all equal to the hypotenuse of the right triangle (or use SAS), so the shape is a square.

b) i) 3 different sizes

4 like this: 

1 like this: 

1 like this: 

There are 4 + 1 + 1 = 6 squares altogether.

ii) 5 different sizes

2 like this: 

9 like this: 

4 like this: 

1 like this: 

4 like this: 

2 like this: 

There are 9 + 4 + 1 + 4 + 2 = 20 squares altogether.

iii) 8 different sizes

16 like this: 

9 like this: 

4 like this: 

1 like this: 

9 like this: 

8 like this: 

2 like this: 

1 like this: 

There are 16 + 9 + 4 + 1 + 9 + 8 + 2 + 1 = 50 squares altogether.

17. Remember, a percentage increase is the percentage of the original number that the change in the amount represents.

a) An elastic is 10 cm long. When stretched, it becomes 13 cm. What was the percentage increase in length?

b) An elastic is 20 cm long. When stretched, its length increased by 50%. What is the new length of the elastic?

c) A square is cut in half across the diagonal. By what percentage did the total perimeter increase? Write your answer to the nearest whole number percentage.

d) A cube is cut in half vertically. By what percentage did the total surface area increase? Write your answer to the nearest whole number percentage.

Answers: a) 30%; b) 30 cm; c) If the side length is \(x\), the perimeter increased by \(2 \times \sqrt{2x^2}\). Since each hypotenuse has length \(\sqrt{2x^2}\), the original perimeter is \(4x\), so the increase, as a fraction of the original perimeter, is \(2\sqrt{2}/4 \approx 0.71\) or 71%; d) If the side length is \(x\), the original surface area is \(6x^2\), and the new surface area is \(8x^2\), so it increased by \(2x^2\), which is one third of the original value. The percentage increase is about 33%. 
18. Ronin left his bike at school for the weekend. On Monday, Ronin walked to school at a speed of 5 km/hr. After school, he biked home at a speed of 15 km/hr, taking the same route. Will his average speed be more than or less than 10 km/hr? Hint: Assume a distance, such as 5 km, and figure out the amount of time spent at each speed.

**Solution:** If the distance from home to school is 5 km, he took 1 hour to get to school and 20 minutes to get home. His average speed is \((\text{total distance}) ÷ (\text{total time}) = \frac{10 \text{ km}}{\frac{4}{3} \text{ hr}} = 7.5 \text{ km/hr}\). This is closer to 5 km/hr, which makes sense because he spent more time walking at 5 km/hr than he did biking at 15 km/hr.

19. In the star:
- \(\angle BEC = \angle EBD\)
- \(\angle ADB = \angle ACE\)
- \(BD = CE\)

Show that \(AC = AD\).

**Solution:** Label the unnamed corners of the star as follows:

\[\triangle BDF \cong \triangle ECG\] by ASA, so \(FD = GC\) and \(\angle EGC = \angle BFD\), since they are corresponding sides and angles of congruent triangles. Then \(\angle AFG = \angle AGF\), since they are supplementary to equal angles. Then \(\triangle AFG\) is isosceles and so \(AF = AG\). But \(GC = FD\), so \(AD = AC\).

20. A point is located inside a rectangle, and the distances from the point to each corner of the rectangle are measured. Going in clockwise order around the rectangle, the distances are \(a\), \(b\), \(c\), and \(d\).

a) Show that \(a^2 + c^2 = b^2 + d^2\).

b) If the point were located outside the rectangle instead, would the same result hold?
Answers: a) Draw a picture.

From the picture, \( a^2 + c^2 = m^2 + n^2 + p^2 + q^2 \) and \( b^2 + d^2 = n^2 + p^2 + m^2 + q^2 \), so indeed, \( a^2 + c^2 = b^2 + d^2 \).

b) yes

21. Can the quadrilateral exist? Hint: Divide the shape into a triangle and square.

Answer: Dividing the shape into a triangle and a square makes a right triangle with side lengths 4, 3.5, and 8, but these are not the sides of a right triangle (alternatively, \( 4 + 3.5 < 8 \), so the triangle does not exist), so the quadrilateral cannot exist.

NOTE: Students who have difficulty with Problem Bank 22 may benefit from drawing the shapes on grid paper (e.g., BLM 1 cm Grid Paper) for parts c) and e) or isometric grid paper (e.g., BLM Isometric Grid Paper) for part d) and cutting out five copies of each shape.

22. One way to create a tessellation from a shape is to make a larger copy from multiple copies of the shape. Example:

Show how to make a larger copy of the shape using four of the original shape.

a) an equilateral triangle  b) half a regular hexagon  c) this L-shape

d) an equilateral triangle with a rhombus attached  e) this L-shape
23. a) Decide how many equilateral triangles you would need to fill all the space around a common vertex if you start with …
   i) two regular hexagons
   ii) two squares
   iii) one square and one regular 12-gon
   iv) one regular octagon and one regular 24-gon
   v) one regular heptagon and one regular 42-gon
b) Draw a rough sketch of how triangles can tessellate with …
   i) regular hexagons
   ii) squares
   iii) squares and regular 12-gons

Selected solution: a) iii) The 12-gon has a 150° angle and the square has a 90° angle, leaving 120° to be filled, which is room for two 60° angles, so two equilateral triangles are needed.

Answers:
a) i) 2, ii) 3, iv) 1, v) 1
b) Sample drawings:

24. Can you use squares and regular hexagons to tessellate the plane? Explain how you know. Hint: How many of each would you need to put around a vertex?

Answer: If you use one hexagon, the squares would have to use $360 - 120 = 240°$, but this is not a multiple of 90°. If you use two hexagons, the squares would have to use $360 - 240 = 120°$, which is again not a multiple of 90°. If you use three hexagons, there is no room for any square, so hexagons and squares cannot tessellate the plane together (although each can tessellate the plane separately).

25. How could you use regular 12-gons and equilateral triangles to tessellate the plane? Hint: How many of each would you put around each vertex?
Answer: You could use two 12-gons and one triangle to make $360^\circ$, since each angle in the 12-gon is $150^\circ$ and $150 + 150 + 60 = 360$. So:

26. Show how to use regular hexagons, squares, and equilateral triangles to tessellate the plane.
Sample answer:

27. Show how to use regular hexagons, squares, and regular 12-gons to tessellate the plane.
Sample answer:
1 cm Grid Paper
Isometric Grid Paper
In this unit students will compare measures of central tendency and use them to compare and describe sets of data. They will also express probability as fractions, ratios, and percents, and find the probability of simple and compound events using various methods. Students will also conduct probability experiments and will determine the experimental probability of various events and compare it to theoretical probability. Students will investigate the effect of sample size on experiments and surveys and connect sample size to probability. They will also learn about applications of probability in real life.

**Meeting your Curriculum**

This unit contains some material studied in Grade 7, both for Ontario and students. The table below identifies which lessons are optional, which are review, and which cover core material. We recommend that you use the Workbook pages for lessons that are marked as review for review and assessment, and re-teach the material as needed.

For students following the WNCP curriculum, the material in lessons PDM8-15 through PDM8-17 was studied in Grade 7 and the material in lessons PDM8-25 through PDM8-28 will be studied in depth in Grade 9. These lessons are thus optional for WNCP students in Grade 8.

<table>
<thead>
<tr>
<th>Lesson</th>
<th>WNCP</th>
<th>Ontario</th>
</tr>
</thead>
<tbody>
<tr>
<td>PDM8-15</td>
<td>Review</td>
<td>optional</td>
</tr>
<tr>
<td>PDM8-16</td>
<td>Measures of central tendency</td>
<td>optional</td>
</tr>
<tr>
<td>PDM8-17</td>
<td>Measures of central tendency</td>
<td>optional</td>
</tr>
<tr>
<td>PDM8-18</td>
<td>Probability of simple events</td>
<td>review</td>
</tr>
<tr>
<td>PDM8-19</td>
<td>Expressing probability in different forms</td>
<td>review</td>
</tr>
<tr>
<td>PDM8-20</td>
<td>Tree diagrams</td>
<td>review</td>
</tr>
<tr>
<td>PDM8-21</td>
<td>Charts for sample space</td>
<td>review</td>
</tr>
<tr>
<td>PDM8-22</td>
<td>Compound events</td>
<td>core</td>
</tr>
<tr>
<td>PDM8-23</td>
<td>Probability of compound events</td>
<td>core</td>
</tr>
<tr>
<td>PDM8-24</td>
<td>Experimental probability</td>
<td>core</td>
</tr>
<tr>
<td>PDM8-25</td>
<td>Complementary events</td>
<td>optional</td>
</tr>
<tr>
<td>PDM8-26</td>
<td>Sample size in surveys and experiments</td>
<td>optional</td>
</tr>
<tr>
<td>PDM8-27</td>
<td>Bias in collecting data</td>
<td>optional</td>
</tr>
<tr>
<td>PDM8-28</td>
<td>Surveys and experiments</td>
<td>optional</td>
</tr>
</tbody>
</table>
**Goals**

Students will organize data using stem and leaf plots and find the range, median, mean, and mode of data sets.

**Prior Knowledge Required**

- Can order numbers from smallest to largest, including decimals
- Can add and subtract decimals and fractions
- Can multiply and divide decimals and fractions by whole numbers

**Materials**

- Counters

**Introduce the stem and leaf of a number.** The *leaf* of a number is its rightmost digit and the *stem* is the number formed by all the remaining digits. Point out that the stem is simply the number of tens in a number.

Ask volunteers to underline the leaf and circle the stem for various 2-, 3-, and 4-digit numbers. **Examples:** 25, 30, 230, 481, 643, 3210, 5403, 5430.

**Ask:** What should be the stem of a 1-digit number? (0, because the number of tens is 0) Students can underline the leaf but the 0 isn’t written, so there is nothing to circle. Have students identify the stems and leaves in more numbers, including 1-digit numbers.

Have students identify the pairs in each set that have the same stem:

- a) 45  46  63  
- b) 79  80  81  
- c) 435  475  431  
- d) 701  70  707  
- e) 642  642  649

**Bonus**

23  235  253  2530  2529

**Answers:**

- a) 45, 46
- b) 80, 81
- c) 435, 431
- d) 701, 707
- e) 642, 649

**Bonus**

235, 239

Ask students to put these numbers in order: 5, 19, 23, 90, 107, 86, 21, 45, 98, 102, 43.

Then, in the original list of numbers, demonstrate circling the stems and writing them in the order in which they appear: 0, 1, 2, 9, 10, 8, 4. When a stem repeats, such as the 2 in 23 and 21, don’t write it again. Have students put the stems in order. **Ask:** Which was easier, writing the numbers in order or writing the stems in order? (writing the stems in order) Why? (There are fewer numbers and the numbers are smaller.) When the numbers are in order, are the stems in order too? (yes)

**Introduce stem and leaf plots.** Show students how to make a stem and leaf plot of the original data:
STEP 1: Write the stems in order in the left column of a T-table.

STEP 2: Write each leaf in the second column in the same row as its stem. Add the leaves in the order in which they appear. For numbers that have the same stem, put the second leaf next to the first. Think aloud as you do the first few numbers, then have students help you do the rest.

STEP 3: Put the leaves in each row in order, from smallest to largest.

<table>
<thead>
<tr>
<th>STEP 1</th>
<th>STEP 2</th>
<th>STEP 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>stem</td>
<td>leaf</td>
<td>stem</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

Have students read the numbers from the finished plot. **ASK:** What are the numbers with stem 0? (just 5) What are the numbers with stem 1? (just 19) With stem 2? (21 and 23) Are these two numbers in order? Were they in order in the original list of data values? What did we do to make sure they would be in order in the stem and leaf plot? (We put the leaves in order. Because the numbers have the same stem [the same number of tens], the number with the larger leaf [more ones] is larger.)

Continue reading and writing all the numbers from the plot. Compare the complete set of numbers (now in order) to the original set (unordered). Explain that the numbers are now in order because we first put them in order according to the number of tens (stems) and then, within groups with the same number of tens, we ordered the ones. Ordering the numbers was easier because we did it in two smaller steps.

**Practise making stem and leaf plots. EXAMPLES:**

a) Time spent on homework (minutes): 43, 45, 86, 64, 61, 59, 58, 44

b) Monthly earnings from babysitting ($): 98, 107, 104, 93, 89, 111, 100, 95, 98

c) Distance run in a minute (m): 341, 326, 305, 333, 329, 341, 339, 327, 321, 312, 336, 340, 341, 338

**Compare stem and leaf plots to other types of graphs.** Explain that stem and leaf plots are used to organize numerical quantities related to individual items when the items themselves do not have a natural order to them. For example, air temperatures taken at different times of day have a natural order to them, but the boiling temperatures of various liquids do not; water does not come before oil in any natural sense. A stem and leaf plot rearranges the data so that it is in numerical order, making it easy to find the maximum value, minimum value, range, mode, and median. In which of the following situations would you use a stem and leaf plot?
a) running times of different people in a class
b) running times of one person over the course of a training program

**ANSWER:** a), not b)

**Introduce range.** Explain to students that the *range* is the difference between the largest and the smallest data values. Ask students to tell you the smallest and largest numbers from each of the stem and leaf plots they have drawn, and the range. **ASK:** Is it easier to find the range for an unorganized set of data or for data in a stem and leaf plot? Why?

**Introduce mode.** Write a stem and leaf plot where one number occurs twice (see margin). Have students identify the number that repeats. (109) Then write a stem and leaf plot for a set in which one number appears twice and another number appears three times. **EXAMPLE:** 28, 37, 29, 38, 37, 32, 40, 41, 44, 34, 41, 41. Have students identify the numbers that repeat. **ASK:** Which number appears most often? (41) Explain that this number—the value that occurs most often—is called the *mode.* Ask if anyone knows the French phrase “à la mode.” Tell students that it means in style or popular. Things that are popular occur often. Similarly, the mode is the most “popular” number in a set. **ASK:** Does every set have to have a mode? Remind students of earlier sets in which every value occurred only once—those had no mode.

**Introduce median.** Explain that the number in the middle of a set of data is called the *median.* Find the median of the following temperatures by first ordering them and then circling the one in the middle:

10, 11, 14, 12, 17, 15, 12

10, 11, 12, (12), 14, 15, 17

The number 12 is in the middle of the set; there are 3 numbers before it and 3 after it.

Now write this set of six temperatures: 10, 11, 12, 14, 15, 21.

**SAY:** When the number of terms in a set is even, there are two numbers in the middle and the median is the number halfway between them. The two numbers in the middle of this set are 12 and 14 (circle them) and the number halfway between 12 and 14 is 13. So the median is 13. Have students find the numbers halfway between 2 and 4, 6 and 10, 13 and 19, then 2 and 3, 12 and 15. Then have students find the sum of the two numbers.

**EXAMPLE:**

<table>
<thead>
<tr>
<th>stem</th>
<th>leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>9</td>
<td>1 8 9</td>
</tr>
<tr>
<td>10</td>
<td>0 2 9 9</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
</tr>
</tbody>
</table>
numbers in each case and compare it to the number halfway between the numbers they found. What do they notice? (the number in the middle is half the sum of the numbers) Have students use this method to find the number halfway between 12.4 and 12.6, 31.5 and 31.8, 216.45 and 217.91, 1.23 and 2.34. (ANSWERS: 12.5, 31.65, 217.63, 1.785)

**Why the number halfway between two numbers is half their sum.**

Give each pair of students an even number of counters, and ask them to divide the counters so that one person has a lot more than the other. Ask students to trade counters one at a time, until they have the same number of counters. Draw a model like the one below on the board to show what is happening. The dots on the number lines show the number of counters each partner has after each trade.

Point out that by trading counters students get closer to the number halfway between the two numbers they started with. **ASK:** How many counters do you have in total? (the sum of the numbers, 9 + 15 in this case) How can you find ahead of time the number of counters you will have after trading? (divide the sum by two)

Give students some unordered sets of data and ask them to order the sets first, then find the median. Include sets in which numbers repeat and sets with decimals. **EXAMPLES:**

a) 5, 3, 5, 5, 9, 7, 11  
b) 21.34, 54.32, 34.65, 33.78

**Introduce mean.** Give five volunteers tokens and write how many each has on the board (EXAMPLE: 5, 10, 7, 7, 6). Tell students that the volunteers need to share the tokens so that they all have the same number. Brainstorm ways to do that. Lead students to the following method: combine the tokens and divide by the number of people. Explain that mathematicians would say that the volunteers took the **mean**, or average, of the number of tokens. **ASK:** How can we find the mean of a set of numbers without using tokens? How many groups did we divide the tokens into? (5) How can we find that number from the set of numbers? (the number of groups is the number of data values) How many tokens do we have altogether? How can we find that total from the data values? (add them together) How can we find the number of tokens in each group if we know the total number of tokens and the number of groups? (divide)

**ASK:** What will the mean of a set of two numbers be? (half the sum between the two numbers, the median)
Write several sets of numbers on the board and have students practise finding the mean. **EXAMPLES:**

4, 7, 2, 1, 1 (mean 3); 10, 12, 3, 7 (mean 8)

**Bonus**

250, 258, 335, 266, 266, 269, 344 (mean 284)

Present a larger set of data (see the example below) and have students use a stem and leaf plot to organize it. Then ask students to find the mean, median, mode, and range of the set of data. **ASK:** Which of these values does the stem and leaf plot help you to find? (median, mode, and range)

**EXAMPLE:** Length of phone calls (in seconds) from a cell phone bill:

Answers Mean = about 52.89, median = 40, range = 231, mode = 3.

**Extension**

Stem and leaf plots can also be used to organize a set of decimals. You might use the whole part of the number as a stem and the decimal part as a leaf, but you need to keep in mind that since 64.32 is smaller than 64.5, the leaf .32 should come before the leaf .5.

a) Organize the following set of data into a stem and leaf plot using the whole part as a stem.

64.32  64.5  64.3  63.9  61.7  61.54  64.47  64.39  64.4  64.07

b) Why it does not make sense to take the last digit as a leaf and all the rest as a stem?

c) What could you do to be able to use the last digit as a leaf and the rest as a stem? Redraw the stem and leaf plot that way.

**ANSWERS:**

b) If we choose only the last digit as a leaf, the last digits will correspond to different place values, which would be confusing. Consider three numbers: 64.5, 64.47, and 64.07 What should come first as a stem: 64 or 64.0? 64 or 64.4?

c) Add zeros after the last decimal digit to make all the decimal parts the same length so that every leaf and stem have the same place values. See the new plot in margin.
Review mean, range, and mode. Tell your students that you are going on a trip, and you found out that the temperatures at your destination (in °C) are predicted to be 15, 14, 11, 12, 11, 10 and 18. Have students find the highest and the lowest temperatures and the range of the temperatures. **ASK:** Do you need to know all the temperatures to decide if you will need snow pants on your trip, or is knowing the lowest temperature enough? Do you need all the temperatures to decide if you will need a pair of shorts, or is knowing the highest enough? Suppose you knew only the range of temperatures at your destination (8°C). What does this mean? (The temperature doesn’t change very much; it’s nearly the same every day.) How can this help you decide what kinds of clothes to pack? (You might need either snow pants or shorts, but not both!) What if the range at your destination is 20°C? What does that tell you? What will you pack? (The temperature changes a lot. You will need clothes for very different temperatures.)

Have students find the mean of the set of temperatures. (13°C) If you know the range and the mean, but not the exact temperatures, do you know what clothes to pack? Will you need snow pants? (no) Will you need a bathing suit? (no) How do you know? (The temperatures are around 13°C. They cannot be below 5°C or above 21°C because the mean is 13°C—so 13°C is somewhere in the middle of the set—and the range is 8°C.) Repeat with the median and the mode of the set. (Given the range and the median/mode we know enough about the temperatures to choose the clothes because the range is small.)

**Outliers and measures of central tendency.** Present another set of temperatures at a different destination: 10, 11, 11, 12, 14, 15, and 32°C.
How are the sets the same? How are they different? (The sets of numbers are the same except for one number, the order differs, the largest number in the second set is larger than the largest number in the first set.) Remind students that any number that is very different in size from the other data values in a set is called an outlier. What is the outlier in this set? (32) Are the clothes that you should pack different? (yes, you will need something lighter for the last day) Ask students to find the mean, median, mode, and range of the new set. Which central tendency measures stayed the same? (median and mode) How did the range and the mean change? (both increased—the mean is now 15°C and the range is 22°C) Which measure of central tendency best reflects the change in the set? (mean) The range also strongly reflects the difference between the two sets.

This is a good time to do the Activity.

**How changing data values affects measures of central tendency and range.** Have students do Investigation 1 on Workbook page 194. Then ask students to find the mode, median, and range of the sets in part A of the Investigation. How does adding 1 to all the data values affect the median, the mode, and the range? (The mode and median increase by 1, but the range stays the same.) Ask students to explain why this happens. What will happen if they add 5 to all data values? Why?

Have students do Investigations 2 and 3 on Workbook pages 194–195. They can use the same structure as in Investigation 2 for Investigation 3.

**Finding mean, median, and mode from a circle graph.** Have students do Question 1 on Workbook page 195. Then ask students what the most common number of cars (per family) on the street is. (1) How do they see that on the circle graph? (It is the largest section) Tell students that you want to find the median of the set as well. Suggest that students look at the way Sally writes the number of cars in a family in order—first 10 zeros, then 25 ones, and finally 15 twos. How many data values do we have in total? (50) Will there be one number in the middle or two numbers? (two) Why? (the total number of data values is even) Between which two terms will the median be? (25th and 26th) How many cars does the 25th family have? (1) How many cars does the 26th family have? (1) So what is the median number of cars in a family?

Another way to visualize the median would be to pretend that all the data values are written in order along a line segment and to divide the line segment into fractions that represent each number:

```
0 1 2
```

The median is the midpoint of the line segment, and one can see from the picture that the median is in the “1 car” region.

Ask students to pretend that the line segment above is a string that is rolled into a circle, and the dividers between 0, 1, and 2 are joined to the centre of the circle. What will the result look like? (a circle graph) Draw the graph
at left to show the result. Where would the median be? (at the bottom of
the circle) Explain that since the numbers on the circle graph are going in
order, the median (or the 50% mark) can be found by extending the divider
between the last and the first numbers to a diameter. The extension of the
divider is in the “1 car” region, so the median of the set is 1.

Have students find the median of the following sets of data:

a) Number of siblings

b) Number of books read
   last month

c) Time spent on homework
   last week (hours)

d) Yearly salaries in a company
   ($1 000)

Present the graph at left and tell students that these are marks out of 50 on
a test. Say that you want to find the median. What mark should you extend
to a diameter? (the divider between 18 and 50) Does the divider fall into
a region? (no) What does this mean? (The median is between the values
27 and 37) What is the median of the set of data? (The number halfway
between 37 and 27, (37 + 27) ÷ 2 = 32)

Have students complete Question 2 on Workbook page 195 individually,
then check answers as a class.

Compare two sets using measures of central tendency and range.
Ask students to compare the measures of central tendency they got
in Questions 1 and 2 on Workbook page 195. The median is 1 in both
questions, the mode is also 1 in both groups surveyed, but the mean is
higher in the second group (1.1 in group 1, 1.25 in group 2). The range is
larger in group 2 (3 vs. 2 in group 1). ASK: What can you tell about the two
surveyed groups without looking at the data sets themselves? We cannot
see any difference from looking at the mode and the median alone, but
we see from the mean that there are more cars per family in the whole city.
Does this mean that there are more cars in the city? (no, because we do
not know what the size of the data set is) We also know that there is a
greater difference between the smallest and the largest number of cars
in the city because of the difference in the range. Can we tell from the
measures of central tendency alone which group has a greater fraction
of families with one car? With two cars? With zero cars? (no) Can we tell
that from a circle graph? (yes)

Creating sets with given measures of central tendency. Review with
students the fact that the mean and median are not necessarily members
of the set, but a mode, if it exists, is always a member of the set. Tell students that you want to create a set that has mode 3, median 4, and mean 5. Can the set have only two data values? Why not? (One answer could be that a mean and a median of a set of two numbers are equal, and $4 \neq 5$. Prompt students to think about how they find the median and the mean of a set of two numbers.) Could there be three data values in the set? To check, draw three blanks to fill in with the possible data values. If there are three numbers in the set, does the median have to be one of them? (yes) If we list the numbers in order, which one will be the median? (the number in the middle) Circle the blank in the middle and write 4 in this blank. If the mode is 3, how many data values have to be 3? (two) Write 3 in the other two blanks. **SAY:** So these are the data values in order. But they are not in order! This means you cannot make a set with three data values to fulfil the requirements.)

**ASK:** Could there be four data values in such a set? Draw four spaces for the data values. **ASK:** How many 3s does the set contain? (at least two) Could it be three 3? (no, then the median would be 3) Suppose we write the numbers from least to greatest. Can the two 3s go in the two middle spaces? Why not? (the median would be 3) Where should the 3s go? (in the first two spaces) If the median is 4, what should the third number be? (5) What should the fourth number be? How do you know? Discuss possible strategies. Make sure to discuss the algebraic solution, which is to denote the missing number by a variable, such as $x$, then write an equation for the mean and solve it: \((3 + 3 + 5 + x) \div 4 = 5\), so \(3 + 3 + 5 + x = 20\), and \(x = 20 - 11 = 9\). Write on the board: 3, 3, 5, 9. **ASK:** Are these numbers in order? (yes) What is the mean? (20 \(\div\) 4 = 5) What is the median? (4) So this is a set with mode 3, median 4, and mean 5.

**EXTRA PRACTICE:**

a) Find a set with six data values, mean 45, median 37, and mode 54.

b) Find a set with six data values, mean 4.5, median 3.7, and mode 5.4.

**SAMPLE ANSWERS:** a) 0, 12, 20, 54, 54, 130 b) 0, 1.2, 2, 5.4, 5.4, 13.

c) Explain why you cannot find a set of five data values with mean 4.5, median 3.7, and mode 5.4.

**SOLUTION:** Try to find the set as required. The median is 3.7 and there are 5 values in the set, so the set (written in order) is ____, ____, 3.7, ____, ____. Since the mode is 5.4, at least two data values should be 5.4, and they are the largest values, so the set is ____, ____, 3.7, 5.4, 5.4. The sum of the values should be \(5 \times 4.5 = 22.5\). So the remaining two values should add to \(22.5 - (3.7 + 5.4 + 5.4) = 8\). However, these values are the smallest in the set and should be smaller than 3.7, so they cannot add to a number that is more than 7.4. This means there is no such set.

Have students solve the following problem before they do Question 4 on Workbook page 195:
a) Find the mean and the median of these sets:

   Set A: 3, 5, 7       Set B: 3, 5, 7, 9, 11

b) Move one data value from Set A to Set B. Find the mean and the median of the new sets. Did the mean and the median of each set increase, decrease, or stay the same?

c) Repeat b) with a different data value from Set A. Is your answer the same as in b)?

d) Explain why the mean and the median of set B cannot increase if a data value from set A is added.

e) Moving which value from Set A to Set B will not affect the mean and the median of Set B? Will it affect the mean and the median of set A? How?

f) Moving which value from Set B to Set A will not affect the mean and the median of Set A? Will it affect the mean and the median of set B? How?

ANSWERS:

a) Set A: mean and median 5. Set B: mean and median 7.

b) Moving 3 will decrease the mean and the median of B and increase the mean and the median of A. Moving 5 will decrease the mean and the median of B, but will not change the mean and the median of A. Moving 7 will decrease the mean and the median of A, and will not affect the mean and the median of B.

c) No—see above.

d) The mean and the median of B are 7. All data values of A are at most 7. Adding a data value smaller than the mean or the median cannot increase the measures of central tendency.

e) Moving 7 will not affect the mean and the median of B, but it will decrease the mean and the median of A, because it is the largest value of the set, more than the mean and the median of A.

f) Moving 5 will not affect the mean and the median of A, but it will increase the mean and the median of B, because we remove a value that is less than the mean/median.

Average can mean different things. Explain to students that the mean, median, and mode are all called measures of central tendency, because they all give a value that is the middle, or centre, of a set in some sense. Explain that the word “average” might mean median, mean, or mode, depending on the situation, but it is most often used to denote the mean. Remind students that they have already seen sets where there is no mode. Can there be a set where there is no mean or no median? Tell students that you checked the eye colours of 15 people and found that 5 people have hazel eyes, 3 have black eyes, 4 have brown eyes, and 3 have blue eyes. Ask: Can we find the mean of this set? The median? Why not? (the data values are not numbers) Can we find a mode? (yes) What is the mode?
(hazel—the data value that appears the most often, the most common eye colour) If students can’t explain why there is no mean or median for the set of eye colours, ask them to think of survey responses. Point out that the data values in surveys are not always numbers. If the data values are not numbers, there is no mean and no median, so the mode will be the only valuable measure.

Here is a problem with numerical data values, describing a situation where the mean has no real meaning and the median does not have much meaning either.

**PROBLEM:** A store owner is deciding what sizes of shoes to order. He looks at his sales data.

<table>
<thead>
<tr>
<th>Shoe Size</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number Sold</td>
<td>2</td>
<td>12</td>
<td>35</td>
<td>28</td>
<td>20</td>
<td>9</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

a) How many shoes did he sell altogether? (111)
b) Which size did he sell the most of? (7) What is the mode of the set of shoe sizes sold? (7)
c) The number of data values above the median is the same as the number of data values below the median. How many data values should be below the median? (55)
d) Think of the data values as if they were all written out in order: 5, 5, 6, 6, 6, and so on. Can the size 6 be the median? Why not?
e) Can the size 7 be a median?
f) What is the median of the set of shoe sizes sold? (8) Can you buy a pair of shoes of this size? (yes)
g) The mean size of the shoes sold is about 7.9. Can you buy a pair of shoes of this size? (no)
h) To decide what size of shoes to order the most of, which is most relevant: the mean, median, or mode? Why? (the mode, the size that is sold the most often)

Then present several situations and have students discuss which measure of central tendency is most appropriate in each case and why.

• A retailer thinks about what size of pants to stock the most of. (mode—the most common size)

• A scientist studying obesity checks the average size of pants in Canada today and 10 years ago. (All measures of central tendency will make sense. Moreover, it might happen that there is no change in the mode but the mean and the median show changes. An increase in the mean or the median will mean the population is becoming more obese.)

• A mother wants to check whether her baby is taller or shorter than average. (She should use a median. Students might be interested
to see a growth chart for ages 0 to 19. Such charts can be obtained from the Web by searching for terms such as “height weight chart children.” The charts usually have a thick line in the middle that shows the median.)

**EXTRA PRACTICE:**

1. A karate school will allow you to pass the test for the next belt if you score more than 70 on at least half of the tests and have an average of at least 75.
   a) Which measures of central tendency are used here and what are their values? (median 70, mean 75)
   b) Is the third measure of central tendency useful here? Explain. (No, using the mode does not make sense. You can have scores such as 10, 11, 12, 80, 80, 21, and 17 (out of 100) and get a mode of 80, but this does not accurately reflect your overall performance.)

2. In a company there are 20 employees. One person has a yearly salary of $200 000, two people have salaries of $75 000, and all the rest have salaries of $17 500.
   a) What is the mean salary? The median salary? The mode salary? (mean $32 375, median and mode $17 500)
   b) The company advertises the average salary using the mean to try to find new employees. Explain why this is misleading. What would be a better choice? (The mean salary is almost twice as large as the salary of the majority of employees, due to the presence of the outlier. The median and mode better reflect the actual salaries.)
   c) For budget reasons, the company needs to write a report explaining the average money spent on wages per employee. Which average should the company use? Explain. (The mean—it takes all the values into account and all salaries should be shown in the budget.)

3. In a company there are 20 employees. One person has a salary of $300 000, two people earn $70 000, another one earns $5 000, and the rest have different salaries from $17 000 to $20 000, with no two salaries the same.
   a) Is there enough information to find the mean salary? The median salary? The mode of the salaries? The range of salaries? (not enough information for the mean and the median; the mode is $70 000, and the range is $295 000.)
   b) Which of the mean, the median, and the mode best reflects the salaries in the company? Explain. (The median. Most salaries are low, but the outlier will increase the mean significantly, making the salaries look larger than they are. The mode is also much higher than most of the salaries in the company.)
ACTIVITY

Show students two groups of four envelopes (Group A and Group B). Explain that in each envelope there is a cheque for a certain amount of “money.” One of the envelopes in one of the groups contains an outlier. A volunteer can pick one envelope at random from either group. The goal is to try and choose an envelope with the most money. Tell students that the mean of Group A is $328, the mean of Group B is $107, and the median for both groups is $106. **ASK:** From which group will you pick an envelope? Why? (Both groups have four envelopes, and both groups have median $106, so the median does not help to make the choice. The mean is larger in Group A, so choose from Group A.)

Show the contents of all of the envelopes:

**Group A:** $100, $104, $108, $1000
**Group B:** $100, $104, $108, $116

**ASK:** How are the sets different? (A has an outlier; A has a larger mean; the difference between the median and the mean is large for A and small for B.) What creates the big difference between the median and the mean in set A? (the outlier) Explain that when there is an outlier in a set, the mean and the median tend to be different, and the bigger the outlier, the larger this difference usually is. Moreover, when the median and the mean are very different, you can expect the set to have an outlier.

Present two more groups of four envelopes and provide the following data for each:

**Group A:** mean $778, median $1002
**Group B:** mean $993, median $994

**ASK:** From which group will you pick an envelope? (The difference between the medians is small, and the mean of Group A is much lower than the mean of Group B. Pick from Group B.) Ask students whether either of these groups could contain an outlier, and, if yes, which group it would be in. Is the outlier smaller than the other data values in the group or larger? (The outlier creates a difference between the median and the mean, so Group A is likely to contain an outlier. The mean is smaller than the median, so the outlier is smaller than the rest of the values in the set.) Show students the contents of the envelopes to check their guess:

**Group A:** $100, $1000, $1004, $1008
**Group B:** $984, $992, $996, $1000

You could also discuss with the students what could they do with the money in the envelopes. (E.g., donate to a charity. Which charity would students pick and why?)
Extensions

1. Find the mean. Write your answer as a mixed number.
   a) 3, 4, 6, 8  b) 4, 7, 8  c) 0, 3, 2, 7, 6  d) 1, 2, 5, 2, 6, 3
   **ANSWERS:** a) $5\frac{1}{4}$  b) $6\frac{1}{3}$  c) $3\frac{3}{5}$  d) $3\frac{1}{6}$

2. In a co-operative running meet, all participants’ times are averaged together for a total score. Individual scores are not tracked, to prevent participants from comparing results with each other. Participants try to improve by having a better average score in the next meet. But there is a twist: instead of taking the average as the total time divided by the number of runners, the average is
   \[
   \text{total time} \div (\text{# of people who actually run} - \text{# of people who could have run but didn't}).
   \]
   In a meet with 10 people, here are the 10 running times in seconds:
   
   20.8  21.2  21.6  20.9  21.0  23.6  25.4  21.7  20.3  22.3
   
   a) Find the mean score for the 10 runners. Is it the same as the “twisted” average?
   b) The slowest runner decides not to run next time in order to improve (i.e., lower) the team’s score. Is this likely to work? Explain.
   c) Why do you think co-operative running meets use this “twisted” average instead of the regular average?

   **ANSWERS:**
   a) 21.88 is both the regular mean and the “twisted” average, because all people present run.
   b) Let’s assume that the slowest runner hadn’t run in the first race. The “twisted” average for the remaining nine times is $193.4 \div (9 - 1) = 24.7$. The twisted average is higher (worse) if one of the people present does not run!
   c) The “twisted” average encourages all people present to run.
Review measuring length, mass, and temperature. Show students two pencils of different lengths. Ask students how they could determine which pencil is longer. Then present two measurements where direct comparison is impossible, such as the length of a ruler and the circumference of a cup. (Students might suggest using a measuring tape to compare them indirectly.) Ask students how they could compare the weight of two objects, say a book and a cup, or the temperature in two different places. Point out that in all cases they tried to attach a number to the characteristic or quantity and to compare the numbers. They used different tools—a measuring tape, a scale, or thermometer—to get a number, that is, a measurement.

Measuring likelihood. ASK: What would you do to compare the likelihood of two events, such as the likelihood of rolling 8 on a pair of dice and the likelihood that your favourite hockey team wins 5 to 3 in its next game? Is there a tool to measure likelihood? (no) Explain that probability is the branch of mathematics that studies the likelihood of events and expresses this likelihood in numbers. The measure of likelihood of an event is called probability of the event.

Outcomes. Hold up a die and ask students to predict what will happen when you roll it. ASK: Can it land on a vertex? On an edge? No, the die will land on one of its sides. Ask students to predict which number you will roll. Then roll the die (more than once, if necessary) to show that while it does land on a side, it doesn’t necessarily land on the number students predicted. Explain that the possible results of rolling the die are called outcomes, and to make predictions students must learn to identify which outcomes of various actions are more likely to happen and which are not. But first, they must learn to identify outcomes correctly.

Hold up a coin and ASK: What are the possible outcomes of tossing a coin? How many outcomes are there? Show a spinner with three equal but differently coloured regions and a set of marbles. What are the possible outcomes of spinning the spinner of picking a marble with your eyes closed?
Ask students to identify the possible outcomes of a soccer game. How many outcomes are there? (3 outcomes: team A wins, team B wins, a draw)

**Equally likely outcomes.** Draw two spinners as in the margin and ask students how they are the same and how they are different. Are you more likely to spin blue on one of the spinners than on the other? (no) Are you more likely to spin yellow on one of the spinners than on the other? (no) Why? (the spinners have identical colouring, the blue region is just split in two in one of the spinners) Ask students to identify the outcomes of each spinner. Point out that the spinner with three regions has three different outcomes, one for each region. On this spinner there are two different ways to spin blue.

Ask students to draw two spinners, each with six possible outcomes, such that one has equally likely outcomes and the other has outcomes that are not equally likely to happen.

**Events.** Explain that when you describe a specific outcome or set of outcomes, such as rolling a 6, rolling an even number, tossing a head, or spinning blue, you identify an event. Ask students to identify all nine possible outcomes in the following situation: Rina and Edith play Rock, Paper, Scissors. (You can use the Scribe, Stand, Share strategy to check students’ answers.) Then **ASK:** Is “Rina wins” an outcome or an event? (Students can signal their answer by making an O or an E with their fingers.) Ask students to explain their answer. (Rina wins is an event that consists of three possible outcomes: Rina has rock, Edith has scissors; Rina has paper, Edith has rock; Rina has scissors, Edith has paper.)

**Probability of events with equally likely outcomes.** Explain that in mathematics when the outcomes are equally likely, we describe the probability of an event as a fraction or a ratio:

\[
P(\text{Event } A) = \frac{\text{Number of outcomes that suit } A}{\text{Number of all outcomes}}\]

If the outcomes are not equally likely, this formula does not work, and we need to change the problem so that the outcomes will be equally likely (but we will deal with it later). For the blue and yellow spinner with three regions, above, there are three possible and equally likely outcomes, so the probability of spinning yellow is \(\frac{1}{3}\) or 1 : 3 and the probability of spinning blue is \(\frac{2}{3}\) or 2 : 3.

**ASK:** Which fraction of the spinner is coloured yellow? Which fraction of the spinner is coloured blue? What do you notice? (the fraction of the spinner that is a particular colour is the probability of spinning that colour)

Have students find the probability of events that consist of equally likely outcomes, as in Questions 6 and 7 on Workbook page 198.

**EXTRA PRACTICE:**
Find the probability of drawing one marble of each colour separately from this collection of 24 marbles: 12 red, 8 blue, 3 yellow and 1 white. What is the probability of drawing a marble that is not yellow? What is the probability of drawing a marble of a colour that is on the Canadian flag?
When events are not equally likely. Return to the two blue and yellow spinners shown earlier in the lesson. Ask students how they could find the probability of spinning blue or spinning yellow on the spinner with two unequal regions. Prompt students to think about whether the probability of spinning either blue or yellow will be different for the spinners. It is not, so can we use one of the spinners instead of the other to determine the probability? (yes)

Present the spinner at left and ask students how they could find the probability of spinning each colour on this spinner. **ASK:** Are the outcomes equally likely? (no) What could you do to make the outcomes equally likely? (split the large region into 4 equal regions that are the same as the other smaller regions) Have students do so and find the probability of spinning each colour.

Have students practise finding probabilities of events with outcomes that are not equally likely, as in Question 8 on Workbook page 198. Then have the students decide whether the outcomes of certain events are equally likely or not and find the probability of the events. **EXAMPLE:**

a) Are the outcomes of the spinners equally likely?

i) ii) iii)

b) Find the probability of each outcome in spinners in i) and iii).

c) Why is it hard to find the probability of each outcome on the spinner in ii)?

**EXTRA PRACTICE:**

Find the probability that a dart lands on each colour. Assume that a dart always lands on the board at left, with equal likelihood of landing in regions of equal area.

**Expressing probability in different ways.** Remind students that decimals, fractions, percents, and ratios are all numbers, and students have learned to convert between them. Use Workbook page 200 Question 1 to check students’ ability to identify various representations of the same number. As well, have students write several fractions as decimals and percents to check that they remember how to convert between representations.

**EXAMPLES:** \(\frac{1}{2} = 0.5 = 50\%; \frac{3}{4} = 0.75 = 75\%; \frac{3}{5} = 0.6 = 60\%; \frac{9}{10} = 0.9 = 90\%\)

Look at the definition of the probability of an event together (on Workbook page 198). **ASK:** What is larger, the number of all possible outcomes or the number of outcomes that suit an event? Is the fraction that you get more than 1 or less than 1? (less) Can it be 1? (yes) What do we call events with probability 1? (certain) Can it be 0? (yes) What do we call events with probability 0? (impossible) Can it be less than zero? (no) Ask students
to give examples of an event that has probability 1 and an event that has probability 0. Use Scribe, Stand, Share to check the answers.

**Applications of probability.** Ask students to think about when people use probability or probability-related knowledge in real life. Possible examples: playing the lottery, gambling, weather forecasting (e.g., probability of precipitation). Knowledge of probability also applies to any game that depends on chance, such as most card games. Probability is also used by scientists studying population, plant growth, genetics, and more.

**ACTIVITY**

**Using probability to decode messages.** The Roman emperor Julius Caesar developed a way to encode messages known as the **Caesar cipher**. His method was to shift the alphabet and replace each letter by its shifted letter. A shift of 3 letters would be:

\[
\begin{align*}
A &\rightarrow D \\
B &\rightarrow E \\
C &\rightarrow F \\
&\vdots \\
Z &\rightarrow C
\end{align*}
\]

Look at the following message: *What's gone with that boy, I wonder?* Using the shift of 3 letters, the message would begin: *Zkdw'v jrqh*

a) Finish encoding the message.

b) Tally the occurrence of each letter in the original message and in the encoded text.

| Letter | a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
| Original |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| Encoded |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |

c) Which letters occur most often in the original message? Which letters occur most often in the encoded message? How are these letters related?

d) In the English language, the letter that occurs most often is almost always “e.” Why did you not get “e” as the letter occurring most often in the original message? (the message is too short, the sample is not representative enough)

e) Use a Caesarean shift cipher encoder (find one on the Web by searching the keywords “Caesarean shift cipher encoder”) to decode and encode messages.

If you encode a message with a shift of 3, you will need to decode the message using a shift of \(26 - 3 = 23\) to get back your original message. What shift would you need to use to decode a message that was encoded using a shift of 7? A shift of 16? A shift of 24?

f) To decode the following message (available in text form on the JUMP website), first determine which letter occurs most often by using an online letter frequency counter (search for “online letter frequency counter”).
“ZUS!”
Tu gtyckx.
“ZUS!”
Tu gtyckx.
“Cngz’y mutk cozn zngz hue, O cutjkx? Eua ZUS!”
Tu gtyckx.
Znk urj rgje varrkj nkx yvkizgirkj juct gtj ruuqkj ubkx znks ghuaaz znk xuus; znkt ynk vaz znks av gtj ruuqkj uaz atjkx znks. Ynk ykrjus ux tkbkx ruuqkj znxuamn znks lux yu ysgrr g znotm gy g hue; znke ckxx nkx yzgzk vgox, znk vxojk ul nkx nkgxz, gtj cckk haoorz lux “yzerk,” tuz ykkboik – ynk iuarj ngbk ykkt znxuamn g vgox ul yzubk-rojy payz gy ckrr. Ynk ruuqkj vxxvrdkj lux g susktz, gtj znkt ygoj, tuz lokuikre, haz yzorr ruaj kwuamn lux znk laxtozak zu nkgx: “Ckrr, O rge ol O mkz nurj ul eua O’rr – “
Ynk joj tuz lotoyn, lux he znoy zosk ynk cyg hktjotm juct gtj vatinotm atjkx znk hkj cozn znk hxuus, gtj yu ynk tkkjkh hxkgzn zu vatizagzk znk vatinky cozn. Ynk xkyaxxkizkj tuznotm haz znk igz.
“O tkbkx joj ykk znk hkgz ul zngz hue!”

Which letter do you think represents the letter “e”? Why? (k, because that is the most common letter)

What shift do you think was used? (6-shift) What shift do you think will decode the message? (20-shift) Check your answer using an online Caesarean shift cipher encoder. (The decoded message is the beginning of The Adventures of Tom Sawyer by Mark Twain.)
Introduce tree diagrams. Explain to students that mathematicians often use tree diagrams when they have to make choices and want to keep track of all the possible combinations. For example, Katie has 3 pairs of mittens and 2 hats. How many different outfits can she wear? Show students how Katie can build a tree diagram to keep track of her choices (see margin).

The first two branches lead to the first choice: white hat or blue hat. Once she’s chosen a hat, Katie can choose from 3 pairs of mittens: green, white, or blue. Each path along the diagram is a different outfit. For example, the highlighted path shows a combination of a white hat and blue mittens.

The number of branches at the last level is the total number of possible combinations. In this case, Katie has a total of 6 different outfits.

Expand the tree diagram by adding a third choice: a scarf. Katie has 2 scarves, a white scarf and a green scarf. First she chooses the hat, then the mittens, and then the scarf. Add 2 new branches to each branch to show her third choice.

ASK: How many different combinations does Katie have? (12) How many of these combinations have 3 colours? (3) How many of the combinations have only 1 colour? (1) Two colours? (12 – 1 – 3 = 8) Let your students count the different paths to find the answers.

Explain to students that tree diagrams make it easy to see all the different outcomes of an action or series of actions, such as putting together an outfit. Each path in the diagram corresponds to one outcome (in this case, one outfit). When you know the total number of outcomes, you can quickly identify the number of suitable outcomes for a particular problem or question. SAY: Pretend that Katie chooses her clothes at random, so choosing an outfit becomes a probability experiment. Is it likely or unlikely that Katie will wear an outfit in two colours? (likely) Is it likely or unlikely that Katie’s outfit will be all white? (unlikely)

Combinations of experiments. Explain that we can think of choosing an outfit (or ordering dishes in a restaurant) as performing one or more experiments. Experiments can be associated with different events and have many outcomes. EXAMPLE:
experiment: roll 2 dice  
event: roll a total of 6  
outcomes: 3 3, 2 4, 4 2, 1 5, 5 1

Point out that in the example above the outcomes 2 4 and 4 2 are different. This is easy to see from the tree diagram. Draw the tree diagram and follow the paths for 2 4 and 4 2 (see the incomplete tree diagram in the margin). Point out as well that 3 3 occurs only once (the end of that path is circled), and even if we switch the numbers we do not change the outcome—it is the same.

Experiments can be performed separately or combined. For example, we can roll a die and spin a spinner at the same time; one outcome for this combination of experiments is rolling 3 and spinning red. In the first extra practice question, below, rolling each die is a separate experiment (even though we may have rolled the dice at the same time). Choosing each article of clothing when you choose an outfit is also a separate experiment, and we can use tree diagrams to find all the outcomes for the combination of experiments. The choices at each level are the outcomes of each individual experiment.

Students can practise finding all possible outcomes of combined experiments using Questions 1 through 3 on Workbook page 201.

**EXTRA PRACTICE:**

1. Draw a tree diagram to show the results of tossing a pair of regular dice. Is it likely or unlikely that one of the numbers is twice as large as the other?

2. A restaurant offers 3 main courses (chicken, fish, vegetarian) and 5 desserts (cake, ice cream, cookies, fruit, pie). Draw a tree diagram to show all the possible dining combinations.

**PROCESS EXPECTATION**

Connecting

**Connecting tree diagrams to probability.** Remind students that when outcomes are equally likely, the probability of an event is a ratio of the number of outcomes that suit the event to the total number of outcomes. 

**ASK:** How can we find the total number of outcomes from a tree diagram? (count the total number of paths in the diagram or the number of branches at the last level)

Ask students to complete a chart like the one in Question 4 on Workbook page 201 for all the tree diagrams they have made during the lesson. 

**ASK:** How can you find the total number of outcomes for a combination of experiments (i.e., the total number of paths in the diagram) from the number of outcomes for each individual experiment, (i.e., the number of branches at each level)? (multiply them) Why do you need to multiply the number of branches at each level? (each choice at one level has the same number of choices at the next level)

Ask students to check whether the rule they discovered works in a situation with three events, such as in the example of choosing a hat, mittens, and scarf. Then present several problems where it is hard to draw a tree diagram because the number of choices is too large, and ask students to
find the total number of outcomes using the rule. **EXAMPLES:**

1. Rolling two dice with 12 faces each. \((12 \times 12 = 144\) outcomes\)

2. Pulling a card from a deck of 52 cards and spinning a 3-part spinner. \((52 \times 3 = 156\) outcomes\)

3. Rolling three regular dice. \((6 \times 6 \times 6 = 216\) outcomes\)

**Bonus**

Tossing 10 coins \((2 \times 2 \times \ldots \times 2\) (ten 2s) \(= 2^{10} = 1024\) outcomes\)

**Extensions**

1. The colour of a flower is determined by its genes. A red rose, for example, has two \(R\) genes, a white rose has two \(r\) genes and a pink rose has one \(R\) gene and one \(r\) gene.

   If two pink roses crossbreed, the "child rose" can be red, white, or pink. The possible results are shown below.

<table>
<thead>
<tr>
<th>Gene from parent 1</th>
<th>Gene from parent 2</th>
<th>Resulting genes</th>
<th>Colour</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R)</td>
<td>(R)</td>
<td>(RR)</td>
<td>red</td>
</tr>
<tr>
<td>(r)</td>
<td>(R)</td>
<td>(Rr)</td>
<td>pink</td>
</tr>
<tr>
<td>(R)</td>
<td>(r)</td>
<td>(Rr)</td>
<td>pink</td>
</tr>
<tr>
<td>(r)</td>
<td>(r)</td>
<td>(rr)</td>
<td>white</td>
</tr>
</tbody>
</table>

   a) When two pink roses crossbreed, what is the probability the resulting rose will be pink? (50%)

   b) When a pink rose crossbreeds with a red rose, what is the probability the resulting rose will be red? Pink? White?

   **SOLUTION:**

<table>
<thead>
<tr>
<th>Gene from parent 1</th>
<th>Gene from parent 2</th>
<th>Resulting genes</th>
<th>Colour</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R)</td>
<td>(R)</td>
<td>(RR)</td>
<td>red</td>
</tr>
<tr>
<td>(R)</td>
<td>(R)</td>
<td>(RR)</td>
<td>red</td>
</tr>
<tr>
<td>(R)</td>
<td>(r)</td>
<td>(rR)</td>
<td>pink</td>
</tr>
<tr>
<td>(R)</td>
<td>(r)</td>
<td>(rR)</td>
<td>pink</td>
</tr>
</tbody>
</table>

   \(P(red) = 50\%, \ P(pink) = 50\%, \ P(white) = 0\)

   c) When a pink rose crossbreeds with a white rose, what is the probability that the resulting rose will be pink? (50%) Red? (0) White? (50%)

   d) What happens when a red rose crossbreeds with a white rose? (The child rose is always pink.)
2. A certain type of antigen called the Rh factor can be present or absent in the blood of each person. If you have the Rh factor, your blood type is Rh\(^+\); if you don’t, your blood type is Rh\(-\). You inherit your blood type from your parents. Each person can have one of three combinations of genes: ++, +−, or −−. If a + gene comes from one (or both) parents, a baby will have Rh\(^+\) blood. The baby will only have Rh\(^−\) blood if both parents are Rh\(^−\).

If a pregnant woman has Rh\(^−\) blood and her baby has Rh\(^+\) blood, the baby can develop severe health problems. Scientists have found that certain medications can help to solve these problems if the mother takes them during pregnancy.

a) If both parents have the gene combination +− (and so have Rh\(^+\) blood), what are the chances that their baby will have Rh\(^−\) blood (the gene combination −−)?

b) A father has Rh\(^+\) blood and a mother has Rh\(^−\) blood. What are the chances that their baby has Rh\(^+\) blood? Take both possible combinations of genes for the father into account.

**SOLUTION:**

a) Father’s gene     Mother’s gene     Baby’s genes     Baby’s blood type

| + | − | + | ++ | Rh\(^+\) |
| − | + | − | +− | Rh\(^+\) |
| − | − | + | −+ | Rh\(^+\) |
| − | − | − | −− | Rh\(^−\) |

The chances the baby will have Rh\(^−\) blood type are 1 out of 4, or 25%.

b) Father can have genes ++ or +−. Mother can have only −−.

| Father’s gene     Mother’s gene     Baby’s genes     Baby’s blood type |

| Father can be      |
| +− | − | + | −− | Rh\(^+\) |
| +− | − | − | +− | Rh\(^−\) |
| +− | − | + | −+ | Rh\(^+\) |
| +− | − | − | −− | Rh\(^−\) |

The chances the baby will have Rh\(^+\) blood type are 6 out of 8, or 75%.
Introduction.

Remind students that in the previous lesson they used tree diagrams to keep track of all possible outcomes of several experiments or a number of choices. Ask students to think about the convenience of using a tree diagram to figure out all possible outcomes of two experiments such as, say, rolling a 12-sided die and a 6-sided die. Demonstrate trying to draw the tree diagram to show how crowded it is. Ask: Why will the tree diagram be inconvenient? Tell students that another way to keep track of the outcomes for two or more experiments is to use a chart.

Show students the spinners at left. Draw a two-column chart with headings Colour Spinner and Number Spinner.

Spin the colour spinner and list its outcome in the appropriate column. Say: I am going to spin the second spinner now. What are the possible outcomes? List all the possible outcomes of the second spinner and fill in the colour you spun on the colour spinner next to each one. If you spun “red” on the colour spinner, your chart will look like the chart in the margin.

Say: I am going to spin the colour spinner again. Suppose I get a different colour. What are the possible outcomes of the number spinner? Are they different from the previous spin? (No, we have the same three outcomes: 1, 2, 3.) Continue filling in the table for different outcomes. How many outcomes are there in total? (12)

Ask: How many times did we write each outcome of the first spinner? (3) How is this number related to the outcomes of the second spinner? (it is the same as the number of possible outcomes of the second spinner) If the second spinner had 5 numbers, how many times would you write each outcome of the first spinner? (5)

Return to the example of the two dice above. Start drawing the chart using the 12-sided die as the first die. Ask: How many times do I need to write the number 1? (6 times) Why? (because there are 6 outcomes for the second die) Write the first 12 rows of the chart, then point out that though writing all the possible outcomes still takes up a lot of space, at least the rows of the chart do not crowd each other the way the branches...
of a tree diagram do. **ASK:** How many rows will my chart have (without the headings)? \((12 \times 6 = 72\) rows) How do you know? (There are 6 rows for each number 1 to 12.)

**Practice finding the sample space.** Review the steps for systematically listing all of the different outcomes (the sample space) for two experiments as on Workbook page 202. Have students create more charts to answer questions such as these:

a) Find all possible outcomes of rolling a regular die and spinning a spinner with three colours.

b) Find all possible outcomes of rolling two dice, one with eight sides marked 1 to 8, and another with four sides marked 1 to 4.

c) Find all possible outcomes of flipping two coins.

d) Find all possible outcomes of Rock, Paper, Scissors for two players. (Remind students that “I win” is not an outcome, it is an event! It can happen in three different ways: I have rock, you have scissors; I have scissors, you have paper; I have paper, you have rock.)

**PROCESS ASSESSMENT**

8m2, [R, C] Workbook Question 8c)

Have students decide whether certain events related to the charts they drew are likely or unlikely. **EXAMPLES:** Rolling 3 in b) (unlikely, 2 outcomes out of 32), flipping at least one head in c) (likely, 3 out of 4), John winning when John and Bob play Rock, Paper, Scissors (unlikely, 3 out of 9).
Compound events. Show your students three playing cards, two red (hearts or diamonds) and one black (spades or clubs). Explain that you are planning two probability experiments. For Experiment A, you will draw a card (without looking), write down the colour, and put the card back in. Then you will draw another card and write down the colour. The result of your experiment is two colours. Discuss with students what the possible results of the experiment are. (RR, RB, BR, BB) Write the rules for Experiment A on the board and the possible results for it.

For Experiment B, you will draw a card, write down the colour, and leave it out of the deck. Then you will draw another card and write down the colour. The result of Experiment B will be two colours as well. Discuss the possible results. (RR, RB, BR) Can you get two black cards? Why not? (there is only one black card, so if it is out, you can only get a red card on the second draw) Write the rules and results for Experiment B on the board as well.

Explain to students that these experiments actually consist of two separate experiments: two drawings of a card from a deck. We call such experiments compound experiments. Events that are combinations of outcomes from a compound experiment are called compound events. Today students will learn to find the probability of compound events.

Independent events. Point out that each outcome of a compound event consists of two parts, each related to a different experiment (e.g., rolling a die and spinning a spinner). So we can look at each compound event as
consisting of two different events too. Explain that two parts of a compound event are called independent events if the results of the experiments do not depend on each other. For example, if I roll a die two times, the number I roll the first time doesn’t affect the number I will roll the second time; the die rolls are independent. Are the draws of cards in Experiments A and B independent? Have students signal the answer for each experiment separately, then ask them to explain their answer. (Yes for A and no for B; in B, the possible results for the second round depend on the card drawn in the first round, but in A, you have the same deck regardless of the results of the first round.) Present several compound events and have students identify the sub-events in each. Then ask students to decide whether the events within each compound event are independent or not. Students can signal the answers again, and explain their thinking afterwards. **EXAMPLES:**

a) Rolling two dice. (independent)

b) Roll a regular die. If the number is even, roll it again. If it is odd, roll a 12-sided die. (dependent)

c) Draw two cards from a deck, one after the other, without replacing. (dependent)

d) Draw two cards from a deck, one after the other, replacing the card in the deck after the first draw. (independent)

**Bonus**
Use 3 regular dice of different colours (say, red, white, and blue). Roll the red die and record the number. If you roll a number that is 3 or less, roll the white die and record the number (but not the colour). If you roll a number that is 4 or more, roll the blue die and record the number (but not the colour). (Since you do not record the colour and the dice are identical, the results of the second roll do not depend on the results of the first roll. The events are independent.)

**Find the theoretical probability of both experiments.** Ask students to draw tree diagrams to show the possible outcomes for card experiments A and B (see answer in margin). Ask students to circle the ends of the paths for all the outcomes that produce two red cards (see answer in margin). What is the probability of RR in each experiment? (4 : 9 for Experiment A, 1 : 3 for Experiment B)

**EXTRA PRACTICE:** Students can use the following questions to practise calculating the probabilities of compound events using tree diagrams, lists of outcomes, or charts. Encourage students to write the answers using fractions, ratios, and percents.

1. Jennifer rolls two dice, one red and one blue. She uses the result of the red die as the numerator of a fraction and the result of the blue die as the denominator. Have students list all possible combinations. Get them started:

<table>
<thead>
<tr>
<th>Red</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Have students find:

a) the probability of rolling a fraction already reduced to lowest terms.
b) the probability of rolling a fraction greater than 1.
c) the probability of rolling a fraction less than 1.

**ANSWERS:**
a) \(\frac{23}{36}\)  
b) \(\frac{5}{12}\)  
c) \(\frac{5}{12}\)

2. Samantha tosses 3 coins: 2 pennies and 1 nickel. Use a tree diagram to find

a) the probability of tossing 2 heads and 1 tail.
b) the probability of tossing 2 tails and 1 head. Compare this answer to your answer in a).
c) the probability of tossing tails on coins with total denomination of 6 cents

**ANSWERS:**
a) \(\frac{3}{8}\)  
b) \(\frac{3}{8}\)  
c) \(\frac{1}{4}\)

After students solve this problem, discuss why the answers in a) and b) are the same. Tossing two tails and one head is exactly the opposite of tossing two heads and one tail. To see this, suppose that you tossed two tails and one head some way. If you turn the coins over, you see that the coins that showed tails now show heads and those that showed heads now show tails. This means that the number of outcomes suiting a) is the same as the number of outcomes suiting b). As well, discuss whether the answers to a) and b) would change if the coins were the same—if you were tossing, say, three quarters instead of two pennies and a nickel. Since questions a) and b) do not involve the denominations of the coins, the answers should stay the same.

**Develop the formula for theoretical probability of compound events with independent components.** Explain to students that it is inconvenient to draw a tree diagram or to count combinations every time you need to find the probability of a compound event. However, when the events are independent, there is a simpler way to find the probability. For that you will need to look at the probabilities of the independent events that make up the compound event.

Review multiplying fractions before assigning Investigation 1 on Workbook pages 204–205. Instruct students to use the fractional form for probability, because this will make seeing the pattern in the answers easier. Discuss the results as a class. Then have students find the probability of drawing each colour playing card in one round of Experiment A and check that the formula \(P(A \text{ and } B) = P(A) \times P(B)\) works for Experiment A.

Point out that just multiplying the probability of drawing a red card out of three cards by itself gives you the probability of drawing two red cards in Experiment A, but not in Experiment B. What is \(P(R) \times P(R)\)? \(\left(\frac{2}{3}\times\frac{2}{3} = \frac{4}{9}\right)\)

Is that the same as probability of drawing two red cards in Experiment B? (no) The answer cannot be the same because the probability of drawing a
red card in the second part of Experiment B depends on the result of the first part of the experiment.

**EXTRA PRACTICE:**

Refer back to extra practice problem 1, above (Jennifer’s experiment). What is the probability of rolling $\frac{5}{6}$? Can you apply the formula for the probability of compound events? Explain. **ANSWER:** You have to roll 5 on the red die and 6 on the blue die. The rolls are independent, and the formula for the probability of compound events applies. The probability for each of the rolls is $\frac{1}{6}$, so $P(\frac{5}{6}) = \frac{1}{36}$.

**Extensions**

1. **What is the probability of winning Lotto 6/49?**

   In the lottery game Lotto 6/49, balls numbered 1 to 49 are randomly mixed in a machine. Six balls are then chosen at random from the machine. Balls are not put back, so the same number cannot be chosen twice. Players pick their own numbers on lottery tickets, hoping to match those that come out of the machine. Even if more tickets are sold than the number of different combinations possible, it’s possible that no one wins because many people might have picked the same numbers. The probability of winning doesn’t depend on how many tickets are sold, but the amount you win does. This is the opposite of a raffle, where the probability of winning does depend on the number of tickets sold (the more tickets sold, the lower your probability of winning).

   a) If you pick 6 different numbers from 1 to 49 in Lotto 6/49, how do you think a game called Lotto 2/7 would work? (pick 2 different numbers from 1 to 7)

   b) The only possible combination in Lotto 2/2 is 1 2. Lotto 2/3 has 3 possible combinations: 1 2, 1 3, 2 3. Use an organized list to find the possible combinations for:

   Lotto 2/4   Lotto 2/5   Lotto 2/6   Lotto 2/7

   **HINT:** Start by listing all the combinations that start with 1, then the remaining combinations that start with 2, and so on.

   c) Complete the following chart:

<table>
<thead>
<tr>
<th>$N$</th>
<th>Lotto</th>
<th>Number of combinations</th>
<th>Probability of winning</th>
<th>$N \times (N - 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2/2</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2/3</td>
<td>3</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>2/4</td>
<td>6</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>2/5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
d) By comparing the last two columns of the chart in c), write down a rule for the probability of winning the Lotto 2/N in terms of \(N\) (where \(N\) is the numbers you have to choose from). (The number of combinations is given by the formula \(N \times (N - 1) / 2\), and the probability of winning is \(\frac{2}{N(N - 1)}\).)

e) Find the probability of winning
i) Lotto 2/10  ii) Lotto 2/49

**ANSWERS:** i) 1/45  ii) \(1/(49 \times 24) = 1/1176\)

f) Mathematicians have shown that the chances of winning Lottos 2/49, 3/49, and 4/49 are as shown in margin.

Extend the pattern to predict the chances of winning Lotto 6/49.

\[
\left( \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{49 \times 48 \times 47 \times 46 \times 45 \times 44} \right)
\]

g) Check www.lotterycanada.com to find the odds of winning Lotto 6/49. Does your prediction match what is published on the website? If not, check your calculation.

Students can now try to answer one of the following questions and write a concluding report describing their position.

i) What does it cost to play one combination of numbers in Lotto 6/49? How much money do you need to buy all possible combinations so as to ensure that you win?

Assume it takes 5 seconds to mark each combination on a lottery ticket. How much time will it take you to mark all the possible combinations? Assuming you spend 12 hours a day marking lottery tickets, how many days will it take? Will you be done in 3 days? In a week?

Assume you have enough time and money to buy all the possible combinations (e.g., you get a computer to help you mark all the combinations quickly), guaranteeing that you will win. Other people can buy tickets too, and some of them might win as well, in which case the jackpot will be divided among all the winners. If the jackpot is $30 000 000, would you buy all the possible combinations? Why or why not?

**ANSWER:** At the time of writing, the price of one game is $2, so you will need \(13 983 816 \times $2 = $27 967 632\) to buy all the possible combinations. It would take about 19 432 hours to mark all the combinations by hand (5 seconds per mark), which is about 1 618.5 days. This works out to over four years if you do not take any vacations and work 12 hours a day.

Even if you manage to buy all the combinations, and you are the only winner, your gain would be about $2 000 000 (jackpot – amount spent to buy tickets). However, there is a chance that someone else will win as well, in which case the jackpot is split...
between you and the other person. If that happens, you would win only $15 000 000 after spending about $28 million to buy tickets, so the loss is large. The business is too risky…

ii) Bill buys a lottery ticket every week for 25 years. How many tickets does he buy in total?

Bill’s chances of winning the big jackpot at least once in 25 years are about 1 : 10 000. Assuming the price of a lottery ticket does not change, how much money does he spend on lottery tickets in 25 years?

Joanna saves $2 per week for 25 years. She can invest the money different ways. Use the Internet to find the interest rate for any three ways Joanna could invest her money. Use an online compound interest calculator to see how much money she will have in total after 25 years, depending on the way she invested her money.

Would you rather try to win a lottery or invest your money?

ANSWER: Students should be able to see that any way of investing money is more profitable than trying to win a lottery.

iii) Tom buys a lottery ticket twice every week for 25 years. How many tickets does he buy in total? Assuming the price of lottery tickets does not change, how much money does he spend on lottery tickets in 25 years?

Check the website http://www.lotterycanada.com to see the odds of winning each type of prize. About how many times during these years could Tom win each prize?

For each prize, use the payout information from the last week to see how much money Tom could win during all these years. Did he get all the money he spent back in prizes?

Is it worth Tom’s money to buy lottery tickets?

PARTIAL ANSWERS: In 25 years, Tom bought 2 600 tickets at a total cost of $5 200.

<table>
<thead>
<tr>
<th>Numbers matched</th>
<th>Probability of winning</th>
<th>Number of expected wins</th>
<th>Sample prize size</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 out of 6</td>
<td>1 : 13 983 816</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5 out of 6</td>
<td>1 : 2 330 636</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5 out of 6 + Bonus</td>
<td>1 : 55 492</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4 out of 6</td>
<td>1 : 1 033</td>
<td>3</td>
<td>$75.40</td>
</tr>
<tr>
<td>3 out of 6</td>
<td>1 : 57</td>
<td>46</td>
<td>$10</td>
</tr>
<tr>
<td>2 out of 6 + Bonus</td>
<td>1 : 81</td>
<td>32</td>
<td>$5</td>
</tr>
</tbody>
</table>

Total Tom’s winnings $846.2

NOTE: Many people understand that their chances of winning the big jackpot in a lottery are very low, but they still play in the hopes...
of winning “something.” The calculations above show that this “something” is far less than what people may spend to buy tickets.

2. a) Chris has a box with 3 red marbles, 2 purple marbles, and 1 brown marble. Ashley has a box with 100 red marbles, 2 purple marbles, and 1 brown marble. Chris and Ashley each pick two marbles from their box, blindly. Without finding the exact probability, say who has a greater chance of picking two red marbles. (Ashley has mostly red marbles in her box, so her chances of picking two red marbles are greater than Chris’s.)

b) Ashley thinks that the equally likely outcomes of picking the two marbles in both experiments are:

- two red
- two purple
- one red and one purple
- one red and one brown
- one purple and one brown

This means the probability of picking two red marbles is 0.2 in both experiments. Is Ashley correct? Explain. (Ashley is wrong. There are many ways to pick two red marbles in both experiments, with many more ways to do so in Ashley’s experiment, so these outcomes are not equally likely.)

c) Is picking two marbles simultaneously the same as

- picking a marble, listing the colour, replacing the marble in the box, and taking out another marble? (no)
- picking a marble, leaving it out of the box, and taking out another marble? (yes)

d) Label the red marbles in Chris’s box R1, R2, and R3, the purple marbles P1 and P2, and the brown marble B. List all the possible ways to draw two marbles (at the same time) from Chris’s box. HINT: Use your answer in c).  ANSWER:

<table>
<thead>
<tr>
<th>First marble</th>
<th>R1</th>
<th>R1</th>
<th>R1</th>
<th>R1</th>
<th>R1</th>
<th>R2</th>
<th>R2</th>
<th>R2</th>
<th>R2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Second marble</td>
<td>R2</td>
<td>R3</td>
<td>P1</td>
<td>P2</td>
<td>B</td>
<td>R1</td>
<td>R3</td>
<td>P1</td>
<td>P2</td>
</tr>
<tr>
<td>First marble</td>
<td>R3</td>
<td>R3</td>
<td>R3</td>
<td>R3</td>
<td>R3</td>
<td>P1</td>
<td>P1</td>
<td>P1</td>
<td>P1</td>
</tr>
<tr>
<td>Second marble</td>
<td>R1</td>
<td>R2</td>
<td>P1</td>
<td>P2</td>
<td>B</td>
<td>R1</td>
<td>R3</td>
<td>P2</td>
<td>P2</td>
</tr>
<tr>
<td>First marble</td>
<td>P2</td>
<td>P2</td>
<td>P2</td>
<td>P2</td>
<td>P2</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>Second marble</td>
<td>R1</td>
<td>R2</td>
<td>R3</td>
<td>P1</td>
<td>B</td>
<td>R1</td>
<td>R2</td>
<td>R3</td>
<td>P1</td>
</tr>
</tbody>
</table>

e) Find the probability of picking these combinations from Chris’s box:

i) two marbles of the same colour ii) two marbles that are not red

ANSWERS:

i) \( \frac{8}{30} = \frac{4}{15} \)  ii) \( \frac{6}{30} = \frac{1}{5} \)
Review Experiments A and B from the previous lesson. Review how the cards were drawn in each case and what the theoretical probability of drawing two red cards is (4/9 in Experiment A and 1/3 in Experiment B). Write the theoretical probabilities as a decimal and a percent (approximately 0.44 and approximately 44.4% for Experiment A; approximately 0.33 and approximately 33.3% for Experiment B).

Perform Experiments A and B to see how often you actually draw two red cards. Divide students into two groups, one group to perform Experiment A and the other to perform Experiment B. Have students work in pairs: one partner shuffles the cards, the other draws and records the answers (as R and B). Partners switch roles after performing the experiment five times. Have students perform the experiment 25 times, tally the results, and fill in the first five columns of the table below:

<table>
<thead>
<tr>
<th>Number of trials</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of times two red cards were drawn</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then ask students to form groups of four and combine their results to fill in the last column of the table.

Introduce experimental probability. Explain that the experimental probability of an event A is a ratio, often expressed as a fraction:

\[
\text{Exp P(Event A)} = \frac{\text{Number of times that A happened}}{\text{Number of all experiments performed}}
\]
Have students add a row with the heading Exp P(RR) to the table above, find the experimental probability of RR for each multiple of five trials, and fill in that row. Finally, combine the numerical results of the whole class for each of Experiment A and Experiment B and have students find the experimental probability of RR for both.

**Compare theoretical and experimental probabilities.** Explain that you want students to graph the results of the experiment they performed using a line graph. Have students copy the template below and explain the features on each axis. Why does the vertical axis go from 0 to 1? (Probability has values between 0 and 1) Why is there a zigzag on the horizontal axis? (The number of trials for the whole class would be so large compared to 50 that using a continuous scale on the horizontal axis would make it too large to fit on the page.) Point out as well that students will plot points for 5, 10, 15, 20, 25, and 50 trials, but not for 30, 35, 40, 45. Have students plot the results of the experiments.

![Graph template](image)

Ask students to draw a horizontal line at the height of the theoretical probability of drawing two red cards in the experiment the students performed (A or B). Ask students to compare this line to the line that shows the results of their experiment. What do students notice about their lines? When are they closer together? When are they farther apart? Explain to students that, in general, the more trials that are performed, the closer the experimental probability becomes to the theoretical probability.

**ASK:** Is anyone’s experimental probability for 25 trials closer to the theoretical probability than the experimental probability for 50 trials is? If so, explain that this is not a mistake. Invite any student who answers yes to write his or her results for 25 and 50 trials on the board, writing the probabilities in terms of percents. **ASK:** What is the theoretical probability of your experiment expressed in percents? (44.4% in Experiment A, 33.3% in Experiment B) Point out that the experimental probability after 50 trials is the mean of the two experimental probabilities after 25 trials. So if both pairs were off in the same direction (both had experimental probability higher than the theoretical one, or both lower), the combined result will be between the two, so for one of the pairs the experimental probability would be closer to the theoretical probability after 50 trials, and for the other the experimental probability would be farther from the theoretical one after 50 trials. Have students check if there are any other situations that would cause a similar pattern. Have students try to explain the results in the other
cases too (e.g., if one pair in Experiment A had Exp P(RR) = 32% (lower than the theoretical probability), and the other had Exp P(RR) = 44% (higher than the theoretical probability), then Exp P(RR) after 50 trials will be 38%. The difference between the experimental and the theoretical probability for the second pair was a lot larger than the same difference for the first pair, so the mean was farther from the theoretical probability than the result of the first pair alone.).

**Probability as the average result.** Show students the spinner at left. **ASK:** What fraction of the spinner is blue? Green? Red? What are the possible outcomes of spinning this spinner? How many outcomes are there? (3) What is the probability of spinning each of the colours? Are the chances of spinning blue the same as the chances of spinning red? Why? What about the chances of spinning green and the chances of spinning blue? (they are all the same because all three regions have the same central angle)

**SAY:** I am going to spin the spinner 12 times. How many times do you expect me to spin blue? Why? Write the calculation on the board:

\[ \frac{1}{3} \text{ of } 12 = 4 \quad \text{OR} \quad 12 \div 3 = 4 \text{ times} \]

Remind students that actual outcomes (in experimental probability) usually differ from expected outcomes (in theoretical probability). You might not get blue 4 times every time you make 12 spins, but 4 is the most likely number of times you will spin blue. Remind students that when we say “the chances of spinning blue are 1 out of 3,” we mean that on average, 1 out of 3 spins is blue. What does this mean? Suppose 4 people each spin this spinner 3 times. They write the number of times they got blue. We expect the mean of this set of data to be 1.

Review with students that the sum of the central angles in a circle is 360°, so the angle between the boundaries of the colours is 360° ÷ 3 = 120°. To make a spinner, students can draw a copy of the spinner using a protractor (the circle doesn’t have to be perfect, but the angles need to be exact) and spin a paper clip around the point of a pencil (hold the pencil upright in the centre of the circle). Have students spin their spinners three times and write down the number of times they spin blue. Then put students them into groups of 4 and ask them to find the mean of the number of times they spun blue. (A group of four will have, say, these results: 1, 1, 1, 0. The mean is 0.25, so the group spun blue, on average, less than 1 out of 3 times.) Is the actual mean the same as the expected mean? Combine the results of the whole class. Is the class mean closer to 1? Why?

**Expected outcome of \( n \) experiments.** Review with students how to find a fraction of a set and a fraction of a number, as in Lesson NS6-23. Show the spinner at left and **ASK:** Which fraction of this spinner is blue? What are the chances of spinning blue? (3 out of 4, 3/4, 75%) What is the probability of spinning blue on this spinner? If I spin the spinner 20 times, how many times should I expect to get blue? (3/4 of 20, 15 times)

Have students practise calculating expected outcomes with these and similar questions:
a) If you flip a coin 16 times, how many times do you expect to get a tail?

b) Hong wants to know how many times he is likely to spin green if he spins this spinner 24 times. He knows that 1/3 of 24 is 8 (24 ÷ 3 = 8). How can he use this information to find how many times he is likely to spin green?

c) If you roll a die 18 times, how many times do you expect to get a 4? To get a 1?

d) How many times would you expect to spin blue if you spin this spinner 50 times? How many times would you expect to spin green?

e) If you roll a die 30 times, how many times do you expect to roll an even number? How many times do you expect to roll either 4 or 6?

f) You flip two coins, a nickel and a dime, 12 times. List the possible outcomes. How many times do you expect to get one head and one tail?

g) Jack and Jill play Rock, Paper, Scissors 18 times. List all possible outcomes of the game. (HINT: “Jack has rock and Jill has paper” is different from “Jack has paper, Jill has rock”! Why?) What is the probability of a draw? How many times do you expect to see a draw during 18 games?

Activity

Each student receives a copy of the game board below and a pair of dice. Play the game as follows: Eleven runners are entered in a race. Each runner is given a number from 2 to 12. Runners move one place forward every time their number is the total number rolled on a pair of dice. Which number wins the race? Have students predict the answer, then roll the dice 20 times and move the runners accordingly. The runner who gets the farthest is the winner. Was the students’ prediction correct?

<table>
<thead>
<tr>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

After performing the experiment, have students make a chart to show all possible outcomes of the game and to find the theoretical probabilities of each total. Using the list of outcomes of rolling two dice as in Question 3 on Workbook page 206, we have the probabilities of each total as following:

<table>
<thead>
<tr>
<th>Total</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>1/36</td>
<td>1/18</td>
<td>1/12</td>
<td>1/9</td>
<td>5/36</td>
<td>1/6</td>
<td>5/36</td>
<td>1/9</td>
<td>1/12</td>
<td>1/18</td>
<td>1/36</td>
</tr>
</tbody>
</table>

Discuss the results of the game as a class. What was the most common number to win in the class? (The answer will most likely be 7 because
7 is the total you are most likely to roll, so runner 7 will move most often and is most likely to be “farthest” after 20 rolls.) Does that agree with the theoretical probability? (yes) Was 7 always the winner? (no) Explain why the actual winner wasn’t always the expected winner. (experimental probability is not the same as theoretical probability)

Extensions

1. You have 50 coins with a total value of $1.00. Exactly two of the coins are nickels. If you lose one coin what is the chance it is a dime?

   **Answer:** The only way to have 50 coins with a total value of $1.00, including exactly 2 nickels, is to have 1 quarter, 2 dimes, 2 nickels, and 45 pennies. The chance that the coin you lost is a dime is 2 : 50, or 4%.

2. Sally asks her classmates when their birthdays are. There are 366 possible outcomes.

   a) Is the outcome August 31 equally likely as the outcome February 29, more likely, or less likely? Why?

   b) February 29 is one outcome out 366. Is the probability of being born on February 29 equal to $1/366$? Why or why not?

   c) Find the probability of being born on February 29. ** Hint:** How many days are there in any four consecutive years? How many of those days are February 29?

   d) Assume that the Greater Toronto Area (GTA) has 6 000 000 people. How many people in the GTA would you expect to have their birthday on February 29?

   **Answers:**

   a) August 31 is more likely because there are four days with this date in four years, and only one February 29. 
   b) No. There is only one such day in 4 years, not one every year. 
   c) 1 : 1 461 
   d) 4 107

3. Look at the following question.

   In what year did women in Canada gain the right to vote?

   1915  1916  1917  1918  1919

   a) If you guess randomly, what is the probability of answering correctly?

   b) On a test of 30 similar questions, how many questions would you expect to guess correctly? Incorrectly? (Assume you answer each question blindly.)

   c) You get 4 points (+4) for each correct answer and you lose 1 point (−1) for each incorrect answer. What do you expect your final score to be?
ANSWERS: a) 20%  b) 6 correctly, 24 incorrectly  c) $6 \times 4 - 24 \times 1 = 0$

4. At a fair there is a game with the following rules: You roll a regular die. If you roll a prime number, you win and get $5. If you roll a composite number, you lose and pay $10.

   a) What prime numbers can you roll on a regular die? (2, 3, 5)  
   What composite numbers can you roll on a regular die? (4, 6)

   b) What is the theoretical probability of winning? (1/2)  Of losing? (1/3)  
   Would you play this game? Explain.

   c) If you play the game 6 times, how much money do you expect to gain and to lose? Will you play this game 6 times? Is your decision different from your decision in b)? (You expect to win 3 times, thus gaining $15, and to lose 2 times, thus losing $20, so in total you lose $5. You should not play.)

   d) During the day the game is played 3 000 times. Did the game organizers lose money or win money? How much money do you expect them to win or to lose? (Expected winnings for players: $1/2 \times 3,000 = 1,500$ times, so the loss for the organizers is $7,500. Expected losses for players: $1/3 \times 3,000 = 1,000$ times, so the gain for the organizers is $10,000. The organizers gain $2,500.)
**Goals**

Students will identify events complementary to given events, and find the probability of complementary events.

**Prior Knowledge Required**

- Understands ratios, percents, and fractions
- Can convert between ratios, fractions, and percents
- Can find a fraction or percentage of a number
- Can find the theoretical probability of simple and compound events
- Can multiply fractions

**Materials**

- Playing cards
- Protractors and paper clips (see below)
- Dice
- Game board (see below)

**Introduce Complementary Events.** Explain that two events that have no outcomes in common but cover all possible outcomes together are called **complementary events**. For example, “rolling an even number” and “rolling an odd number” are complementary events. Ask students to list all possible outcomes of rolling a regular die. Which outcomes fall into “rolling an even number”? Which fall into “rolling an odd number”? Then consider the following pairs of events using regular dice, and check as a class which outcomes fall into each one. Are the events complementary?

a) “rolling 3” and “rolling any number other than 3” (yes, you either roll a 3 or roll anything else, there are no other possibilities)

b) “rolling a number greater than 5” and “rolling a number smaller than 5” (no, rolling 5 does not fall into either one of these categories, so not all possible outcomes are covered)

c) “rolling a number greater than 2” and “rolling a number smaller than 4” (no, rolling 3 is in both events)

d) “rolling a number greater than 3” and “rolling a number smaller than 4” (yes, 4, 5, 6 are in one event and 1, 2, 3 are in the other)

Ask students to decide which events would be complementary to each of the events in b) and c). **(Answers:** b) rolling 5 or less; rolling 5 or more; c) rolling 1 or 2; rolling 4 or more) Then present several more pairs of events and have students decide whether the pairs are complementary, and, if they aren’t, determine complementary events for each one. **Examples:**
1. You have a bag with 2 red, 3 white, and 1 green marble.
   a) drawing a red marble and drawing a white marble (no, drawing green is not in any event)
   b) drawing a green marble and drawing a marble with a colour on the Canadian flag (yes)

2. Winning a soccer game and losing a soccer game (no, a tie does not fall into any category)

3. Flipping heads on a coin and flipping tails on a coin (yes)

4. Flipping two heads on two coins and flipping two tails on two coins. (no, you can flip a head on one and a tail on the other)

5. Rolling an even number on a regular die + spinning red on the spinner in margin and rolling an odd number on a regular die + spinning yellow. (no, rolling an odd number and spinning red or rolling an even number and spinning yellow are in neither category)

Bonus

In a bag there are tiles with letters, at least one for each English letter. Is drawing a vowel complementary to drawing a consonant? (no, y is both)

Bonus

Are “scoring 4 goals in a soccer game” and “losing the game” complementary events? (no, it is possible, though unlikely, to lose a game even if you team scored 4 goals)

**Probabilities of complementary events.** Have students find the probabilities of pairs of complementary events and record the data in a table similar to that used in the Investigation on Workbook page 209. Then ask students to look for a pattern in the table. To explain why the probabilities always add to 1, review with students what certain events are—events that have to happen because they cover all possible outcomes. Since they cover all possible outcomes, their probability is 1—the numerator of the probability fraction (all suitable outcomes) is equal to its denominator (all possible outcomes). Since complementary events together cover all possible outcomes, and do not have any outcomes in common, the sum of the number of outcomes in the numerators should be equal to the number of all possible outcomes.

**Finding the probability of events using complementary events.** Tell students that you want to find the probability of rolling a number smaller than 11 on a pair of regular dice. One way to do it would be to draw a tree diagram for or list all the possible combinations of rolling two dice, and to choose those that add to numbers less than 11. Have students solve the problem that way. Then point out that using an event complementary to this event would give a different way of solving the problem. **ASK:** Which event is complementary to rolling a number smaller than 11 on a pair of regular dice? (Rolling 11 or 12) Which combinations will produce 11? (5 and 6, 6 and 5) Which combinations will produce 12? (6 and 6) How
many combinations in total is that? (3) How many different outcomes does rolling two dice have? (36) How do you know? (Each die has 6 outcomes, and for each outcome of the first die there are 6 outcomes of the second die, producing a total of $6 \times 6 = 36$ outcomes.) So what is the theoretical probability of rolling a sum of 11 or more? (3 out of 36, or 1/12) What is the probability of the event complementary to this one, which is the event we are interested in? ($1 - 1/12 = 11/12$) Does this answer agree with the answer students obtained using a list of combinations or a tree diagram? Which way of solving the problem was easier?

Tell students that you want to know the probability of rolling a number more than 3 when rolling three dice. How is this problem different from the previous one? Does it make sense to make a list of combinations? Why not? How many combinations will be in the list? ($6 \times 6 \times 6 = 216$) Suggest that students look at the complementary event. What would the complementary event be? (PROMPT: What is the smallest number you can roll on three dice? The answer is 3, so the complementary event is rolling 3.) How many ways can you roll a 3 on two dice? (one way: $1 + 1 + 1$) What is the probability of rolling 3? (1 : 216) What is the probability of rolling a number more than 3? (215 : 216)

Have students find the complementary event, its probability, and the probability of the given event in questions such as these:

1. What is the probability of tossing at least 1 tail when tossing 4 coins? (Complementary event: tossing 4 heads, $P(4$ heads) = 1/16, so $P$(at least one tail) = 15/16)

2. What is the probability that a family of 7 children has at least 2 boys? (Complementary event: 0 boys or 1 boy, $P$(0 boys or 1 boy) = $8/128 = 1/16$, so $P$(at least 2 boys) = 15/16)

Should you use the complementary event to find the probability? Present this question: What is the probability that when you roll a die you get 3? What is the complementary event? Does it make sense to use the probability of the complementary event to find the probability of rolling 3? Why not? (It is easier to find the probability of rolling 3 than it is to find the probability of the complementary event.) Tell students that in the next few questions they will have to decide which one is easier to find, the probability of the event itself or the probability of the complementary event. Then have students find the probability using the method they selected.

1. You will spin red or yellow on the two-part spinner above. (the event itself is about as easy as the complementary event)

2. A family with three children will have two girls. (event itself)

3. A family with twelve children will have at least one boy. (complementary event)

4. A family of three children will have at least one boy. (complementary event)
5. A family of three children will have at least two boys. (event itself or complementary event—the probabilities are the same)

6. You will roll 7 on a pair of regular dice. (event itself)

7. You will roll a total more than 5 on a pair of regular dice. (complementary event)

**Bonus** You will roll a total more than 5 on a pair of tetrahedral (4-sided) dice. (event itself)
A census or a sample? Tell students that sometimes they will want to know things about really large groups of people but they may not be able to gather data for everyone in the population. For example, if I want to know how many people in Canada have read *The Wizard of Oz*, it wouldn’t be practical to ask everyone if they’ve read it. Tell students that you want to find the average shoe size of everyone in Ontario. Ask if you should survey everyone in Ontario or only a sample. Have students explain their answer. Repeat with various situations. *(EXAMPLE:)* If I want to know which books my classmates read last week, should I survey the whole class or only a sample? If I want to know the average shoe size of people on my block, should I survey everyone on the block or only a sample? If I want to know how many people in Canada watch a particular television show, should I ask everyone in Canada or just a sample?)

A large sample provides better results than a small sample. Bring in a large bowl or jar of beans in two different colours (say, red and white). Make 40% of the beans one colour and 60% the other colour, but don’t tell students. Mix the beans thoroughly. Tell students that you want to figure out which proportion of beans are red and which are white without counting every single bean. Invite students to describe how they might do this.

Have students each choose 10 beans with their eyes closed, and record the results. Invite volunteers to tell how many of each colour they chose. **ASK:** Do you think we can estimate the fraction of red and white beans in the whole bowl based on the fraction in a sample of 10? *(no)* Why not? (Everyone got different answers, so we wouldn’t know whose answer to take.) Have students pool their results in pairs and then groups of 4 (pairs pair up). Repeat with groups of 8, 16, and then the whole class. Students can record their results in a table:
Tell students the exact proportion of beans of each colour in the bowl and **ASK**: Which sample produced the best estimate? How large did our sample need to be before we started getting really close to the actual proportion?

Now tell the students that you want to know whether the people in a town are in favour of or against a proposal (say, to open a new library). If the red beans represent those in favour and the white beans represent those against, how many people are actually in favour of the proposal? Ron said that he asked 10 people he met at random and found that 7 people they liked the idea of a new library and 3 people didn’t. How did that happen? Did anyone pick 7 red and 3 white beans even though only 40% are red and 60% are green? To make conclusions about a large population, you need to ask a large sample. Is 10 people a large enough sample? Is 300 people better? Explain that larger samples, whether of beans or people in a town, will give more accurate information about the whole population.

Have students do the Investigation on Workbook pages 211–212. Discuss students’ findings. Part L: Point out that if you expect 80% of people in the town to say “yes” in the survey (instead of 20%), then you could perform the same experiment but look at the numbers that are not multiples of 5. From the probability point of view, these two situations—80% say yes and 20% say yes—are like mirror images; this event is “opposite” to the one you started with. Part M: If the town’s population is larger, you would change the upper limit in the random number generator—instead of picking numbers from 1 to 10 000 you would pick numbers from 1 to 100 000. But since both populations are very large, the number of multiples of 5 will be close enough to 20% at about the same point in both cases, so the same sample size should work in both.

**Sample size and probability.** **ASK:** Let’s think of the experiment that you performed in terms of probability. Asking a question or picking a random number is the probability experiment you perform. Each person asked is a trial of experiment. What are the favourable outcomes in your experiment? (a number is divisible by 5) In the survey? (a person answers yes) What is the theoretical probability that a randomly picked number is divisible by 5? (20%, or 1 in 5) When you performed the experiment, what did you find in terms of probability? (the experimental probability of a number being divisible by 5) What do you know about experimental probability, theoretical probability, and the number of trials performed? (the larger the number of trials, the closer experimental probability is to theoretical probability) In terms of a survey, what does this mean? (the larger the sample, the closer the answer is to the answer you would get in a census)
**Surveys and complementary events.** Return to the results of the experiment in the Investigation. In terms of probability, what are “a person answers yes” and “a person answers no”? (complementary events) Now think of a survey that has more than two answers, say, “yes”, “no” and “I don’t know”. Are “yes” and “no” still complementary events? (no) What would the complementary event to “yes” be? (“no” or “I do not know”)

**Extension**

Distribute a page of French text. Put students into the same number of groups as there are paragraphs in the text, and assign a paragraph to each group. Ask students to identify the most common letter in one sentence in their paragraphs (make sure all sentences are used; depending on the length of the text, some sentences may be assigned to more than one student). Is the most common letter the same for all students/sentences? How many letters do students identify as the most common? Which letters do students identify as most common?

Now have each group identify the most common letter in its paragraph. Finally, have students identify the most common letter on the page.

**ASK:** Do you think the most common letter on this page will be the same as the most common letter on another page? in a whole book? in the French language overall? Why is it better to use a large sample size to calculate the most common letter in a language than a small sample (such as a sentence or paragraph)?
Bias in a sample. Tell students that you want to know whether grade 8 students prefer action movies or comedies. **ASK:** What would happen if I asked only students in Timmons, Ontario? What would happen if I asked only boys?

Tell students that a *biased sample* is not similar to the whole population because some part of the population is either not represented or overrepresented. In the above example, "grade 8 students in Canada" is the whole population. If only boys are surveyed, then girls are not represented. If only Timmons students are surveyed, then people from other cities and towns are not represented. If only public school students are surveyed, then private school students are not represented. In some questions, the whole population is very specific. For example, if I want to know if boys prefer action movies or comedies, I only have to ask boys—it wouldn’t make sense to ask girls.

Tell students that an elementary school (grades 1–6) is planning a games party and wants to decide what games to buy. Would the sample be biased if the school surveyed:

a) all students in grade 2? (yes)
b) all students in grades 3 and 5? (yes)
c) all boys in grades 3 and 4? (yes)
d) every tenth student, when listed in alphabetical order? (no)
e) 3 people chosen at random from each classroom? (no)

Tell students that a sample that is similar to the whole population is called *representative*. Finding a representative sample is often the most difficult part of conducting a survey. Have students think about the following situation:

An apartment building has a games room and would like to reserve it once a week for a dance. Discuss the bias in the samples if, to decide what kind of music to play at the dance, the building manager

a) asks the bridge club. (People in the bridge club are likely to be older.)
b) asks the soccer club. (People in the soccer club are likely to be younger and there might be more males than females.)
c) asks the book club. (More women tend to join book clubs than men.)
d) asks the teen movie club. (Only teenagers will be represented.)
e) puts a survey under every tenth door by apartment number. (no bias)
f) lists the names of people living in the building in alphabetical order and picks every tenth person to ask. (no bias)
g) asks people at the playground. (Teenagers are less likely to be represented and families with small children are more likely to be overrepresented.)

Timing can create bias. Explain that even the same location, at different times, can produce different biases. Ask: How would your samples be different if you surveyed people at the mall on a weekday morning and a weekend morning? If a mall wants to decide whether to rent space to a jewellery store or a video gaming store, how and when should they conduct their survey? Some points to discuss:

- During a weekday morning, teenagers are definitely excluded as are many working people. People who go to malls during weekday mornings are either unemployed, retired, or very well off. They might also work part-time or have variable work schedules (i.e., not 9 to 5). University students might also have class schedules that allow them to go the mall on a weekday morning.

- People who can go to the mall on weekdays might not go on the weekends, to avoid the crowds. Thus, any group that was overrepresented on weekday mornings may be under-represented on the weekend.

- The mall should probably take samples at different times and give them different weightings based on how crowded the mall is. If more customers come during the weekend, the weekend should get a higher weighting.

- The mall is interested in biasing their sample to their own customers. If it happens that more elderly people go to the mall, they want to target older people.

- Suppose there are 3 large apartment buildings adjacent to the mall. The mall decides to survey every tenth apartment in these 3 buildings instead of asking their customers at different times of day. Is this a representative sample? (No. Only people living in apartments are represented, and this will bias the sample against people who live in nearby houses. Also, not all the people who live in the apartments necessarily like visiting malls, so this survey will not help the mall.)

The wording of a question can affect the results of the survey. Discuss the following two cases where the sample is representative, but the results are biased.
1. A town council is thinking of selling a city park and allowing a department store to build in its place. Two groups ask different questions:

   Question A: Are you in favour of having a new store that will provide jobs for 50 people in our town?

   Question B: Are you in favour of keeping our parks quiet and peaceful?

   a) Which question do you think was proposed by someone in favour of selling the park? Which was proposed by someone against selling the park?

   b) Write a survey question that is more neutral and does not already suggest an answer.

2. Tell students that you want to survey two different younger classes (students can actually conduct the surveys if you want) using two different questions:

   Question A: Is it okay to watch television while having a conversation with a friend?

   Question B: Is it okay to have a conversation with a friend while watching television?

   **ASK:** How are the questions the same? How are they different? Which question do you think will get more “yes” answers? Why? (Even though the logical content of the questions is the same, the pictures you get in your mind when answering the two questions are different. When answering Question A, you are likely to picture someone not really paying attention to their friend because they are so interested in the television show. When answering Question B, you may picture two people who are talking with the television on in the background, or two people who are talking about the television show they are watching together. No one will offend the television show by not paying attention to it.)

**Extension**

Jowan wants to know whether more people support the local team or the guest team at a hockey game. He stands at the entrance to the rink and counts the number of people wearing jerseys for the home team and the number wearing jerseys for the guest team. He finds that 40% of people wearing jerseys are wearing jerseys for the guest team, and 60% are wearing jerseys for the home team. He thinks that 60% of the spectators are cheering for the home team.

Is Jowan’s sample representative or biased? Can he trust his survey result? Explain.

**ANSWER:** Jowan’s sample is biased. People coming from afar are more likely to wear their team’s jersey than not, so there might be more supporters of the home team among those who are not wearing jerseys than among those who are.
Choices for survey questions. Conduct a survey with your students by asking them what their favourite flavour of ice cream is. Do not limit their choices at this point. Write each answer with a tally, then ask students how many bars will be needed to display the results on a bar graph. How can the question be changed to reduce the number of bars needed to display the results? How can the choices be limited? Should choices be limited to the most popular flavours? Why is it important to offer an “other” choice?

Explain to students that the most popular choices to a survey question are predicted before a survey is conducted. Why is it important to predict the most popular choices? Could the three most popular flavours of ice cream have been predicted?

Have your students predict the most popular choices for the following survey questions:

- What is your favourite type of music?
- What means of transportation do you use to get to school?
- What country where you born in?
- What is your first language?
- What is your favourite sport?

Students may disagree on the choices. Explain that a good way to predict the most popular choices for a survey question is to ask the survey question to a few people before asking everyone.

Each person should give a unique answer. Emphasize that the question has to be worded so that each person can give only one answer. Which of the following questions are worded so as to receive only one answer?

a) What is your favourite ice cream flavour?
b) What flavours of ice cream do you like?
c) Who will you vote for in the election?
d) Which of the candidates do you like in the election?
e) What is your favourite colour?
f) Which colours do you like?

“Other” category. Have students think about whether or not an “other” category is needed for the following questions:
1. What is your favourite food group?
   Vegetables and Fruits   Meat and Alternatives
   Milk and Alternatives   Grain Products

2. What is your favourite food?
   Pizza   Burgers   Tacos   Salad

Ask students how they know when an “other” category is needed. Continue with these examples:

   What is your favourite day of the week? (List all seven days)
   What is your favourite day of the week? (List only Friday, Saturday and Sunday)
   What would you like to be when you grow up? (List doctor, lawyer, teacher)
   How many siblings do you have? (List 0, 1, 2, 3, 4 or more)
   Who will you vote for in the election? (List all candidates)

**Choosing a good survey question.** Tell students that a school wants to determine how much students enjoy reading. The principal decides to survey students to find out. Have students explain what they like and don’t like about each of the following proposals for a survey question.

1. Do you like to read for fun?   YES   NO
2. How many books have you read in the last year for fun? ____
3. How often do you read for fun?
   Any chance I get   Often   Sometimes   Not very often   Never
4. How many books have you read in the last year for fun?
   0   1–5   6–10   11 or more

Emphasize the importance of using a question that people are able to answer (e.g., some people may not know how many books they read last year) and providing various choices as answers (e.g., some people might feel uncomfortable answering “yes” or “no” if they only read for fun “sometimes”). Question 3 is probably the best question in this list. There are various reasonable answers to choose from, though some people may have trouble choosing among them. Also, Question 3 asks about how frequently people read rather than how much they read. How often people read for fun says more about how much they read for fun than the number of books they read. Someone who likes to read may read only a few long books, while someone who doesn’t might read the same number of short books. And someone who reads for fun may read only magazines, and no books at all!

Have students make up good survey questions to find out
a) where Grade 8 students should go on a school trip. Would it make sense to include Hawaii in the survey choices? Should the school provide an "other" category or force students to choose from among destinations that parents could possibly afford?

b) what kind of books students in our province like best.

c) the future career plans of students in your class.

Ask students to decide who they would survey for each situation. For a), would it make sense to survey only students in your class or should other grade 8 students be included? Why or why not?

Designing your own survey. Tell students that they will be designing their own survey. Have students brainstorm topics for a survey. (EXAMPLES: family size, career choices, how long it takes people to get ready for school in the morning, height of students in different grades)

Have students conduct their surveys. Ask students to represent their survey results in two different ways and explain the advantages and shortcomings of each representation.

Bias in an experiment. Explain that sometimes the trickiest part of doing an experiment is making sure that you are really testing for what you want to be testing and that nothing else influences your results. Discuss the following experiments with students. Which experiments use representative samples and which use biased samples?

a) On a weekend, a model airplane club tests two plane designs to see how long they can remain in the air.

   - On each day, they test 5 planes of each design. (representative)

   - On the first day they test 10 planes of one design, and on the second day they test 10 planes of the other design. (biased)

   (It might be windier on one day, causing the planes to stay in the air longer on that day than on the other day. The conditions are different on the two days, so the sample is biased.)

b) A class is testing two brands of seeds to find out what percent will germinate. They plant:

   - 10 seeds from each package
   - the 10 largest seeds from each package

Designing an experiment. Tell your students that you want to answer the question: Do ice cubes made from the same amount of water but in containers of different shapes melt at the same rate? Discuss the different types of containers that could be used for this experiment and how students would make sure the same amount of water is put into each container. How would they measure the rate of melting? What other equipment would they need? Why is it important to think about the kind of equipment they need ahead of time? Would the experiment be fair if some ice cubes were put in the sun and others were put in the shade? How would that affect the results?
of the experiment? Have students predict the results of the experiment—will the container’s shape affect the rate of melting? What type of graph would students use to display their results? Is there a natural ordering of the containers? (no) Is there a possibility of data in between the data values? (No, not unless they choose their containers in a very structured way, i.e., by increasing the length and decreasing the width, in which case there would be a natural ordering of the containers.)

If time permits, have volunteers bring in the equipment and conduct the experiment with the class. Demonstrate the entire process, from doing the experiment to displaying the results (on a bar graph, since it doesn’t make sense to speak about trends in the shape of the container).

Brainstorm factors that could influence the results of the following experiments:

A. Using 3 paper rectangles with the same area but different perimeters to build paper airplanes. Which rectangle makes an airplane that flies the farthest?

B. How does adding sugar to strawberries affect how long the strawberries stay fresh?

C. How does adding salt to ice affect the rate at which the ice melts?

In A, the thickness and quality of paper, the environment (such as the existence of wind), the design of the airplanes, and the quality of bending (how precise the creases are) could affect the results, so these should be kept the same for all three models. In B, the initial freshness of the strawberries, how evenly the sugar is spread, the container (metal, plastic, glass, airtight, shape and size), the surrounding temperature, the type of strawberries, and the type of sugar should be the same in each sample to which different amounts of sugar are added. In C, the size and shape of the ice, the type of salt (i.e., sea salt, table salt), the temperature of the surrounding areas, the colour of the container (e.g., white or black colours will absorb heat at different rates), and the source of the water should be kept the same as the amount of salt increases. Note that in B and C, it makes sense to look for trends and so a line graph is more appropriate. In A, a bar graph is more appropriate.

**Extensions**

1. Conduct a survey of your class. Have students answer this question: Including yourself, how many people are in your family and live together with you? Then ask students to predict whether the result for your classroom will be higher or lower than the average size of families with children in Canada. The average is 3.5, and it is likely that your class will have an average family size that is larger than this. Ask students to think about why this might be the case. (One explanation: Children in grade 7 are likely to have more siblings than younger children, just because there has been more time for the family to choose to have
another child; a 4-year-old is more likely to be an only child than a 14-year-old.) Students can pursue this reasoning by asking which percentage of children in each grade are children without siblings (only children). Ask students to predict whether this percentage will increase with age, decrease with age, or stay the same. (The percentage of only children should decrease with age.) What type of graph would show this trend best? Why? (a line graph because line graphs show trends)

2. a) Elections are held in different ways in different countries. Let’s look at four voting systems. In every case, citizens 18 years of age and older can vote. Citizens can choose one candidate or nobody. To vote, citizens need a card sent by the election committee. Cards are used to prevent double voting. Citizens cast their votes when they are alone in a room.

• In country A, there are many ballots, each with one candidate’s name. Citizens choose the ballot with the name of the candidate they support. Nobody can see which ballot is chosen.

• In country B, there are two candidates, the ballots are numbered, and the numbers are recorded on a list with the names of the citizens.

• In country C, there is only one candidate, and the voting is either for or against the candidate. The ballots have the name of the candidate and two options—for or against the candidate—and nothing else on them. Nobody can see what is marked on the ballot.

• In country D, there is only one candidate and the voting is either for or against the candidate. The ballots are numbered and the numbers are recorded on a list with the names of the citizens.

In which countries are the elections biased and how? Explain your thinking.

b) These elections use a census—everyone over a certain age has the right to vote. In practice, however, everyone might not have an equal opportunity to vote. Discuss sources of sample bias (EXAMPLE: if there is no way for people to vote from home, those with mobility problems might be under-represented; if voting places are all located in urban centres, those from rural areas might be under-represented).
Nets of 3-D Shapes (1)
Nets of 3-D Shapes (3)
Nets of 3-D Shapes (4)
Nets of 3-D Shapes (5)
Nets of 3-D Shapes (6)
Nets of 3-D Shapes (7)
Nets of 3-D Shapes (8)
Nets of 3-D Shapes (9)
Nets of 3-D Shapes (10)
Nets of 3-D Shapes (11)
Nets of 3-D Shapes (12)
Nets of 3-D Shapes (13)
Nets of 3-D Shapes (14)
Nets of 3-D Shapes (15)
Nets of 3-D Shapes (16)
Nets of 3-D Shapes (17)
Nets of 3-D Shapes (18)
Nets of 3-D Shapes (19)
Nets of 3-D Shapes (20)
Nets of 3-D Shapes (21)
Nets of 3-D Shapes (22)
Nets of 3-D Shapes (23)
Nets of 3-D Shapes (24)
A Net or Not a Net?

Does this picture make a net of a 3-D shape? If yes, what shape? Predict, then cut out the net and fold it to check your prediction.
Is It a Net?

A

B
Answer Keys for AP Book 8.2

Number Sense – AP Book 8, Part 2: Unit 1

AP Book NS8-75

1. a) 5
   b) 1 + 4
   c) 1 + 8

2. a) \(24 + 2 = (3 \times 8) + 2 = 3 \times (8 + 2)\)
   b) \(3 + 8 = (3 \times 1) + 8 = 3 \times (1 + 8)\)
   c) From above:
   \(3 + 8 = 3 \times (1 + 8) = \frac{3}{8}\)
   d) From above:
   \(3 + 8 = 3 \times (1 + 8) = \frac{3}{8}\)

3. Teacher to check long division.
   a) 0.2
   b) 3 + 5; 0.6
   c) 4 + 10; 0.4
   d) 2 + 4; 0.5
   e) 4 + 8; 0.5
   f) 2.5
   g) 1.75
   h) 0.9
   i) 0.8
   j) 0.6

4. a) Teacher to check long division.
   0.125, 0.25, 0.375
   b) Start at 0.125 and add 0.125 each time.
   c) 0.5, 0.625, 0.75, 0.875

5. Teacher to check calculator checks.
   a) 0.85
   b) 85, 0.85
   c) 152, 0.152
   d) \(\frac{68}{100} = 0.68\)
   e) \(\frac{35}{100} = 0.35\)
   f) \(\frac{546}{1000} = 0.546\)
   g) \(\frac{444}{1000} = 0.444\)

BONUS \(\frac{5625}{10000} = 0.5625\)

AP Book NS8-76

1. a) 0.333 333 33
   b) 0.033 333 33
   c) 0.003 333 33
   d) 0.525 252 52
   e) 0.817 817 81
   f) 0.817 171 71
   g) 0.926 262 62
   h) 0.253 737 37
   i) 7.233 333 33
   j) 8.253 953 95

2. a) \(24 \div 2 = (3 \times 8) \div 2 = (3 \times (1 + 8)) \div 2\)
   b) \(3 \div 8 = (3 \times 1) \div 8 = (3 \times (1 + 8)) \div 8\)

3. Teacher to check long division.
   a) \(\frac{5}{6} (0.83), \frac{4}{5} (0.8)\)
   b) \(\frac{56}{75} (= 0.76), \frac{13}{17} (= 0.765)\)

4. a) i) \(0.57 = 7\frac{57}{100} < 0.60 = \frac{3}{5}\)
    ii) \(0.83 = 8\frac{3}{100} > 0.80 = \frac{4}{5}\)
    iii) \(2\frac{3}{300} > 1\frac{111}{300} = 0.37\)
   b) i) \(0.57 < 0.6\)
    ii) \(0.83 > 0.8\)
    iii) \(0.6 > 0.37\)

5. a) \(23\frac{2}{5}\)
   b) \(0.28\overline{5}\)

6. Answers may vary slightly – teacher to check.
   a) \(\frac{1}{2}\)
   b) \(\frac{1}{3}\)
   c) \(\frac{3}{10}\)

AP Book NS8-77

1. a) 0.2
   b) 0.75; 0.7
   c) 0.2; 0.25
   d) 0.4; 0.42

2. a) \(0.700, 0.667; \frac{2}{3}\)
   b) \(0.143, 0.125, 0.111; \frac{1}{8}\)
   c) \(3.500, 3.333, 2.667; \frac{10}{3}\)

3. \(\frac{5}{6} (0.83), \frac{4}{5} (0.8)\)

4. a) \(0.57 = \frac{57}{100}\)
   b) \(0.83 = \frac{83}{100}\)
   c) \(2\frac{3}{300}\)

5. \(\frac{10}{1000}\)

INVESTIGATION

A. Answers will vary – teacher to check.

B. If a decimal terminates, it has a finite number of decimal places, and this number of decimal places gives the number of 10s in the “power of 10” denominator. So one decimal place refers to tenths (denominator 10), two decimal places refer to hundredths (denominator 100), and so on.

C. a) \(0.625 (T) = 625 \div 1000\)
   b) \(0.583 (R)\)
   c) \(0.461538 (R)\)
   d) \(0.46 (R)\)
   e) \(0.571428 (R)\)
   f) \(0.0065 (T) : 65 \div 10000\)

Answer Keys for AP Book 8.2

V-1
D. \[ 100 = 10 \times 10 \]
\[ = 2 \times 5 \times 2 \times 5 \]
\[ 1000 = 10 \times 100 \]
\[ = 2 \times 5 \times 2 \times 5 \times 2 \times 5 \]
E. Answers will vary –
teacher to check.
NOTE: Correct answers will always terminate.
Sample answer:
\[ \frac{11}{2 \times 2 \times 5} = \frac{11}{20} = 0.55 \]
F. \[ \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{7}{12} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{11}{12} \cdot \frac{12}{12} \]
Once simplified, only \( \frac{3}{6} \) has a “2 and/or 5 product”
denominator – therefore
has a “2 and/or 5 product”
denominator – therefore
will only terminate.

1. a) 3, 9, 27, 81, ...
   b) Yes, they repeat:
   their denominators
cannot be written as
products of 2s/5s.

2. a) \[ \frac{1}{12} \cdot \frac{1}{6} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{5} \cdot \frac{1}{12} \cdot \frac{1}{2} \]
   \[ \frac{7}{12} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{11}{12} \cdot \frac{12}{12} \]
   b) Teacher to check.
   Correct prediction:
   Only \( \frac{3}{6} \) & \( \frac{9}{12} \)
terminate: just their
denominators are
products of 2s/5s.

c) \[ \frac{1}{12} = 0.083 \]
   \[ \frac{2}{12} = 0.16 \]
   \[ \frac{3}{12} = 0.25 \]
   \[ \frac{4}{12} = 0.3 \]
   \[ \frac{5}{12} = 0.416 \]
   \[ \frac{6}{12} = 0.5 \]
   \[ \frac{7}{12} = 0.583 \]
   \[ \frac{8}{12} = 0.6 \]
   \[ \frac{9}{12} = 0.75 \]
   \[ \frac{10}{12} = 0.83 \]
   \[ \frac{11}{12} = 0.916 \]

   d) Teacher to check.

3. Since their numerators
all have a factor of 3 too,
the 3 “cancels out” and
each one can be reduced
to a denominator that
terminates.

### AP Book NS8-79
page 6

1. Teacher to check that
lining up is correct.
   b) 0.3472
   c) 0.95
   d) 0.59385749
   e) 0.30
   f) 0.2

2. a) i) 0.3
   ii) 0.7
   iii) 0.8

   b) i) \( \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = 0.3 \)
   ii) \( \frac{2}{9} + \frac{9}{5} = \frac{7}{9} = 0.7 \)
   iii) \( \frac{4}{9} + \frac{4}{8} = \frac{8}{9} = 0.8 \)

c) Answers will vary –
teacher to check.

3. a) i) 1
   ii) 1.1
   iii) 1.11
   iv) 1.111

   b) Teacher to check.
   Correct prediction: 1.1

c) They both have an
infinite number of
decimal places – so
there’s nowhere to
start adding.

   d) \[ 3 \cdot \frac{7}{9} = \frac{10}{9} = \frac{1}{9} \]
   \[ = 1 + 0.1 = 1.1 \]

### AP Book NS8-80
page 7

1. a) Teacher to check long division.
   b) 0.3472
   c) 0.95
   d) 0.59385749
   e) 0.30
   f) 0.2

2. a) i) 0.3
   ii) 0.7
   iii) 0.8

   b) i) \( \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = 0.3 \)
   ii) \( \frac{2}{9} + \frac{9}{5} = \frac{7}{9} = 0.7 \)
   iii) \( \frac{4}{9} + \frac{4}{8} = \frac{8}{9} = 0.8 \)

c) Any fraction with an
equal numerator and
denominator = 1.

   So: \( \frac{9}{9} = \frac{11}{11} = 1 \)

   \[ \therefore \text{ we know that } 0.\overline{9} \text{ must equal } 1 \]

   d) 0.45; 0.4545; 0.454545; 0.45

   e) 0.09 \times 5 = \frac{9}{99} \times 5

   \[ = \frac{45}{99} \]

\[ = \frac{0.45}{0.99} \]

### AP Book NS8-81
page 8

1. a) 7

   b) 0.34

   c) 0.006

   d) 0.734

   e) 0.046

   f) 0.075

   g) 0.423

2. a) Teacher to check.

   b) 0.17; 0.1717;
   0.171717; 0.17

   c) \( \frac{17}{99} = 17 \times \frac{1}{99} \)

   \[ = 17 \times 0.01 \]

   \[ = 0.17 \]

3. a) 0.25
1. a) 461
   b) 999
   c) 38
   d) 99
   e) 61

2. a) 254.4...
   b) 266.666...
   c) 2.49191...
   d) 32.32
   e) 0.0032
   f) 543.136
   g) 0.0341
   h) 0.007432

3. a) 0.4444...; 0.00444...
   b) 0.6666...; 0.000666...

4. a) 0.13, 0.143, 0.1443
   b) Teacher to check.

5. a) 1/3 = 33 1/3%
   b) 2/3 = 66 2/3%
   c) 3/4 = 75%
   d) 1/5 = 20%

6. a) 1/9 = 11.1%
   b) 2/9 = 22.2%
   c) 3/9 = 33.3%
   d) 4/9 = 44.4%
   e) 5/9 = 55.5%
   f) 6/9 = 66.6%
   g) 7/9 = 77.8%
   h) 8/9 = 88.9%

7. Teacher to check drawings.

<table>
<thead>
<tr>
<th></th>
<th>52%</th>
<th>67%</th>
<th>18%</th>
</tr>
</thead>
<tbody>
<tr>
<td>52</td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>67</td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

AP Book NS8-83

1. Teacher to check colouring.

2. a) 40, 30, 70, 70%
   b) 60, 10, 70, 70%
   c) 60, 35, 25, 25%
   d) 80, 40, 40, 40%

3. a) 47%
   b) 90%
   c) 40%

4. a) 21%
   b) 1%
   c) 41%

5. a) 1%; 10, 10%; 25, 25%
   b) 35%; 52%
   c) 74%

AP Book NS8-84

1. a) 6 squares shaded
   b) 4 squares shaded

2. 50%; 50%

3. a) 7 dots shaded
   b) 30%

4. 90 are not blue

5. a) 80, 0.80
   b) 75, 0.75
   c) 21, 0.21
   d) 9, 0.09
   e) 15, 0.15

6. a) 12
   b) 7
   c) 49
   d) 3
   e) 100

7. a) 0.25
   b) 0.75
   c) 0.13
   d) 0.40
   e) 0.07
   f) 0.09
   g) 0.70
   h) 0.01

8. a) 50%
   b) 70%
   c) 40%
   d) 10%
   e) 90%
   f) 27%
   g) 60%
   h) 53%
   i) 7%
   j) 99%

9. a) ≈ 38%
   b) ≈ 93%
   c) ≈ 38%

10. a) 70 are classical;
    b) 30% are jazz

AP Book NS8-85

1. a) 80, 0.80
   b) 75, 0.75
   c) 15
   d) 32

2. a) 0.42
   b) 0.07
   c) 1.00

3. a) 0.25
   b) 0.75
   c) 0.13
   d) 0.40
   e) 0.07
   f) 0.09
   g) 0.70
   h) 0.01

4. a) 50%
   b) 70%
   c) 40%
   d) 10%

5. a) 0.13, 0.143, 0.1443
   b) Teacher to check.

6. Correct prediction: 0.1444... = 0.4444...

7. Teacher to check drawings.

8. Teacher to check colouring.

9. a) 0.13, 0.143, 0.1443
   b) Teacher to check.
2. 40% ordered pizza
b) \( \frac{1}{5} = \frac{20}{100} = 20\% \)
c) \( \frac{1}{2} = 50\% \)
d) \( \frac{1}{4} = 25\% \)
e) 40%
f) 70%
g) 40%
h) 85%
i) 25%

5. a) 0
b) 0

c) 0
d) 0
e) 0
f) 0
g) 0
h) 0
i) 0

6. a) 75%

7. Estimates may vary – teacher to check.

8. a) 42%
b) 0.3 = 0.33 = 33%
c) 0.6 = 0.67 = 67%
d) 0.2 = 0.22 = 22%
e) 0.83 = 0.83 = 83%
f) 0.14 = 0.14 = 14%

AP Book NS8-86

page 13

1. a) 50%
b) 25%
c) 20%
d) 50%
e) 75%
f) 50%
g) 60%
h) 50%
i) 20%
j) 25%
k) 25%
l) 20%
m) 20%

2. Teacher to check decimal fractions.
b) <
c) <
d) <
e) >
f) <
g) <
h) =
i) >
j) >
k) >
l) >

AP Book NS8-87

page 14

1.  

<table>
<thead>
<tr>
<th>F</th>
<th>D</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{4} )</td>
<td>0.25</td>
<td>25%</td>
</tr>
<tr>
<td>( \frac{35}{100} ) = ( \frac{7}{20} )</td>
<td>0.35</td>
<td>35%</td>
</tr>
</tbody>
</table>

AP Book NS8-88

page 15

1. a) 0.7
b) 1.0
c) 3.5
d) 21.0
e) 0.64
f) 5.06
g) 0.1
h) 0.39
i) 0.405
j) 0.674
k) 0.009
l) 6.008

3. a) 1.5
b) 35.3

c) 2.7, 0.27
d) 62.6

e) 17, 1.7
f) 0.7, 0.07

g) 3.0, 0.21

4.  

<table>
<thead>
<tr>
<th>Percent</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>10</td>
</tr>
<tr>
<td>10%</td>
<td>20</td>
</tr>
<tr>
<td>100%</td>
<td>420</td>
</tr>
</tbody>
</table>

5. a) 3
b) 0.01 \times 2000 = 20

c) 0.01 \times 15 = 0.15
d) 0.01 \times 60 = 0.60

6. a) 4
b) 6
c) 24

7. a) 32
b) 1

c) 6.6
d) 0.08
e) 3.15

8.  

<table>
<thead>
<tr>
<th>Percent</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>10</td>
</tr>
<tr>
<td>4%</td>
<td>20</td>
</tr>
<tr>
<td>8%</td>
<td>40</td>
</tr>
<tr>
<td>10%</td>
<td>50</td>
</tr>
<tr>
<td>20%</td>
<td>120</td>
</tr>
<tr>
<td>50%</td>
<td>375</td>
</tr>
<tr>
<td>25%</td>
<td>187.5</td>
</tr>
<tr>
<td>100%</td>
<td>500</td>
</tr>
</tbody>
</table>

9. a) 18
b) 4
c) 160
Answer Keys for AP Book 8.2

Number Sense – AP Book 8, Part 2: Unit 1 (continued)

d) 65

10. $9 tip

11. a) 25% of $40 = $10; 30% of $30 = $9; 75% of $32 or $24 so the sale price is approximately $32 or $24

AP Book NS8-89

page 17

1. Teacher to check Step 1 multiplication.
   a) $4 \times 23 = 782; 782 \div 100 = 7.82; 23\% \text{ of } 34 \text{ is } 7.82$
   b) 85 \times 17 = 1445;
      1445 \div 100 = 14.45; 7\% \text{ of } 85 \text{ is } 14.45

2. a) 7.26
    b) 6.72
    c) 19.8
    d) 30.36
    e) 62.25
    f) 7.2
    g) 16.12
    h) 15.3

3. a) i) $4 \times 35 = 1400$
      $1400 \div 100 = 14$
      35\% \text{ of } 40 \text{ is } 14
    ii) 25\% \text{ of } 40 \text{ is } $10$
      40 \text{ of } 4 \text{ is } 10
      35\% \text{ of } 40 \text{ is } 14

   b) Yes
   c) $\frac{1}{2}$; half of 40 (20);
      Yes, it is reasonable.
      Sample explanation: $14 = 40 + 3; \text{ just as } 35\% = 100\% + 3$

AP Book NS8-90

page 18

1. b) $\frac{5}{10}$ are shaded
      $\frac{1}{2}$ are shaded
      5 is $\frac{1}{2}$ of 10
      $5:10 = 1:2$

   c) $\frac{3}{4}$ are shaded
      $\frac{1}{3}$ are shaded
      4 is $\frac{1}{3}$ of 12
      $4:12 = 1:3$

   d) $\frac{6}{8}$ are shaded
      $\frac{3}{4}$ are shaded
      6 is $\frac{3}{4}$ of 8
      $6:8 = 3:4$

AP Book NS8-91

page 20

1. Teacher to check arrows.
   a) 15
   b) 3
   c) 21
   d) 28
   e) 9
   f) 12
   g) 65
   h) 40

6. P W | % | F
   b) 7 | 20 | $\frac{7}{20} = \frac{20}{100}$
   c) ? | 24 | $\frac{6}{24} = \frac{6}{100}$
   d) 3 | ? | $\frac{3}{12} = \frac{12}{100}$
   e) 4 | 90 | $\frac{4}{90} = \frac{90}{100}$
   f) ? | 18 | $\frac{7}{18} = \frac{52}{100}$
   g) 7 | 25 | $\frac{7}{25} = \frac{25}{100}$

AP Book NS8-92

page 21

4. P W | % | F
   a) 4 | 20 | $\frac{4}{20} = \frac{20}{100}$
   b) 6 | ? | $\frac{6}{25} = \frac{25}{100}$
   c) ? | 10 | $\frac{17}{100}$
   d) 12 | 30 | $\frac{12}{50} = \frac{50}{100}$
   e) 80%
   f) 24
   g) 30

5. $25 \times 4 = 100$ so you just multiply 3 by 4 to get x.

6. a) $\frac{3}{15} = \frac{1}{5}$
   b) $\frac{6}{24} = \frac{1}{4}$
   c) $\frac{12}{30} = \frac{4}{10} = \frac{2}{5} = \frac{x}{100}$

7. P W | % | F
   b) ? | 48 | $\frac{48}{75} = \frac{75}{100}$
   c) ? | 30 | $\frac{3}{45} = \frac{60}{100}$
   d) 45 | 30 | $\frac{3}{4} = \frac{3}{40}$
   e) 45 | ? | $\frac{60}{100}$

8. a) 15
   b) 36
   c) 75%
   d) 9.6
   e) $\frac{7}{10} = \frac{63}{100}$
   f) $\frac{2}{3} = \frac{24}{100}$

9. a) 20
   b) 18
   c) 20
   d) 25%

Answer Keys for AP Book 8.2

V-5
10. 16% are blue;  
  84% are not blue.  
11. 81 students  
12. 20 students  

**AP Book NS8-92**  
**page 22**  
1. a) $a + b = c$  
   b) $a = b 	imes c = c 	imes b$  
2. c) $x = 11 \times 3$  
   d) $3x = 21$  
   e) $12 = 11x$  
   f) $x = 7 \times 9$  
   g) $24 = 8x$  
   h) $6 \times 7 = x$  
3. b) $x = 40$  
   c) $x = 2.5$  
   d) $x = 30$  
   e) $x = 4$  
   f) $x = 72$  
   g) $x = 55$  
   h) $x = 9$  
4. b) $3 \times 8 = 6 \times 4$  
   24 = 24  
   c) $1 \times 10 = 2 \times 5$  
   10 = 10  
   d) $2 \times 12 = 3 \times 8$  
   24 = 24  
   e) Answers will vary – teacher to check.  
5. a) $\neq$; no  
   b) $2 \times 25 = 10 \times 5$; yes  
   c) $9 \times 100 \neq 10 \times 81$; no  
   d) $5 \times 35 \neq 28 \times 7$; no  
   e) $3 \times 20 = 4 \times 15$; yes  
   f) $5 \times 42 = 6 \times 35$; yes  
   g) $91 \times 120 = 104 \times 105$; yes  
   h) $14 \times 48 \neq 30 \times 21$; no  
6. c) $11 \times 2 = 5x$  
   d) $4 \times 3 = 9x$  
   e) $5x = 3 \times 21$  

**AP Book NS8-93**  
**page 25**  
1. $\frac{1}{1000}$  
2. a) 0.3%, 0.003  
   b) 0.9%, $\frac{9}{1000}$  
   c) 0.25, 25%  
3. a) 27.3%  
   b) 84.8%  
   c) 36.9%  
   d) 40.5%  
   e) 0.5%  
   f) 12.5%  
   g) 7.7%  
   h) 24.2%  
4. Teacher to check.  
5. a) $\frac{35}{200} = \frac{7}{40}$  
   b) $\frac{7}{1000}$  
   c) $\frac{64}{1000} = \frac{8}{125}$  
   d) $\frac{4}{10000} = \frac{1}{2500}$  
   e) $\frac{3}{33} \times \frac{3}{3} = \frac{33}{100} \times \frac{3}{3} = \frac{1}{3}$  
6. Estimates may vary a bit – teacher to check.  
   a) $\approx 135\%$  
   b) $\approx 350\%$  
7. a) 110, 110%  
   b) 350, 350%  
   c) 261, 261%  
8. a) $\frac{3}{10}$  
   b) $\frac{3}{4}$  
   c) $\frac{2}{25}$  
   d) $\frac{15}{20}$  
   e) $\frac{17}{20}$  
9.  

<table>
<thead>
<tr>
<th>Percent</th>
<th>Mixed #</th>
<th>Decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>920%</td>
<td>$\frac{20}{100}$</td>
<td>9.2 = 9.20</td>
</tr>
<tr>
<td>232%</td>
<td>$\frac{32}{100}$</td>
<td>2.32</td>
</tr>
<tr>
<td>190%</td>
<td>$\frac{90}{100}$</td>
<td>1.90</td>
</tr>
<tr>
<td>535%</td>
<td>$\frac{35}{100}$</td>
<td>5.35</td>
</tr>
<tr>
<td>340%</td>
<td>$\frac{40}{100}$</td>
<td>3.4 = 3.40</td>
</tr>
</tbody>
</table>
9. continued

| 176% | $ \frac{176}{100} | 1.76 |
| 108% | $ \frac{108}{100} | 1.08 |

10. a) $= 438\%$
   b) $= 593\%$
   c) $= 501\%$
   d) $= 300\%$

11. a) $3.50 \times \frac{50}{100} = 3.5$
   b) $5.40 \times \frac{40}{100} = 2.16$
   c) $2.75 \times \frac{75}{100} = 2.0625$
   d) $3.60 \times \frac{60}{100} = 2.16$
   e) $5.15 \times \frac{15}{100} = 0.7725$

12. a) $250\%$
   b) $375\%$
   c) $830\%$
   d) $120\%$
   e) $2015\%$
   f) $1736\%$

13. a) $342\%$
   b) $3 + 0.3 = 3.33$
   c) $4 + 0.6 = 4.67$
   d) $1 + 0.2 = 1.22$
   e) $2 + 0.14 = 2.14$

14. a) $1.5, 1.73, 1.8$
   b) $1.6, 1.57, 1.62$
   c) $6.25, 6.09, 6.15$

15. a) $60$
   b) $125$
   c) $6.6$
   d) $6.6$

16. a) 50, 500
   b) 500
   c) 8

18. Teacher to check
   closeness of estimates.
   a) $150\%$
   b) $193\%$
   c) $850\%$
   d) $186\%$
   e) $293\%$
   f) $5391\%$

AP Book NS8-95

1. a) $37\%$
   b) $59\%$
   c) $71\%$
   d) $76\%$

2. Answers will vary based
   on where you live.
   Sample answers:
   Ontario's sales tax is $13\%$.
   a) $\$1.95$
   b) $\$5.20$
   c) $\$8.74$
   d) $\$10.73$

3. 43\% voted for Yen.

4.

<table>
<thead>
<tr>
<th>F</th>
<th>%</th>
<th>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drywall</td>
<td>$\frac{11}{20}$</td>
<td>55%</td>
</tr>
<tr>
<td>Paint</td>
<td>$\frac{3}{20}$</td>
<td>15%</td>
</tr>
<tr>
<td>Wallpaper</td>
<td>$\frac{3}{10}$</td>
<td>30%</td>
</tr>
</tbody>
</table>

5. He still has 80\% to raise.

6. Discount ($\$)$

| Shoes | $\$12.48$ | $\$37.44$ |
| CD | $\$4.47$ | $\$10.43$ |

7. Clare saved $533.33$.

8.

<table>
<thead>
<tr>
<th>F</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chair</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Table</td>
<td>$\frac{91}{200}$</td>
</tr>
<tr>
<td>Sofa</td>
<td>$\frac{59}{200}$</td>
</tr>
</tbody>
</table>

9. Erik now has
   580 Canadian stamps.
   They make up $\frac{580}{1700} = \frac{29}{85}$
   or $34\%$ of his collection.

AP Book NS8-96

1. a) $76\%$
   b) $18\%$
   c) $9\%$
   d) $15\%$
   e) $10\%$
   f) $21\%$

2. A: 24 vs 16 girls
   B: 12 vs 8 girls

3. Ron: $4 \times \frac{7}{4} = \$20$
   Ella: $3 \times \frac{7}{3} = \$15$

4.

<table>
<thead>
<tr>
<th>F</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soccer</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td>Swimming</td>
<td>$\frac{2}{5}$</td>
</tr>
<tr>
<td>Baseball</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Gymnastics</td>
<td>$\frac{3}{20}$</td>
</tr>
</tbody>
</table>

To find the number who
chose swimming, you can
simply double the number
who chose soccer.

AP Book NS8-97

1. a) $10$
   b) $21$
   c) $55$

2. a) $8$
   b) $20$
   c) $25$
   d) $25$

3. a) $\frac{3}{5}$
   b) $\$40$

4. $800$ students

5. a) $\frac{1}{12}$
   b) $\frac{7}{12}$
   c) $72$

6. $1250$ stamps

7. $500$ lights in total
1. a) $16; 80 – 16 = $64
   b) $64

2. $20.18

3. No, you need $28.50.

4. b) $64

5. $20.18

6. 25%

7. a) 200
   b) 220

8. $6.37

9. 9.5%

10. b) 6%
    c) 24%

11. He sold 8%.

AP Book NS8-100 page 34
1. a) 2 : 2 : 4,
    4 : 2 : 8
   b) 4 : 2 : 1,
    1 : 7 : 4

2. a) i) 3 : 5
    ii) 5 : 4
    iii) 4 : 3
    iv) 5 : 3
    v) 4 : 5
    vi) 3 : 4
   b) 20
   c) 9
   d) red = 15
    blue = 20
    green = 25
   e) green

3. a) 6 : 21
   b) 21 : 15
   c) 6 : 15
   d) 15 : 21
   e) 15 : 6
   f) 21 : 6

4. a) 4 : 8
   b) 16 : 9
   c) 3 : 15
   d) 7 : 8
   e) 11 : 30
   f) 72 : 6

5. a) $1.67 vs $2
    :: 1st offer is better.
   b) $16 vs $17
    :: 1st offer is better.
   c) $18.26 vs $15.40
    :: 2nd offer is better.

6. 7.5 min

7. 34,545.45 s ≈ 9.6 h

8. a) 21 mL
   b) 7 mL
   c) 210 mL

9. 7.5 km

10. a) 6 km walked / 1 h
    b) 15 km rowed / 1 h
    c) 1 cup of flour : 20 cups of flour
    d) 7 km / 30 L of gas
    e) 17 mL of ginger ale : 3 mL of orange juice

11. 12

12. $738.75

13. $1050

14. $4840

AP Book NS8-101 page 35
1. b) 6 km
   c) 5 km
   d) 14 km
   e) 1 kg

2. Measurements may vary; teacher to check.

Distance (map) | Distance (real life)
---|---
E – C | 1.7 cm | 204 km
E – J | 2.0 cm | 240 km
E – FM | 2.3 cm | 276 km

AP Book NS8-103 page 37
1. a) 6 km walked / 1 h
    b) 15 km rowed / 1 h
    c) 1 cup of flour : 20 cups of flour
    d) 7 km / 30 L of gas
    e) 17 mL of ginger ale : 3 mL of orange juice

2. a) 12
    b) $50 / 5 = 250
    c) $39 / 2 = 19.5
    d) $60 / 2 = 30
    e) $72 / 3 = 24
    f) $54 / 6 = 9

3. a) 72 km
    b) 72 days
    c) 1166.67 m
    d) $1 / 2 cup

4. David: 572.5 km
   Felicity: 638 km
   Jack: 1900 km

5. $318.2 s

6. $160 + 12 = 13.3
   :: he will need 14 cans of paint.

7. Grapes: $8.40
   Watermelon: $9.00
   Peaches: $16.20
   TOTAL: $33.60
1. a) 50  
b) 15  
c) 30; The number of people doubled, so you can expect the number choosing “Action” to also double (15 \times 2 = 30).  
d) To get frequency, divide each percent by 5 (100 ÷ 20).

<table>
<thead>
<tr>
<th>Type</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comedy</td>
<td>16</td>
</tr>
<tr>
<td>Horror</td>
<td>8</td>
</tr>
<tr>
<td>Action</td>
<td>15</td>
</tr>
<tr>
<td>Other</td>
<td>11</td>
</tr>
</tbody>
</table>

2. 

<table>
<thead>
<tr>
<th>Mark</th>
<th>Tally</th>
<th>Freq</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>///</td>
<td>11</td>
<td>55</td>
</tr>
<tr>
<td>C</td>
<td>///</td>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>D</td>
<td>//</td>
<td>2</td>
<td>10</td>
</tr>
</tbody>
</table>

3. a) Pop 7  

<table>
<thead>
<tr>
<th>Type</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pop</td>
<td>7</td>
</tr>
<tr>
<td>Hip Hop</td>
<td>9</td>
</tr>
<tr>
<td>Rock</td>
<td>3</td>
</tr>
<tr>
<td>Other</td>
<td>1</td>
</tr>
</tbody>
</table>

b) i) Frequency  
ii) Relative frequency; may not play 20 songs – want the correct proportion.

AP Book PDM8-7

1. a)  

<table>
<thead>
<tr>
<th>C</th>
<th>A</th>
<th>H</th>
<th>O</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>40</td>
<td>30</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>38</td>
<td>19</td>
<td>34</td>
<td>9</td>
<td>100</td>
</tr>
</tbody>
</table>

b) 100%; Yes, since the circle graphs include all the responses.

2. a) Yes  

b) No; A higher percentage at School A chose “Action,” but this doesn’t necessarily mean more people.  
c) No; The statement is false since less than half prefer “Action.”  
d) Yes

3. 48 + 21 + 26 + 9 = 104% but, to be correct, the percents must add to 100.

4. a) Yes  

b) No; A higher percentage at School A chose “Action,” but this doesn’t necessarily mean more people.  
c) No; The statement is false since less than half prefer “Action.”  
d) Yes

AP Book PDM8-8

1. 

<table>
<thead>
<tr>
<th>Type</th>
<th>Fraction</th>
<th>%</th>
</tr>
</thead>
</table>
| Mystery  | 16/40 = 2/5 | 40  
| Fantasy  | 4/40 = 1/10 | 10 |
| Romance  | 12/40 = 3/10 | 30 |
| Other    | 8/40 = 1/5 | 20  

b) Entertainment 45%, 162°  

AP Book PDM8-9

1. 

<table>
<thead>
<tr>
<th>Type</th>
<th>Percent</th>
<th>Angle</th>
</tr>
</thead>
</table>
| Walk     | 25%     | 90°   
| Bike     | 20%     | 72°   
| Bus      | 35%     | 126°  
| Car      | 15%     | 54°   
| Other    | 5%      | 18°   |

b) 25 + 20 + 35 + 15 + 5 = 100%  
c) 90 + 72 + 126 + 54 + 18 = 360°  
d) 72°; 20%; 126°; 35%; 54°; 15%; 18°; 5%  
e) Teacher to check.

3. 

<table>
<thead>
<tr>
<th>Habit</th>
<th>Percent</th>
<th>Angle</th>
</tr>
</thead>
</table>
| Delivered | 40%     | 144°  
| Occasionally | 50% | 180°  
| Never    | 10%     | 36°   |

Daily Newspaper Habit

4. a) 30, 30%  
b) 28, 28%  
c) 45, 45%  
d) 11, 44, 44%  
e) 4, 40, 40%  
f) 13, 65, 65%

5. a) 162  
b) 117  
c) 7, 63  
d) 7, 70

6. a)  

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Angle</th>
</tr>
</thead>
</table>
| Board    | 11/36 | 110°  
| Card     | 4/36  | 40°   
| Video    | 18/36 | 180°  
| Other    | 3/36  | 30°   |

Favourite Indoor Games
b) Fraction Angle

- Hockey 8/20 = 2/5 144°
- Swimming 7/20 126°
- Running 3/20 54°
- Other 2/20 = 1/10 36°

AP Book PDM8-9

Page 43

1. a) 25%
   b) 10%
   c) 77.7%

2. Answers may vary slightly – teacher to check.
   a) A: 35° = 9.72%
      B: 145° = 40.27%
      C: 127° = 35.27%
   b) PT FT
      A 29 271
      B 12 18
      C 21 39
   c) Company A: Circle graphs only show percents, not numbers. Although Company A only has 10% part-time staff (while B has 40%), it has 10 times as many employees.

3. a) Company A: 40
   - under 20: 108 30 12
   - 20–34: 108 30 12
   - 35–49: 144 40 16
   - 50–65: 0 0 0
   - over 65: 0 0 0

   Company B: 200
   - under 20: 20 5.6 11
   - 20–34: 105 29.2 58
   - 35–49: 145 40.3 81
   - 50–65: 35 9.7 19
   - over 65: 55 15.3 31

   AP Book PDM8-10

   Page 45

   1. a) Circle: the dot in the top/right hand corner of the plot, at (14,170).
      b) Kevin and Melanie: Their dots are right beside each other.
      NOTE: If they were actually one on top of the other, it would only look like one dot!

   INVESTIGATION

   A. a) 3
      b) Taller
      c) No
      d) Increase
      e) The plot shows a definite trend: it goes upward as it moves to the right.

   B. a) Not affected by age
      b) There is no pattern to the dots – just randomly placed.

   2. a) Decreases
      b) Is not affected
      c) Increases
      d) Is not affected
      e) Decreases
      f) Decreases
      g) Increases
      h) Increases
      i) Decreases (assuming the numerator remains the same)

   5. Selections will vary – teacher to check.

   6. a) Generally, that as a country’s area increases, so does its population.
   b) – e) see marks below

   7. Age vs Height
      Gender vs Height
      Gender vs Age

   V-10 Answer Keys for AP Book 8.2
Probability and Data Management – AP Book 8, Part 2: Unit 2 (continued)

a) Agree; In the 2nd plot, male heights are generally more than female heights.
b) Agree; In the 1st plot, there is an upward pattern.
c) Disagree; The 3rd plot shows that male ages are the same as female.

8. a) $1 200 000
b) Housing Prices in Toronto (Tasfia)
   ![Graph of Housing Prices in Toronto (Tasfia)]
   - [Graph Description]
   - [Analysis]
c) There is generally a price increase as the number of bedrooms increases.
d) Housing Prices in Toronto (Ahmed)
   ![Graph of Housing Prices in Toronto (Ahmed)]
   - [Graph Description]
   - [Analysis]
e) There is generally a price decrease as the subway distance increases.
f) In Q7, we saw that height relates to both age and gender, but that age and gender themselves had no relationship. This may be true here as well: that the two factors related to house price (e.g. # of bedrooms and subway distance) don’t relate to one another. Ahmed should compare these factors directly (ideally with a larger data sample) to see if any relationship exists.

g) Between 30 and 45, and between 105 and 120 – so the biggest change in gas consumption occurs between these points.
h) The line graph; the line shows a distinct pattern – one that isn’t clear from the scatter plot.

AP Book PDM8-11
page 48

1. a) Scatter plot:
   ![Graph of Gas Usage in Sarah’s Car]
   - [Graph Description]
   - [Analysis]
   - [Conclusion]
   - [Sample explanation]
b) They are identical except for the joining line in the 2nd graph.
c) The scatter plot shows no obvious relationship: gas doesn’t clearly increase or decrease in relation to speed.
d) ![Graph of Gas Usage in Sarah’s Car]
   - [Graph Description]
   - [Analysis]
   - [Conclusion]
   - [Sample explanation]

AP Book PDM8-12
page 49

1. a) Scatter plot:
   ![Graph of Sarah’s foot size and height]
   - [Graph Description]
   - [Analysis]
   - [Conclusion]
   - [Sample explanation]
   - [Graph of Sarah’s foot size and height]
   - [Graph Description]
   - [Analysis]
   - [Conclusion]
   - [Sample explanation]
b)  
<table>
<thead>
<tr>
<th>Interval</th>
<th>Freq</th>
</tr>
</thead>
<tbody>
<tr>
<td>0‒299</td>
<td>3</td>
</tr>
<tr>
<td>300‒599</td>
<td>6</td>
</tr>
<tr>
<td>600‒899</td>
<td>6</td>
</tr>
<tr>
<td>900‒1199</td>
<td>3</td>
</tr>
<tr>
<td>1200‒1499</td>
<td>2</td>
</tr>
</tbody>
</table>

c)  
Interval Freq  
0‒299 3  
300‒599 6  
600‒899 6  
900‒1199 3  
1200‒1499 2

d)  
Frequency of Weekly Salaries

<table>
<thead>
<tr>
<th>Hours spent on homework each week</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>0‒5 hours</td>
<td>2</td>
</tr>
<tr>
<td>5.1‒10 hours</td>
<td>5</td>
</tr>
<tr>
<td>10.1‒15 hours</td>
<td>4</td>
</tr>
<tr>
<td>15.1‒20 hours</td>
<td>5</td>
</tr>
<tr>
<td>20.1‒25 hours</td>
<td>3</td>
</tr>
<tr>
<td>25.1‒30 hours</td>
<td>2</td>
</tr>
<tr>
<td>30.1‒35 hours</td>
<td>1</td>
</tr>
</tbody>
</table>

e) The new data is "between" intervals: > $899 but < $900.

e) Answers may vary – teacher to check.  
Sample: “Frequency of Weekly Salaries”

f)  
Frequency of Weekly Salaries

<table>
<thead>
<tr>
<th>Weekly Salary ($)</th>
<th>Number of employees</th>
</tr>
</thead>
<tbody>
<tr>
<td>0‒499</td>
<td>2</td>
</tr>
<tr>
<td>500‒999</td>
<td>5</td>
</tr>
<tr>
<td>1000‒1499</td>
<td>3</td>
</tr>
<tr>
<td>1500‒1999</td>
<td>1</td>
</tr>
<tr>
<td>2000‒2499</td>
<td>2</td>
</tr>
<tr>
<td>2500‒2999</td>
<td>1</td>
</tr>
<tr>
<td>3000‒3499</td>
<td>1</td>
</tr>
<tr>
<td>3500‒3999</td>
<td>2</td>
</tr>
</tbody>
</table>

3. a)  
Histogram

b)  
Bar Graph

c)  
Both

d)  
Both

e)  
Histogram

f)  
Both

4. Answers will vary – teacher to check.  
Sample answers:

Similarities:
- Have labels, titles and a scale;
- Bars represent frequency of category;
- Bars are equal width.

Differences:
- In a histogram:
  - Bars touch each other;
  - Categories are numeric intervals, listed in order, and they're continuous and the same size.
- In a bar graph:
  - Bars have space between them;
  - Categories are discrete;
  - Order of categories doesn't matter.

5. a)  
Average

<table>
<thead>
<tr>
<th>Average</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>20‒39</td>
<td>1</td>
</tr>
<tr>
<td>40‒59</td>
<td>4</td>
</tr>
<tr>
<td>60‒79</td>
<td>9</td>
</tr>
<tr>
<td>80‒99</td>
<td>3</td>
</tr>
</tbody>
</table>

b)  
17

c)  
5

d)  
Has too few intervals

3. Intervals may vary – teacher to check.

<table>
<thead>
<tr>
<th>Times</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>90‒100</td>
<td>///</td>
<td>3</td>
</tr>
<tr>
<td>100‒110</td>
<td>/// ///</td>
<td>4</td>
</tr>
<tr>
<td>110‒120</td>
<td>/// /// /</td>
<td>6</td>
</tr>
<tr>
<td>120‒130</td>
<td>/// /// /</td>
<td>6</td>
</tr>
<tr>
<td>130‒140</td>
<td>/// /// /</td>
<td>6</td>
</tr>
<tr>
<td>140‒150</td>
<td>/// /// /</td>
<td>5</td>
</tr>
</tbody>
</table>

6. a)  
Average

<table>
<thead>
<tr>
<th>Average</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>40‒50</td>
<td>1</td>
</tr>
<tr>
<td>50‒60</td>
<td>1</td>
</tr>
<tr>
<td>60‒70</td>
<td>///</td>
</tr>
<tr>
<td>70‒80</td>
<td>/// ///</td>
</tr>
<tr>
<td>80‒90</td>
<td>/// ///</td>
</tr>
<tr>
<td>90‒100</td>
<td>///</td>
</tr>
</tbody>
</table>

b)  
17

c)  
5

d)  
Has too few intervals

4. Intervals may vary – teacher to check.

<table>
<thead>
<tr>
<th>Times</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0‒2</td>
<td>///</td>
<td>2</td>
</tr>
<tr>
<td>2‒4</td>
<td>/// ///</td>
<td>5</td>
</tr>
<tr>
<td>4‒6</td>
<td>/// ///</td>
<td>7</td>
</tr>
<tr>
<td>6‒8</td>
<td>/// ///</td>
<td>4</td>
</tr>
<tr>
<td>8‒10</td>
<td>///</td>
<td>2</td>
</tr>
</tbody>
</table>

7. a)  
Marks

<table>
<thead>
<tr>
<th>Marks</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>40‒50</td>
<td>/</td>
<td>1</td>
</tr>
<tr>
<td>50‒60</td>
<td>/</td>
<td>1</td>
</tr>
<tr>
<td>60‒70</td>
<td>///</td>
<td>3</td>
</tr>
<tr>
<td>70‒80</td>
<td>/// ///</td>
<td>5</td>
</tr>
<tr>
<td>80‒90</td>
<td>/// ///</td>
<td>3</td>
</tr>
<tr>
<td>90‒100</td>
<td>///</td>
<td>1</td>
</tr>
</tbody>
</table>

b)  
See grey box on page 49: it's on the interval's border.

c)  
AP Book PDM8-13 page 52

1.  
AP Book PDM8-13 page 52

<table>
<thead>
<tr>
<th>Age</th>
<th>Number of people</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

2. a)  
Doesn't cover the entire range of data

b)  
Contains intervals of different sizes

c)  
Has too many intervals

d)  
Has too few intervals

3. Intervals may vary – teacher to check.

<table>
<thead>
<tr>
<th>Hours</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0‒2</td>
<td>///</td>
<td>2</td>
</tr>
<tr>
<td>2‒4</td>
<td>/// ///</td>
<td>5</td>
</tr>
<tr>
<td>4‒6</td>
<td>/// ///</td>
<td>7</td>
</tr>
<tr>
<td>6‒8</td>
<td>/// ///</td>
<td>4</td>
</tr>
<tr>
<td>8‒10</td>
<td>///</td>
<td>2</td>
</tr>
</tbody>
</table>

5. a)  
Frequency Histogram

b)  
Relative Frequency Histogram

b) Answers will vary – teacher to check.
### Time to School (in minutes)

<table>
<thead>
<tr>
<th>Time to School</th>
<th>%</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>50–60</td>
<td>4</td>
<td>14.4°</td>
</tr>
<tr>
<td>40–50</td>
<td>8</td>
<td>28.8°</td>
</tr>
<tr>
<td>30–40</td>
<td>16</td>
<td>57.6°</td>
</tr>
<tr>
<td>20–30</td>
<td>20</td>
<td>72°</td>
</tr>
<tr>
<td>10–20</td>
<td>24</td>
<td>86.4°</td>
</tr>
<tr>
<td>0–10</td>
<td>28</td>
<td>100.8°</td>
</tr>
</tbody>
</table>

### AP Book PDM 8-14 page 54

1. **a)**

**Spending by Two Charities**

<table>
<thead>
<tr>
<th>Spending Category</th>
<th>Charity A</th>
<th>Charity B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fundraising</td>
<td>30%</td>
<td>25%</td>
</tr>
<tr>
<td>Medical Supplies</td>
<td>20%</td>
<td>15%</td>
</tr>
<tr>
<td>Administration</td>
<td>15%</td>
<td>40%</td>
</tr>
<tr>
<td>Supplies</td>
<td>35%</td>
<td>30%</td>
</tr>
</tbody>
</table>

2. **b)**

**Spending by Charity A**

- **Charity B % Angle**
  - Fundraising: 25% 90°
  - Shipping: 25% 90°
  - Admin: 10% 36°
  - Supplies: 40% 144°

3. **c) i) Less than 20 minutes (either circle or histogram)**

4. **d) No:**
   - the provinces are discrete categories

5. **e) Circle;**
   - care less about the amount of $ spent and more about the allocation of funds

6. **f) Charity B;**
   - a larger percentage will go to supplies

### Additional Information

- **Graph A**
- **Graph B**
- **Spending by Charity B**
- **Graph A**
- **Graph B**
- **Graph A**
- **Graph B**
- **Graph A**
- **Graph B**
Geometry – AP Book 8, Part 2: Unit 3

IMPORTANT NOTE:
For many questions in this unit, there are multiple correct answers, e.g. line segment AB can be written as BA, \( \angle RST \) is the same as \( \angle STR \), etc.
Where appropriate, teachers should be sure to check for similar equivalent answers in their students’ work.

AP Book G8-15
page 57
1. Answers will vary – teacher to check.
2. a) BA
   b) DE, ED
   c) ST, TS
3. a) Teacher to check.
   b) Line segment
4. 6.9 cm = 69 mm
5. Teacher to check.
6. a) For endpoint, circle:
   i) S
   ii) K
   iii) H
   iv) G
   b) ST
   iii) KJ
   iv) HG
7. a) RK
   b) BM, BQ
   c) DA, XD
8. Answers will vary – teacher to check.
9. a) AB, E
    b) RS, FG, C
   c) MN, Ji, I
10. a) EF, UT, S
    b) AH, DF, P
    c) IE, IG, I
11. Answers will vary – teacher to check.

Sample answers:

AP Book G8-16
page 59
1. Teacher to check.
2. b) Circle: \( \triangle VW, \triangle WWU \)
   c) Circle: E
   d) Circle: M
3. a) \( \angle HIJ \)
   b) \( \angle ACD \)
   c) \( \angle RMK \)
BONUS
There are 6 unique angles:
- 2 acute: \( \angle AEC, \angle DEB \)
- 2 obtuse: \( \angle AED, \angle CEB \)
- 2 straight: \( \angle AEB, \angle CED \)
4. Circle two:
   \( \angle ATB \) or \( \angle BTA \), \( \angle CTB \) or \( \angle BTC \)
5. a) Circle: B, C
   b) Circle: L, K, J
6. a) QR
   b) AB, BC, CA
   c) WX, XY, YW
7. b) KML
   c) YVZX
   d) DEF
8. a) Circle three:
   b) Answers will vary – teacher to check.
BONUS
Answers will vary – teacher to check.

Sample answers:

AP Book G8-17
page 61
1. a) Acute
   b) Obtuse
   c) Acute
   d) Obtuse
2. a) Acute; 30°
   b) Obtuse; 150°
   c) Obtuse; 130°
   d) Acute; 40°
3. a) 30°
   b) 130°
   c) 78°
4. a) 65°, 25°; 90°
   b) 30°, 60°; 90°
5. Teacher to check.

AP Book G8-18
page 63
1. Teacher to check.
   Sample sketches:
   a) \( \triangle ABC \)
   b) \( \triangle ABD \)
2. a) \( \triangle ABC \)
   b) \( \triangle ABD \)
3. a) \( \triangle ABC \)
   b) \( \triangle ABD \)
   c) \( \triangle ABC \)
4. a) 2 cm
   b) 2 cm
5. a) 2 cm
   b) \( \angle C \)
   c) \( \angle D \)
   d) \( \triangle ABC \)
   e) \( \triangle ABD \)
6. a) \( \triangle ABC \)
   b) \( \triangle ABC \)
    * not necessarily a rectangle
7. a) \( \triangle ABC \)
   b) \( \triangle ABC \)
   c) \( \triangle ABC \)
8. a) \( \triangle ABC \)
   b) \( \triangle ABC \)

BONUS
Answers will vary – teacher to check.

Sample answers:

AP Book G8-19
page 65
1. Teacher to check.
   Sample sketches:
   a) \( \triangle ABC \)
   b) \( \triangle ABD \)
2. a) \( \triangle ABC \)
   b) \( \triangle ABD \)
3. a) \( \triangle ABC \)
   b) \( \triangle ABD \)
4. a) 2 cm
   b) 2 cm
5. a) 2 cm
   b) \( \angle C \)
   c) \( \angle D \)
   d) \( \triangle ABC \)
   e) \( \triangle ABD \)
6. a) \( \triangle ABC \)
   b) \( \triangle ABC \)
    * not necessarily a rectangle
7. a) \( \triangle ABC \)
   b) \( \triangle ABC \)
8. a) \( \triangle ABC \)
   b) \( \triangle ABC \)

BONUS
Answers will vary – teacher to check.

Sample answers:

AP Book G8-20
page 71
1. Teacher to check.
   Sample sketches:
   a) \( \triangle ABC \)
   b) \( \triangle ABD \)
2. a) \( \triangle ABC \)
   b) \( \triangle ABD \)
3. a) \( \triangle ABC \)
   b) \( \triangle ABD \)
4. a) 2 cm
   b) 2 cm
5. a) 2 cm
   b) \( \angle C \)
   c) \( \angle D \)
   d) \( \triangle ABC \)
   e) \( \triangle ABD \)
6. a) \( \triangle ABC \)
   b) \( \triangle ABC \)
    * not necessarily a rectangle
7. a) \( \triangle ABC \)
   b) \( \triangle ABC \)
8. a) \( \triangle ABC \)
   b) \( \triangle ABC \)

BONUS
Answers will vary – teacher to check.

Sample answers:
Geometry – AP Book 8, Part 2: Unit 3 (continued)

12. Answers will vary – teacher to check.
   Sample answers:
   a) 
   ![Diagram A]
   b) 
   ![Diagram B]
   c) 
   ![Diagram C]
   d) 
   ![Diagram D]

13. Teacher to check additional information.
   a) 
   \[c^2 = a^2 + b^2\]
   \[= 3^2 + 4^2 = 25\]
   \[c = 5 \text{ cm}\]
   b) 
   \[c^2 = 5^2 + 12^2 = 169\]
   \[c = 13 \text{ cm}\]

14. a) No;
   B doesn’t show that the wall and ground are perpendicular.
   b) Says: wall = vertical, ground = horizontal.
   \[\therefore \] perpendicular
   c) A
   d) A;
   It’s simpler, with only the essential details.
   e) 
   \[x^2 + 5^2 = 13^2;\]
   \[x^2 = 13^2 - 5^2\]
   \[= 169 - 25 = 144\]
   \[x = 12 \text{ m}\]
   f) No; 12.5 m

AP Book G8-19
page 66
1. a) CD
   b) EF \perp KJ
   c) PQ \perp DE
   d) CB \perp AB

2. Teacher to check.
   a) The 4 middle angles are 90°.
   b) The 4 corner angles are 90°.
   c) The 4 corner and the 4 middle angles are all 90°.

3. B; D; A; C
4. Teacher to check.
5. a) Teacher to check.
   Sample explanation:
   The two lines are perpendicular because the angle was constructed to be 90°.
   b) Teacher to check.
6. a) 
   ![Diagram E]
   b) \[AB \perp BC\]
   \[BC \perp CD\]
   \[AD \perp AB\]
   \[AD \perp DC\]
   c) Yes;
   Since its angles are all 90°, any pair of adjacent sides will be perpendicular.
7. a) 
   \[40°;\]
   \[\angle ABC = \angle ABD + \angle DBC\]
   \[\therefore \angle DBC = (90 – 50)°\]
   b) Teacher to check.
8. a) \[\angle 1, \angle 2\]
   b) \[\angle B, \angle C\]
   c) \[\angle B, \angle C\]
   d) \[\angle 1, \angle 2, \angle 3, \angle 4\]
   e) \[\angle 1, \angle 5, \angle 5, \angle 1\]
   f) \[\angle 3, \angle 2, \angle 3, \angle 2\]
9. a) 45°
   b) 53°
   c) 50°

AP Book G8-20
page 69
1. a) \[\angle COB; \angle AOD;\]
   \[\angle BOD; \angle BOC\]
   b) \[\angle 1, \angle 5; \angle 3, \angle 7;\]
   \[\angle 1, \angle 5; \angle 3, \angle 7\]
   c) \[\angle ACB;\]
   \[\angle ACB + \angle BCE;\]
   \[\angle DCE + \angle BCE;\]
   \[\angle ACD + \angle DCE\]
2. a) 60°
   b) 115°
   c) 125°
   d) 112°, 72°
   e) 65°, 121°
   f) 52°
   g) 90°
   h) 65°, 65°
3. a) \[\angle 1, \angle 3, \angle 4, \angle 2;\]
   \[\angle 4, \angle 1; \angle 3, \angle 2\]
   b) \[180°, 180° - \angle 3;\]
   \[180°, 180° - \angle 3;\]
   \[\text{They are equal.}\]
   c) As straight angles, \[\angle 2 + \angle 3 = 180°;\]
   \[\text{and} \angle 2 + \angle 4 = 180°,\]
   \[\text{which means that:}\]
   \[\angle 2 + \angle 3 = \angle 2 + \angle 4\]
   \[\therefore \angle 3 = \angle 4\]
4. a) \[\angle DOB; \angle COB\]
   b) \[\angle APD = \angle CPB;\]
   \[\angle APC = \angle DPB\]
   c) \[\angle 4, \angle 3, \angle 8, \angle 7\]
   d) \[\angle 2 = \angle 5, \angle 1 = \angle 4;\]
   \[\angle 3 = \angle 6\]
   e) \[\angle DCE = \angle ACB;\]
   \[\angle DCA = \angle ECB\]

AP Book G8-21
page 71
1. Teacher to check line extensions; the lines will intersect.
2. C, A, B
3. Answers will vary – teacher to check.
4. a) 
   ![Diagram F]
   b) 
   ![Diagram G]
   c) 
   ![Diagram H]
   d) 
   ![Diagram I]
5. a) XY
   b) \[FG \parallel RS\]
   c) \[LM \parallel GH\]
6. \[AD = BE = CF = 1.5 \text{ cm};\]
   They are all equal.

AP Book G8-22
page 72
1. \[\angle 9, \angle 3 \angle 4, \angle 10\]
   \[\angle 13, \angle 7 \angle 8, \angle 14\]
   \[\angle 3, \angle 9 \angle 4, \angle 10\]
   \[\angle 13, \angle 7 \angle 8, \angle 14\]

INVESTIGATION 1
A. \[\angle 3, \angle 4, \angle 7, \angle 8\]
B. \[100°, 80°, 100°, 80°\]
   \[80°, 100°, 80°, 100°\]
C. equal
Geometry – AP Book 8, Part 2: Unit 3

INVESTIGATION 2

A. \( \angle 3, \angle 4 \)
B. \( 45^\circ, 135^\circ, 75^\circ, 105^\circ \)
C. No (since the rays aren’t parallel)

3. a) \( 115^\circ \)
   b) \( C \)
   c) \( \) No

INVESTIGATION

1. a) 180°, 180°; 180°, 150°; 30°
   b) 70°
   c) 180°, 90°, 40°; 50°
   d) 35°; 60°; 85°
   e) 55°; 35°
   f) 43°; 43°; 69°
   g) 3; equilateral
   h) 2; isosceles

AP Book G8-24

1. 180°, 180°; 180°, 150°; 30°
2. a) 70°
   b) 30°; 30°; 120°
   c) 180°, 90°, 40°; 50°
   d) 35°; 60°; 85°
   e) 55°; 35°
   f) 43°; 43°; 69°
3. a) 90°
   b) 90°
   c) 60°
4. a) \( \angle DCA \); equal; \( \angle DCA \)
   b) \( \angle ECB \); equal; \( \angle ECB \)
   c) \( \angle ACD + \angle ACB + \angle BCE = 180° \)
5. a) 93°
   b) 55°; 125°
   c) 45°
   d) 70°; 60°
   e) 120°
6. a) \( \angle 1 + \angle 3 = 180° \)
   b) \( \angle 1 + \angle 2 = 180° \)
   c) \( \angle A + \angle B + \angle C = 180° \)

AP Book G8-25

page 78

INVESTIGATION

A. G
   B. D, F
   C. E, G
   D. A, B

C. Isosceles; equilateral

1. a) 120°; obtuse
   b) 90°; right
   c) 80°; acute
2. a) no (or 0); scalene
   b) 2; isosceles
   c) 3; equilateral
3. 4 m or 7 m
   (see G8-18 #9 for details)
4. Teacher to check.
5. a) Right acute obtuse
   b) Neither an obtuse or a right equilateral triangle is possible;
      An equilateral triangle always has 3 equal angles, each
      one measuring 60° (which is acute).
6. \( \triangle ABC \):
      right, scalene
      \( \triangle DEF, \triangle GEF \):
      right, scalene
      \( \triangle DEF \):
      obtuse, isosceles
      \( \triangle HJK \):
      right, isosceles
### Answer Keys for AP Book 8.2 V-17

**Geometry – AP Book 8, Part 2: Unit 3**

1. **ΔNLM:**
   - right, scalene
2. **ΔNLK:**
   - right, isosceles
3. **ΔKMN:**
   - obtuse, scalene

   Since $\angle KNM = 105^\circ$

4. a) Teacher to check.
   b) i) Yes; 2 options
      ii) No
   iii) Yes; Infinite options as $AB$’s length changes

5. **Explanations and sketches may vary – teacher to check.**

6. a) No
   
   **Sample answer:**
   
   ![Diagram](image)

   In a triangle, the angles must add to $180^\circ$ so, if one is greater than $90^\circ$ (obtuse), the sum of the other two must be less than $90^\circ$ – meaning they’re both acute.

   b) No
   
   **Sample answer:**
   
   ![Diagram](image)

   Again, the angles’ sum must be $180^\circ$ so, if one is $90^\circ$, the other two must add to $90^\circ$ – so they must both be acute.

7. a) Triangle 1: $30^\circ$, $120^\circ$
   b) $35^\circ$, $35^\circ$
   c) In a), we are given an acute angle.
      In b), we are given an obtuse angle.
      They have a different number of solutions because, if a given triangle has an obtuse angle,
      its other two angles must be acute – this is why b) has only one solution.
      In a), we’re given an acute angle, which might either be one of the two equal angles in the isosceles triangle, or the unequal angle – this is why it has two possible solutions.

8. **AP Book G8-26 page 80**

   1. b) $57^\circ$ (OAT)
      $80^\circ$ (SATT)
      c) $55^\circ$ (CAT)
      $65^\circ$ (SATT)
      d) $75^\circ$ (AAT)
      $45^\circ$ (AAT)
      $60^\circ$ (SAT or SATT)
      e) $119^\circ$ (SA)
      $119^\circ$ (AAT)
      $119^\circ$ (CAT)
      f) $55^\circ$ (OAT)
      $95^\circ$ (AAT)
      $30^\circ$ (SATT)
      g) $115^\circ$ (CAT)
      $65^\circ$ (SA)
      $115^\circ$ (OAT)
      $65^\circ$ (CAT or SA)
      h) $35^\circ$ (SATT)
      $35^\circ$ (AAT)
      $55^\circ$ (AAT or SATT)
      $125^\circ$ (SA)
      i) $81^\circ$ (AAT)
      $39^\circ$ (SATT)
      $39^\circ$ (AAT)
      $60^\circ$ (SAT)

   2. Specific theorems used by students may vary – teacher to check.
   
   Also teacher to check any numbered “helper” angles marked by students.

   a) $\angle 1 = 67^\circ$ (EAT)
   $\angle 2 = 83^\circ$ (SATT)
   b) $\angle 1 = 32^\circ$ (CAT)
   $\angle 2 = 58^\circ$ (CAT)
   c) $\angle 1 = 75^\circ$ (AAT/SAT)
   $\angle 2 = 130^\circ$ (SAT)

   3. Again, specific theorems used may vary – teacher to check.
      a) $\angle 1 = \angle 2 = 37^\circ$ (ITT)
      b) $\angle 1 + \angle 2 = 60^\circ$ (EAT)
      $\angle 1 = \angle 2 = 30^\circ$ (ITT)
      c) We know that the other two angles in the triangle are each $55^\circ$ (ITT), so:
      $\angle 1 = \angle 2 = 125^\circ$ (SAT)
      d) $\angle 1 = 80^\circ$ (EAT)
      $\angle 2 = 50^\circ$ (ITT)

   4. **NOTE:**
      In c) and d), the missing angles can be found using ITT only – students don’t necessarily have to solve for $x$.
      b) by SATT:
      $70 + (x + 20) + (3x + 10) = 180$
      $4x + 100 = 180$
      $4x = 80$
      $x = 20$
      \[ \therefore \text{angles are } 70^\circ, 40^\circ \text{ and } 70^\circ. \]
      c) by ITT:
      $6x + 5 = 35$
      $6x = 30$
      $x = 5$
      \[ \therefore \text{angles are } 35^\circ, 35^\circ \text{ and } 110^\circ. \]

   5. **Similarities:**
      ✓ Both are quadrilaterals.
      **Differences:**
      ✓ A parallelogram has two pairs of parallel sides; a trapezoid has only one.
      ✓ A parallelogram has equal opposite sides and angles; this isn’t true for a trapezoid.

   6. a) It’s a parallelogram with 4 right angles.
      b) Their sides are not all equal.
      c) A trapezoid has one pair of parallel sides; a parallelogram has two.
      d) It’s a parallelogram with all sides equal.

   7. **AP Book G8-27 page 82**

      1. Teacher to check marked angles and sides.
         a) rectangle
         b) parallelogram
         c) square
         d) (right) trapezoid
      2. a) rhombus
         b) rectangle
      **BONUS**
      **Trapezoid**
      3. a) all
         b) no
         c) no
         d) some
      4. Teacher to check.
      5. **Similarities:**
         ✓ Both are quadrilaterals.
      **Differences:**
         ✓ A parallelogram has two pairs of parallel sides; a trapezoid has only one.
         ✓ A parallelogram has equal opposite sides and angles; this isn’t true for a trapezoid.
      6. a) It’s a parallelogram with 4 right angles.
         b) Their sides are not all equal.
         c) A trapezoid has one pair of parallel sides; a parallelogram has two.
      d) It’s a parallelogram with all sides equal.
      7. **Teacher to check for proper sketches.**
         a) One square plus many different rhombuses

---

**AP Book G8-27 page 82**

1. Teacher to check marked angles and sides.
2. a) It’s a parallelogram with 4 right angles.
   b) Their sides are not all equal.
   c) A trapezoid has one pair of parallel sides; a parallelogram has two.
   d) It’s a parallelogram with all sides equal.
3. Teacher to check for proper sketches.
   a) One square plus many different rhombuses
INVESTIGATION

A. Teacher to check.
   In the top parallelogram, the opposite angles are 130° and 50°.
   In the other, the opposite angles are 110° and 70°.

   a) It doesn't talk about a rectangle being a parallelogram.
   b) No (185°; 4 e/a A, C
      2 pairs e/a B, E, I
      No e/a F, G
      4 r/a A, C
      No r/a B, D, F, H
   c) The kites and any shapes with an indentation aren't parallelograms.
   d) No (185°; No r/a B, D, E, G, I
   e) Trapezoid

   a) Teacher to check.
   b) Yes, this will always be true. (A square is also a rectangle.)
   c) Using the following sketch,
      \[ \triangle \]
      and the reminder at the top of page 83:
      \[ \angle A + \angle D = 180^\circ \]
      \[ AB \parallel DC \]
      \[ \angle A + \angle B = 180^\circ \]
      \[ AD \parallel BC \]
      So \( ABCD \) must be a parallelogram.
      Since \( ABCD \) is a parallelogram and it has 4 right angles, we know \( ABCD \) is a rectangle.
      \( A, B, E \)

   B. 2; 2
   C. 180°; 180°; \( \angle BAD; 180^\circ; \angle ADC \)

1. Explanations and (counter) examples may vary – teacher to check.
   a) True;
      Sample explanation:
      In a parallelogram, the opposite sides are always parallel.
      \[ \triangle \]
      So, from the reminder box on page 83, we know that each pair of adjacent angles \( a/b, b/c, c/d \) and \( d/a \) adds to 180°.
   b) False;
      Sample counter-example:
      Isosceles trapezoids have 2 pairs of equal angles but are not parallelograms.
   c) False;
      Sample counter-example:
      In the above trapezoid, we see that \( \angle a + \angle b \neq 180^\circ \).
   d) False;
      Sample counter-example:
      Here, the two sets of opposite angles add to 180° but this quadrilateral is a kite, not a trapezoid or a parallelogram.
   e) Trapezoid
   f) No (185°; No r/a B, D, E, G, I
   g) Yes; Yes
   h) Two correct options:
      \( \triangle ABC \) and \( \triangle BCD \) or \( \triangle ADC \) and \( \triangle BAD \)

9. a) Two correct options:
    \( \triangle ABC \) and \( \triangle BCD \) or \( \triangle ADC \) and \( \triangle BAD \)
    b) Yes; Yes
    c) Two correct options:
       \( \triangle ABC \) and \( \triangle BAD \) or \( \triangle ADC \) and \( \triangle BCD \)
    d) No (185°); No r/a B, D, E, G, I
    e) Trapezoid
   f) No (185°; No r/a B, D, E, G, I
   g) Yes; Yes
   h) Two correct options:
      \( \triangle ABC \) and \( \triangle BCD \) or \( \triangle ADC \) and \( \triangle BAD \)

10. a) Teacher to check.
    b) 

AP Book G8-28
page 84

BONUS

If a quadrilateral has two pairs of equal angles (let's call them pair \( a \) and pair \( b \)), then only two possible cases exist:
1. the pairs of equal angles are opposite each other;
2. the pairs of equal angles are adjacent to each other.

Case 1: Opposite
Looking clockwise, the angle order is \( \angle a, \angle b, \angle c, \angle d \).
As a quadrilateral, we know:
\[ \angle a + \angle b + \angle a + \angle b = 360^\circ \]
\[ 2(\angle a + \angle b) = 360^\circ \]
\[ \angle a + \angle b = 180^\circ \]
From this, as in #2 c) above, we see that all four adjacent angle pairs add to 180°, so opposite sides will be parallel.
\( A, B, E \)

Case 2: Adjacent
Looking clockwise, the angle order here is \( \angle a, \angle b, \angle c, \angle d \).
As a quadrilateral, \( \angle a + \angle b = 180^\circ \) also applies here. So, with two adjacent (and distinct) angle pairs adding to 180°, this shape will have one set of parallel sides.
\( A, B, E \)

Teacher to check drawn diagonals.
The kites have diagonals that intersect at a right angle.

6. a) \( a = 5 \) units 
   \( c = 2 \) units 
   \( b = d = 3 \) units 
   b) Correct prediction: A kite's diagonals are perpendicular and one of the diagonals bisects the other.
   Teacher to check drawings of the two other kites.
c) Teacher to check. Students should notice that kites always have one pair of equal, opposite angles.

7. a), b) Rhombus
b) Rhombus

8. a) Square
b) Rhombus
c) Square
d) Rhombus

AP Book G8-29
page 86

1. 6 cm; 3 cm; 3 cm; 6 cm

2. Midpoint at 4 cm; Teacher to check.

3. a) C, DB; N, AB
b) No, there is nothing to indicate that FE = ED.

BONUS
No, it can’t. By definition, a line is infinite and has no endpoints. It can have no midpoint.

4. Teacher to check.

5. a) 

b) Kite

BONUS
\(O^2 + O^2 = O^2 + O^2\)
\(X^2 + Y^2 = Z^2 + Y^2\)
\(X = Y \)
\(O^2 + O^2 = O^2 + O^2\)
\(X = Z \)

AP Book G8-30
page 88

1. Teacher to check.

2. EF, FG, EG

3. \(\angle P = \angle R, \angle Q\)

4. \(\angle D = \angle S, \angle E = \angle L, \angle F = \angle M\)

5. \(\angle W = \angle R, \angle X = \angle T, \angle Y = \angle S; WY = RS, WX = TR, XY = TS\)

6. No; she forgot to check if the corresponding sides were equal.

7. a), b) 

7. The third angle in any triangle will be equal to \(180° - \angle 1 - \angle 2\).

In this case, \(\angle B = \angle S\) and \(\angle C = \angle T\), so:
\(\angle A = 180° - \angle B - \angle C\)
\(= 180° - \angle S - \angle T\)
\(= \angle R\)
\(\angle A = \angle R\), even though they aren’t marked.

8. 5: \(\triangle WXY \cong \triangle RTS\)
7: \(\triangle ACB \cong \triangle RTS\)

9. a) i) \(\angle C = \angle F\)
These four angles are equal.
ii) Sample:
\(\triangle CBA \cong \triangle FED\)
iii) Sample:
\(\triangle ABC \cong \triangle DEF\)

b) i) \(\angle G = \angle J\)
All six angles are equal.
ii) Sample:
\(\triangle HGF \cong \triangle LJK\)
iii) Sample:
\(\triangle GHI \cong \triangle JKL\)

10. Corresponding sides
\(\triangle PQR \cong \triangle KLM\)

Corresponding angles
\(\angle P = \angle K, \angle Q = \angle L, \angle R = \angle M\)

11. VW = YZ \(\angle V = \angle Y\)
UV = XZ \(\angle W = \angle Z\)

12. None of Tom’s statements are correct;
\(\triangle ABC \cong \triangle XWV\)

AP Book G8-31
page 91

1. a) SAS; \(\triangle DEF\)
b) ASA; \(\triangle RST \cong \triangle XYZ\)
c) SSS; \(\triangle MNO \cong \triangle FGH\)

2. a) No;
They aren’t equal.
b) Yes
9. a) \[ \begin{array}{c} \text{a)} \hfill \hfill \\
\text{b)} \hfill \hfill \\
\text{c)} \hfill \hfill \\
\text{d)} \hfill \hfill \\
\end{array} \]

b) \[ \begin{array}{c} \text{a)} \hfill \hfill \\
\text{b)} \hfill \hfill \\
\text{c)} \hfill \hfill \\
\text{d)} \hfill \hfill \\
\end{array} \]

10. No, it depends where the equal side and angle are positioned within the triangles.

Case 1:

Here, \( \triangle ABC \cong \triangle DEF \).

Case 2:

Here, \( \triangle ABC \not\cong \triangle DEF \).

INVESTIGATION

A. \( \angle A = \angle D, \ CB = EF \) and \( AB = DE \)

B. No: the corresponding angle isn’t between the corresponding sides.

C. No

D. No, neither has its corresponding angle between its corresponding sides — only SAS is a proper congruence rule.

11. Yes, by SSS and the Pythagorean Theorem; The third side of both triangles is \( \sqrt{13^2 - 5^2} = 12 \) cm long.

12. Yes, by SSS and the Pythagorean; Both triangles have sides that are 9 cm, 12 cm and 15 cm in length.

13. a) No, by SSS and the Pythagorean; One’s sides are 1, 2 and \( \sqrt{5} \), and the other’s sides are 1, \( \sqrt{2} \) and \( \sqrt{3} \).

b) No, by SATT/ITT; One has angles that are 80°, 50° and 50°. The other’s are 70°, 55° and 55°.

AP Book G8-32

page 94

1. a) \( 53^\circ; \) \( 53^\circ + 53^\circ = 106^\circ \)

b) \( 75^\circ; \) \( 75^\circ + 75^\circ = 150^\circ \)

c) \( 20^\circ; \) \( 20^\circ + 20^\circ = 40^\circ \)

2. a) No; In this rectangle, the diagonal is at a \( 20^\circ/70^\circ \) angle to the vertex.

b) Yes; A square’s diagonal is always at a 45° angle to the vertex. (Teacher to check drawing.)

3. a) ASA

b) SAS

c) SSS

d) SSS

4. \( CB, DC; \) \( CBD; \) \( CBD, SSS; \) \( CBD; \) \( \angle CBD, 90^\circ; \) \( AC; \) \( AC \)

5. a) \[ \begin{array}{c} \text{A)} \hfill \hfill \\
\text{b)} \hfill \hfill \\
\text{c)} \hfill \hfill \\
\text{d)} \hfill \hfill \\
\end{array} \]

6. a) F

b) T

c) F

d) T

e) T

BONUS

Explanations will vary — teacher to check.

7. They are equal.

a) \( DC; \) \( \angle DAC = \angle DCA \)

b) \( BC; \) \( \angle BCA = \angle DAC \)

c) \( \triangle ABC \cong \triangle DCA \) (ASA)

d) \( BC = DA, AB = CD; \) Yes

8. a) Measurements may vary slightly but relationships must remain to be correct.

\( AE = CE = 2.8 \) cm

\( BE = DE = 1.5 \) cm

AP Book G8-33

page 97

INVESTIGATION 1

A. \( AC = BC \)

B. They will be equal (think of an isosceles triangle).

C. \( \triangle ACM \cong \triangle BCM \) by SAS: \( AM = BM \) (given), \( \angle AMC = \angle BMC \) (given), and side CM is common. This congruency means that corresponding sides \( AC \) and \( BC \) are equal.

INVESTIGATION 2

A. \( BC \)

B. Isosceles

C. 

D. No, \( AB \) and \( CD \) would have to be perpendicular; \( \triangle ADC \cong \triangle BCD \) (SSS)

\( \angle CDA = \angle CDB \), but this can only happen if both angles equal 90°, meaning \( AB \perp CD \).
1. a) All three intersect at the same point.

2. a) In all cases, the centre O lies on the perpendicular bisector of AB. This happens because OA = OB (both are radii) and all points equidistant from A and B lie on their perpendicular bisector.

3. a) All three intersect at the same point.

4. NO: By drawing ΔKLM and its three sides' perpendicular bisectors, you identify the centre of the circle through K, L and M.

BONUS
Teacher to check drawings.
NOTE:
The centre of a right triangle’s circumcentre is at the midpoint of the triangle’s hypotenuse.

AP Book G8-34

1. a) \( \text{AB}: \text{EF} = 2 : 6 = 1 : 3 \)
   \( \text{BC}: \text{FG} = 1 : 3 \)
   \( \text{CD}: \text{GH} = 2 : 6 = 1 : 3 \)
   \( \text{AD}: \text{EH} = 1 : 3 \)
   b) Yes
   c) By definition, all of a rectangle’s angles are 90°.

2. Side ratio = 1 : 2 = 2 : ?
So the length of the unknown side, indicated by \( ? = 2 \times 2 \text{ cm} = 4 \text{ cm} \).

3. a) 6 cm
   b) 12 cm
   c) 15 cm

4. No; A square has all sides equal, while a rectangle only needs its opposite sides to be equal.

5. No; A square always has two pairs of parallel sides but, by definition, a trapezoid has exactly one pair of parallel sides.

AP Book G8-35

13. a) NOTE:
Measurements may vary slightly but the 2 : 3 proportion must remain.

14. a) \( x = 20, y = 25 \)
b) \( x = 16, y = 12 \)

15. a) No: they have only one equal angle.
b) No: not all of the corresponding sides are proportionate.

INVESTIGATION

A. Teacher to check.
In all cases, the resulting triangles should be similar with a scale factor of 2.
Sample answer:
   a) \( \text{AB} = 3 \text{ cm} \)
   \( \text{AC} = 5 \text{ cm} \)
   \( \text{BC} = 5.7 \text{ cm} \)
   b) \( \text{AB}' = 6 \text{ cm} \)
   \( \text{AC}' = 10 \text{ cm} \)
   \( \text{BC}' = 11.4 \text{ cm} \)
   c) \( \angle A = \angle A' = 86° \)
   \( \angle B = \angle B' = 62° \)
   \( \angle C = \angle C' = 32° \)
   Yes
d) Scale factor = 2

B. a) \( \text{AB} = 4.8 \text{ cm} \)
   \( \text{AC} = 2.2 \text{ cm} \)
   \( \text{BC} = 5.4 \text{ cm} \)
b) Teacher to check
ΔA'B'C':
A'B' = 14.4 cm
A'C' = 6.6 cm
c) B'C' = 16.2 cm
d) No, since they meet the SSS rule in A above, we know they are similar.
e) Teacher to check.

C. a) Teacher to check.
b) Teacher to check.
c) Yes:
\[ \angle M = \angle M' = 80^\circ \]
\[ \frac{K'M'}{KM} = \frac{10.8}{2.7} \]
\[ \frac{M'L'}{ML} = \frac{8}{2} \]
d) Yes:
\[ \angle M = 80^\circ, \angle M' = 80^\circ \] (sum of the triangles' angles must be 180°)
d) Yes;
Scale factor will vary – teacher to check.
e) Teacher to check.

D. a) Teacher to check.
b) While ΔK'L'M' side lengths will vary, all three ratios will be equal if correct.
c) Yes;
Scale factor will vary – teacher to check.

1. a) SAS
b) AA
c) SSS
2. a) \[ \frac{A'B'}{AB} = \frac{1.65}{3.3} \]
\[ \frac{B'C'}{BC} = \frac{2.35}{4.7} \]
Once reduced, both equal 0.5.
b) \[ \angle C = \angle C' = 19^\circ \]
c) No
d) No

3. No;
In Question 2, we showed that ASS isn’t sufficient so AS certainly isn’t.

4. Teacher to check.
Sample example:
\[ \begin{array}{c}
A' \\
\downarrow \\
\rightarrow \\
B' \\
\end{array} \]
Here we can see that:
\[ \frac{A'B'}{AB} = \frac{10.8}{2.7} = 2 \]
but obviously \( A' \neq A \) so the triangles aren’t similar.

5. a) \[ \begin{array}{c}
A \\
\downarrow \\
\rightarrow \\
B \\
\end{array} \]
b) \[ \triangle A \]
\[ \angle A \]
is common
\[ \angle ABD = \angle ACE \]
\[ \angle ADB = \angle AEC \]
c) They are similar.

6. a) \( \triangle K \)
b) 2 : 1 ; 2 : 1
c) They are similar, by SAS.
d) \[ KN = 2 \times KO \]
\[ \frac{KO}{KN} = \frac{1}{2} \]
e) Two correct options:
\[ \angle KLO \text{ and } \angle M \text{ or } \angle KOL \text{ and } \angle N \]
Each corresponding pair is equal.
f) \[ MN \parallel LO \text{ (CAT); } \]
LMNO is a trapezoid.

7. Answers may vary – teacher to check.
NOTE:
Teachers should see an example used in each student explanation.
Sample answer - SSSS:
This square and rhombus have a side proportion of 1:2 but their angles are different.

NOTE: 2a = b

INVESTIGATION 2

A. \[ \triangle ABC: \] AC = 3.3 cm
BC = 4.3 cm
AB = 6.8 cm

\[ \triangle DEF: \] FD = 5.1 cm
DE = 3.2 cm
FE = 2.5 cm
B. \[
\frac{AB}{FD} = \frac{6.8}{5.1} = 1.33 \\
\frac{AC}{FE} = \frac{3.3}{2.5} = 1.32 \\
\frac{BC}{DE} = \frac{4.3}{3.2} = 1.34
\]
\[
\therefore \text{yes, they are similar} \quad \text{their corresponding sides are proportionate.}
\]
C. \[
s = \frac{4}{3} = 1.3
\]
D. 14.4 cm, 10.8 cm
E. \[
P(DABC) = 14.4 \\
\frac{P(DEF)}{P(DABC)} = \frac{10.8}{14.4} = 0.75 = s
\]
\[
\therefore P(DEF) = \frac{P(DABC)}{s}
\]
F. Teacher to check pair of shapes drawn.

Perimeter ratio
\[
= \frac{\text{Per A}}{\text{Per B}} = 1 : 5
\]

Sample answer:

```
A (original) : B (image) 
= 2 : 10 = 3 : 15 
= 1 : 5 
\therefore their scale factor is 5.
```

Perimeter A (original) = 10
Perimeter B (image) = 50
\[
\therefore \text{the ratio between their perimeters is also 1 : 5.}
\]

**General rule:**
For similar shapes, the ratio between the perimeters is the same as the ratio between the corresponding sides.

4. a) \[
\begin{align*}
\angle BAC &= \angle ACD \text{ (AAT)} \\
\angle BAC &= \angle BCA \text{ (ITT)} \\
\text{Since it's given that} \angle BCD &= 2 \angle ADC, \text{we also know that} \angle ADC &= \angle ACD. \\
\therefore \triangle ABC \text{ and } \triangle DAC \text{ are similar (AA)}.
\end{align*}
\]

b) \[
\begin{align*}
\angle BAC &= \angle ACD \text{ (AAT)} \\
\angle BAC &= \angle BCA \text{ (ITT)} \\
\text{Since it's given that} \angle BCD &= 2 \angle ADC, \text{we also know that} \angle ADC &= \angle ACD. \\
\therefore \triangle ABC \text{ and } \triangle DAC \text{ are similar (AA)}.
\end{align*}
\]

c) Yes: \(\triangle ABD\) and \(\triangle BCD\) are similar, since they are both similar to \(\triangle ABC\).

5. a) 90°; 90°; \(\angle BCA\); They're similar (AA).

b) \[
\begin{align*}
\angle C &= \angle C \text{ (common)} \\
\angle ABC &= \angle BDC = 90°; \\
\text{They're also similar (AA).}
\end{align*}
\]

c) Yes: \(\triangle ABD\) and \(\triangle BCD\) are similar, since they are both similar to \(\triangle ABC\).

6. 2.88 m = 288 cm

7. a) \[
\begin{align*}
\angle PQR &= \angle PSQ \\
\angle P &= \angle P \text{ (shared)} \\
\therefore \triangle PQS \text{ and } \triangle PRQ \text{ are similar (AA).}
\end{align*}
\]

b) \[
\begin{align*}
\angle PQR &= \angle PSQ \\
\angle PRO &= \angle PQS, \\
\angle RPQ &= \angle QPS \\
\therefore \triangle PQS \text{ and } \triangle PRQ \text{ are similar (AA).}
\end{align*}
\]

c) \[
\begin{align*}
\frac{QR}{PS} &= \frac{PR}{PS} = \frac{PR}{PS} \\
\therefore \triangle PQS \text{ and } \triangle PRQ \text{ are similar (AA).}
\end{align*}
\]

d) \[
\begin{align*}
PQ &= 10 \text{ cm} \\
\frac{PQ}{PS} &= \frac{10}{8} = 1.25 \\
\end{align*}
\]

e) \[
\begin{align*}
24 \text{ cm}^2 \\
(1.25)^2, 37.5 \text{ cm}^2 \\
\end{align*}
\]

f) \[
\begin{align*}
24 \text{ cm}^2 \\
\text{Explanations may vary -- teacher to check.} \\
\angle PSQ &= \angle QSR = 90° \text{ (given)} \\
\angle PQS + \angle QRS &= 90° \text{ (given)} \\
\angle PQS + \angle QPS &= 90° \text{ (SATT)} \\
\therefore \angle QPS &= \angle QRS \therefore \triangle PQS \text{ and } \triangle PRQ \text{ are similar (AA).}
\end{align*}
\]

From this similarity, we also know that \(\angle SQP = \angle SRQ\).
1. a) 1
   b) –8
   c) –3
   d) –5
   e) 2
   f) 8
   g) –17
   h) –15
   i) 26
   j) 19
   k) –33
   l) 39

2. a) –6
   b) –12
   c) 12
   d) –63
   e) –8
   f) 5
   g) –6
   h) –2
   i) 18
   j) 8
   k) –33
   l) –7

3. b) –3
   c) –18
   d) 4
   f) 6
   g) –2
   h) –19
   i) –14
   j) –51
   k) –9
   l) –26
   m) 8
   n) 9
   o) 25

BONUS 30

q) –2
r) –1
s) –9
t) –5
u) 2

9. a) False; Counter-example will vary – teacher to check.
   Sample: If \( a = –2 \), then
   \[ 3a = –6 > –10 = 5a \]
   b) True
c) True
d) False;
   If \( a = 0 \), \( 3a = 0 = 5a \)

AP Book PA8-17

1. a)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( -(3)n )</th>
<th>( -(3)n - 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>–5</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>–4</td>
<td>12</td>
<td>7</td>
</tr>
<tr>
<td>–3</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>–2</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

4. Teacher to check guess and check.
   a) –2
   b) 2
   c) –3
   d) –6

AP Book PA8-18

1. b) + (–3)
c) + (–5)
d) + (+5)
e) – (–7)
f) \( \times (–9) \)
g) \( \times (–5) \)
h) + (–7)
i) + 8

2. Circle 3 expressions:
   \(-6m + (–8), m + 6 \times 6 \text{ and } 6 + m + (–6)\)
   If \( m = –5 \), values from left to right are:
   \(-36, –5, –5, –30, –5, 17\)

3. Teacher to check student checks.
   b) \( x = 15 \)
c) \( x = –5 \)
d) \( x = –12 \)
e) \( x = 3 \)
f) \( x = –3 \)
g) \( x = 100 \)
h) \( x = –8 \)
i) \( x = –21 \)
j) \( x = –35 \)

4. Teacher to check substitution checks.
   Answers may also be given in decimal form.
   a) \( s = \frac{29}{3} \)
b) \( t = \frac{11}{2} \)
c) \( y = \frac{8}{7} \)
d) \( x = \frac{21}{5} \)
e) \( a = \frac{9}{6} = \frac{3}{2} \)
Patterns and Algebra – AP Book 8, Part 2: Unit 4 (continued)

5. ii)
6. Teacher to check substitution checks.
   b) 3; –36; –36 + 4; h = 9
   c) s = –11
   d) t = –7
   e) x = 36
   f) x = –22
   g) s = \( \frac{32}{3} \)
   h) t = \( \frac{11}{3} \)
   i) y = \( \frac{46}{5} \)
   j) Z = \( \frac{2}{6} \) or \( \frac{1}{3} \)

AP Book PA8-19
page 111

1. b) \( \Delta = \frac{10}{3} \) circles
   c) \( \Delta = \frac{6}{3} \) or 1.5 circles
2. a) No
   b) Yes; No; Method 1
   c) On the left side of Line 2, the 3 should subtract from x, not add to it.
   d) \( (–2)x + 6 = –4 \)
      \( (–2)(x – 3) = –4 \)
      \( (–2)(x – 3) + (–2) = –4 + (–2) \)
      \( x – 3 = 2 \)
      \( x = 5 \)
3. a) Incorrect
   b) Incorrect
   c) Correct
   d) Incorrect
4. a) Line 2 should read:
    \( (–3)(x – 2) = –21 \)
   b) Line 2 should read:
    \( (–3)(x – 2) = –21 \)
   c) Correct
   d) Line 4 should read:
    \( (–3)x + (–3) = –27 + (–3) \)
5. \( x = –26^\circ \)
   \( y = +1^\circ \)
   \( z = –23^\circ \)

<table>
<thead>
<tr>
<th>AP Book PA8-20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Page 112</td>
</tr>
<tr>
<td>1. a) ( 5s + t )</td>
</tr>
<tr>
<td>b) ( 7s + t )</td>
</tr>
<tr>
<td>c) ( 6s + t )</td>
</tr>
<tr>
<td>d) ( 9s + t )</td>
</tr>
<tr>
<td>e) ( 6s + t )</td>
</tr>
<tr>
<td>f) ( 7s + t )</td>
</tr>
<tr>
<td>2. a) ( 10s + t )</td>
</tr>
<tr>
<td>b) ( 2s + t )</td>
</tr>
<tr>
<td>c) ( 3s + t )</td>
</tr>
<tr>
<td>d) ( 5s + t )</td>
</tr>
<tr>
<td>3. a) ( s + 7 + t )</td>
</tr>
<tr>
<td>b) ( s + 5 + t )</td>
</tr>
<tr>
<td>c) ( s + 3 + r )</td>
</tr>
<tr>
<td>d) ( s + 1 + t )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>AP Book PA8-21</th>
</tr>
</thead>
<tbody>
<tr>
<td>Page 115</td>
</tr>
<tr>
<td>1. b) ( FN )</td>
</tr>
<tr>
<td>c) ( FN + 4 )</td>
</tr>
<tr>
<td>d) ( FN + 8 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>AP Book PA8-22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Page 117</td>
</tr>
<tr>
<td>1. b) ( FN )</td>
</tr>
<tr>
<td>c) ( FN + 4 )</td>
</tr>
<tr>
<td>2. a) ( FN + 8 )</td>
</tr>
<tr>
<td>b) ( FN + 12 )</td>
</tr>
<tr>
<td>3. a) ( 2 )</td>
</tr>
<tr>
<td>b) ( 6 )</td>
</tr>
<tr>
<td>c) ( 10 )</td>
</tr>
</tbody>
</table>

Answer Keys for AP Book 8.2

V-25
c) $8, 8 + 2 \times (n - 1)$
   When simplified, $8 + 2 \times (n - 1) = 8 + 2n - 2 = 2n + 6$

4. Answers may vary but likely students will say the formula is easier.
   Sample explanation:
   I would set the formula to 125 and then solve for TN.

5. a) i) $2, 4, 6, 8, \ldots$
   Gaps: +2
   Start at 2, then add 2 each time.
   ii) $3, 6, 9, 12, \ldots$
   Gaps: +3
   Start at 3, then add 3 each time.
   iii) $6, 12, 18, 24, \ldots$
   Gaps: +6
   Start at 6, then add 6 each time.

b) Yes, they do.
   Formula:
   There is no constant added or subtracted to the multiple of the Term Number.
   Stepwise rule:
The starting number is equal to the gap in the sequence.

AP Book PA8-24

1. In each case, the number in the circles will equal the gap.
   a) $11, 15$
   Gap: +4
   b) $7, 15, 23$
   Gap: +8
   c) $4, 10, 16$
   Gap: +6
   d) $-8, -18, -28$
   Gap: -10
   The gap equals the number in the formula that is multiplied by the Input.

2. Teacher to check if students can predict the gaps (prompt: how many new blocks are added each time?).
   a) i) $\text{FN} \ # \ # \ B$

   b) i) $2 \times \text{FN} + 2$

   c) $n \times G$

   d) $n \times G$

   e) $n \times G$

   f) $n \times G$

   g) $2n - 1$

   h) $9n$

   i) $8n - 1$

   j) $3n + 6$

   k) $12n + 10$

   l) $2n + 54$

3. a) It’s the number multiplied by the Term Number (TN).

   b) Substitute TN = 1 and solve.

   c) Substitute TN = 15 and solve.

4. “Input $\times$ Gap”
1. a)  
<table>
<thead>
<tr>
<th>OP</th>
<th>1st</th>
<th>2nd</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3,1)</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>(5,3)</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>(7,5)</td>
<td>7</td>
<td>5</td>
</tr>
</tbody>
</table>

b)  
<table>
<thead>
<tr>
<th>OP</th>
<th>1st</th>
<th>2nd</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2)</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>(3,4)</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>(5,6)</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

c)  
<table>
<thead>
<tr>
<th>OP</th>
<th>1st</th>
<th>2nd</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2,3)</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(4,4)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(6,5)</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

2. Teacher to check the marked grid points.

a)  
<table>
<thead>
<tr>
<th>OP</th>
<th>1st</th>
<th>2nd</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,1)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(1,3)</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(2,5)</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>(3,7)</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

b)  
<table>
<thead>
<tr>
<th>OP</th>
<th>1st</th>
<th>2nd</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,5)</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>(2,6)</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>(4,7)</td>
<td>4</td>
<td>7</td>
</tr>
</tbody>
</table>

c)  
<table>
<thead>
<tr>
<th>OP</th>
<th>1st</th>
<th>2nd</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(1,3)</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(2,6)</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

3. a)  
<table>
<thead>
<tr>
<th>Line A</th>
<th>Line B</th>
<th>Line C</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>O</td>
<td>I</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>7</td>
</tr>
</tbody>
</table>

b)  
<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>3I - 1</td>
<td>I + 2</td>
<td>I</td>
</tr>
</tbody>
</table>

4. a) Teacher to check.

5. a)  
<table>
<thead>
<tr>
<th>L (min)</th>
<th>C (¢)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
</tr>
</tbody>
</table>

b) \( C = 20L \)

c) \( 200¢ = $2.00 \)

d) 5 minutes

e) There is no Term Number “0” since the term numbers are based on the counting numbers.

6.  
<table>
<thead>
<tr>
<th>1st</th>
<th>2nd</th>
<th>OP</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>(2,1)</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(3,3)</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>(4,5)</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>(5,7)</td>
</tr>
</tbody>
</table>

7. a)  

b) Teacher to check the marked points. Students should not use the point (0,3). Note that T-tables will vary depending on the points selected.

<table>
<thead>
<tr>
<th>TN</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
</tr>
</tbody>
</table>

c)  
<table>
<thead>
<tr>
<th>OP</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
</tr>
<tr>
<td>(2,5)</td>
</tr>
<tr>
<td>(3,9)</td>
</tr>
<tr>
<td>(4,13)</td>
</tr>
</tbody>
</table>

8. a)  

i)  
<table>
<thead>
<tr>
<th>TN</th>
<th>T</th>
<th>OP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>(1,5)</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>(2,8)</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>(3,11)</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>(4,14)</td>
</tr>
</tbody>
</table>

b)  
<table>
<thead>
<tr>
<th>OP</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,3)</td>
</tr>
<tr>
<td>(2,5)</td>
</tr>
<tr>
<td>(3,7)</td>
</tr>
<tr>
<td>(4,9)</td>
</tr>
<tr>
<td>(5,11)</td>
</tr>
</tbody>
</table>

c)  
<table>
<thead>
<tr>
<th>OP</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,41)</td>
</tr>
<tr>
<td>(2,38)</td>
</tr>
<tr>
<td>(3,35)</td>
</tr>
<tr>
<td>(4,32)</td>
</tr>
<tr>
<td>(5,29)</td>
</tr>
</tbody>
</table>

d)  
<table>
<thead>
<tr>
<th>OP</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,10)</td>
</tr>
<tr>
<td>(2,20)</td>
</tr>
<tr>
<td>(3,40)</td>
</tr>
<tr>
<td>(4,80)</td>
</tr>
<tr>
<td>(5,160)</td>
</tr>
</tbody>
</table>

9. a)  

A: 0, 6, 12, 18, 24
B: 64, 32, 16, 8, 4
C: 64, 32, 16, 32, 64
D: 4, 9, 4, 9, 4, 9, 4, 9
E: 55, 46, 37, 28, 19
F: 3, 6, 11, 18, 27

b)  
<table>
<thead>
<tr>
<th>OP</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>D</td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td>F</td>
</tr>
</tbody>
</table>
Patterns and Algebra – AP Book 8, Part 2: Unit 4 (continued)

Sample descriptions:
- Increasing sequences go up to the right; decreasing sequences go down to the right.
- If the gaps in a sequence are equal (i.e., they increase or decrease by the same amount each time), the line is straight.
- A repeating sequence makes a zig-zag pattern.

AP Book PA8-27

page 126

1. a) i)\[ (1, 0), (2, 1), (3, 3), (4, 6), (5, 10), (6, 15) \]
   ii)\[ (1, 1), (2, 3), (3, 5), (4, 7), (5, 9), (6, 11) \]
   iii)\[ (1, 0), (2, 3), (3, 6), (4, 9), (5, 12), (6, 15) \]
   iv)\[ (1, 1), (2, 5), (3, 9), (4, 12), (5, 14), (6, 15) \]
   v)\[ (1, 4), (2, 2), (3, 0), (4, –2), (5, –4), (6, –6) \]
   vi)\[ (1, 14), (2, 10), (3, 8), (4, 4), (5, 2), (6, 0) \]

2. a) (1, 3), (2, 2), (3, 1), (4, 0), (5, –1)
   b) (1, 1), (2, –2), (3, 3), (4, –4), (5, 5), (6, –6)
   c) (1, –8), (2, –5), (3, –2), (4, 1), (5, 4)

INVESTIGATION 1

A. ii) 1, 3, 5, 7, 9, 11
   Gaps: +2
   iii) 0, 3, 6, 9, 12, 15
   Gaps: +3
   v) 4, 2, 0, –2, –4, –6
   Gaps: –2

B. i) 0, 1, 3, 6, 10, 15
   Gaps: +1, +2, +3, +4, +5
   iv) 1, 5, 9, 12, 14, 15
   Gaps: +4, +4, +3, +2, +1
   vi) 14, 10, 8, 4, 3, 0
   Gaps: –4, –2, –4, –1, –3

C. If all the gaps are equal, the sequence is linear.
4. a) B is linear because the gaps are all equal to \(-3\).

b) A:

\[
\begin{array}{c|c}
\hline
n & 3n - 2 \\
\hline
1 & 1 \\
2 & 4 \\
3 & 7 \\
4 & 10 \\
5 & 13 \\
\hline
\end{array}
\]

c) B:

\[
\begin{array}{c|c}
\hline
n & 2n - 1 \\
\hline
1 & 1 \\
2 & 3 \\
3 & 5 \\
4 & 7 \\
5 & 9 \\
\hline
\end{array}
\]

INVESTIGATION 2

A. i) 3, 5, 7, 9
   Gaps: +2

ii) 6, 7, 8, 9
   Gaps: +1

iii) 1, 4, 9, 16
   Gaps: +3, +5, +7

iv) 1, 4, 7, 10
   Gaps: +3

v) 0.25, 0.5, 0.75, 1
   Gaps: +0.25

vi) 0, 2, 6, 12
   Gaps: +2, +4, +6

B. i), ii), iv) and v)
C. Choices will vary – teacher to check.
D. The two non-linear sequences both multiply the Term Number with itself, rather than just with a constant.

AP Book PA8-28
page 128

2. a) i)

\[
\begin{array}{c|c|c}
\hline
n & 2n + 3 & (n, 2n + 3) \\
\hline
1 & 5 & (1, 5) \\
2 & 7 & (2, 7) \\
3 & 9 & (3, 9) \\
4 & 11 & (4, 11) \\
5 & 13 & (5, 13) \\
\hline
\end{array}
\]

b) In each case, \(n\) has the same coefficient (i.e. the number in front of it) and only the “added number” changes (3, -2, 0).

c) Teacher to check combined graph.
No, the lines will never intersect: they are parallel.
In the formula, this is shown by the common \(n\) coefficient.

3. a) i)

\[
\begin{array}{c|c|c}
\hline
n & 3n + 2 & (n, 3n + 2) \\
\hline
1 & 5 & (1, 5) \\
2 & 8 & (2, 8) \\
3 & 11 & (3, 11) \\
4 & 14 & (4, 14) \\
5 & 17 & (5, 17) \\
\hline
\end{array}
\]

b) i)

\[
\begin{array}{c|c|c}
\hline
n & 4n - 2 & (n, 4n - 2) \\
\hline
1 & 2 & (1, 2) \\
2 & 6 & (2, 6) \\
3 & 10 & (3, 10) \\
4 & 14 & (4, 14) \\
5 & 18 & (5, 18) \\
\hline
\end{array}
\]

b) i)

\[
\begin{array}{c|c|c}
\hline
n & 4n - 3 & (n, 4n - 3) \\
\hline
1 & 1 & (1, 1) \\
2 & 5 & (2, 5) \\
3 & 9 & (3, 9) \\
4 & 13 & (4, 13) \\
5 & 17 & (5, 17) \\
\hline
\end{array}
\]

b) i)

\[
\begin{array}{c|c|c}
\hline
n & 5n - 1 & (n, 5n - 1) \\
\hline
1 & 4 & (1, 4) \\
2 & 9 & (2, 9) \\
3 & 14 & (3, 14) \\
4 & 19 & (4, 19) \\
5 & 24 & (5, 24) \\
\hline
\end{array}
\]
1. The following values are missing:
   a) Input: 7
      Output: 5
   b) Term Number: 4, 7
      Term Value: 6, 18

2. Teacher to check lines.
   a) 18
   b) 55

3. a) i) $n$ Term
       1 2
       2 6
       3 10
       4 14
       \[ \therefore \text{7th term} = 26 \]
   ii) $n$ Term
       1 1
       2 4
       3 7
       4 10
       \[ \therefore \text{8th term} = 33 \]
      b) $n$ # TP
         1 5
         2 9
         3 13
         \[ 4n + 1 \]
   c) $n$ # TP
      1 4
      2 7
      3 10

5. b) +4
   c) +2
   d) −1
   e) +5
   The gap equals the coefficient of $n$.

6. a) 3, 4, 10, 12
   Gaps: +1, +6, +2
   b) The term values
   c) Vertical

7. a) i) +4
    ii) +2
    iii) +3
    b) A: (1, 1), iii), i), ii)
    c) No such figure.
       Sample: No multiple of 5 equals 101.
   iii) a) 9, 12, 15
    b) 3n + 6
    c) No such figure.
       Sample: No multiple of 3 equals 94.
   iv) a) 4, 8, 12
    b) 4n
    c) Figure 25

8. a) i) 6
     ii) 4
     iii) 1
     b) A: (1, 1), iii)
       B: (1, 4), ii)
       C: (1, 6), i)

9. B, C, A

10. Answers will vary – teacher to check.

11. a) 60
    b) $h$ D (km)
       1 60
       2 120
       3 180

12. d) 60(4,5) = 270 km
     e) Teacher to check extended line (as shown above).
        6 hours
     f) 60h = 360
        \[ \therefore h = 6 \]
Patterns and Algebra – AP Book 8, Part 2: Unit 4 (continued)

Sample explanation:
Formula for graph is $4n + 2$, which fits the skate scenario ($\$2$ flat fee + $\$4/hr$) whereas, for volume of a cube, outputs would be $1^3 = 1, 2^3 = 8, 3^3 = 27$, etc.

4. a) $2^2 - 1^2 = 4 - 1 = 3$
   
   b) $(n + 1)^2 - n^2$
   or $2n + 1$
   
   c) 2009th row
   Sample explanation:
   Using the 1st formula above, $n + 1 = 2010$ and $n = 2009$.
   
   Teacher to check student checks.

5. a) $2n + 10$
   
   b) Figure 33

AP Book PA8-31
page 133

1. b) $m + 3 - 3 = m$
   
   c) $7 + m - m = 7$
   
   d) $m + 2 \times 2 = m$
   
   e) $m - 5 + 5 = m$
   
   f) $9 - m + m = 9$
2. a) Tara is correct for all values of $a$ except $a = 0$.
   
   b) Answers will vary – teacher to check.

3. b) 15
   
   c) $12 \times 3 = 36$
   
   d) $12 + 3 = 4$
   
   e) $3 \times 12 = 36$
   
   f) $3 + 12 = \frac{1}{4} = 0.25$

4. Circle: c), e)
   
   a) Yes: $-28$
   
   b) Yes: $-15$
   
   c) Yes, though answers will vary – teacher to check.

5. $a \times b = b \times a$
   
   6. b) $3 \times (-5 + 3) = 3 \times (-5 + 3) = 3 \times (-2) = -15 + 9 = -6$
   
   c) $3 \times (-5 + 4) = 3 \times (-5 + 4) = 3 \times (-1) = -15 + 12 = -3$
   
   d) $3 \times (-5 + 5) = 3 \times (-5 + 5) = 3 \times 0 = -15 + 15 = 0$

   7. 3, 5, 5; 3, 5, 5
   
   8. a) $5, -2, 7$
   
   b) $3, 2, -5$
   
   c) $-4, 12, 83$
   
   9. a) $5, 1$
   
   b) $4, 4$
   
   c) $(-7) \times 0 + (-7) \times 3$

   10. a) 8, 8
   
   b) $1, -1$
   
   c) $-70, -70$

   The two answers are equal, so we know that:
   
   $(a + b) \times c = a \times c + b \times c$

   11. a) $5, 3$
   
   b) $8, 5, -3;$
   
   c) $7, -2, 9;$
   
   $7 \times 9 + (-2) \times 9$
   
   d) $-3, 2, 0;$
   
   $(-3) \times 0 + 2 \times 0$

   12. $b) 5x - 2x = 9$
   
   c) $3x = 9$
   
   d) $x = 3$

   e) $8x - 3x = 20$

   13. b) $4x - 2x = 8 + 2$
   
   c) $-3x - 2x = -14 + 4$

   14. b) $2x = 8$
   
   c) $3x = 9$
   
   d) $x = 3$

   15. b) $x = 2$
   
   c) $x = 3$
   
   d) $x = 4$

   16. b) red: $x$
   
   yellow: $x - 3$

   c) red: $x$

   yellow: $x - 5$

   17. b) red: $x$
   
   yellow: $x + 6$

   so $x + x + 6 = 20$

   $x = 7$

   so there are 7 red and 13 yellow beads
**Number Sense – AP Book 8, Part 2: Unit 5**

### AP Book NS8-104

**page 137**

1. b) 3, 2  
   c) 7, 4  
2. b) $7 \times 7 \times 7$  
   c) $8 \times 8 \times 8 \times 8$  
3. a) $3^3$  
   b) $4^4$  
   c) $9^2$  
   d) $8^4$  
4. a) 8  
   b) 81  
   c) 16  
   d) 25  
   e) 16  
   f) 125  
5. Circle: c) and e)

**INVESTIGATION**

A. i) 8, 9  
   ii) 243, 125  
   iii) 100, 1024  
B. Yes  

6. a) 3  
   b) 5  
   c) 8  
   d) 13  
   e) 2057  
7. a) $3 \times 3 \times 3 \times 3$  
   b) $3 \times 3 \times 3 \times 3 \times 3$  
   c) $3 \times 3 \times 3 \times 3 \times 3 \times 3$  
   d) $5 \times 5 \times 5 \times 5$  
   e) $4 \times 4 \times 4 \times 4 \times 4$  
   f) $8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8$  
8. a) $8 \times 8 = 64$  
   b) $3 \times 3 \times 3 = 27$  
   c) $2 \times 2 \times 2 \times 2 \times 2 = 32$  
9. a) $1^2$  
   b) $1^3$  
   c) $1^4$  
   d) $1^5$  
10. a) 1  
    b) 1  
    c) 1  
    d) 1  
    e) 1

### AP Book NS8-105

**page 139**

11. a) i) 6  
   ii) 2  
   iii) 4  
   iv) 14  
   b) i) $6 \times 6 \times 6 = 216$  
   ii) $4 \times 4 \times 4 \times 4 \times 4 = 1024$  
   iii) $7 \times 7 \times 7 \times 7 = 2401$  
12. b) $3 \times 8 = 24$  
   c) $16 + 2 = 8$  
   d) $2 \times 25 = 50$  
   e) $4 \times 9 = 36$  
   f) $9 \times 8 = 72$  
   g) $100 + 25 = 4$  
   h) $4 + 9 = 13$  
   i) $49 + 36 = 85$  
   j) $64 - 4 = 60$

### AP Book NS8-106

**page 141**

1. d) 100 000  
   e) 10 000 000  
   f) 100 000 000  
2. b) $10^5$  
   c) $10^7$  
3. b) $10 000 000 000 = 10^{10}$  
   c) $10 000 = 10^4$  
   d) $100 000 000 = 10^8$  
4. b) $3 \times 10 000 + 6 \times 1000 + 9 \times 100 + 8 \times 10 + 2$  
   = $3 \times 10^4 + 6 \times 10^3 + 9 \times 10^2 + 8 \times 10 + 2$  
   c) $4 \times 1 000 000 + 2 \times 1000 + 5 \times 1000 + 9 \times 100 + 1$  
   = $4 \times 10^6 + 2 \times 10^5 + 5 \times 10^3 + 9 \times 10^2 + 1$  
5. b) 434 519  
   c) 3050  
   d) 905 037
Number Sense – AP Book 8, Part 2: Unit 5 (continued)

AP Book NS8-107
page 142
1. a) 
(−2)^3 = −8 2^3 = 8
(−2)^4 = 16 2^4 = 16
(−2)^5 = −32 2^5 = 32
(−2)^6 = 64 2^6 = 64
(−2)^7 = −128 2^7 = 128
b) 2, 4, 6
c) 8
d) (−2)^n = 2^n
when n is even.
(−2)^n = −(2^n)
when n is odd.
e) −8 192; 16 384
2. a) −1, 1, −1, 1, −1, 1
b) (−1)^73 = −1 since 973 is odd.
3. (−2)^1, (−2)^2, (−2)^3, (−2)^4, (−2)^5
4. a) Teacher to check student predictions.
Solution:
(−3)^1, (−3)^2, (−3)^3, (−3)^4, (−3)^5
b) −2187, −243, −27, −3, 9, 81, 729
AP Book NS8-108
page 143
1. a) 2^2 = 8
b) (−2)^2 = 4
(c) 0^2 = 0
d) 81 + ∴1 = 162
e) 0^2 = 0
f) 32 − 32 = 0
(g) (−2)^3 = −8
h) −5 + 27 = 22
2. b) 7^2 = 7^1 = 7
(c) (−2)^2 = (−2)^3
= −8

d) Subtract 9 from 7. Divide by 2. Raise the result to the third power.

AP Book NS8-109
page 144
1. a) 200; 2 000; 20 000; 200 000
b) 2^2 × 5^2; 2^2 × 5^3; 2^3 × 5^3; 2^3 × 5^5
20 000 000; follow the pattern or notice that:
2^a × 5^b = 2 (2^a × 5^b)
= 2 (10^b)
2. a) 2, 4, 8, 16, 32, 64, 128, 256
b) 2, 4, 8, 6, repeat
2. b) 3, 9, 7, 1, repeat
3. 2, 4, 16, 32, 64, 256; 2 000; 20 000; 200 000
i ) 3
ii) 9
iii) 3
iv) 3
v) 9
vii) 3
4. a) 3^a = 1s digit
3^4 = 81 1
3^5 = 243 3
3^6 = 729 9
3^7 = 2187 7
3^8 = 6561 1
3^9 = 19683 3
3^10 = 59049 9
3^11 = 177147 7
3^12 = 531441 1
b) 3, 9, 7, 1, repeat
5. 2^3, 2^6, 2^9
2^2, 2^5, 2^8
2^1, 2^4, 2^7
2^0, 2^3, 2^6
6. a) i) 30
ii) 9
iii) 3
iv) 3
v) 3
vi) 3
b) Answers may vary – teacher to check.
	i) 3^2 × 3
	ii) 3^2 × 3
	iii) 3 × (3 + 3)
	iv) 3 × 3 × 3
	v) 3 × (3 − 3)
1. b) Teacher to check marks.
   \[ \frac{8}{10} \cdot \frac{8}{10} \]

2. Teacher to check marks.
   a) E, B, F, C, D, A, G
   b) G, A, D, C, F, B, E
   c) The lists are the reverse of each other.

3. Teacher to check number line and marks.
   A \( -2.8 \)
   B \( \frac{4}{5} \)
   C \( -\frac{2}{5} \)
   D \( -\frac{3}{4} \)

4. a) 0.032, \( \frac{1}{8} \), 0.134
   b) \( -\frac{14}{5} \), \( -2\frac{3}{4} \), \( -2.715 \)
   c) \(-9.126, -9\frac{1}{8}, -0.2, 0.3, 7\frac{1}{2}, 7.56 \)
   d) \(-3\frac{1}{3}, -3.257, -3\frac{1}{4}, 3\frac{1}{4}, 3.257, 3\frac{1}{3} \)
Measurement – AP Book 8, Part 2: Unit 6

AP Book ME8-9
page 146

1. a) Teacher to check shading of base.
   i) hexagonal
   ii) pentagonal
   iii) triangular
b) rectangles
c) i) 6
   ii) 5
   iii) 3
2. a) $\sqrt{2}$ = 2
   b) $\sqrt{3}$
c) $\sqrt{5}$ = 2
3. a) Right prisms: A, C, E
   Not prisms: B, D
b) B isn't because it only has one base (it's a pyramid); D isn't because all its sides are pentagons.
4. Teacher to check.
5. Teacher to check.
6. Teacher to check.
7. C; A; B
   Teacher to check that prism dimensions are marked correctly.
8. a) Answers may vary – teacher to check.
   Sample answers:
   200 = 1 × 1 × 200
   200 = 2 × 10 × 10
   200 = 4 × 5 × 10
b) Teacher to check.

AP Book ME8-10
page 148

1. a) Rectangular (square) prism
   b) triangular prism

2. Teacher to check that bases are shaded correctly.

3. b) Position of faces may vary – teacher to check.

4. All measurements below are in metres.

5. a) Circle: the first net only
   b) Circle: the middle net only
   c) In a): the 2nd net has 6 side faces; the 3rd net has side faces that are attached to the incorrect edges of the base faces.
   For b): in the 1st net, the middle side face isn't wide enough – it should match the hypotenuse of the base triangle; in the 3rd net, the bases don't face the same direction.

6. a) rectangular prism
   b) rectangular prism
   c) triangular prism
   d) rectangular prism

7. Position of added face may vary – teacher to check.

8. Nets may vary slightly – teacher to check.

INVESTIGATION
A. ii) 6 cm$^2$
   iii) 15 cm$^2$
   iv) 12 cm$^2$
B. ii) 6 cm$^3$
   iii) 15 cm$^3$
   iv) 12 cm$^3$
C. The numbers are the same but the units change from cm$^2$ to cm$^3$.
D. i) 2
   ii) 1
   iii) 2
   iv) 4
E. i) 2 cm
   ii) 1 cm
   iii) 2 cm
   iv) 4 cm
F. The numbers are the same but the units change from layers to cm.
G. i) 24 cm$^3$
   ii) 6 cm$^3$
   iii) 30 cm$^3$
   iv) 48 cm$^3$
H. height

4. $15; 4; 15 \times 4 = 60$
5. $12 \times 2 = 24; 12 \times 2 = 24$
6. a) $15; 15$
   The numbers are the same, but the units aren't (cm$^2$ vs cm$^3$).
   BONUS B, A
b) 8; 8;
The numbers are the same but the units aren't (layers vs cm).
c) Answers may vary – teacher to check.
Sample explanation:
Put simply, we know the number of blocks is the same so the volume must also be the same.
Put more formally, the "numbers" are the same in both equations – just the units change. The resulting product of the units, however (cm³), is the same.

7. a) height
b) length
c) height
d) width
e) length
f) width
8. a) 10 × 7 = 70
b) width; 35 × 2 = 70
c) length; 14 × 5 = 70
Yes, you get the same answer.
9. a) 70 cm³
b) 40 cm³
c) 40 cm³
d) 60 cm³
10. a–d) Teacher to check.
e) The prisms and their dimensions/volume don't change – they have simply been rotated in space.
11. a) i) 20; 3; 60
ii) 12; 5; 60
iii) 15; 4; 60
b) All 3 prisms have the same volume; Their dimensions are all the same – just their "direction" is different.

12. A and B have the same volumes;
All of their dimensions are the same – they've just been rotated. In C, there is a 3 cm edge where the others have a 2 cm edge.

AP Book ME8-12

INVESTIGATION 1

A. | ii) | iii) |
---|---|---|
11 | 8 |
6  | 3 |
66 | 24 |
11 | 8 |
6  | 3 |

B. Yes

INVESTIGATION 2

A. a) one half
b) one half
c) one half
Sample explanation:
A diagonal cuts a rectangle into two equal (but inverted) triangles.
B. a) one half
b) one half
c) one half
Sample explanation:
From A, we know that the base area of each triangular prism is half of its "cut" rectangular prism. Since their heights are the same, the volume is simply being halved.
C. 2
D. height; triangle; height; base; height

1. Any edge that is parallel to those marked below is also correct.
a) b)

2. a) You can think of this prism as two separate prisms put together: a triangular prism (at left) and a rectangular prism.
Volume of TP = area of base × h = 15 × h
Volume of RP = area of base × h = 20 × h
c) Yes;
Can see this by the distributive property or by substitution:
h = 1 → V = 35 cm³
h = 2 → V = 70 cm³
h = 3 → V = 105 cm³
d) Volume of prism = (15 × h) + (20 × h) = (15 + 20) × h = 35 × h
But 35 is the area of the prism's base, so: volume = area of base × height.
C. \( \frac{V}{h} \)

3. a) Base area: 15 cm²
Volume: 75 cm³
b) Base area: 36 cm²
Volume: 108 cm³
4. Estimates will vary – teacher to check.
a) Base area: 42.5 cm²
Volume: 425 cm³
b) Base area: 33.6 cm²
Volume: 336 cm³
c) Base area: 112.53 cm²
Volume: 1125.3 cm³
5. Answers will vary – teacher to check.

6. V = base area × h
∴ base area = V + h
= 600 + 15
= 40 cm²

AP Book ME8-13

page 157

1. a) 24 cm³
b) 30 cm³
c) 28 cm³
d) 18 cm³

INVESTIGATION

A. Volume = base × height;
A cylinder is a prism too, just with a round base.
B. He did this to account for the thickness of the can; 5 cm;
Again, to account for the thickness of the can.
C. \( \frac{r^2 \times h}{hr} \)

23.04 253.44 = 3.141
16.81 218.53 = 3.112
16 172.8 = 3.125
6.25 50 = 3.12

D. The last column: \( \frac{V}{hr} \)

E. \( hr^2 \), \( hr \)
F. \( \pi r^2 \)
G. height
H. They both multiply the area of the base by the height. This makes sense since a cylinder is like a prism with a round base.
I. Answers will vary, depending on prediction.

2. a) 36 cm³
b) 120 cm³
c) 120 cm³
3. a) base = 64 \( \pi \) cm²
≈ 201 cm²
height = 10 cm
volume = 2010 cm³
b) base = 49\pi \text{ cm}^2 \\
= 154 \text{ cm}^2 \\
height = 20 \text{ cm} \\
volume = 3080 \text{ cm}^3 

4. a) 30 \times 20 = 600 

b) 30 \times 20 = 600 

Yes, they are the same; Regardless of shape, both jars have same base area and height – and, as such, the same area. 

. they will hold the same number of candies.

5. a) NOTE: Student measurements may vary a bit. Teacher to check.

Sample answers:

Diameter = 19 mm; 
Radius = 9.5 mm; 
Height of 1 coin = 1.5 mm; 
Height of 10 coins = 14.5 mm.

The height found by stacking 10 coins and dividing will be more accurate; It’s hard to measure anything involving a part of a mm – it’s too small to see clearly. The 10-coin method allows these “parts” to accumulate enough to be measurable.

\[ V = \pi r^2 h \]

\[ = \pi(9.5)^2(14.5) \]

\[ = 410.91 \text{ mm}^3 \]

\[ = 0.411 \text{ cm}^3 \]

b) The volume of 10 pennies is about 4.11 cm\(^3\) or 4.11 mL. 
: the water level should rise to just above 34 mL.

b) 1000 mL 

2. a) 10 mm 

b) 100 mm\(^2\) 

2. 1890 cm\(^3\) 

3. a) 10 mm 

b) 100 mm\(^2\) 

3. \[ V = 2000 \text{ cm}^3 \]

: the area of the base 
\[ = 2000 + 25 = 80 \text{ cm}^2 \]. 

4. 250 mL = 250 cm\(^3\) 

For the bottom section of the carton:

\[ V = l \times w \times h \]

so:

\[ 250 = 49h \]

\[ h = 5.1 \text{ cm} \]

: total height = 9.6 cm

5. a) \[ V = \pi r^2 h \]

\[ = \pi(3.4)^2(12.2) \]

\[ = 442.84\text{ cm}^3 \]

b) 8 cans used in total so \[ = 3.54 \text{ L of juice} \]

6. a) Total base area of:

2 circular pans

\[ = 2 \times \pi r^2 \]

\[ = 2 \times \pi(4.5)^2 \]

\[ = 127.17 \text{ inch}^2 \]

1 rectangular pan

\[ = 13 \times 9 \]

\[ = 117 \text{ inch}^2 \]

For the same volume, the one with the lower base area needs more height. 
: the cake mix will be higher in the rectangular pan.

b) rectangular pan

7. Volume of mixture

\[ = 11.25 \text{ cups} = 2700 \text{ mL} \]

Base area of the pan

\[ = 20 \times 30 = 600 \text{ cm}^2 \]

: the mixture will be 4.5 cm high in the pan.

8. a) Tegan cannot carry the full aquarium: the water alone (i.e. not including the aquarium itself) weighs 162 kg.

b) She will need to make 27 trips.

c) Answers will vary – teacher to check.

AP Book ME8-16

page 161

1. b) 

2. a) 1 cm 

b) 3 cm 

c) 5 cm 

3. He forgot to convert to the same units first:

Incorrect \[ \times 4 \times 80 \times 50 = 16000 \text{ cm}^3 \]

Correct \[ \checkmark 4 \times 0.8 \times 0.5 = 1.6 \text{ m}^3 \]

or \[ 400 \times 80 \times 50 = 1600000 \text{ cm}^3 \]
3. Teacher to check that students have shaded the face opposite to one that is marked.

4. a) back: 6 cm²
   bottom: 12 cm²
   left: 8 cm²
   b) back: 15 cm²
   bottom: 6 cm²
   right: 10 cm²
   c) back: 12 m²
   bottom: 18 m²
   left: 6 m²

5. top + bottom = 15 cm² × 2 = 30 cm²
   right + left = 10 cm² × 2 = 20 cm²

6. a) 22 cm²
   b) 52 m²
   c) 76 mm²

7. Miki forgot to include the “invisible” sides in her calculation. She needs to multiply her answer by two: 80 cm².

8. a) She is correct; the front/back face areas = the right/left face areas
   b) 2(8 × 8) + 4(5 × 8) = 288 cm²

9. Exact assignment of face names may vary – teacher to check.

   a) back: 1 cm
   top: 3 cm
   front: 4 cm
   left: 1 cm
   bottom: 4 cm
   right: 4 cm
   c) 5 cm

10. a) 38 cm²
    b) 60 m²
    c) 73.86 m²

11. a) 4 m
    b) 3 m
    c) 7 m

12. a) 5 m
    b) 5 m
    c) 4 m

13. There is only one length combination that works:
    a = 6 m
    b = 3 m
    c = 2 m

14. Surface area
    = 2lw + 2lh + 2wh
    = 2(lw + lh + wh)

15. In cm: 22 000 cm²
    In m: 2.2 m²

16. NOTE:
   Do not include the floor or ceiling, just the walls.
   Surface area, including door/windows = 66 m²
   Door = 2 m²
   Windows = 6 m²
    area to paint = 58 m², which would cost $23.20.

17. a) 70 cm
   b) 50 cm
   c) 100 cm
   40 cm
   150 cm

18. a) SA = (4n + 2) cm²
    b) 82 cm²

AP Book ME8-17
page 164

1. a) rectangle
   b) parallelogram
   c) i) base = 15 cm
      ii) base = 15 cm
   d) Yes;
      The two tubes are identical (their areas are the same) – they are just cut differently.

2. a) 21 cm²
    b) 125.6 cm²
    c) 621.72 cm²

3. a) SA = C × h
    b) SA = πd × h
    c) SA = 2πr × h

4. A, B, C, E;
   The four correct nets are made from rectangles and/or parallelograms.
   The two others aren’t (both include a trapezoid).

5. a) 244.4 rectangle
    b) Net 2;
    The dimensions are correct in both cases but in Net 1, the circumference and height have been reversed.

6. a) B, D, E
   b) A:

   c) The surface area of a cylinder is equal to the area of its net;
      The circles in the net are the same size as the two bases of the cylinder, and the rectangle of the net – when “glued” – creates the curved face of the cylinder.

7. a) 157.0 rectangle
    b) 25π = 78.5 top circle
       + = 82.5 bottom circle
       = 314.0 cm²
    c) 56.52 rectangle
       π(4.5)² = 63.59 top circle
       + = 63.59 bottom circle
       = 183.70 cm²
    d) 113.04 rectangle
       4π = 12.56 top circle
       + = 12.56 bottom circle
       = 138.16 cm²

8. Surface area of the can
    = 2πr² + 2πrh
    = 2π(r + h)
### Measurement – AP Book 8, Part 2: Unit 6 (continued)

**AP Book ME8-18**

#### page 166

1. **Answers will vary – teacher to check.**
   
   **Sample answers:**
   - a) \( 2 \text{ cm} \times 2 \text{ cm} \times 3 \text{ cm} \)
   - b) \( 1 \text{ cm} \times 2 \text{ cm} \times 4 \text{ cm} \)
   - c) \( 2 \text{ cm} \times 3 \text{ cm} \times 3 \text{ cm} \)

2. **Answers will vary – teacher to check.**
   
   **Sample answer:**
   - \( 2 \text{ cm} \times 12 \text{ cm} \times 1 \text{ cm} \);
     surface area = 76 \( \text{ cm}^2 \)
   - \( 2 \text{ cm} \times 2 \text{ cm} \times 6 \text{ cm} \);
     surface area = 56 \( \text{ cm}^2 \)
   - \( 1 \text{ cm} \times 3 \text{ cm} \times 8 \text{ cm} \);
     surface area = 70 \( \text{ cm}^2 \)
   
   Of these, the 2\(^{nd}\) uses the least amount of material.

   **NOTE:**
   The closer the prism is to a cube, the smaller its surface area will be.

3. **a)** \( 42 \text{ cm}^2 \)
   **b)** \( 18 \text{ cm}^3 \)

4. **Volume = 240 \( \text{ m}^3 \)**
   
   **Surface area = 280 \( \text{ m}^2 \)**
   
   **Strategies will vary – teacher to check.**

5. **b) Explanations will vary – teacher to check.**
   
   **Sample explanation:**
   Given a circle and square of the same area, the circle’s circumference will be less than the square’s perimeter.
   This means that the area of the cylinder’s curved face will be smaller than the sum of the side face areas in the prism.
   \( \therefore \) the cylinder will require less material since its overall surface area is less.

6. **a) The volumes of the two containers are equal. From the first container, we know**
   \( V = 98\pi \text{ cm}^3 \).
   
   \( \therefore h = 3.92 \text{ cm} \)

   **b)** \( \text{SA #1} = 252.77 \text{ cm}^2 \)
   **SA #2 = 280.09 \text{ cm}^2 \)

   **c) Cost #1 = $0.20 or 20¢**
   **Cost #2 = $0.22 or 22¢**
   
   \( \therefore \) the first can is cheaper to make.

7. **a) Area L = 7.065 \( \text{ cm}^2 \)**
   **b) Area S = 7.065 \( \text{ cm}^2 \)**
   **c) They hold the same volume: their height and base area are both equal.**

8. **She can multiply 20 \( \text{ cm}^2 \) by 2 to get 40 \( \text{ cm}^2 \) since the surface area of the front = back, top = bottom and right = left.**

9. **Capacity = 10,000 \( \text{ mL} \) or 10 \( \text{ L} \)**
   
   **Volume = 10,000 \( \text{ cm}^3 \) or 0.01 \( \text{ m}^3 \)**
   **Surface area = 4,200 \( \text{ cm}^2 \) or 0.42 \( \text{ m}^2 \)**

10. **a) Cylinder A:**
    \( V = 508.68 \text{ cm}^3 \)
    **SA = 621.72 \( \text{ cm}^2 \)**

    **Cylinder B:**
    \( V = 628 \text{ cm}^3 \)
    **SA = 408.2 \( \text{ cm}^2 \)**

   **b) B**
   **c) A**

11. **a) SA = 160 \( \text{ cm}^2 \)**
    **V = 100 \( \text{ cm}^3 \)**

    **b) A cube that is 5 cm \times 5 cm \times 5 cm:**
    **SA = 150 \( \text{ cm}^2 \) (<160)**
    **V = 125 \( \text{ cm}^3 \) (>100)**

12. **b) Volume of rect prism = 300 \( \text{ cm}^3 \)**
    **Volume of cylinder = 282.6 \( \text{ cm}^3 \)**

    **c) Satya should pour the water from the full cylinder into the empty prism.**
    *If the prism overflows, the cylinder is larger.*
    *If there is still space in the prism, it is larger.*

13. **a) Teacher to check.**

    **b) Tube A:**
    We first calculate the radius of the base = 3.50 cm
    **V = 1077.02 \( \text{ cm}^3 \)**
    **SA = 692.37 \( \text{ cm}^2 \)**

    **Tube B:**
    We first calculate the radius of the base = 4.46 cm
    **V = 1374.11 \( \text{ cm}^3 \)**
    **SA = 741.11 \( \text{ cm}^2 \)**

    **c) B**
    **d) B;**
    to solve, use SA ÷ V

14. **42\(\pi \) \approx 131.88 \( \text{ cm}^2 \),
    when radius = 3 cm and height = 4 cm**
1. a) Teacher to check.
   b) Teacher to check.
2. Answers will vary – teacher to check.
   a) Triangle
   b) Teacher to check.
3. a) i) Reflection
   ii) Translation
   iii) Rotation
   b) Yes; 180° rotation around point Q, translate 2 units up
   c) 180° rotation around point R
   d) Yes, it tessellates.
   Descriptions will vary – teacher to check.
   Sample description: 180° rotation around point P (onto shape 4).
   Together, 1 and 4 create a rectangle, which tessellates using translations.
4. Explanations may vary – teacher to check.
Sample strategy explanation:

Start with shape 1: rotate shape 180° around point A, then translate it 2 units left.

Rotate two shapes together 90° CCW around point B.

Follow this same process with shape 2, translating 4 units down.

Follow this same process with shape 3, translating 8 units down.

Continue with ever larger Ls to tessellate further.

6. Answers may vary – teacher to check.
Sample answers:

(i) Reflection in mirror line $M_1$
(ii) Reflection in mirror line $M_2$

b) The two reflections above (in order) will take shape A to shape C.

A single rotation of 120° CCW about centre O will have the same result.

7. Answers may vary – teacher to check.
Sample answers:

b) Answers will vary based on labelling.

AP Book G8-39

page 174

1. a) $(r + s + t)$
    + $(u + v + w)$
    + $(x + y + z)$
    $= 3 \times 180°$
    $= 540°$

Measured angles:

$\angle A = 54°$

$\angle B = 142°$

$\angle C = 64°$

$\angle D = 150°$

$\angle E = 130°$

Sum $= 540° \checkmark$

b) Answers will vary based on labelling.

2. $\angle A = 103°$
   $\angle B = 139°$
   $\angle C = 24°$
   $\angle E = 46°$

The fifth angle equals 540° minus the sum of the other four angles, so:

$\angle D = 540° - (103° + 139° + 24° + 46°)$

$= 540° - 312°$

$= 228°$

3. 360°;

Explanations will vary – teacher to check.
Sample explanation:

Four right angles “fit” around a point, and 4 × 90° = 360°.

4. Sidra’s total is 360° greater than the sum of the pentagon’s interior angles alone.

Rather than 3 triangles, she divided the pentagon into 5 triangles.

Correct total: $3 \times 180° = 540°$.

Sidra’s total:

$5 \times 180° = 900°$.

The 900° comes from the sum of the interior angles plus the extra 360° in the five angles around the point at the centre of the shape.

5. a) 60°

b) We know from Question 3 that the sum of angles around a point or vertex is 360°, and $6 \times 60° = 360°$.

c) Four squares will fit: $4 \times 90° = 360°$.

d) 

e) 

6. Teacher to check.

NOTE:
Divisions can originate at any vertex.

b) 4 triangles

c) 5 triangles

d) 6 triangles
e) 7 triangles
f) 8 triangles

<table>
<thead>
<tr>
<th>S</th>
<th>T</th>
<th>Exp IA</th>
<th>Sum IA</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>180° × 2</td>
<td>360°</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>180° × 3</td>
<td>540°</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>180° × 4</td>
<td>720°</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>180° × 5</td>
<td>900°</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>180° × 6</td>
<td>1080°</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
<td>180° × 7</td>
<td>1260°</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>180° × 8</td>
<td>1440°</td>
</tr>
<tr>
<td>n</td>
<td>n–2</td>
<td>180°(n–2)</td>
<td>180°(n–2)</td>
</tr>
</tbody>
</table>

**INVESTIGATION**

**A.**

<table>
<thead>
<tr>
<th>V</th>
<th>Sum IA</th>
<th>Each IA</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>180°</td>
<td>180° + 3 = 60°</td>
</tr>
<tr>
<td>4</td>
<td>360°</td>
<td>360° + 4 = 90°</td>
</tr>
<tr>
<td>5</td>
<td>540°</td>
<td>540° + 5 = 108°</td>
</tr>
<tr>
<td>6</td>
<td>720°</td>
<td>720° + 6 = 120°</td>
</tr>
<tr>
<td>7</td>
<td>900°</td>
<td>900° + 7 = 128.6°</td>
</tr>
<tr>
<td>8</td>
<td>1080°</td>
<td>1080° + 8 = 135°</td>
</tr>
</tbody>
</table>

**B. Increase;**
Yes. As the chart above shows, the interior angles increase as the number of sides/vertices in the regular polygon increases.

**C.**

a) It gives the number of copies of the regular polygon that “fit” around a common vertex.

b) Recall that, to tessellate, a polygon must fit around a common vertex with no gaps or overlaps. For this to happen, its interior angle $x°$ must divide evenly into 360°.

**D.**

<table>
<thead>
<tr>
<th>$x°$</th>
<th>$360° + x°$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60°</td>
<td>6</td>
</tr>
<tr>
<td>90°</td>
<td>4</td>
</tr>
<tr>
<td>108°</td>
<td>$\approx 3.3$</td>
</tr>
</tbody>
</table>

**AP Book G8-40 page 177**

1. Like a square, a rectangle has four 90° angles. Since four rectangles fit evenly around a common point, it tessellates:

<table>
<thead>
<tr>
<th>Adjacent Angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>120°</td>
</tr>
<tr>
<td>135°</td>
</tr>
</tbody>
</table>

2. a) The adjacent angles in any parallelogram add to 180°, so it can tessellate using only translations.

**Sample answer:**
First, translate the parallelogram below $a$ units to the right. Then translate this whole row down $b$ units in the direction of the slanted side.

b) Rotation; Specifically, a 180° rotation around the midpoint of one of its sides:

<table>
<thead>
<tr>
<th>Equal sides are marked in a) above.</th>
</tr>
</thead>
</table>

**AP Book G8-40 page 177**

3. a) In i) and ii) below, students can use a few transformation combinations that are correct, such as:

- a translation then a reflection, or
- a 180° rotation followed by a translation, etc.

Teacher to check.

iii) |
|---|

b) Rotation; Specifically, a 180° rotation around the midpoint of one of its sides:

<table>
<thead>
<tr>
<th>Equal sides are marked in a) above.</th>
</tr>
</thead>
</table>

**AP Book G8-40 page 177**

4. a) |
|---|

b) Like in a), the transformations used may vary – teacher to check.

**AP Book G8-40 page 177**

5. a) No; Angles in a regular octagon equal 135°, and $360° – (2 × 135°) = 90°$. This is not enough space to accommodate a third octagon.
6. a) 135°
   b) 90°
   c) Square; 1 cm
   d) Kong also needs to use a square, but his will have sides that are 2 cm long.

7. a) Since the sum of interior angles in a pentagon is 540°, we know that:
   \[ \angle B + \angle D = 540° - 180° - 50° = 310° \]
   But they are equal, so \( \angle B = \angle D = 155° \).
   b) \( \angle A \) and \( \angle E \)
   c) Around \( \angle A \) and \( \angle E \):
   d) \( \angle B \) and \( \angle D \) don’t divide into 360° but:
   \[ \angle B + \angle D + \angle C = 155° + 155° + 50° = 360° \]
   e) Prediction: Yes √

8. a) \[ \angle a = \frac{540° - 2(120°)}{3} \]
    \[ = 300° \]
    \[ = 100° \]

9. a) \[ \angle E = 540° - (140° + 90° + 90° + 50°) = 170° \]
   This pentagon will tessellate since 140° + 170° + 50° = 360°.
   b) \[ \angle D = \angle E = \frac{540° - 2(120°) - 70°}{2} = \frac{230°}{2} = 115° \]
   This pentagon won’t tessellate: its angles can’t be combined in any way to add up to 360°.
   c) \[ \angle C = 540° - 4(120°) = 60° \]
   This pentagon will tessellate since 120° + 60° + 180° = 360°.
   e) Prediction: Yes √

10. a) i) Unknown angles \[ = \frac{720° - 2(180°)}{3} = 132° \]
   b) i) Correct prediction: It tessellates since 60° + 150° + 150° = 360°.
   c) No, she can’t.
   The angles in the pentagons are 100° and 120°, which can’t be combined in any way to add up to 160°:
   \[ \begin{align*}
   100° + 100° &= 200° \\
   100° + 120° &= 220° \\
   120° + 120° &= 240°
   \end{align*} \]
   d) Descriptions may vary – teacher to check.

11. a) i) \[ a = \frac{1080° - 480°}{4} = 150° \]
   Since 60° + 120° = 180° and 30° + 150° = 180°, four Bs and one A form a rectangle, which will tessellate.
   b) A and E:
   A is an octagon with all equal angles \[ : a = 135° \]
   E is a right isosceles triangle \[ : \text{its two other angles} = 45° \]
   Since 135° + 45° = 180°, four Es and one A form a rectangle, which will tessellate.
   B and C:
   B is a right scalene triangle with one 30° angle \[ : \text{its 3rd angle} = 60° \]
   C is an octagon with four 120° angles \[ : b = \frac{1080° - 480°}{4} = 150° \]
   Since 60° + 120° = 180° and 30° + 150° = 180°, four Bs and one C form a rectangle, which will tessellate.
   D and F:
   D is an equilateral triangle \[ : \text{all three angles} = 60° \]
   F is a regular hexagon \[ : c = 120° \]
   Since 60° + 120° = 180°, D and F will form a variety of shapes that will tessellate.
   c) B, D, E and F

12. a) Yes
b) Yes

c) No, her shape won’t tessellate; in order to tessellate (that is, to fit the gap with angle $c$), Nellie must place the two “cut out” shapes so they will end up side by side when tessellated ($a + b = c$). This is impossible here; the rectangles will overlap.

**AP Book G8-41**

*page 180*

1. Teacher to check.

2. a) Teacher to check.
   b) First, students must rotate their shape 180° around the midpoint from a). This will result in a parallelogram.
   
   After creating this initial parallelogram, students may choose to use a variety of transformations to form the tessellation. Teacher to check.
   
   **Sample tessellation:**
   ✓ To form a parallelogram, I rotated my shape 180° around the point marked.
   ✓ From there, I used translations to create my full tessellation.

3. a)  
   b) Teacher to check.

4. a)  
   b) Teacher to check.
   c) To eliminate the curved sides, you have to place the shapes using a 120° rotation. However, when you place six shapes (eliminating all the curved sides), they make a loop with a hexagonal hole in the middle:

   The hole cannot be filled with this shape, so the shape does not tessellate.

5. A will tessellate:

   B is congruent to A (just rotated CCW slightly) so, yes, it will tessellate.

   C will tessellate:

**AP Book G8-42**

*page 181*

1. Teacher to check.

2. Teacher to check.

3. b)  
   c)  
   d)  

4. Teacher to check shading.
   b)  
   c)  
   d)  
   e)  
   f)  
   g)  
   h)  

5. a)  
   b)  
   c)  
   d)  

6. Teacher to check shading.
   b)  
   c)  
   d)  
   e)  
   f)  
   g)  
   h)  

7. Teacher to check shading. C, D, A, B

8. b)  
   c)  
   d)  

9. Answers will vary – teacher to check.

**AP Book G8-43**

*page 184*

1. Teacher to check shading.
   b)  
   c)  
   d)  

2. a) Circle: the last (4th) view
   b) The right side is shaded here:

   From this sketch, we can see that the layer heights, from front to back, are 2, 1, 3.
   
   We can also see that only the two bottom corners have multiple layers.

3. Teacher to check shading.
   b)  
   c)  
   d)  

4. Teacher to check shading.
   b)  
   c)  
   d)  

5. front, top, left
6. b) top view

    front view right side view
c) top view

    left side view front view
d) top view

    front view right side view
e) top view

    front view right side view

7. Answers will vary – teacher to check.

AP Book G8-44
page 186

1. b) top view

    A

    B

    top view

    A

    B

    front view right side view
c) top view

    A

    B

    front view right side view
d) top view

    A

    B

    front view right side view
e) top view

    A

    B

    front view right side view

2. b) width height

    top view: 1 cm 1 cm
    front view: 1 cm 2 cm
    right side view: 1 cm 2 cm
c) width height

    top view: 2 m 1 m
    front view: 2 m 2 m
    right side view: 1 m 2 m
d) width height

    top view: 2 cm 25 mm
    front view: 2 cm 7 mm
    right side view: 25 mm 7 mm

3. a) top view

    5 cm 25 mm

    5 cm 3 cm 2 cm

    front view right side view
e) top view

    5 cm 25 mm

    front view right side view

4. b) top view

    8 m 10 m

    25 m 25 m

    left side view front view
c) top view

    8 m 10 m

    25 m 25 m

    front view right side view
d) top view

    8 m 10 m

    25 m 25 m

    front view right side view
e) top view

    8 m 10 m

    25 m 25 m

    front view right side view

BONUS

5. a) top view

    10 m

    8 m

    10 m

    25 m

    bottom view

    8 m

    25 m

    front view right side view
b) Circle:  
\[ \text{the structure in the top, right corner} \]

3. a) 

b) Explanation and “most helpful view” will vary – teacher to check.

i)  

ii)  

iii)  

4. a) 

b) 

c) i) 

ii)  

iii) 

d) 

5. Turn the shape vertically 90° CCW (so that the front face becomes the bottom face). Part d).

6. Teacher to check drawings.

Without the thick lines, the left and right side views are reflections of one another in a vertical line.

For example:

a)  

b)  

c)  

7. Answers will vary – teacher to check.

INVESTIGATION 2

A. Teacher to check built structure:

B. and C.

L F R Bk  

before  

90° CW  

180° CW  

270° CW  

90° CCW  

D. In each row, the right and left side views (without the thick lines) are reflections of each other in a vertical line.

This is also true of the back and front views.

E. When you move down from one row to the next (not including the last row), the views all shift 1 cell to the right.

This makes sense since the structure is rotating 90° each time.
F. A 180° CCW rotation has the same effect as a 180° CW rotation (since 180° + 180° = 360°), so its views will match the 3rd row.

A 270° CCW rotation has the same effect as a 90° CW rotation (since 270° + 90° = 360°), so its views will match the 2nd row.
1. a) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>1 5</td>
</tr>
<tr>
<td>34</td>
<td>1 5 8</td>
</tr>
<tr>
<td>35</td>
<td>2 6</td>
</tr>
</tbody>
</table>

321, 325, 341, 345, 348, 352, 356

b) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>3 9</td>
</tr>
<tr>
<td>2</td>
<td>0 5 8</td>
</tr>
</tbody>
</table>

7, 13, 19, 20, 25, 28

2. a) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7 8</td>
</tr>
<tr>
<td>1</td>
<td>0 3 8</td>
</tr>
</tbody>
</table>

b) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>7 9 9</td>
</tr>
<tr>
<td>10</td>
<td>1 3</td>
</tr>
</tbody>
</table>

3. a) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1 3 4 6 7 9 9 9</td>
</tr>
<tr>
<td>3</td>
<td>0 1</td>
</tr>
</tbody>
</table>

Range = 10
Mode = 29

b) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 0 0 1 2 3 5</td>
</tr>
<tr>
<td>1</td>
<td>0 2 2 3</td>
</tr>
<tr>
<td>2</td>
<td>1 5 9</td>
</tr>
</tbody>
</table>

Range = 29
Mode = 0

c) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>741</td>
<td>6 8</td>
</tr>
<tr>
<td>742</td>
<td>1</td>
</tr>
<tr>
<td>743</td>
<td>0 2 5</td>
</tr>
<tr>
<td>744</td>
<td>4</td>
</tr>
<tr>
<td>745</td>
<td>6</td>
</tr>
</tbody>
</table>

Range = 40
Mode = none

d) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5 8</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

Median = 25
Mode = none
Range = 25

3. a) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1 3 4 6 7 9 9 9</td>
</tr>
<tr>
<td>3</td>
<td>0 1</td>
</tr>
</tbody>
</table>

Range = 10
Mode = 29

b) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 0 0 1 2 3 5</td>
</tr>
<tr>
<td>1</td>
<td>0 2 2 3</td>
</tr>
<tr>
<td>2</td>
<td>1 5 9</td>
</tr>
</tbody>
</table>

Range = 29
Mode = 0

c) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>741</td>
<td>6 8</td>
</tr>
<tr>
<td>742</td>
<td>1</td>
</tr>
<tr>
<td>743</td>
<td>0 2 5</td>
</tr>
<tr>
<td>744</td>
<td>4</td>
</tr>
<tr>
<td>745</td>
<td>6</td>
</tr>
</tbody>
</table>

Range = 40
Mode = none

d) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5 6 7</td>
</tr>
<tr>
<td>2</td>
<td>1 2 7 9</td>
</tr>
</tbody>
</table>

Median = 25
Mode = none
Range = 25

4. a) 3 or 5
b) 2 or 4
c) Anything higher than 5
d) 0, 1 or anything higher than 5

5. a) 20
b) 40
c) 20

d) 

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>7</td>
</tr>
<tr>
<td>21</td>
<td>6 9 9</td>
</tr>
<tr>
<td>22</td>
<td>2 5 7</td>
</tr>
<tr>
<td>23</td>
<td>0 1 5</td>
</tr>
<tr>
<td>24</td>
<td>1 8</td>
</tr>
</tbody>
</table>

Median = 226
Mode = 219
Range = 41

7. a) 30; 6; 5
b) 20; 4; 5
c) 7

d) 2.5
f) 9
g) 25.2
i) 11.75

8. a) Noa should give John 3 apples, and Cynthia 1 apple.

b) The mean is 6, which is the number of apples each person will have after they’re shared equally.

c) Since Bilal already has the mean number of apples, he doesn’t need to give or receive any for the sharing to remain equal.

INVESTIGATION 1

A. i) 4
   ii) 5
   iii) 6

Each new set is created by adding 1 to all the data points in the previous set:

Sample solution:
New set: 3, 4, 6, 8, 9
Prediction of new mean: 6 (original mean) + 1 = 7
Check:
(4 + 5 + 7 + 9 + 10) + 5 = 35 + 5 = 7

B. Answers will vary – teacher to check.

Sample solution:
Original set: 7, 5, 3, 1, 4
New set: 7, 7, 5, 3, 3, 1, 1, 4, 4
Prediction of new mean: will remain 4 since the “doubling” cancels out
Check:
(7 + 7 + 5 + 5 + 3 + 3 + 1 + 1 + 4 + 4) + 10 = 40 + 10 = 4

INVESTIGATION 2

A. a) 24 ÷ 4 = 6
   b) 48 ÷ 8 = 6
   c) 72 ÷ 12 = 6
   d) 12 ÷ 4 = 3
   e) 36 ÷ 12 = 3

B. Answers will vary – teacher to check.

Sample solution:
Original set: 7, 5, 3, 1, 4
New set: 7, 7, 5, 3, 3, 1, 1, 4, 4
Prediction of new mean: will remain 4 since the “doubling” cancels out
Check:
(7 + 7 + 5 + 5 + 3 + 3 + 1 + 1 + 4 + 4) + 10 = 40 + 10 = 6
C. No; Repeating all the data values the same number of times is like multiplying the sum of the data values by a number. However, this means you’re also multiplying the total number of data values by the same number. In this way, when we find the mean by dividing the sum by the number of data values, this factor will cancel out.

INVESTIGATION 3
Answers will vary – teacher to check. NOTE: The mode and median will always remain unchanged.

1. a) Yes, they both get the same answer (mean = 1.1) since Sally’s data is simply Tina’s data repeated five times – and then divided out by that same multiple of five in the end.

b) Answers may vary – teacher to check.

c) 2.75; This doesn’t make sense because the mean can’t be higher than the data values.

d) 0.55; This doesn’t make sense because the majority of families have at least 1 car – this tells us that 0.55 is too low.

2. a) | # cars | * | % |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>90</td>
<td>25</td>
</tr>
<tr>
<td>1</td>
<td>125</td>
<td>35</td>
</tr>
<tr>
<td>2</td>
<td>110</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td>35</td>
<td>10</td>
</tr>
</tbody>
</table>

   b) Repeating values the same number of times doesn’t affect their proportional relationship. It is this proportional relationship (which is the one shown in a circle graph) that defines the data values’ mean, median and mode.

   3. Answers will vary – teacher to check.

   Sample answer:
   The data set 4, 6, 7, 7, 7, 8 and 10 gives mean = mode = median = 7.

   4. Answers will vary – teacher to check.

   Sample answer:
   Team A has 5 players, aged 6, 7, 7, 8 and 8 (mean = 7.2).
   Team B has 4 players, aged 4, 5, 5 and 6 (mean = 5).
   If the 6-year-old from Team A moves to Team B, the new mean of A is 7.5 and the new mean of B is 5.2 – both means have increased.

  5. a) 2000;
     data set = (0, 0, 500, 500, 500, 500)
  
     b) 3000;
     data set = (500, 500, 500, 500, 500, 500)
  
     c) 1006;
     data set = (0, 1, 2, 3, 500, 500)
  
     d) Answers will vary – teacher to check.

   AP Book PDM8-17
   page 196
   1. Mean = 83.86
      Median = 84
      Mode = 86
      Store B is using the mode in their claim. This is misleading because the price is higher than both the median and the mean, so the client would pay less in more than half the stores. Also, in terms of money, “average” usually suggests “mean.”

   2. Mean;
      Some days sales will be higher than others (e.g. weekends vs weekdays) so this accounts for any fluctuation.

   3. Mean;
      Grade averages are always found using the mean.

   4. Mode;
      It gives the most common size, which lets the owner match his stock to his customers’ needs.

   5. a) 2 000;
     data set = (0, 0, 500, 500, 500, 500)
   
     b) 3 000;
     data set = (500, 500, 500, 500, 500, 500)
   
     c) 1 006;
     data set = (0, 1, 2, 3, 500, 500)
   
     d) Answers will vary – teacher to check.

   6. a) 80 + 91 + 93 + x = 360
     ∴ x = 96%
   
     b) 72 + 86 + 92 + 73 + 76 + x = 510
     ∴ x = 111%
     A grade over 100% isn’t possible so no, Bob can’t do it.

   7. 31 + 2x = 75
     ∴ x = 22 kg

   8. a) 660 km
   
     b) 240 km
   
     c) 900 km
   
     d) 900 km/h = 90 km/h
   
     e) 110 km/h;
     This makes sense because they spent more time travelling at this speed.
d) \( \frac{6}{8} = \frac{3}{4} \)

e) \( \frac{2}{8} = \frac{1}{4} \)

f) \( \frac{5}{8} \)

10. a) 1, 2, 3, 4, 5, 6
b) 6

c) \( \frac{3}{6} = \frac{1}{2} \)

d) 2, 4, 6
b) 3
c) \( \frac{2}{6} = \frac{1}{3} \)

e) 5, 6
b) 2
c) 2
f) \( \frac{3}{5} \)

13. D, A; C, B

14. a) \( \frac{1}{5} \)
b) \( \frac{2}{5} \)
c) \( \frac{1}{5} \)
d) \( \frac{1}{5} \)
e) \( \frac{4}{5} \)
f) \( \frac{3}{5} \)

15. Ella is right. There are 6 possible rolls on a die (1, 2, 3, 4, 5, 6) and each one is equally likely, so the probability of any roll (including 5 or 1) is \( \frac{1}{6} \).

16. Answers will vary – teacher to check.

### AP Book PDM8-19

**Page 200**

1. \( \text{Circle:} \)
   - \(30 \div 50 = 60\% \), 3 out of 5,
   - \(3 \times 2 = 9\) out of 15, \(60\% \)

2. a) \( P(W) = \frac{1}{4} = 0.25 \)
   - \( P(\text{any colour}) = 1 = 1.0 = 100\% \)
   b) \( P(R) = \frac{2}{5} = 0.4 \)
   - \( P(W) = \frac{3}{5} = 0.6 \)
   - \( P(G) = \frac{0}{5} = 0 = 0\% \)

3. \( \text{Circle:} \)
   - 40\% \), \( \frac{1}{3} \), 0.58, \( \frac{4}{7} \), 1:5, 1,
   - 57.3\%, \( \frac{31}{100} \), 0.35, 0,
   - \( +0.25, 2:7 \)

4. Answers will vary – teacher to check.

   **Sample answers:**
   - a) rolling a 1, 2, 3, 4, 5 or 6 on a regular die
   - b) flipping heads on a regular coin
   - c) rolling a 7 on a regular die

5. a) \( \frac{60}{100} = \frac{3}{5} \)
   b) \( \frac{35}{100} = \frac{7}{20} \)
   c) \( \frac{75}{100} = \frac{3}{4} \)

6. a) \( \frac{125}{1000} = \frac{1}{8} \)
   b) \( \frac{300}{1000} = \frac{3}{10} \)
   c) \( \frac{425}{1000} = \frac{17}{40} \)
   d) \( \frac{256}{1000} = \frac{32}{125} \)
   e) \( \frac{324}{1000} = \frac{81}{250} \)

### AP Book PDM8-20

**Page 201**

1. \( \text{a)} \)
   - H
   - R
   - G
   - Y
   - T
   - Y
   - R
   - B
   - Y
   - S
   - R
   - B
   - Y

   **drama**
   **visual arts**
   **dance**
   **cr.wrtg**
   **dance**
   **cr.wrtg**

2. a) 2: (2,2), (3,1)
   b) 2: (2,2), (4,1)

3. \( \text{Circle:} \)
   - \( \frac{2}{3} \)
   - \( \frac{3}{4} \)

   **3. a)\( 2 \times 2 = 4 \)**
   b) \( 2 \times 2 = 4 \)
   c) \( 2 \times 3 = 6 \)

4. a) \( Q \)
   - 1st
   - 2nd
   - T
   - 4
   - 2
   - 2
   - 3
   - 2
   - 6
   b) You multiply them.
   c) It will have \( 20 \times 12 = 240 \) paths.

### AP Book PDM8-21

**Page 202**

1. \( \text{a)} \)
   - W W W W R R R R
   b) \( \frac{4}{6} = \frac{2}{3} \)
   c) \( \frac{4}{6} = \frac{2}{3} \)
   d) \( \frac{4}{6} = \frac{2}{3} \)
   e) \( \frac{4}{6} = \frac{2}{3} \)

2. a) Player C
   - (0.32 vs 0.315 and 0.33)
   b) Player A
   - (0.25 vs 0.24 and 0.22)

3. b) 3
   c) 3 times each
   d) 9

4. a) \( \text{Morning Afternoon} \)
   - painting drama
   - painting creative writing
   - painting dance
   - music drama
   - music creative writing
   - music dance
   b) Because there are 3 options for the afternoon.

5. \( \text{C H H H T T T T S R B Y R B Y} \)

   The 6 outcomes are: \( (H,R), (H,B), (H,Y), (T,R), (T,B), (T,Y) \)

6. \( \text{RP LP Value} \)
   - Q D 35¢
   - Q N 30¢
   - D D 20¢
   - D N 15¢

7. There are 9 combinations (3 \( \times 3 \)):

   **Morning**
   - badminton
   - canoeing
   - badminton
   - swimming
   - badminton
   - diving
   - squash
canoeing
   - squash
diving
tennis
canoeing
tennis
swimming
tennis
swimming

**Copyright © 2011, Jump Math. Not to be copied.**
8. a) | D1 | D2 | Score |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

b) 1: (0,1), (1,0)
2: (0,2), (1,1), (2,0)
3: (0,3), (1,2), (2,1), (3,0)
4: (1,3), (2,2), (3,1)
5: (2,3), (3,2)

c) 3 combinations give a total of 4; No, since each region is a different (and unknown) size.

9. There are 16 combinations (4 x 4):

<table>
<thead>
<tr>
<th>1st</th>
<th>2nd</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

10. a) 6 x 6 = 36
   b) 6 x 35 = 210

AP Book PDM8-22

page 204

1. a) ii) G, G, B, B
   b) Yes

2. a) i) B, G, G, B, B
    ii) B, G, G, B, B
   b) No

3. Jade’s

4. a) Yes, since the draw doesn’t affect the spin.
   b) No, since the coin is not replaced after the first draw.
   c) No, since the same student can’t be picked as captain twice (the team picking second has fewer choices).

5. a) (1,R), (2,R), (3,R), (1,B), (2,B), (3,B)
   b) 6
   c) i) 1
      ii) 2
   d) i) 1
      ii) 2
   e) i) 3
      ii) 3

INVESTIGATION 1

A. P(R) = 1/4
   P(W) = 2/4 = 1/2
   P(Y) = 1/4

B. P(A) = 1/3
   P(B) = 1/3
   P(C) = 1/3
   P(A and R) = 1/12
   P(A and W) = 2/12 = 1/6
   P(A and Y) = 1/12
   P(B and R) = 1/4
   P(B and W) = 2/4 = 1/2
   P(B and Y) = 1/12
   P(C and R) = 1/3
   P(C and W) = 1/3
   P(C and Y) = 1/3

Conjecture should be to multiply the probabilities of the individual events – teacher to check.

P(B and Y) = 1/12
P(B and Y) = 1/12

E. P(Red) = 2/3
   P(Red and Red) = 4/9

For Bob’s way: No
For Nick’s way: Yes

The formula is correct for Nick’s way since his draws are independent events; Bob’s are not.

INVESTIGATION 2

A. Bob:

B. Nick
(9 vs 6 outcomes)

C. Nick’s

D.

E. P(Red) = 2/3
   P(Red and Red) = 4/9

For Bob’s way: No
For Nick’s way: Yes

The formula is correct for Nick’s way since his draws are independent events; Bob’s are not.

AP Book PDM8-23

page 206

<table>
<thead>
<tr>
<th>1st</th>
<th>2nd</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
1. a) 

<table>
<thead>
<tr>
<th>Joe</th>
<th>Ida</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
<td>$10</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>$10</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>$10</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>$10</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>$10</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>$10</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>$15</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>$15</td>
</tr>
</tbody>
</table>

b) Five combinations will be enough:

(5,10), (5,10), (10,5), (10,5), (10,10)

\[ P(\text{sum} \geq $15) = \frac{5}{9} \]

c) Yes, the events are independent.

From formula:

\[ P(10,10) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \]

From table:

\[ P(10,10) = \frac{1}{9} \]

3. She wrote the number 1 six times because there are 6 sides on a die so 6 possible rolls for the second die.

Completed table:

<table>
<thead>
<tr>
<th>1st</th>
<th>2nd</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

b) Circle, as shown:

(1,4), (2,3), (3,2), (4,1)

c) \[ \frac{4}{32} = \frac{1}{8} \]

d) There are 2 ways:

(2,8) and (4,8)

\[ P(\text{8,even}) = \frac{2}{32} = \frac{1}{16} \]

e) You could use it for part d):

\[ P(\text{8}) = \frac{1}{8} \]

\[ P(\text{even}) = \frac{2}{4} = \frac{1}{2} \]

\[ P(\text{8,even}) = \frac{1}{16} \]

4. Simone’s table shows 36 combinations (6 x 6):

a) i) \[ \frac{3}{36} = \frac{1}{12} \]

ii) \[ \frac{5}{36} \]

iii) \[ \frac{6}{36} = \frac{1}{6} \]

iv) \[ \frac{2}{36} = \frac{1}{18} \]

v) \[ \frac{1}{36} \]

b) i) \[ \frac{18}{36} = \frac{1}{2} \]

ii) \[ \frac{9}{36} = \frac{1}{4} \]

iii) \[ \frac{10}{36} = \frac{5}{18} \]

c) \[ P(\text{sum of 2}) = \frac{1}{36} \]

\[ P(\text{sum of 12}) = \frac{1}{36} \]

d) \[ P(\text{sum of 7}) = \frac{6}{36} = \frac{1}{6} \]

5. a) \[ B \left( \frac{1}{16} \text{ vs } \frac{1}{36} \right) \]

b) \[ B \left( \frac{1}{4} \text{ vs } \frac{1}{18} \right) \]

c) Answers will vary – teacher to check.

Answer Keys for AP Book 8.2

INVESTIGATION

A. \[ P(H,H) = \frac{1}{4} \]

Students can find this by using a T-table or by knowing that coin tosses are independent events.

B. Answers will vary – teacher to check.

C. Answers will vary – teacher to check.

D. Answers will vary – teacher to check.

NOTE: The horizontal line should be drawn at \( y = 0.25 \). In theory, student line graphs will zig-zag around this line, getting closer as the number of coin tosses increases.

E. Answers will vary – teacher to check.

NOTE: Students should not expect to get the same results as everyone else.

F. Answers will vary – teacher to check.

G. Answers will vary – teacher to check.

NOTE: The one using 50 tosses will likely be closer to the theoretical possibility.

2. a) \[ 1 : 3 \]

b) i) 5

ii) 11

iii) 20
iv) Although very small, it is possible for this to happen since, for each spin, there is a 1 in 3 chance of spinning yellow.

d) \( P(Y) = \frac{8}{15} \)
e) Although Karin’s experimental probability for spinning yellow was 8 : 15, we expect this ratio to approach 1 : 3 (the theoretical probability) as the number of spins increases. Instead, she should expect to get about 500 yellow spins in 1500 spins, since 1 : 3 = 500 : 1500.

3. a) Experiment 1 has one component: drawing 2 marbles together.
   Experiment 2 has two independent experiments: drawing each marble separately.
   In both cases, two marbles are drawn as the results, but the 2nd experiment has more outcomes.
   \( \therefore \) we expect the theoretical probabilities to be different.
   b) Answers will vary – teacher to check.
   c) Answers will vary – teacher to check.

INVESTIGATION

<table>
<thead>
<tr>
<th>a)</th>
<th>1/2</th>
<th>1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>b)</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>c)</td>
<td>3/7</td>
<td>4/7</td>
</tr>
<tr>
<td>d)</td>
<td>1/3</td>
<td>2/3</td>
</tr>
<tr>
<td>e)</td>
<td>1/6</td>
<td>5/6</td>
</tr>
<tr>
<td>f)</td>
<td>26/52 = 1/2</td>
<td>26/52 = 1/2</td>
</tr>
</tbody>
</table>

B. The probability of an event plus the probability of its complementary event equals 1.

C. \( P(\text{certain event}) = 1; \) Examples will vary – teacher to check.
   Sample answer:
   Event A: rolling a 1, 2, 3, 4, 5 or 6 on a regular die.
   Complement to Event A: not rolling a 1, 2, 3, 4, 5 or 6 on a regular die.
   \( P(A) + P(\text{not } A) = 0 \)
   \( \therefore \) Yes: \( P(A) + P(\text{not } A) = 1 \)

D. By definition, an event and its complementary event have no common events but, together, account for all possible outcomes.
   That means it is certain that either the event or its complement will occur each time. Therefore, together, their probability must be 1.

INVESTIGATION

A. 2 000

B. Random numbers will vary;
   It’s like randomly choosing 5 people because each number represents one person.

3. a) \( P(\text{no rain}) = \frac{2}{5} \)
   b) \( P(\text{no rain}) = \frac{3}{10} \)
C. We assigned numbers to people, so choosing the same number twice would be like surveying the same person twice. As such, this person would get to express their opinion more than once, which is unfair to other participants. This should not be allowed in a survey, so you should not allow repeated numbers.

D. The one in part (ii)

<table>
<thead>
<tr>
<th>Size</th>
<th>#</th>
<th>Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>1/5</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>2/5</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1/10</td>
</tr>
</tbody>
</table>

F. Answers will vary – teacher to check.

G. Answers will vary – teacher to check.

H. Answers will vary – teacher to check.

I. Answers will vary – teacher to check.

J. Yes, in theory.

K. Answers will vary, depending on the results of the group. Some classes may need to form a group of 32 and have 640 numbers.

L. In part D ii) above, for example, the fraction of multiples of 5 in the set would be the same as the fraction of “no” answers, rather than of “yes” answers.

M. I wouldn’t change the investigation at all. I would expect the same fraction of multiples of 5 in any set of 320 values, regardless of whether the 320 were chosen from 1 to 10,000 or from 1 to 100,000. So the same fraction of “yes” answers would appear in a sample of 320 people chosen from 10,000 people as a sample of 320 people chosen from 100,000 people.

AP Book PDM8-27

1. a) The first group isn’t a random sample: it just includes those who came early.
   b) B
   c) 200 students

2. A is biased since early arrivers are more likely to prefer an earlier start time.

3. a) biased (only older students); representative
   b) biased (more fit); representative
   c) biased (too close); representative; biased (too far)

4. The shopping mall since it is representative rather than biased.

   Beach = swimmers ∶ more likely to prefer a pool
   Music store = music fans ∶ more likely to prefer a concert hall
   Soccer game = sports fans ∶ more likely to prefer a baseball stadium

5. a) Mickey’s sample is more biased. He only asked the people in his class, who know him so are more likely to vote for him than for someone else.

   b) Helen, since her sample represents the whole school and not a biased sample (students not in Mickey’s class will be less likely to vote for him).

6. The selected hour ranges are too large: it is unlikely that many students will play video games for more than 8 hours/day. Rewritten surveys will vary – teacher to check.

   Sample answers:
   a) “During the week, what time do you usually get up?”
      - 6:00–6:59 am
      - 7:00–7:59 am
      - 8:00–8:59 am
      - other, please say what time: ______
   b) I would survey a group as varied as possible: younger, older, professionals, retirees, students, shift workers, males, females, etc.

   To select my sample, I would assign numbers to the houses in my neighbourhood and then choose from them randomly.

   c) I would use a circle graph because it will clearly represent the proportion who get up at each time.

7. a) A; B
   b) Answers will vary – teacher to check.

   Sample question: “Are you in favour of replacing the city park with a department store?”

8. a) The questions
   b) The order of the questions
   c) Survey A, because that question immediately follows the question about music, which most people enjoyed.
   d) It can put particular thoughts in your mind as you move into a new question. This may influence your original/natural response.

AP Book PDM8-28

1. Answers will vary – teacher to check.

   Sample answers:
   a) 1. “How do you get to school?”
   2. “How long does it take?”
   b) I would only include walkers and bus riders in my survey, not cyclists, those dropped off by parents, etc.

   To pick the sample group, I could choose students randomly from the school’s class lists.
Probability and Data Management – AP Book 8, Part 2: Unit 8 (continued)

3. Answers will vary – teacher to check.

Sample answers:
   a) Two
   b) 1. “How tall are you, in cm?”
      2. “What grade point average did you get last term?”
   c) Again, I would choose a random sample from the school (using class lists, as in #2).
   d) I would use a scatter plot to see if there was any relationship between the categories.

4. Answers will vary – teacher to check.

Sample answers:
   a) A scatter plot, with length on one axis and width on the other.
   b) I would expect an “upward” pattern, something like this:

5. Answers will vary – teacher to check.

Sample answers:
   a) I would need to measure each person’s accuracy over time.
   b) I will measure my results based on the scores of 6 darts.
      At the very start, each person has two turns with 3 darts; I will record their scores and find the mean.
      Then I will let each person practice for the same amount of time (say 5 minutes) before they each get two official post-practice turns, which I’ll record for comparison.
      For equipment, I’ll need various dart stations with 3 darts each – and paper/pencils to record the scores.
   c) I would have people help by having them run the different dart stations (e.g. time-keeping, recording, etc.)
   d) Yes, the distance must always be the same – and should also be consistent across all the dart stations. The data would be biased otherwise since, the closer you are to the board, the easier it is.
   e) I would use a bar graph with two bars: one of the average “pre-practice” scores and the second with the average of the “post-practice” scores.

6. Answers will vary – teacher to check.

7. Answers will vary – teacher to check.
1. Convert each fraction to a decimal fraction. Then change the fraction to a decimal.

a) \( \frac{37}{50} \)  
b) \( \frac{27}{60} \)

2. Circle the larger number: 0.354  0.354

3. Add by lining up the decimal places: 0.6 + 0.25 = _________

4. Change each fraction to a repeating decimal.

a) \( \frac{374}{999} \)  
b) \( \frac{74}{990} \)

5. Change each repeating decimal to a fraction.

a) \( 0.\overline{56} \)  
b) \( 0.0\overline{31} \)

**BONUS:** Change 0.1\(\overline{3} \) to a fraction \( \frac{a}{b} \). Check your answer by dividing \( a \div b \) using long division.
Unit 1: Number Sense

Quiz (Lessons 75–81) — ON & WNCP

1. a) \( \frac{74}{100} = 0.74 \)
   b) \( \frac{9}{20} = \frac{45}{100} = 0.45 \)

2. Circle: 0.354

3. 0.916

4. a) 0.374
   b) 0.074

5. a) \( \frac{56}{99} \)
   b) \( \frac{31}{990} \)

BONUS

\[
0.1 + 0.03 = \frac{1}{10} + \frac{1}{30} = \frac{3}{30} + \frac{1}{30} = \frac{4}{30} = \frac{2}{15}
\]

Then divide 2 ÷ 15 using long division. This indeed results in 0.13.
Unit 1: Number Sense
Quiz (Lessons 82–89) — ON & WNCP

1. Write the fraction as a percentage.
   a) \(\frac{73}{100} = \) _____ %
   b) \(\frac{9}{100} = \) _____ %

2. Write the percentage as a fraction.
   a) 48% =
   b) 6% =

3. Find the missing percentage in the circle graph.
   Lacrosse: _____ %

4. Write the percentage as a decimal.
   a) 36% = _____
   b) 2% = _____

5. Write the decimal as a percentage.
   a) 0.04 = _____ %
   b) 0.29 = _____ %

6. Write the fraction as a percentage by changing to a fraction out of 100. Reduce the fraction to lowest terms if necessary.
   a) \(\frac{18}{25} = \) _____ = _____ %
   b) \(\frac{9}{12} = \) _____ = _____ = _____ %
Unit 1: Number Sense

Quiz (Lessons 82–89) — ON & WNCP

7. Write the fraction as a decimal. Round to two decimal places. Then write the approximate percentage.
   a) $\frac{5}{6} = \underline{\hspace{2cm}} \approx \underline{\hspace{2cm}}$  
       $= \underline{\hspace{2cm}}\%$
   b) $\frac{3}{16} = \underline{\hspace{2cm}} \approx \underline{\hspace{2cm}}$  
       $= \underline{\hspace{2cm}}\%$

8. What percentage of the figure is shaded?
   a)  
       b)  
       $\underline{\hspace{2cm}}\%$  
       $\underline{\hspace{2cm}}\%$

9. Write “<,” “>,” or “=” between each pair of numbers to compare the numbers. First change the numbers to a pair of decimal fractions.
   a) $\frac{3}{4} \quad 70\%$  
   b) $\frac{2}{5} \quad 45\%$  
   c) $\frac{18}{25} \quad 72\%$

10. Calculate each percentage by first finding 1% of the number.
    a) 40% of 500  
       b) 37% of 2,000
       
       $1\%$ of 500 = \underline{\hspace{2cm}}  
       $1\%$ of 2,000 = \underline{\hspace{2cm}}
       
       $40\%$ of 500 = \underline{\hspace{2cm}}  
       $37\%$ of 2,000 = \underline{\hspace{2cm}}
11. Multiply to find 62\% of 35.

**BONUS:** Find 2\% of 1,000,000.
Unit 1: Number Sense

Quiz (Lessons 82–89) — ON & WNCP

1. a) 73%
   b) 9%
2. a) \(\frac{48}{100}\)
   b) \(\frac{6}{100}\)
3. 10%
4. a) 0.36
   b) 0.02
5. a) 4%
   b) 29%
6. a) \(\frac{72}{100} = 72\%\)
   b) \(\frac{3}{4} = \frac{75}{100} = 75\%\)
7. a) 0.83333...
   \(\approx 0.83 = 83\%\)
   b) 0.1875
   \(\approx 0.19 = 19\%\)
8. a) 75%
   b) 40%
9. a) \(\frac{3}{4} = \frac{75}{100}\)
   \(70\% = \frac{70}{100}\)
   \(\frac{3}{4} > 70\%\)
   b) \(\frac{2}{5} = \frac{40}{100}\)
   \(45\% = \frac{45}{100}\)
   \(\frac{2}{5} < 45\%\)
   c) \(\frac{18}{25} = \frac{72}{100}\)
   \(72\% = \frac{72}{100}\)
   \(\frac{18}{25} = 72\%\)
10. a) 5
    b) 20
11. 21.70

BONUS

1% of 1 000 000
\(= 10 000\)
2% of 1 000 000
\(= 20 000\)
1. Convert each fraction to a decimal fraction. Then change the fraction to a decimal.
   a) \( \frac{13}{25} \)  
   b) \( \frac{137}{200} \)

2. Change the decimals to fractions and then subtract the fractions. Then change the fraction to a decimal by dividing.
   \( 0.75 - 0.\overline{3} = \)

3. Change each fraction to a repeating decimal.
   a) \( \frac{14}{99} \)  
   b) \( \frac{8}{90} \)

4. Change each repeating decimal to a fraction.
   a) \( 0.\overline{703} = \)  
   b) \( 0.0\overline{23} = \)

5. Complete the chart.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>19 ( \frac{25}{25} )</th>
<th>7 ( \frac{20}{20} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decimal</td>
<td>0.45</td>
<td>0.80</td>
</tr>
<tr>
<td>Percent</td>
<td>30%</td>
<td>22%</td>
</tr>
</tbody>
</table>

6. Order the numbers from least to greatest: \( \frac{2}{5} \), 31\%, 0.46  
   \( ______ < ______ < ______ \)

7. a) What is 6\% of 300?  
   b) If 25\% is 14, what is 100\%?

8. Find 42\% of 15.
Unit 1: Number Sense

Test (Lessons 75–89) — ON & WNCP

9. 3 out of 5 players on a basketball team are girls. What fraction and percent of the players are girls?

10. A local shelter has 3 dogs and 7 cats for adoption. What fraction of the pets are dogs? What percent are cats?

11. You buy a book that costs $12. Sales tax is 15% of the cost of the book. How much sales tax will you pay? Hint: 15% = 10% + 5%.

12. A 5% initial payment on a new car is $1 200. What is the total price of the car?

13. A dress that usually costs $50 is on sale for 30% off. What is the sale price of the dress?

**BONUS:** a) Add 0.27 + 0.72 ...

i) by lining up the decimal places.  

ii) by converting each decimal to a fraction, then adding the two fractions.

b) Based on your answers from part a), how do the values 0.9 and 1 compare?
1. a) \( \frac{52}{100} = 0.52 \)
   
b) \( \frac{685}{1000} = 0.685 \)

2. \( \frac{3}{4} - \frac{1}{3} = \frac{5}{12} \)
   
   and \( 5 + 12 = 0.416 \)

3. a) 0.74
   
b) 0.08

4. a) \( \frac{703}{999} \)
   
b) \( \frac{23}{990} \)

5. \( \frac{3}{10} = 0.3 = 30\% \)
   
   \( \frac{45}{100} = \frac{9}{20} = 0.45 = 45\% \)

   \( \frac{19}{25} = 0.76 = 76\% \)

   \( \frac{11}{50} = 0.22 = 22\% \)

   \( \frac{7}{20} = 0.35 = 35\% \)

   \( \frac{4}{5} = 0.8 = 80\% \)

6. \( 31\% < \frac{2}{5} < 0.46 \)

7. a) 18
   
b) 56

8. 6.3

9. \( \frac{3}{5} \) or 60\% are girls.

10. \( \frac{3}{10} \) are dogs
    
    70\% are cats

11. \$1.20 + \$0.60 = \$1.80 tax

12. If 5\% is \$1 200, then 10\% is \$2 400.
    
    \( \therefore \) the car costs \$24 000.

13. \$35.00

   bonus

   a) i) 0.272727...

   \[
   \begin{array}{c}
   \hline
   0.27 \hfill \\
   + 0.727272 \hfill \\
   \hline
   0.999999 \hfill \\
   \end{array}
   \]

   ii) \( \frac{99}{99} = 1 \)
Unit 1: Number Sense

Quiz (Lessons 90–96) — ON & WNCP

Name: ______________________

Date: ________________

1. Solve the proportion.
   a) \( \frac{30}{5} = \frac{7}{7} \)
   b) \( \frac{28}{45} = \frac{63}{63} \)

2. Write a proportion to represent the percent problem. Then solve the proportion.
   a) What is 30% of 50?
   b) What percent of 50 is 30?
   c) If 30 is 50%, what is 100%?

3. Which part of Question 2 asks to solve the same proportion as the word problem below?
   “A shirt that costs $50 is $30 off. What percent sale is that?” part ______

4. Write the numbers in order.
   a) 0.009, 0.065, 0.5%
   b) 4.5, 387%, 22

   ______ < ________ < ________

   ______ < ________ < ________

5. Cross-multiply to write an equation. Then solve for \( x \).
   a) \( \frac{x}{10} = \frac{9}{15} \)
   BONUS: \( \frac{10}{7} = \frac{3}{x} \)
1. a) 42  
   b) 20  

2. a) \( \frac{30}{100} = \frac{x}{50} \)  
    \( \therefore x = 15 \)  
   b) \( \frac{30}{50} = \frac{x}{100} \)  
    \( \therefore x = 60 \)  
   c) \( \frac{30}{x} = \frac{50}{100} = \frac{1}{2} \)  
    \( \therefore x = 60 \)  

3. b)  

4. a) 0.5% < 0.009 < 0.065  
   b) 387% < 4.5 < 22  

5. a) \( x = 6 \)  
   BONUS  
   \( x = 2.1 \)
1. Find the number.
   a) \( \frac{3}{4} \) of a number is 12.
   \( \frac{3}{4} = _____ \)

   b) \( \frac{2}{3} \) of a number is 18.
   \( \frac{2}{3} = _____ \)

   \( x = _____ \)

2. A box holds red and blue marbles. \( \frac{2}{5} \) of the marbles are red. There are 14 red marbles.
   a) How many marbles are there altogether?
   b) How many blue marbles are there?

3. Find the new price.
   a) A video game costs $80. Now the video game is 15% off.
   b) A collectible card sells for $20. Now the card sells for 8% more.

4. Pure gold is 24 karat. A gold ring with a mass of 12 g has 9 g of pure gold. How many karat gold is the ring?
5. Find the unit rate.
   a) $32 \text{ km} / 8 \text{ h}$
   b) $24 \text{ } / 3 \text{ h}$

6. On a map, the straight line distance from Montreal to Vancouver is 5 cm. The actual straight line distance is 3 680 km.
   a) What does 1 cm on the map represent in actual distance?

   b) On the same map, the distance from Toronto to another city is 3 cm. What is the actual distance?

7. Solve the proportion.
   a) $\frac{5}{0.4} = \frac{12}{\phantom{0}}$
   b) $\frac{0.8}{\phantom{0}} = \frac{10}{16}$

BONUS: Solve the proportion.
   $\frac{10 000 000}{15 000 000} = \frac{2}{\phantom{0}} = \frac{\phantom{0}}{12 000 000}$
1. a) \( \frac{3}{4} \cdot \frac{12}{x} \)  
   \( x = 16 \)

b) \( \frac{2}{3} \cdot \frac{18}{x} \)  
   \( x = 27 \)

2. a) \( \frac{2}{5} \cdot \frac{14}{x} \)  
   \( x = 35 \)
   35 marbles altogether

b) \( x = 35 - 14 \)  
   = 21 blue marbles

3. a) discount = 0.15 \times 80  
   = 12
   price = 80 - 12  
   = $68

b) tax = 0.08 \times 20  
   = 1.60
   price = 20 + 1.60  
   = $21.60

4. \( \frac{9}{12} = \frac{x}{24} \)  
   \( x = 18 \)
   18 karat gold

5. a) 4 km / 1 h

b) $8 / 1 h

6. a) 3680 km / 5 cm  
   = 736 km / 1 cm

b) 3 \times 736 = 2208 km

7. a) \( \frac{5}{0.4} = \frac{50}{4} = \frac{150}{12} \)

b) \( \frac{0.5}{0.8} = \frac{5}{8} = \frac{10}{16} \)

BONUS

\[ \frac{10\,000\,000}{15\,000\,000} = \frac{2}{3} \]

\[ \frac{8\,000\,000}{12\,000\,000} \]
1. Find the number.
   a) \(\frac{3}{4}\) of a number is 12.
   \[
   \frac{3}{4} = \_\_\_
   \]
   x = __
   
   b) \(\frac{2}{3}\) of a number is 18.
   \[
   \frac{2}{3} = \_\_\_
   \]
   x = __

2. A box holds red and blue marbles. \(\frac{2}{5}\) of the marbles are red. There are 14 red marbles.
   a) How many marbles are there altogether?
   b) How many blue marbles are there?

3. Find the new price.
   a) A video game costs $80. Now the video game is 15% off.
      
   b) A collectible card sells for $20. Now the card sells for 8% more.

4. Pure gold is 24 karat. A gold ring with a mass of 12 g has 9 g of pure gold. How many karat gold is the ring?
5. Find the unit rate.
   a) 32 km / 8 h       b) $24 / 3 h

6. On a map, the straight line distance from Montreal to Vancouver is 5 cm. The actual straight line distance is 3,680 km.
   a) What does 1 cm on the map represent in actual distance?

   b) On the same map, the distance from Toronto to another city is 3 cm. What is the actual distance?

7. Solve the proportion.

   a) \( \frac{5}{0.4} = \frac{\text{______}}{12} \)
   
   b) \( \frac{10}{0.8} = \frac{\text{______}}{16} \)

   c) _____ : 16 = 3 : 8 = 18 : _____

**BONUS:** Solve the proportion. \( \frac{10\,000\,000}{15\,000\,000} = \frac{2}{12\,000\,000} \)
1. a) \[ \frac{3}{4} = \frac{12}{x} \]
   \[ x = 16 \]

   b) \[ \frac{2}{3} = \frac{18}{x} \]
   \[ x = 27 \]

2. a) \[ \frac{2}{5} = \frac{14}{x} \]
   \[ x = 35 \]
   
   35 marbles altogether

   b) \[ x = 35 - 14 \]
   \[ = 21 \text{ blue marbles} \]

3. a) discount = 0.15 \times 80
   \[ = 12 \]
   
   price = 80 - 12
   \[ = 68 \]

   b) tax = 0.08 \times 20
   \[ = 1.60 \]
   
   price = 20 + 1.60
   \[ = 21.60 \]

4. \[ \frac{9}{12} = \frac{x}{24} \]
   \[ x = 18 \]
   
   18 karat gold

5. a) 4 km / 1 h
   
   b) $8 / 1 h

6. a) 3 680 km / 5 cm
   \[ = 736 \text{ km} / 1 \text{ cm} \]

   b) \[ 3 \times 736 = 2 208 \text{ km} \]

7. a) \[ \frac{5}{0.4} = \frac{50}{4} = \frac{150}{12} \]

   b) \[ \frac{0.5}{0.8} = \frac{5}{8} = \frac{10}{16} \]

   c) \[ 6 : 16 = 3 : 8 = 18 : 48 \]

BONUS

\[ \frac{10 000 000}{15 000 000} = \frac{2}{3} \]

\[ = \frac{8 000 000}{12 000 000} \]
1. Cross-multiply to decide if each pair of ratios is equivalent.
   a) \( \frac{5}{7} \) and \( \frac{8}{11} \)  
   b) \( \frac{3}{5} \) and \( \frac{12}{20} \) 

   **BONUS:** \( \frac{4}{9} \) and \( \frac{2.8}{6.3} \)

2. Cross-multiply to write an equation. Then solve for \( x \).
   a) \( \frac{15}{x} = \frac{6}{8} \)  
   b) \( \frac{10}{9} = \frac{8}{x} \) 

   **BONUS:** Solve for \( x \) and \( y \).
   \( \frac{3}{5} = \frac{12}{x} = \frac{y}{35} \)

3. Solve the percent problem by first writing a proportion.
   a) 7 is 4% of what number?  
   b) 7 is what percent of 8?

4. Terri bought a new computer at a 30% discount. She saved $400. What was the original price?

5. The price of a pair of shoes after a 5% tax is $58.80.
   a) What percent of the original price is $58.80?

   b) What is the original price of the shoes?
6. Write the decimal as a percent.
   a) 0.347 = ______ %                  b) 0.004 = ______ %                  c) 6.05 = ______ %

7. Write the percent as a proper or improper fraction in lowest terms.
   a) 3.2%                                        b) 0.05%                       c) 655%

8. Find 100% if …
   a) 25% is 40.                                      b) 300% is 15.                   c) 0.5% is 2.

   Which is the better buy?

**BONUS:** In a fish tank, \( \frac{2}{5} \) of the fish are red, \( \frac{1}{4} \) are yellow, and the rest are green.

There are 8 more red fish than green fish. How many fish are in the tank in total?
1. a) \(5 \times 11 = 55\) 
\(7 \times 8 = 56\) 
\(\therefore\) no, they aren’t equivalent.

b) \(3 \times 20 = 60\) 
\(5 \times 12 = 60\) 
\(\therefore\) yes, they are equivalent.

**BONUS**

\(4 \times 6.3 = 25.2\)
\(9 \times 2.8 = 25.2\)
\(\therefore\) yes, they are equivalent.

2. a) \(15 \times 8 = 6x\)
\(120 = 6x\)
\(\therefore\) \(x = 20\)

b) \(10x = 72\)
\(\therefore\) \(x = 7.2\)

**BONUS**

\(3x = 60\)
\(\therefore\) \(x = 20\)
\(3(35) = 5y\)
\(105 = 5y\)
\(\therefore\) \(y = 21\)

3. a) 175

b) 87.5%

4. $1\,333.33$

5. a) 105%

b) $56$

6. a) 34.7%

b) 0.4%

c) 605%

7. a) \(\frac{4}{125}\)

b) \(\frac{1}{2000}\)

c) \(\frac{131}{20}\)

8. a) 160

b) 5

c) 400

9. the 4 L bag

**BONUS**

160
1. Cross-multiply to decide if each pair of ratios is equivalent.
   a) \( \frac{5}{7} \) and \( \frac{8}{11} \)  
   b) \( \frac{3}{5} \) and \( \frac{12}{20} \)  
   BONUS: \( \frac{4}{9} \) and \( \frac{2.8}{6.3} \)

2. Cross-multiply to write an equation. Then solve for \( x \).
   a) \( \frac{15}{x} = \frac{6}{8} \)  
   b) \( \frac{10}{9} = \frac{8}{x} \)  
   BONUS: Solve for \( x \) and \( y \).
   \[ \frac{3}{5} = \frac{12}{x} = \frac{y}{35} \]

3. Solve the percent problem by first writing a proportion.
   a) 7 is 4% of what number?  
   b) 7 is what percent of 8?

4. Terri bought a new computer at a 30% discount. She saved $400. What was the original price?

5. Solve the 3-term proportion.
   \[ \_\_\_\_ : 5 : 3 = 36 : 20 : \_\_\_\_ \]
6. Kai ordered a meal at a restaurant. The tax is 13% and Kai wants to leave the waiter a 20% tip. If the original price was $22, how much money should Kai leave altogether?

7. Write the decimal as a percent.
   
   a) $0.347 = \underline{\quad} \% \\
   b) $0.004 = \underline{\quad} \% \\
   c) $6.05 = \underline{\quad} \%$

8. Write the percent as a proper or improper fraction in lowest terms.
   
   a) $3.2\%$ \\
   b) $0.05\%$ \\
   c) $655\%$

9. Find 100% if …
   
   a) $25\%$ is 40. \\
   b) $300\%$ is 15. \\
   c) $0.5\%$ is 2.

    Which is the better buy?

**BONUS:** In a fish tank, $\frac{2}{5}$ of the fish are red, $\frac{1}{4}$ are yellow, and the rest are green.

There are 8 more red fish than green fish. How many fish are in the tank in total?
Unit 1: Number Sense

Test (Lessons 90–103) — WNCP

1. a) \(5 \times 11 = 55\)
   \(7 \times 8 = 56\)
   \(\therefore\) no, they aren’t equivalent.

   b) \(3 \times 20 = 60\)
   \(5 \times 12 = 60\)
   \(\therefore\) yes, they are equivalent.

BONUS
   \(4 \times 6.3 = 25.2\)
   \(9 \times 2.8 = 25.2\)
   \(\therefore\) yes, they are equivalent.

2. a) \(15 \times 8 = 6x\)
   \(120 = 6x\)
   \(\therefore x = 20\)

   b) \(10x = 72\)
   \(\therefore x = 7.2\)

BONUS
   \(3x = 60\)
   \(\therefore x = 20\)
   \(3(35) = 5y\)
   \(105 = 5y\)
   \(\therefore y = 21\)

3. a) 175
   b) 87.5%

4. $1333.33

5. \(9 : 5 : 3 = 36 : 20 : 12\)

6. $29.26

7. a) 34.7%
   b) 0.4%
   c) 605%

8. a) \(\frac{4}{125}\)
   b) \(\frac{1}{2000}\)
   c) \(\frac{131}{20}\)

9. a) 160
   b) 5
   c) 400

10. the 4 L bag

BONUS
    160
1. Fifty people were surveyed about their favourite board game. Complete the table below.

<table>
<thead>
<tr>
<th>Favourite Board Game</th>
<th>Tally</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Checkers</td>
<td>☒  ☒  ☒</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Chess</td>
<td>☒  ☒  ☒  ☒</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Backgammon</td>
<td></td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>Dominoes</td>
<td>☒  ☒ ☒</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>☒  ☒  ☒  ☒  ☒  ☒</td>
<td>50</td>
<td>100%</td>
</tr>
</tbody>
</table>

2. Measure the angle of each part of the circle graph. Then calculate the percent.

<table>
<thead>
<tr>
<th>Favourite Colour</th>
<th>Angle</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Red</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3. Look at the graph below.

![Scatter plot showing the relationship between number of vowels and number of letters for 13 English words.]

a) Circle the data points for the words “idea” and “kangaroo.”
   Hint: a vowel is a, e, i, o, u, and sometimes y.

b) Add a data point for the word “average.”

c) In English, longer words tend to have more vowels than shorter words.
   How does the scatter plot show this?

d) Cross out two data points that show that a shorter word can have more vowels than a longer word. Explain how the points you chose show this.

**BONUS:** Find two English words that satisfy the data you crossed out:

________________________ and _____________________
1. 11, 22%
   17, 34%
   IIII IIII, 18%
   13, 26%
2. 180°, 50%
   108°, 30%
   72°, 20%
3. a) “idea” = 4 letters,
    3 vowels
    “kangaroo” = 8 letters,
    4 vowels
b) “average” = 7 letters,
    4 vowels
c) The points go from the bottom left to the top right.
d) Answers will vary.
   Teacher to check.

BONUS
   Answers will vary.
   Teacher to check.
1. Fifty people were surveyed about their favourite board game. Complete the table below.

<table>
<thead>
<tr>
<th>Favourite Board Game</th>
</tr>
</thead>
<tbody>
<tr>
<td>Game</td>
</tr>
<tr>
<td>Checkers</td>
</tr>
<tr>
<td>Chess</td>
</tr>
<tr>
<td>Backgammon</td>
</tr>
<tr>
<td>Dominoes</td>
</tr>
<tr>
<td><strong>Total</strong></td>
</tr>
</tbody>
</table>

2. Calculate the percent and angle for the survey results below that are going to be put in a circle graph.

<table>
<thead>
<tr>
<th>Favourite Snack</th>
</tr>
</thead>
<tbody>
<tr>
<td>Snack</td>
</tr>
<tr>
<td>Fruit</td>
</tr>
<tr>
<td>Peanuts</td>
</tr>
<tr>
<td>Chips</td>
</tr>
<tr>
<td>Granola Bar</td>
</tr>
</tbody>
</table>

3. Measure the angle of each part of the circle graph. Then calculate the percent.

<table>
<thead>
<tr>
<th>Favourite Colour</th>
</tr>
</thead>
<tbody>
<tr>
<td>Colour</td>
</tr>
<tr>
<td>Blue</td>
</tr>
<tr>
<td>Red</td>
</tr>
<tr>
<td>Other</td>
</tr>
</tbody>
</table>
Unit 2: Probability and Data Management

Quiz (Lessons 6–9, 14) — WNCP

4. The circle graph shows the percentage of Canadians that have a certain blood type.
   a) Which blood type is the most common? _____
   b) Which blood type is the least common? _____
   c) Do more than half of the people have blood type A? Explain.

**BONUS:** A circle graph has 20 equal parts. How many degrees are in each section of the graph?
Unit 2: Probability and Data Management

Quiz (Lessons 6–9, 14) — WNCP

1. 11, 22%
   17, 34%
   18, 18%
   13, 26%

2. 25%, 90°
   50%, 180°
   15%, 54°
   10%, 36°

3. 180°, 50%
   108°, 30%
   72°, 20%

4. a) O
   b) AB
   c) No. The A section of the graph is less than half a circle.

BONUS

\[360° + 20 = 18°\]
1. The table shows the frequency for employee salaries. Draw the histogram for the table.

<table>
<thead>
<tr>
<th>Salary Frequency Table (1 000s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interval</td>
</tr>
<tr>
<td>0–30</td>
</tr>
<tr>
<td>31–60</td>
</tr>
<tr>
<td>61–90</td>
</tr>
<tr>
<td>91–120</td>
</tr>
</tbody>
</table>

2. The histogram shows the results of a survey asking students the amount of money in their bank accounts.

   a) How many students were surveyed?  

   b) How many students had less than $170?  

   c) How many students had $230 or more?  

   d) How many students had between $200 and $230?  

   e) How many students had $170 or more?  

3. Complete the relative frequency table for the hours of homework done each week by students.

<table>
<thead>
<tr>
<th>Hours of homework</th>
<th>0–2</th>
<th>2–4</th>
<th>4–6</th>
<th>6–8</th>
<th>8–10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of students</td>
<td>3</td>
<td>7</td>
<td>6</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Percentage of students</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Twenty students were surveyed about their favourite type of movie. The circle graph shows the results.

a) How many students chose each category?
   i) Comedy _____
   ii) Drama _____
   iii) Horror _____
   iv) Romantic comedy _____

b) We can also show the results using a different type of graph. Explain why it would be better to use a bar graph than a histogram to show the results.

c) Construct a bar graph to display the results of the survey.
1. Teacher to check.
2. 
   a) 13
   b) 3
   c) 4
   d) 1
   e) 10
3. 12%, 28%, 24%, 16%, 20%
4. 
   a) 
      i) 10
      ii) 5
      iii) 4
      iv) 1
   b) The categories are not continuous in a bar graph.
   c) Teacher to check.
1. The data gives the time in seconds for students competing in the 100 m backstroke.

<table>
<thead>
<tr>
<th>Time to complete 100 m backstroke</th>
<th>121</th>
<th>118</th>
<th>135</th>
<th>149</th>
<th>145</th>
<th>133</th>
<th>99</th>
<th>123</th>
<th>108</th>
<th>117</th>
<th>103</th>
<th>137</th>
<th>146</th>
<th>114</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of students</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fraction of students</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) Divide the data into intervals and make a relative frequency table. Put data values on the border of an interval into the higher interval.

b) Draw a histogram from the data. Don’t forget to show that the data does not start at 0.

c) Draw a circle graph from the data.
d) Write a question you would answer using the circle graph.

e) Write a question you would answer using the histogram.

2. Would you use a line graph or a scatter plot to display each data? Explain your choice.

a) 

<table>
<thead>
<tr>
<th></th>
<th>Tasfia</th>
<th>Mark</th>
<th>Nomi</th>
<th>Guled</th>
<th>Bilal</th>
<th>Jeff</th>
<th>Sara</th>
<th>Tom</th>
<th>Bob</th>
<th>Ron</th>
<th>Miki</th>
<th>Lina</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Height (cm)</td>
<td>93</td>
<td>99</td>
<td>108</td>
<td>100</td>
<td>105</td>
<td>115</td>
<td>109</td>
<td>112</td>
<td>119</td>
<td>115</td>
<td>122</td>
<td>128</td>
</tr>
</tbody>
</table>

b) 

<table>
<thead>
<tr>
<th></th>
<th>Tasfia’s Age</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (cm)</td>
<td>105</td>
<td>111</td>
<td>118</td>
<td>123</td>
<td></td>
</tr>
</tbody>
</table>
BONUS: Use the data that you chose “scatter plot” for to draw a line graph. Hint: Use averages.
1. Intervals selected may vary. Teacher to check.
   Sample answers:
   a)  
<table>
<thead>
<tr>
<th>Time</th>
<th>90–99</th>
<th>100–109</th>
<th>110–119</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>3</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Fraction</td>
<td>(\frac{1}{10})</td>
<td>(\frac{2}{15})</td>
<td>(\frac{1}{5})</td>
</tr>
</tbody>
</table>
   
<table>
<thead>
<tr>
<th>Time</th>
<th>120–129</th>
<th>130–139</th>
<th>140–149</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>6</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>Fraction</td>
<td>(\frac{1}{5})</td>
<td>(\frac{1}{5})</td>
<td>(\frac{1}{6})</td>
</tr>
</tbody>
</table>
   
   b)  
   ![Scatter plot]
   Sample reason:
   Because using a line graph would result in points on the same vertical line, but a scatter plot can be used to see a relationship between age and height.

   b)  
   ![Line graph]
   Sample answer:
   Because this is data for a single person, a line graph will show trends and rates of increase (e.g., when does she grow fastest?).

**BONUS**

Average height:
7-year-olds:
\[(93 + 99 + 108) ÷ 3 = 100\text{ cm}\]
8-year-olds:
\[(100 + 105 + 115) ÷ 3 ≈ 107\text{ cm}\]
9-year-olds:
\[(109 + 112 + 119) ÷ 3 ≈ 113\text{ cm}\]
10-year-olds:
\[(115 + 122 + 128) ÷ 3 ≈ 122\text{ cm}\]

2. a) Scatter plot
   Sample reason:
   Because using a line graph would result in points on the same vertical line, but a scatter plot can be used to see a relationship between age and height.

   b) Line graph
   Sample answer:
   Because this is data for a single person, a line graph will show trends and rates of increase (e.g., when does she grow fastest?).

   b)  
   ![Line graph]
   Sample answer:
   Because this is data for a single person, a line graph will show trends and rates of increase (e.g., when does she grow fastest?).
Unit 2: Probability and Data Management

Test (Lessons 6–9, 14) — WNCP

1. a) Using a protractor, determine what fraction of a company’s staff works part-time.

   b) If the company has 90 employees, how many work part-time?

2. Look at the circle graph below.

   Students in Miss B’s class who like skiing

   Girls
   40%

   Boys
   60%

   Tara concluded that 40% of the girls in Miss B’s class like skiing. Is she correct? Explain.

   Our Company’s Staff

   [Graph showing part-time and full-time employees]
3. The following data shows the causes of deaths due to recreational activities in the mountains.

<table>
<thead>
<tr>
<th>Cause of death</th>
<th>Collision</th>
<th>Fall</th>
<th>Avalanche</th>
<th>Hypothermia</th>
<th>Medical</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of deaths</td>
<td>150</td>
<td>40</td>
<td>100</td>
<td>20</td>
<td>41</td>
<td>49</td>
</tr>
</tbody>
</table>

a) How many deaths were recorded in total? _________

b) Tara says that medical causes caused more deaths than falling.
   Ron says that if any of the unknown deaths were due to falling, then more deaths were caused by falling than from medical causes.
   Tom says it is impossible to tell which caused more deaths, falling or medical causes.

Who is right? __________

c) Explain whether or not each statement follows from the data in the chart.
   i) More people died from an avalanche than from falling.

   ii) A skier who is in a collision is more likely to die than a skier who is in an avalanche.

   iii) About one eighth of the deaths are due to unknown causes.
4. Decide which type of graph you would use to display the data below — a line graph, a bar graph, or a circle graph. Write two questions you could answer using your choice of graph.

<table>
<thead>
<tr>
<th>Year</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Company Sales ($1000)</td>
<td>42</td>
<td>48</td>
<td>60</td>
<td>65</td>
<td>80</td>
<td>110</td>
</tr>
</tbody>
</table>

Type of graph: ____________________________

Question 1: ____________________________________________________________

____________________________________________________________

Question 2: ____________________________________________________________

____________________________________________________________

5. Decide which type of graph you would use to answer the question — a line graph, a bar graph, or a circle graph. Explain your choice.

a) What percentage of a charity’s money is spent on administration?

b) Is the percentage that a charity spends on administration increasing or decreasing over time?

c) How much money does a charity spend on administration?
1. a) \(80^\circ = \frac{80}{360} = \frac{2}{9}\)  
b) \(\frac{2}{9} = \frac{20}{90}\)  
   20 employees work part-time

2. No. The graph only shows the people who do like skiing, so it can’t say what percentage of people like or don’t like skiing.
   For example, the class could have 2 girls and 3 boys that like skiing, and 8 girls and 7 boys who don’t. In that case, only 20% of the girls would like skiing. What the graph says is that of the people who do like skiing (2 girls and 3 boys), 40% are girls.

3. a) 400  
b) Tom  
c) i) This is true.
   Even if all the unknown deaths are due to falling, 100 is still greater than 40 + 49 = 89.
   ii) This cannot be deduced.
   There might be many more collisions and that is why there’s a greater number of deaths from collisions than avalanches. For example, maybe there were only 30 skiers caught in avalanches but 1 000 skiers in collisions.
   iii) This is true.
   One eighth of 400 is 50, and there were 49 deaths due to unknown causes. Since 49 is close to 50, about one eighth of the deaths were due to unknown causes.

4. Line graph
   Questions will vary. Teacher to check.
   Sample Questions:
   What trend do the company’s sales show?
   What were the company’s sales in 2010?

5. a) Circle graph, because they display percentages.  
b) Line graph, because we’re looking for a trend.  
c) A bar graph, because it shows the amount of money.  
   A line graph wouldn’t make sense because the categories are discrete.
1. Draw two lines that intersect at 30°. Use a protractor and a ruler.

2. Draw a line perpendicular to $AB$ through point $P$.

3. Two sides of a right triangle have lengths 4 cm and 3 cm. What is the length of the third side?
   a) Make two sketches to show that there are two possible solutions. Label your triangles.
   b) Find a pair of complementary angles in each of the sketches:
      ______ and ______
      ______ and ______
   c) Find the exact length of the third side in one of the cases.

**BONUS:** Find the exact length of the third side in the second case.
Unit 3: Geometry

Quiz (Lessons 15–19) — ON

1. Teacher to check angle.
   Sample answer:

   \[ \angle \text{A} \]

2. Teacher to check angle.
   Sample answer:

   \[ \angle \text{A} \]

3. Answers may vary.
   Teacher to check.
   Sample answers:
   a) \[ \triangle \text{ABC} \]

   \[ \text{AC} = 3 \text{ cm}, \quad \text{BC} = 4 \text{ cm} \]

   b) \[ \angle \text{A} \text{ and } \angle \text{B} \]

   \[ \angle \text{D} \text{ and } \angle \text{F} \]

   c) \[ 5 \text{ cm or } \sqrt{7} \text{ cm} \]

   **BONUS**

   \[ \sqrt{7} \text{ cm or } 5 \text{ cm} \]
1. Complete the sentence for the diagram. Use the words “opposite,” “supplementary,” “corresponding,” or “alternate.”
   a) $a$ and $q$ are ____________________ angles.
   b) $c$ and $u$ are ____________________ angles.
   c) $d$ and $v$ are ____________________ angles.
   d) $q$ and $v$ are ____________________ angles.

2. Write an equation for the diagram below. Then solve the equation and find the angles of the triangle.

3. a) Find the missing angles for the diagram below.
   \[ \angle b = \underline{\quad} \]
   \[ \angle x = \underline{\quad} \]
   \[ \angle u = \underline{\quad} \]
   \[ \angle a = \underline{\quad} \]
   b) Explain how you found the measure of $\angle x$.

**BONUS:** Explain how you found the measure of $\angle a$. 
4.  a) The diagram below is a rhombus. Label the equal sides and the parallel sides.

   ![Rhombus Diagram]

   b) The diagram below is a trapezoid. Label the equal sides and/or the parallel sides.

   ![Trapezoid Diagram]

5.  Circle whether the statement is true or false.

   a) All squares are parallelograms.  
       T  F

   b) All parallelograms are rectangles.  
       T  F

   c) All trapezoids are parallelograms.  
       T  F

6.  Construct a perpendicular bisector of the line segment.
Unit 3: Geometry

Quiz (Lessons 20–29) — ON

1. a) opposite
   b) alternate
   c) supplementary
   d) corresponding

2. \[3x + 4x = 140\]
   \[7x = 140\]
   \[x = 20\]
   \[3x = 60^\circ\]
   \[4x = 80^\circ\]

3. a) \[b = 125^\circ\]
   \[x = 55^\circ\]
   \[u = 55^\circ\]
   \[a = 125^\circ\]
   b) \[x\] and \[c\] are alternate angles

BONUS
   Sample answer:
   \[a\] and \[x\] are supplementary angles

4. a) Teacher to check.
   Two pairs of opposite sides parallel, all sides equal.
   b) Teacher to check.
   One pair of opposite sides parallel.

5. a) T
   b) F
   c) F

6. Teacher to check.
1. Name the special quadrilaterals:
   A _______________________
   B _______________________
   C _______________________
   D _______________________
   E _______________________
   F _______________________

2. a) Use the Pythagorean Theorem to find the sides of \( \text{KLKN} \).
   
   \[ KL = \_ \_ \_ \_ \quad LM = \_ \_ \_ \_ \]
   
   \[ MN = \_ \_ \_ \_ \quad KN = \_ \_ \_ \_ \]

   b) Which sides are equal? ____________________________

   c) What special quadrilateral is \( \text{KLKN} \)? __________________

   d) Which of the following statements are true? _________________
   
   A: The diagonals of \( \text{KLKN} \) are equal.
   
   B: The diagonals of \( \text{KLKN} \) bisect each other.
   
   C: The diagonals of \( \text{KLKN} \) are perpendicular.
   
   D: One of the diagonals of \( \text{KLKN} \) bisects the other diagonal.

**BONUS:** Name two special quadrilaterals for which the statement is true.

   a) The diagonals are equal. _____________________ and _________________

   b) The diagonals are perpendicular. ________________ and ________________

   c) The diagonals bisect each other. ________________ and ________________
3. In this diagram, name two pairs of …
   a) opposite angles. _____ and _____, _____ and _____
   b) supplementary angles. _____ and _____, _____ and _____
   c) corresponding angles. _____ and _____, _____ and _____
   d) alternate angles. _____ and _____, _____ and _____

4. a) Measure the angles of \(ABCD\).
   \[ \angle A = \____\quad \angle B = \____\]
   \[ \angle C = \____\quad \angle D = \____\]
   b) What type of special quadrilateral is \(ABCD\)? Explain using angle measures.

5. a) Find the missing angles in the diagram on the right.
   \[ \angle a = \____\quad \angle b = \____\]
   \[ \angle c = \____\quad \angle d = \____\]
   \[ \angle e = \____\quad \angle f = \____\]
   b) Explain how you found the measure of \(\angle d\).

BONUS: Explain how you found the measure of \(\angle f\).
1. rectangle
   rhombus
   square
   parallelogram
   trapezoid
   kite

2. a) $KL = \sqrt{65}$
    $LM = \sqrt{90}$
    $MN = \sqrt{90}$
    $KN = \sqrt{65}$

   b) $KL = KN$, $LM = MN$

   c) kite

   d) C, D

BONUS

   Students should select any two of the following:
   a) rectangle, square,
      isosceles trapezoid
   b) kite, rhombus,
      square
   c) rectangle,
      parallelogram,
      rhombus, square

3. Students should select any two pairs of the following:
   a) $\angle a$ and $\angle w$,
      $\angle c$ and $\angle u$,
      $\angle b$ and $\angle y$,
      $\angle d$ and $\angle x$
   b) $\angle a$ and $\angle u$,
      $\angle a$ and $\angle c$,
      $\angle w$ and $\angle u$,
      $\angle w$ and $\angle c$,
      $\angle b$ and $\angle x$,
      $\angle b$ and $\angle d$,
      $\angle y$ and $\angle x$,
      $\angle y$ and $\angle d$
   c) $\angle a$ and $\angle b$,
      $\angle c$ and $\angle d$,
      $\angle u$ and $\angle x$,
      $\angle w$ and $\angle y$
   d) $\angle b$ and $\angle w$,
      $\angle c$ and $\angle x$

4. a) $\angle A = 111^\circ$
    $\angle B = 69^\circ$
    $\angle C = 107^\circ$
    $\angle D = 73^\circ$

   b) Since $\angle A + \angle B$
      $= \angle C + \angle D = 180^\circ$,
      $AD \parallel BC$. However,
      $\angle A + \angle D = 184^\circ$,
      so $AB$ and $CD$ are
      not parallel. This
      means $ABCD$ has
      only one pair of
      parallel sides, and
      it is a trapezoid.

5. a) $\angle a = 117^\circ$
    $\angle b = 117^\circ$
    $\angle c = 63^\circ$
    $\angle d = 63^\circ$
    $\angle e = 90^\circ$
    $\angle f = 27^\circ$

   b) Answers may vary.
      Teacher to check.
      Sample answer:
      $\angle x$ and $\angle c$ are
      alternate angles.
      Since lines $m$
      and $n$ are parallel,
      their alternate
      angles are equal,
      so $\angle x = \angle c = 63^\circ$.

BONUS

   Answers may vary.
   Teacher to check.
   Sample answer:
   $\angle x$ and $\angle f$ are
   complementary
   angles (acute
   angles in a right
   triangle), so
   $\angle x + \angle f = 90^\circ$ and
   $\angle f = 90^\circ - 63^\circ$
   $= 27^\circ$. 
Unit 3: Geometry

Quiz (Lessons 30–33) — ON

1. a) Which triangle in the diagram below is congruent to $\triangle KOQ$? $\triangle ________$

   List the sides and/or angles needed to prove that these triangles are congruent.
   
   $\underline{_________} = \underline{_________}$
   $\underline{_________} = \underline{_________}$
   $\underline{_________} = \underline{_________}$

b) Which congruence rule tells you that these triangles are congruent? _________

c) What type of triangle is $\triangle KOM$? Explain.


d) Identify two more pairs of congruent triangles in the picture.

   $\triangle \underline{_________} \cong \triangle \underline{_________}$
   $\triangle \underline{_________} \cong \triangle \underline{_________}$

e) Construct a circle through points $K$, $L$ and $M$ in the diagram above.

   BONUS: Using a compass and a straightedge, construct a circle through points $A$, $B$ and $C$. 

   $\bullet A$
   $\bullet B$
   $\bullet C$
Unit 3: Geometry

Quiz (Lessons 30–33) — ON

1. a) \( \triangle KOQ \cong \triangle MOQ \)
   \( \angle KQO = \angle MQO \)
   \( KQ = MQ \)
   \( QO = QO \text{ (shared)} \)

b) SAS

c) Isosceles.
   Sample explanation:
   From a) and b),
   we know that \( \triangle MOQ \cong \triangle KOQ \),
   which means that \( KO = MO \).

d) \( \triangle LOP \cong \triangle KOP \)
   \( \triangle MOR \cong \triangle LOR \)

e) Teacher to check.
   The circle should have centre \( O \) and pass through the points \( K, L \) and \( M \).

BONUS

Teacher to check.
The centre of the circle should be at the intersection of the perpendicular bisectors to two of the sides of \( \triangle ABC \).
1. Find the pairs of triangles that are similar and the pairs of triangles that are congruent.

   a) \( \triangle \) _____ and \( \triangle \) _____ are _______________________________.

   b) \( \triangle \) _____ and \( \triangle \) _____ are _______________________________.

   c) \( \triangle \) _____ and \( \triangle \) _____ are _______________________________.

2. \( \triangle \text{ABC} \) is similar to \( \triangle \text{DEF} \).

   a) Fill in the missing information in the proportion. \[ \frac{AB}{DE} = \frac{AC}{EF} \]

   b) Fill in the missing information in the equation.

      i) \( \angle \text{ABC} = \) _____  

      ii) _____ = \( \angle \text{EFD} \)  

      iii) \( \angle \text{ACB} = \) _____

3. \( \triangle \text{DEF} \) is similar to \( \triangle \text{GHI} \).

   a) Find the length of \( GH \). _____

   b) Find the measure of \( \angle \text{EDF} \). _____

   c) Find the measure of \( \angle \text{DEF} \). _____

   d) Find the measure of \( \angle \text{GHI} \). _____
4. A street lamp shines on a man to create a shadow. In the diagram, \( \triangle ABC \) is similar to \( \triangle DEC \).

   a) Fill in the missing information in the proportion.

   \[
   \frac{DE}{AB} = \frac{?}{BC}
   \]

   b) Solve the proportion to find the height of the man.

5. Each side in \( \triangle DEF \) is 3 times longer than each side in \( \triangle ABC \).

   The area of \( \triangle ABC \) is 10 cm\(^2\). Find the area of \( \triangle DEF \).

**BONUS:** \( \triangle ABC \) is similar to \( \triangle DEF \). The area of \( \triangle DEF \) is 1,000,000 times larger than the area of \( \triangle ABC \). How many times longer is each side in \( \triangle DEF \) than in \( \triangle ABC \)?
1. a) $\triangle 1$ and $\triangle 6$ are congruent  
b) $\triangle 3$ and $\triangle 5$ are similar  
c) $\triangle 2$ and $\triangle 4$ are similar  
2. a) $BC$, $DF$  
b) i) $\angle DEF$  
ii) $\angle BCA$  
iii) $\angle DFE$  
3. a) 4.2 cm  
b) 45°  
c) 75°  
d) 75°  
4. a) $EC$  
b) \[ \frac{DE}{6} = \frac{3}{9} \]  
\[ DE = \frac{3}{9} \times 6 \]  
\[ = 2 \text{ m} \]  
5. $10 \times 3^2 = 90 \text{ cm}^2$  
BONUS  
1000
Unit 3: Geometry
Test (Lessons 30–36) — ON

1. a) Which rule tells you that these triangles are similar?

b) Find x and y.

c) The area of the smaller triangle is about 8.9 cm². What is the area of the larger triangle? Show your work.

2. a) Which congruence rule tells you that \( \triangle ABD \) and \( \triangle BCD \) are congruent?

   Congruence rule: __________

   Write the congruence statement for the triangles, keeping the order of equal sides and angles.

   \( \triangle ______ \equiv \triangle ______ \)

b) Mark the equal angles in the picture. Which angles are equal in the triangles?

   \( \angle 1 = _____ \quad \angle 2 = _____ \quad \angle 3 = _____ \)

c) What is the sum of the angles \( DAB \) and \( ABC \)? ______

   How do you know?
3. Circle whether the statement is true or false.
   a) In a rhombus, both diagonals bisect each other. T F
   b) In a rhombus, the diagonals are equal. T F
   c) In a rhombus, the diagonals are perpendicular. T F
   d) In a rhombus, both diagonals are angle bisectors. T F

4. The lines $AB$ and $CD$ are parallel.
   a) Mark the angles you know are equal in $\triangle ABE$ and $\triangle CDE$.
   b) Which angles are equal in $\triangle ABE$ and $\triangle CDE$?
      How do you know?
   
   c) Are $\triangle ABE$ and $\triangle CDE$ congruent? _____ similar? _____
      Explain.

BONUS: Use the Pythagorean Theorem to tell which of the triangles below are similar.
   _______ and _______.
   How do you know?
**Unit 3: Geometry**

**Test (Lessons 30–36) — ON**

1. a) AA
   
   b) \( x = 30 \text{ cm} \)
     
   c) \( y = 35 \text{ cm} \)
     
   c) \( \approx 222.5 \text{ cm}^2 \)

2. a) SSS;
     
     \( \triangle ABD \cong \triangle CDB \)

   b) \( \angle 1 = \angle 6 \)
     
     \( \angle 2 = \angle 5 \)
     
     \( \angle 3 = \angle 4 \)

   c) \( 180^\circ \)
     
     \( \angle DAB + \angle ABC = \angle 1 + \angle 2 + \angle 3 \)
     
     \( = \angle 1 + \angle 2 + \angle 4 \)
     
     \( = 180^\circ \), the sum of the angles in \( \triangle ABD \).

3. a) T
   
   b) F
   
   c) T
   
   d) T

4. a) 

   b) \( \angle EAB = \angle ECD \)
     
     \( \angle EBA = \angle EDC \)
     
     because they are corresponding angles at parallel lines \( AB, CD \).
     
     \( \angle AEB = \angle CED \)
     
     (common angle)

   c) The triangles are not congruent, but they are similar by AAA (or AA) similarity rule.

   They are not congruent, because they are of different size (\( AE > CE \)).

**BONUS**

\( \triangle ABC \) and \( \triangle EFG \) are similar.

By the Pythagorean Theorem, \( AB = 8 \text{ cm} \), so by the SAS similarity rule, the triangles are similar with scale factor 2.

By the Pythagorean Theorem, \( KM = \sqrt{5456} = 74 \text{ cm} \), so \( KM \) is the medium side. The ratios between the smallest sides and the largest sides in \( \triangle KLM \) and \( \triangle ABC \) are \( \frac{84}{17} \) and \( \frac{40}{8} \).

Cross-multiplying, \( 84 \times 8 = 672 \neq 40 \times 17 = 680 \), so the ratios are not the same, and \( \triangle KLM \) is not similar to \( \triangle ABC \), and so not similar to \( \triangle EGF \) as well.
Unit 4: Patterns and Algebra

Quiz (Lessons 16–24) — ON

1. Solve for the variable. Check your answer to part a) by substitution.
   a) \(-5y - 12 = 18\)
   b) \(23 + (-3)x = -16\)

2. a) Find a stepwise rule for the sequence 9, 13, 17, 21, 25.

   b) Using the table at right, find a formula for the same sequence.

   c) Would you use a formula or a stepwise rule to find the 20th term of the sequence?

   d) What is the 20th term of the sequence?

   e) What is the term number of 1001?

   BONUS: Is there a term that equals 555? If yes, which one? If no, explain why not.
Unit 4: Patterns and Algebra

Quiz (Lessons 16–24) — ON

1. a) \( y = -6 \)
   Teacher to check solution and substitution.

   b) \( x = 13 \)
   Teacher to check solution.

2. a) Start at 9, add 4 each time.

   b) \[\begin{array}{|c|c|c|}
   \hline
   n & n \times \text{Gap} & \text{Term} \\
   \hline
   1 & 4 & 9 \\
   2 & 8 & 13 \\
   3 & 12 & 17 \\
   4 & 16 & 21 \\
   5 & 20 & 25 \\
   \hline
   \end{array}\]
   Formula: \(4n + 5\)

   c) Formula

   d) \(4(20) + 5 = 85\)
   The 20th term is 85.

   e) \(4n + 5 = 1\,001\)
   \(4n = 996\)
   \(n = 249\)
   1,001 is the 249th term.

BONUS

Answers may vary.
Teacher to check.

Sample answer:
If there is a term equal to 555, there would be an \(n\) such that:
\(4n + 5 = 555\)
\(4n = 550\)
So \(n = 550 \div 4 = 137.5\)
But this means that 555 is between the 137th term (553) and the 138th term (557).
Therefore, there is no term equal to 555.
1. Solve for the variable. Check your answer to part b) by substitution.
   a) \( \frac{m}{2} + 4 = -5 \)
   b) \(-2(3 + x) = 26\)

2. Describe the mistake in the following “solution”.
   \[-2x - 15 = 27\]
   \[-2x - 15 + 15 = 27 + 15\]
   \[-2x = 42\]
   \[-2x + 2 = 42 + 2\]
   \[x = 21\]

3. a) Using the table at right, find a formula for the sequence 9, 13, 17, 21, 25…
   b) Use your formula to find the 10th term.
   c) What is the term number of 1 001?

   **BONUS:** Is there a term that equals 555? If yes, which one? If no, explain why not.

<table>
<thead>
<tr>
<th>Term Number (n)</th>
<th>n \times \text{Gap}</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1. a) \( m = 18 \)
   Teacher to check solution.

   b) \( x = -16 \)
   Teacher to check solution and substitution.

2. Answers may vary.
   Teacher to check.

   Sample answer:
   In the second line from the bottom, the student should have divided by \(-2\) instead of \(+2\), so the answer is actually
   \[ x = -21. \]

3. | \( n \) | \( n \times \text{Gap} \) | Term |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>21</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>25</td>
</tr>
</tbody>
</table>

   a) \( 4n + 5 \)
   b) \( 4(10) + 5 = 45 \)
   The 10th term is 45.

   c) \( 4n + 5 = 1001 \)
   \( 4n = 996 \)
   \( n = 249 \)
   1001 is the 249th term.

   **BONUS**
   Answers may vary.
   Teacher to check.

   Sample answer:
   If there is a term equal to 555, there would be an \( n \) such that:
   \( 4n + 5 = 555 \)
   \( 4n = 550 \)
   \( n = 550 \div 4 = 137.5 \)
   But this means that 555 is between the 137th term (553) and the 138th term (557).
   Therefore, there is no term equal to 555.
1. Find the gap and complete the table. Then write the formula.

   a)
   \[
   \begin{array}{|c|c|c|}
   \hline
   \text{Term (n)} & n \times \text{Gap} & \text{Term} \\
   \hline
   1 & 2 & \\
   2 & 7 & \\
   3 & 12 & \\
   4 & 17 & \\
   \hline
   \end{array}
   \]

   Formula: ______________________

   b)
   \[
   \begin{array}{|c|c|c|}
   \hline
   \text{Term (n)} & n \times \text{Gap} & \text{Term} \\
   \hline
   1 & 5 & \\
   2 & 7 & \\
   3 & 9 & \\
   4 & 11 & \\
   \hline
   \end{array}
   \]

   Formula: ______________________

2. a) Complete a T-table for the set of points on the grid below.

   Input (\(n\)) | Output
   --- | ---
   
   b) Write a formula for the T-table.

3. Use the formula to fill in the table below. Then make a list of ordered pairs.

   Formula: Term Number \( \times 4 - 3 \)

<table>
<thead>
<tr>
<th>Term Number ((n))</th>
<th>Term</th>
<th>Ordered Pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4. a) Graph the sequence of numbers by first making a list of ordered pairs.

   Sequence: \(-6, -1, 2, 3, 2, -1, -6\)

   \( (\ , \ ) \ (\ , \ ) \ (\ , \ ) \ (\ , \ ) \ (\ , \ ) \)

   \( (\ , \ ) \ (\ , \ ) \ (\ , \ ) \)

   b) Is the sequence linear? Explain.

5. Figure 1    Figure 2         Figure 3            Figure 4

   a) Write a sequence for the number of sticks in each figure.

   b) Find the gap in the sequence.

   c) Write a formula for the number of sticks.

6. Solve the equation by first moving all variable terms to the same side.

   \( 3x = -2x + 15 \)

**BONUS:** Solve for \( x \). \( 100x + 3 = 99x + 4 \)
Unit 4: Patterns and Algebra

Quiz (Lessons 24–32) — ON

1. a) Gap = 5
   5, 10, 15, 20
   Formula: 5n - 3

   b) Gap = 2
   2, 4, 6, 8
   Formula: 2n + 3

2. a) 1, 4
   2, 6
   3, 8
   4, 10

   b) 2n + 2

3. 1. (1, 1)
   5. (2, 5)
   9. (3, 9)
   13. (4, 13)

4. a) (1, -6), (2, -1),
   (3, 2), (4, 3), (5, 2),
   (6, -1), (7, -6)
   Teacher to check graph.

   b) no
   If you join the dots,
you don’t get a straight line.

5. a) 6, 11, 16, 21

   b) 5

   c) 5n + 1

6. 3x = -2x + 15
   3x + 2x = 15
   5x = 15
   x = 15 + 5
   x = 3

BONUS

100x + 3 = 99x + 4
100x - 99x = 4 - 3
x = 1
1. Find the gap and complete the table. Then write the formula.

   a)  
   \[ \begin{array}{ccc} 
   \text{Term (n)} & n \times \text{Gap} & \text{Term} \\
   1 & 2 & \ \\
   2 & 7 & \ \\
   3 & 12 & \ \\
   4 & 17 & \ \\
   \end{array} \]

   Formula: _________________________

   b)  
   \[ \begin{array}{ccc} 
   \text{Term (n)} & n \times \text{Gap} & \text{Term} \\
   1 & 5 & \ \\
   2 & 7 & \ \\
   3 & 9 & \ \\
   4 & 11 & \ \\
   \end{array} \]

   Formula: _________________________

2. a) Complete a T-table for the set of points on the grid below.

<table>
<thead>
<tr>
<th>Input (n)</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   b) Write a formula for the T-table.

3. Use the formula to fill in the table below. Then make a list of ordered pairs.

   Formula: Term Number \( \times 4 - 3 \)

   \[ \begin{array}{ccc} 
   \text{Term Number (n)} & \text{Term} & \text{Ordered Pair} \\
   1 & (\ ,\ ) & \\
   2 & (\ ,\ ) & \\
   3 & (\ ,\ ) & \\
   4 & (\ ,\ ) & \\
   \end{array} \]
4. a) Graph the sequence of numbers by first making a list of ordered pairs.

Sequence: –6, –1, 2, 3, 2, –1, –6

( , ) ( , ) ( , ) ( , )
( , ) ( , ) ( , )

b) Is the sequence linear? Explain.

5. Figure 1 Figure 2 Figure 3 Figure 4

a) Write a sequence for the number of sticks in each figure above.

b) Find the gap in the sequence.

c) Write a formula for the number of sticks.

**BONUS:** How many sticks will be in the 1 000th figure?
1. a) Gap = 5
   5, 10, 15, 20
   Formula: $5n - 3$

   b) Gap = 2
   2, 4, 6, 8
   Formula: $2n + 3$

2. a) $1, 4$
   $2, 6$
   $3, 8$
   $4, 10$

   b) $2n + 2$

3. 1. $(1, 1)$
   5. $(2, 5)$
   9. $(3, 9)$
   13. $(4, 13)$

4. a) $(1, -6), (2, -1),$  
    $(3, 2), (4, 3), (5, 2),$
    $(6, -1), (7, -6)$
    Teacher to check graph.

   b) no
   If you join the dots, you don’t get a straight line.

5. a) 6, 11, 16, 21

   b) 5

   c) $5n + 1$

BONUS

5(1 000) + 1

= 5 001
1. a) Mark 3 grid points on the line segment. Then write a list of ordered pairs and complete the T-table.

<table>
<thead>
<tr>
<th>Ordered pairs</th>
<th>First number</th>
<th>Second number</th>
</tr>
</thead>
<tbody>
<tr>
<td>( , )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( , )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( , )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Write a formula showing how to get the second number from the first number.

c) What will the second number be if the first number is 50?

d) What will the first number be if the second number is 102?

2. Solve for the variable. Write your answer as a decimal and as a mixed number.

a) \(4s = 29\)  
b) \(2t + 11 = 34\)  
c) \(15 - 6n = 6\)

3. Evaluate the expression \(6b - 5a + 4\) for \(a = 0.3\) and \(b = \frac{2}{3}\).

4. Decide which of these sequences is linear by finding the gaps between the terms.

A. 17 15 12 8 3  
B. 7 9 11 13 15

___ is linear because ________________________________.
5. Solve for the variable. Check your answer by substitution.
   a) \(4s = -28\)  
   b) \(-2u + 14 = 4\)

6. Fill in the T-table for the following formula. Then make a list of ordered pairs and plot the points on the graph.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(3n - 2)</th>
<th>Ordered pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>( , )</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>( , )</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>( , )</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>( , )</td>
</tr>
</tbody>
</table>

7. These two rectangles have the same area. What are the sides of the rectangles?

   A. \(x\)  
   B. \(x - 4\)  

   BONUS: a) Find the perimeter of each rectangle in Question 7.

   A: ____________________  
   B: ____________________

   b) Write an expression for the perimeter of each rectangle in terms of \(x\).

   A: ____________________  
   B: ____________________

   c) Check your answer by substituting the value for \(x\) you found above.

   A: ____________________  
   B: ____________________
Unit 4: Patterns and Algebra

Test (Lessons 16–32) — ON

1. a) Students to select any 3 points from:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 2)</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(1, 6)</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>(2, 10)</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>(3, 14)</td>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>(4, 18)</td>
<td>4</td>
<td>18</td>
</tr>
</tbody>
</table>

b) $2^{nd}$ number = $4 \times 1^{st}$ number + 2
c) 202
d) 25

2. a) $s = \frac{7 + \frac{1}{4}}{4} = 7.25$
Teacher to check solution.
b) $t = \frac{11 + \frac{1}{2}}{2} = 11.5$
Teacher to check solution.
c) $n = \frac{1 + \frac{1}{2}}{2} = 1.5$
Teacher to check solution.

3. 6.5

4. B is linear because the gaps are the same size.

5. Teacher to check both solutions and substitutions.
   a) $s = -7$
   b) $u = 5$

6. $n$ | $3n - 2$ | O.P.
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>(3, 7)</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>(4, 10)</td>
</tr>
</tbody>
</table>

7. A: $3 \times 10$
   B: $6 \times 5$

BONUS
   a) A: 26 units
      B: 22 units
   b) A: $2(3 + x)$
      $= 6 + 2x$
      B: $2(5 + x - 4)$
      $= 10 + 2(x - 4)$
      $= 2x + 2$
   c) Teacher to check substitution.
1. a) Mark 3 grid points on the line segment. Then write a list of ordered pairs and complete the T-table.

<table>
<thead>
<tr>
<th>Ordered pair</th>
<th>First number</th>
<th>Second number</th>
</tr>
</thead>
<tbody>
<tr>
<td>( , )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( , )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( , )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Write a formula showing how to get the second number from the first number.

2. Solve for the variable. Write your answer as a decimal and as a mixed number.
   a) \(4s = 29\)  
   b) \(2t + 11 = 34\)  
   c) \(15 - 6n = 6\)

3. Solve for the variable. Check your answer by substitution.
   a) \(4s = -28\)  
   b) \(-2u + 14 = 4\)  
   c) \(-2(5 - x) = 6\)

4. Decide which of these sequences is linear by finding the gaps between the terms.
   
   A. 17 15 12 8 3  
   B. 7 9 11 13 15  

   _____ is linear because ______________________________________________________.
5. Fill in the T-table for the following formula. Then make a list of ordered pairs and plot the points on the graph.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$3n - 2$</th>
<th>Ordered pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>( , )</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>( , )</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>( , )</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>( , )</td>
</tr>
</tbody>
</table>

6. Evaluate the expression $6b - 5a + 4$ for $a = 0.3$ and $b = \frac{2}{3}$.

7. Look at the pattern at right.

a) Write a formula for the number of shaded blocks.

b) Write a formula for the total number of blocks.

c) In which formula does the number of blocks vary directly with the figure number?

d) Use your formula to find how many blocks are in Figure 10.

e) Which figure in the pattern has 255 blocks?

BONUS: Explain why there is no figure that has exactly 1 000 blocks.
Unit 4: Patterns and Algebra
Test (Lessons 16–30) — WNCP

1. a) Students to select any 3 points from:
   - (0, 0) 0 0
   - (1,3)  1 3
   - (2,6)  2 6
   - (3,9)  3 9

   b) \(2^{\text{nd}}\) number = \(3 \times 1^{\text{st}}\) number

2. a) \(s = \frac{7 \frac{1}{4}}{4} = 7.25\)
   Teacher to check solution.

   b) \(t = 11 \frac{1}{2} = 11.5\)
   Teacher to check solution.

   c) \(n = \frac{1}{2} = 1.5\)
   Teacher to check solution.

3. Teacher to check both solutions and substitutions.
   a) \(s = -7\)
   b) \(u = 5\)
   c) \(x = 8\)

4. B is linear because the gaps are the same size.

5. | \(n\) | \(3n - 2\) | O.P. |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>(2, 4)</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>(3, 7)</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>(4, 10)</td>
</tr>
</tbody>
</table>

6. 6.5

7. a) \(3n\)
   b) \(3n + 3\)
   c) part a)
   d) \(3(10) + 3 = 33\)
      Figure 10 will have 33 blocks.
   e) \(3n + 3 = 255\)
      \(3n = 252\)
      \(n = 84\)
      Figure 84 will have 255 blocks.

BONUS
Answers may vary.
Teacher to check.

Sample answer:
If there was a figure with 1 000 blocks, there would be an \(n\) such that:
   \(3n + 3 = 1 000\)
   \(3n = 997\)
But \(9 + 9 + 7 = 25\) is not a multiple of 3, so 997 does not divide evenly by 3, which means there is no whole \(n\) that will solve this equation. Therefore, there is no term number for 1 000 and no figure can have 1 000 blocks.
1. Write a power with exponent 4 and base 3.

2. Write the power as a product.
   a) \(3^4 = \) ______________________
   b) \(5^2 = \) ______________________

3. Write the product as a power.
   a) \(7 \times 7 \times 7 \times 7 \times 7 = \) _____
   b) \(4 = \) _____

4. Evaluate the power.
   a) \(2^3 = \) ______________________
   b) \(3^4 = \) ______________________

5. Write the product as a power.
   a) \(3^2 \times 3^4 = \) _____
   b) \(9^3 \times 9^5 = \) _____

6. Write the number as a power of 10.
   a) \(1 000 = \) _____
   b) \(1 000 000 000 = \) _____

7. Write the number in standard form.
   \(5 \times 10^4 + 3 \times 10^2 + 7 \times 10^1 + 6 = \) ______________________

8. Evaluate.
   \((-2)^5 = \) ______________________
   \(= \) _____
9. Calculate using the correct order of operations.

   a) \(3 + 4^2\)  
   \[= \quad \]
   \[= \quad \]
   \[= \quad \]

   b) \((3 + 4)^2\)  
   \[= \quad \]
   \[= \quad \]
   \[= \quad \]

   c) \((-2)^3 + 5^2\)  
   \[= \quad \]
   \[= \quad \]
   \[= \quad \]

10. Calculate \(2^5 \times 5^6\) by regrouping.

11. Write the numbers in order from smallest to largest.

   \[2.1, -1 \frac{1}{2}, -3.2, 5 \frac{1}{4}\]

BONUS: Calculate \((-1)^{105}\).
1. $3^4$
2. a) $3 \times 3 \times 3 \times 3$
   b) $5 \times 5$
3. a) $7^5$
   b) $4^1$
4. a) $2 \times 2 \times 2 = 8$
   b) $3 \times 3 \times 3 \times 3 = 81$
5. a) $3^3$
   b) $9^3$
6. a) $10^3$
   b) $10^9$
7. 50,376
8. $(-2) \times (-2) \times (-2) \times (-2) \times (-2) = -32$
9. a) $3 + 16 = 19$
   b) $7^2 = 49$
   c) $(-8) + 25 = 17$
10. $(2 \times 5) \times (2 \times 5) \times (2 \times 5) \times (2 \times 5) \times 5 = 500,000$
11. $-3.2, -1 \frac{1}{2}, 2.1, \frac{5}{4}$

**BONUS**

$-1$
1. Write the base and the exponent for the power $3^4$.
   base = ______ exponent = ______

2. Write the product as a power: $2 \times 2 \times 2 \times 2 = ______$

3. Write the power as a product: $5^3 = ____________$

4. Evaluate the powers.
   a) $2^3 = ______________$
   b) $3^2 = ______________$
   c) $(-2)^3 = ______________$
   d) $(-3)^2 = ______________$
   e) $10^4 = ______________$
   f) $(-10)^4 = ______________$
   g) $10^7 = ______________$
   h) $(-10)^7 = ______________$

5. Evaluate the powers, then add, subtract, multiply, or divide.
   a) $2^2 + 3^3$
   b) $3^2 - 2^3$
   c) $3 \times 5^2$
   d) $10^3 + 4$

6. a) Evaluate the powers, then divide, and then write your answer as a power of 2.
   
   $2^3 \div 2 = _______ \div 2 = _______ = 2$
   $2^4 \div 2 = _______ \div 2 = _______ = 2$
   $2^5 \div 2 = _______ \div 2 = _______ = 2$
   $2^6 \div 2 = _______ \div 2 = _______ = 2$

   b) Use the pattern in part a) to write $2^{100} \div 2$ as a power of 2.

   BONUS: Explain why your answer in part b) makes sense.
7. Write the numbers in expanded form using powers of ten.
   a) 85 402 = ________________________________________________
   b) 3 970 061 = ______________________________________________

8. Write the number in standard form.
   a) $9 \times 10^5 + 4 \times 10^4 + 6 \times 10^3 + 5 \times 10^2 + 8 \times 10 + 2 =$ _________________
   b) $8 \times 10^7 + 5 \times 10^6 + 9 \times 10^3 + 4 =$ _________________

9. Write > (greater than) or < (less than) in the square to make the statement true.
   a) $5 \times 10^3 + 8 \times 10 + 2$ $\square$ $5 \times 10^3 + 8 \times 10^2 + 2$
   b) $4 \times 10^4 + 9 \times 10^3$ $\square$ $4 \times 10^4 + 9 \times 10^2 + 7 \times 10 + 6$
   c) $7 \times 10^5 + 3 \times 10^4 + 5$ $\square$ $7 \times 10^5 + 8 \times 10^3 + 5$

10. Calculate using the correct order of operations.
    a) $3 \times (-2) + 7$
    b) $3 \times ((-2) + 7)$
    c) $8 \div (-2) \times 3$
    d) $(2 - 3)^2$
    e) $2^2 - 3^2$
    f) $2 - 3^2$

11. Order the numbers from smallest to largest.
    a) $\frac{2}{9}$ $0.27$ $\frac{1}{4}$
    b) $-\frac{2}{9}$ $-0.27$ $-\frac{1}{4}$
        _______ < _______ < _______
        _______ < _______ < _______
    c) $4.2$ $-5$ $\frac{17}{4}$ $\frac{-17}{4}$ $\frac{-17}{3}$ $4\frac{1}{3}$
        _______ < _______ < _______ < _______ < _______ < _______

BONUS: What is the ones digit of $2^{100}$?
Unit 5: Number Sense

Test (Lessons 104–110) — ON

1. base = 3
   exponent = 4
2. \( 2^5 \)
3. \( 5 \times 5 \times 5 \)
4. a) 8
   b) 9
   c) –8
   d) 9
   e) 10 000
   f) 10 000
   g) 10 000 000
   h) –10 000 000
5. a) \( 4 + 27 = 31 \)
   b) \( 9 – 8 = 1 \)
   c) \( 3 \times 25 = 75 \)
   d) \( 1 000 + 4 = 250 \)
6. a) \( 8 \div 2 = 4 = 2^2, \)
    \( 16 \div 2 = 8 = 2^3, \)
    \( 32 \div 2 = 16 = 2^4, \)
    \( 64 \div 2 = 32 = 2^5 \)
   b) \( 2^{99} \)
   BONUS
   \( 2^{100} \) is the product of one hundred 2s.
   Dividing by 2 cancels one of them out, leaving ninety-nine 2s in the product.
7. a) \( 8 \times 10^4 + 5 \times 10^3 + 4 \times 10^2 + 2 \)
   b) \( 3 \times 10^6 + 9 \times 10^5 + 7 \times 10^4 + 6 \times 10 + 1 \)
8. a) 946 582
   b) 85 009 004
9. a) <
   b) >
   c) >
10. a) 1
   b) 15
   c) –12
   d) 1
   e) –5
   f) –7
11. a) \( \frac{2}{9} < \frac{1}{4} < 0.27 \)
   b) \( -0.27 < -\frac{1}{4} < -\frac{2}{9} \)
   c) \( -\frac{17}{3} < -5 < -\frac{17}{4} \)
   \( < 4.2 < \frac{17}{4} < 4 \frac{1}{3} \)

BONUS
The first few powers of 2 are: 2, 4, 8, 16, 32, 64, 128, and 256. The pattern in their ones digits is 2, 4, 8, 6, repeat.
This means that every 4 terms has the same ones digit.
This tells us that \( 2^{100} \) has the same ones digit as \( 2^4 = 16 \), since 100 is divisible by 4.
\( \therefore \) its ones digit is 6.
1. A prism with base shown on the right has height 80 cm.
   a) Calculate the volume of the prism (express your answer in cm³ and in m³).
   b) Find the capacity of the prism (express your answer in L).

2. A cylinder with base of radius 60 cm has height 1.2 m. Calculate the volume of the cylinder.

3. A square has perimeter 3.6 m. What is its area in cm²?

BONUS: A circle has circumference 3.14 m. What is its area in cm²?
Unit 6: Measurement

Quiz (Lessons 9–15) — ON

1. a) $168,000 \text{ cm}^3$
   $= 0.168 \text{ m}^3$

   b) 168 L

2. $1,356,480 \text{ cm}^3$
   $= 1.35648 \text{ m}^3$

3. $8100 \text{ cm}^2$

BONUS

   $7850 \text{ cm}^2$
1. A prism with base shown on the right has height 80 cm. Calculate the volume of the prism (express your answer in cm$^3$ and in m$^3$).

2. A cylinder with base of radius 60 cm has height 1.2 m. Estimate the volume of the cylinder, then find the actual volume using a calculator.

3. Finish the net for the prism on the right.

BONUS: Match the 3-D shapes to the sketches of the nets.

A. B. a) _____ b) _____
Unit 6: Measurement

Quiz (Lessons 9–15) — WNCP

1. \(168 000 \text{ cm}^3 = 0.168 \text{ m}^3\)
2. Estimates will vary.
   Actual:
   \(1356480 \text{ cm}^3 = 1.35648 \text{ m}^3\)
3. BONUS
   a) A
   b) B
1. Calculate the surface area of the prism.

   a) ![Prism](image)

   b) ![Cylinder](image)

2. Calculate the surface area of the cylinder. Use 3.14 for \( \pi \).

   ![Cylinder](image)
3. Find the volume of the cylinder. Use 3.14 for \( \pi \).

**BONUS:** A cylinder with volume 100 cm\(^3\) has its radius doubled. What is its new volume?
1. a) \[ 2 \times 6 \times 3 + 2 \times 6 \times 6 + 2 \times 3 \times 7 \]
\[ = 36 + 84 + 42 \]
\[ = 162 \text{ cm}^2 \]

b) \[ 2 \times 8 \times 6 + 2 \times 6 \times 12 + 8 \times 12 + 10 \times 12 \]
\[ = 48 + 72 + 96 + 120 \]
\[ = 336 \text{ m}^2 \]

2. \[ 2 \times 3.14 \times 5 \times 6 + 2 \times 3.14 \times 5^{2} \]
\[ = 188.4 + 157 \]
\[ = 345.4 \text{ cm}^2 \]

3. \[ 3.14 \times 3^2 \times 10 \]
\[ = 282.6 \text{ cm}^3 \]

**BONUS**

New Volume
\[ = 2^2 \times \text{Old Volume} \]
\[ = 4 \times 100 \]
\[ = 400 \text{ cm}^3 \]
1. The base of a cylinder has an area of 2.8 m\(^2\).
   The cylinder has a volume of 5.6 m\(^3\). What is its height?

2. A cup with capacity 314 mL has radius 4 cm. What is its height in cm?

3. A cylindrical tin can has radius 5 cm and height 10 cm. It has no lid.
   a) What is the capacity of the can? Round your answer to the nearest tenth of a litre.

   b) Tin costs 2 cents per cm\(^2\) of foil. How much does the tin for the can without the lid cost?
      Round your answer to the nearest cent.
4. a) Calculate the volume and surface area of each can. Leave your answer in terms of $\pi$.

A. 

Volume = _________________   Volume = _________________

Surface Area = _________________  Surface Area = _________________

b) Which can has the largest volume? ______

c) Which can has the largest surface area? ______

d) Sketch a net for can B. Mark the dimensions on the net.

BONUS:

e) Find the volume (to the nearest cm$^3$) and the surface area (to the nearest cm$^2$) of can A.

f) Sketch a rectangular prism with larger volume and smaller surface area than can A. Label the dimensions of your prism.
1. 2 m
2. 6.25 cm
3. a) \( \approx 0.8 \text{ L} \)
   b) $7.85
4. a) A:
   \[ V = 300\pi \text{ cm}^3 \]
   \[ SA = 260\pi \text{ cm}^2 \]
   B:
   \[ V = 750\pi \text{ cm}^3 \]
   \[ SA = 350\pi \text{ cm}^2 \]
   b) B
   c) B
   d) 
   
   BONUS
   e) \[ V = 942 \text{ cm}^3 \]
   \[ SA = 816 \text{ cm}^2 \]
   f) Answers may vary.
   Teacher to check.
   Sample answer:
   Cube with sides of length 10 cm.
Unit 6: Measurement
Test (Lessons 9–18) — WNCP

NOTE: Do not use a calculator for this test.

1. Calculate the surface area and volume of the prism shown.

Surface Area = ____________________  Volume = _______________________

2. A right prism with volume 600 cm\(^2\) has the base shown below:

What is the height of the prism? _________

3. Find the volume and surface area of the prism.
4. The base of a cylinder has an area of 2.8 \( m^2 \).
   The cylinder has a volume of 5.6 \( m^3 \). What is its height?

5. a) Calculate the volume and surface area of each can. Leave your answer in terms of \( \pi \).

   A.
   
   ![Diagram of a cylinder with dimensions 10 cm and 3 cm]
   
   Volume = _____________________  Volume = _____________________

   Surface Area = _________________  Surface Area = ___________________

   b) Which can has the largest volume? ______

   c) Which can has the largest surface area? ______

   **BONUS:**
   
   d) Find the volume (to the nearest cm\(^3\)) and the surface area (to the nearest cm\(^2\)) of can A.

   e) Sketch a rectangular prism with larger volume and smaller surface area than can A.
   Label the dimensions of your prism.
1. SA = 172 cm²
   V = 112 cm³

2. 8 cm

3. SA = 4.95 m²
   V = 675 000 cm³
   = 0.675 m³

4. 2 m

5. a) A:
   V = 300π cm³
   SA = 260π cm²

   B:
   V = 750π cm³
   SA = 350π cm²

b) B

c) B

**BONUS**

d) V ≈ 942 cm³
   SA ≈ 816 cm²

e) Answers may vary.
   Teacher to check.
   Sample answer:
   Cube with sides of length 10 cm.
1. a) Find the number of vertices \((V)\), edges \((E)\), and faces \((F)\) for each shape.

   Then complete the chart.

<table>
<thead>
<tr>
<th>(V)</th>
<th>(E)</th>
<th>(F)</th>
<th>(V + E)</th>
<th>(V + F)</th>
<th>(E + F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Cube Diagram]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>![Octahedron Diagram]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>![Pyramid Diagram]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Circle the correct formula.

\[ V + E = F + 2 \quad V + F = E + 2 \quad E + F = V + 2 \]

c) The formula you circled in part b) is called ________________________________.

d) A soccer ball has 32 faces and 90 edges. Use the formula from part b) to decide how many vertices a soccer ball has.

**BONUS:**

a) Alice thinks that since none of the pentagons on a soccer ball meet each other, she can find the number of vertices on a soccer ball as \((\text{number of pentagons}) \times 5\).

   Is she correct? _____

   How many pentagons does a soccer ball have according to Alice’s calculation? _____

   How many hexagons? _____ Hint: Look at Question 1, part d) for the number of vertices and faces.

b) Bruno thinks that since each vertex on a soccer ball belongs to exactly two hexagons, he can find the number of vertices on a soccer ball as \((\text{number of hexagons}) \times 6 \div 2\).

   Is he correct? _____

   How many hexagons does a soccer ball have according to Bruno’s calculation? _____

   How many pentagons? _____
Unit 7: Geometry
Quiz (Lesson 37) — ON

1. a) 

<table>
<thead>
<tr>
<th>V</th>
<th>E</th>
<th>F</th>
<th>V + E</th>
<th>V + F</th>
<th>E + F</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>12</td>
<td>6</td>
<td>20</td>
<td>14</td>
<td>18</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>7</td>
<td>25</td>
<td>17</td>
<td>22</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>5</td>
<td>13</td>
<td>10</td>
<td>13</td>
</tr>
</tbody>
</table>

b) \( V + F = E + 2 \)

c) Euler’s formula

d) 60

BONUS

a) yes
   - 12 pentagons
   - 20 hexagons

b) yes
   - 20 hexagons
   - 12 pentagons
Unit 7: Geometry
Quiz (Lessons 38–41) — WNCP

1.  a) Split the heptagon on the right into triangles.
   What is the sum of the angles in a heptagon?

b) In the heptagon $KLMNOPQ$, $\angle M = \angle N = \angle Q = 120^\circ$.
   Use the sum of the angles in a heptagon to find the measures of the
   missing angles in $KLMNOPQ$.
   $x = ______$

c) Three copies of $KLMNOPQ$ are placed around a common vertex
   and another copy is added. Find the measures of the angles $a$ and $b$.
   $\angle a = ______$  $\angle b = ______$

d) Does the polygon $KLMNOPQ$ tessellate? Explain.

**BONUS:** Add 5 copies of the given shape to the grid on
the right to show that this heptagon tessellates.
1. a) 900°
b) 135°
c) \( \angle a = 90° \)
   \( \angle b = 105° \)
d) No: the angles in the heptagon are too large to fit into the 90° or 105° gap produced when two pentagons are placed together.

**BONUS**

Answers may vary.
Teacher to check.
Sample answer:

```
\begin{verbatim}
\text{Diagram}
\end{verbatim}
```
Unit 7: Geometry

Quiz (Lessons 42–46) — WNCP

1. a) Shade the top faces of the cubes.
   Then draw the top view.

   ![Top View of Cubes](image)

   a) Shade the top faces of the cubes.
   Then draw the top view.

b) Shade the front faces of the cubes.
   Then draw the front view.

   ![Front View of Cubes](image)

2. a) Draw the top view of the structure.
   Include thick lines to show where the depth changes.

   ![Top View of Structure](image)

b) Draw the right side of the structure.
   Include thick lines to show where the depth changes.

   ![Right Side View of Structure](image)
3. Draw the front, top, and right side views for the structure below.

**BONUS:** The front, left side, right side, and top view of a structure are all the same as the picture on the right. What shape is the structure?
Unit 7: Geometry

Quiz (Lessons 42–46) — WNCP

1. Teacher to check.
2. Teacher to check.
3. Teacher to check.

BONUS

cube
1. a) Identify the transformation you would use to take shape A onto the other shape. State the centre, amount and direction of rotation, the amount and direction of translation and/or draw the mirror lines.
   i) B: __________________________________
   ii) C: __________________________________
   iii) D: __________________________________
   iv) E: __________________________________

   b) Describe how you could make this tessellation starting from shape A.

2. Circle the structure that has the side views shown below on the left.
3. Draw and label the top, front and right side views of the chair.

![Diagram of a chair with labels: front, right side]

4. a) Draw and label the top, front and right and left side views of the structure below.

![Grid with a structure on it]

b) Kevin rotates the structure horizontally $90^\circ$ counter-clockwise ($\leftarrow$). Draw the top view of the structure after the rotation.

![Top view of the structure after rotation]

**BONUS:** Draw the front view and the right side view of the structure after the rotation.
1. a) i) 90° CW rotation around the nose of the monster
   ii) Translation 2 units right
   iii) 90° CCW rotation around the nose of the monster
   iv) 180° CW or CCW rotation around the nose of the monster

b) Answers may vary. Teacher to check.
Sample answer:
Rotate shape A 90° CW, 90° CCW and 180° CW or CCW around the nose to obtain a block of four shapes: A, B, D, E.
Translate this block 2 units right, and 2 units down, and then 2 units right & 2 units down.

2.

3.

4. a) top view

   [Diagram of a shape with views labeled: left side view, front view, right side view]

b) [Diagram of a shape with views labeled: front view, right side view]

BONUS
front view right side view
Unit 8: Probability and Data Management

Quiz (Lessons 15–17) — ON

1. The graph on the right shows the number of children in the families of Thianna’s class.
   Use the circle graph to compute:
   a) the mean ______________
   b) the median ______________
   c) the mode ______________

2. Decide which average — mean, median, or mode — you would use to answer each question.
   a) You want to know if your score on a test is in the top half of the class. ________________
   b) You want to know if a store is likely to be out of your shoe size. ________________
   c) You need an average of 80% in all subjects to get into a special high school. ________________

3. Create a set of 6 data values with mode 8, median 7, and mean 11.
   _______    _______    _______    _______    _______    _______

BONUS: Robert needs an average of at least 85 out of 100 to get into an art program. His grades (out of 100) are shown in the table on the right.
   What should his grades in English and Science be in order to get into the program? Explain.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>English</td>
<td></td>
</tr>
<tr>
<td>French</td>
<td>70</td>
</tr>
<tr>
<td>Math</td>
<td>87</td>
</tr>
<tr>
<td>Science</td>
<td></td>
</tr>
<tr>
<td>Social Studies</td>
<td>69</td>
</tr>
<tr>
<td>Music</td>
<td>71</td>
</tr>
<tr>
<td>Art</td>
<td>98</td>
</tr>
</tbody>
</table>
1.  
   a)  2.04  
   b)  2  
   c)  1  

2.  
   a) median  
   b) mode  
   c) mean  

3.  Answers may vary.  
   Teacher to check.  
   Data values should add to 66, the average of the two values in the middle should be 7, two values will equal 8, with no two other values the same.  
   Sample answer:  
   4, 5, 6, 8, 8, 35  

BONUS  
Both grades should be 100.  
The other grades add to 395, and if the average should be 85, the sum of all 7 grades should be 595, so Robert needs the two missing grades to add to 200. This means both should be 100.
1. A marble is drawn from a box containing a green, red, and blue marble. Then a coin is tossed.
   a) Draw a tree diagram to show all the possible outcomes.
   b) How many possible outcomes are there?

2. A game is played with two spinners.
   a) How many outcomes are on spinner B? _____
   b) To make a list of outcomes for spinning both spinners at the same time, how many times should we write the colours R, G, Y, B on the list?
   c) Complete a table of outcomes.

<table>
<thead>
<tr>
<th>Spinner A</th>
<th>Spinner B</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>1</td>
</tr>
<tr>
<td>G</td>
<td>2</td>
</tr>
<tr>
<td>Y</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
</tr>
</tbody>
</table>

   d) How many outcomes does this game have?

3. A coin is tossed and then a spinner is spun. Find the probabilities.
   a) P(Tail)  b) P(Yellow)  c) P(Tail and Yellow)
4. A computer generates a random number chosen from the numbers 1, 2, and 3. It then generates a second random number from the numbers 1 and 2.

   a) Make a T-Table and list all the possible outcomes.
   b) Circle all the outcomes that add to 3.
   c) What is the probability that the computer will generate two numbers that add to 3?
   d) What is the probability that the computer will generate two numbers that do not add to 3?

5. Cindy used two spinners for a game. She wins the game if the number 1 appears on both spinners.
   a) If she plays the game 80 times, how many times can she expect to win?
   b) If she plays the game 120 times, how many times can she expect to win?

**BONUS:** A computer generates two random numbers. The first number is between 1 and 100 (including 1 and 100). The second random number is between 1 and 10,000 (including 1 and 10,000).
What is the probability that the computer will generate the number 1 twice?
1. a) 

b) 6

2. a) 2
b) 2
c) R R G G Y Y B B
   1 2 1 2 1 2 1 2

3. a) \(\frac{1}{2}\)
b) \(\frac{1}{4}\)
c) \(\frac{1}{8}\)

4. a) 1, 1
   1, 2
   2, 1
   2, 2
   3, 1
   3, 2
b) 1, 2
   2, 1
c) \(\frac{2}{6} = \frac{1}{3}\)
d) \(\frac{4}{6} = \frac{2}{3}\)

5. a) 10
b) 15

BONUS

\[
\frac{1}{1 000 000}
\]
1. a) Draw a tree diagram to show all the combinations of outcomes of equal likelihood you could spin on these two spinners.

b) How many ways can you spin an R on the first spinner and a 3 on the second spinner? ________

c) What is the probability of spinning an R on the first spinner? ________

d) What is the probability of spinning a 3 on the second spinner? ________

e) What is the probability of spinning an R on the first spinner and a 3 on the second? ________

f) How can you get your answer in part e) from your answers in parts c) and d)? Explain.

BONUS: What is the probability of getting a colour on the Canadian flag on the first spinner and an even number on the second spinner?

2. a) List all possible outcomes when tossing 2 four-sided dice marked 1 to 4.

b) Linda thinks that since 7 is more than 3, the probability of tossing 7 with the two dice will be more than the probability of tossing 3. Is she correct? Explain.
3. a) What is the theoretical probability of getting two heads when tossing two fair coins? How do you know? Write the probability as a fraction, percent and ratio.

b) Ahmed decides to toss two fair coins. How many times should he expect to get two heads if he tosses the coins...

   i) 20 times? __________
   ii) 40 times? __________
   iii) 100 times? __________

c) Ahmed tossed the coins 60 times and they landed on two heads 12 times. What was Ahmed's experimental probability of getting two heads?

d) Ahmed thinks that if he tosses the coins 200 times, he will get two heads 40 times. Explain Ahmed's mistake.

e) Ahmed tosses two coins 10 times and Katie tosses two coins 500 times. Whose experimental probability do you expect to be closer to the theoretical probability of getting heads? Explain.
1. a) 

\[ \begin{array}{c}
\text{R} \quad 2 \\
\text{W} \quad 3 \\
\text{B} \quad 3 \\
\end{array} \]

b) \( \frac{1}{2} \)

c) \( \frac{1}{4} \) or 25%

d) \( \frac{1}{3} \)

e) \( \frac{1}{12} \)

f) The spins are independent, so the probability of spinning R and 3 is the product of the probabilities in parts d) and c).

\[ \frac{1}{4} \times \frac{1}{3} = \frac{1}{12} \]

**BONUS**

\( \frac{1}{2} \) or 50% 

2. a) \( (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4) \)

b) \( P(3) = \frac{2}{16} \)

\[ P(7) = \frac{2}{16} \]

The probabilities are the same, so Linda is wrong.

3. a) \( \frac{1}{4} = 25\% \)

\( = \) 1 out of 4 or 1:4

Explanations may vary.
Teacher to check.

Sample explanation:
Tossing coins are independent events, and the probability of getting heads on each coin is \( \frac{1}{2} \), so the probability of getting two heads is

\[ \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \]

b) i) \( 5 \)

ii) \( 10 \)

iii) \( 25 \)

c) \( \frac{12}{60} = \frac{1}{5} \) or 20%

d) Answers will vary.
Teacher to check.

Sample answer:
Ahmed thinks the experimental probability will be the same as in his experiment with 60 tosses. However, the more tosses, the closer the experimental probability gets to the theoretical probability, so he should expect to get two heads about 50 times.

e) Katie’s, because she has more trials.
1. The table on the right shows all the possible totals when rolling a pair of dice.

   a) Circle the entries in the table where the total is 7.

   b) How many of the entries are 7? ______

   c) What is the probability of rolling a 7?

   d) What is the probability of rolling a 3?

   **BONUS:** What is the probability of rolling a 1?

<table>
<thead>
<tr>
<th>First Die</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3 4 5 6</td>
</tr>
<tr>
<td>1 2 3 4 5 6 7</td>
</tr>
<tr>
<td>2 3 4 5 6 7 8</td>
</tr>
<tr>
<td>3 4 5 6 7 8 9</td>
</tr>
<tr>
<td>4 5 6 7 8 9 10</td>
</tr>
<tr>
<td>5 6 7 8 9 10 11</td>
</tr>
<tr>
<td>6 7 8 9 10 11 12</td>
</tr>
</tbody>
</table>

2. a) Write a set of ordered pairs to show all the combinations you could spin on the spinners.

   b) How many different possible totals are there? ______

   c) What is the probability of rolling a total of 4?

   d) If you spun both spinners 1 000 times …

      i) which total would you expect to appear the least? ______

      ii) which total would you expect to appear the most? ______
3. Describe the complementary event to the given event.
   a) flipping tails on a coin
   _____________________________________________________
   b) rolling an even number on a die
   _____________________________________________________
   c) drawing a black card from a deck of cards
   _____________________________________________________

4. A basketball player sinks 7 out of every 10 free throws. What is the probability that she will miss a free throw?

5. A sample of 1 000 students showed that 200 students preferred chocolate ice cream over vanilla. In a province of 50 000 students, how many do you expect to prefer chocolate?

6. A survey of 500 people in a home for the aged found that 350 chose The Beatles as their favourite rock and roll band. Using this number, Tony predicted that 700 000 in a city of 1 000 000 people would choose The Beatles as their favourite band. Explain why this prediction may not turn out to be accurate.

BONUS: Out of the 1 000 000 000 people using a particular social media website, 6 000 000 say they use it at least 5 times a day. In a sample of 500 people, how many do you expect will use the website at least 5 times a day?
1. a) Teacher to check.
   b) 6
   c) \( \frac{6}{36} = \frac{1}{6} \)
   d) \( \frac{2}{36} = \frac{1}{18} \)

**BONUS**

0

2. a) (1, 1), (1, 2), (2, 1),
    (2, 2), (3, 1), (3, 2)
   b) 4
   c) \( \frac{2}{8} = \frac{1}{4} \)
   d) i) 5
      ii) 3

3. a) flipping heads
    b) rolling an odd number
    c) drawing a red card

4. \( \frac{3}{10} \)

5. 10 000

6. Sample answer:
   More people in the old age home will know The Beatles than in the rest of the population because The Beatles were popular in the 1960s and 1970s.

**BONUS**

3
1. Using the stem and leaf plot shown on the right, find the range, median and mode.
   
   **Range:** _________
   **Median:** _________
   **Mode:** _________

2. a) What is the theoretical probability of getting heads when tossing a fair coin? Write your answer as a fraction, percent and ratio.

   b) Ahmed decides to toss a fair coin. How many times should he expect to get heads if he tosses the coin …
      
   i) 10 times? _________
   ii) 30 times? _________
   iii) 300 times? _________

   c) Ahmed tossed the coin 30 times and it landed on heads 12 times. What was Ahmed’s experimental probability of getting heads?

   d) Ahmed thinks that if he tosses the coin 100 times, he will get 40 heads. Explain Ahmed’s mistake.

   e) Ahmed tosses a coin 10 times and Katie tosses a coin 500 times. Whose experimental probability do you expect to be closer to the theoretical probability of getting heads? Explain.
3. Describe the complementary event to the given event.
   a) rolling a 6 on a regular die
   b) getting heads when tossing a coin

4. The probability that it will rain is 30%.
   What is the probability that it will not rain? _________

   **BONUS:** The probability that it will rain is 0.472.
   What is the probability that it will not rain? _________

5. You want to find out if students who live closer to school tend to arrive earlier or later than students who live farther from school.
   a) Write two unbiased questions you would ask people in your survey.
   b) Who would you survey? How would you choose your sample?
   c) What type of graph would you use to display your results?
      Explain how you would use your choice of graph to answer the original question.
1. Range: 37  
   Median: 52  
   Mode: 43  

2. a) $\frac{1}{2}$, 50%, 1:2  
   b) i) 5  
      ii) 15  
      iii) 150  
   c) $\frac{12}{30} = \frac{2}{5}$ or 40%  
   d) Answers will vary.  
      Teacher to check.  
      Sample answer:  
      Ahmed thinks the experimental probability will be the same as in his experiment with 30 tosses. However, the more tosses, the closer the experimental probability gets to the theoretical probability, so he should expect about 50 heads.  
   e) Katie’s, because she has more trials.  

3. a) Rolling a 1, 2, 3, 4 or 5  
   b) Getting tails  

4. 70%  

BONUS  
0.528  

5. Answers will vary.  
   Teacher to check.  
   Sample answers:  
   a) How far do you live from school?  
      0–500 m, 500–1 000 m, 1 000–1 500 m, 1 500–2 000 m, or more than 2 000 m  
      When did you arrive to school today?
## Contents

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number Sense and Numeration</td>
<td>3</td>
</tr>
<tr>
<td>Measurement</td>
<td>6</td>
</tr>
<tr>
<td>Geometry and Spatial Sense</td>
<td>8</td>
</tr>
<tr>
<td>Patterning and Algebra</td>
<td>10</td>
</tr>
<tr>
<td>Data Management and Probability</td>
<td>12</td>
</tr>
</tbody>
</table>
Notes

To ensure that the curriculum is fully covered, use the worksheets with the lessons plans in the Teacher’s Guide.

Underlined lesson numbers indicate relevant preparatory exercises.

OCUP: Ontario Curriculum Unit Planner

JUMP Math workbook units are represented by:

- **NS** Number Sense
- **PA** Patterns and Algebra
- **ME** Measurement
- **G** Geometry
- **PDM** Probability and Data Management
Number Sense and Numeration

Overall Expectations
By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>OCUP Code</th>
<th>Overall Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>8m8</td>
<td>represent, compare, and order equivalent representations of numbers, including those involving positive exponents;</td>
</tr>
<tr>
<td>8m9</td>
<td>solve problems involving whole numbers, decimal numbers, fractions, and integers, using a variety of computational strategies;</td>
</tr>
<tr>
<td>8m10</td>
<td>solve problems by using proportional reasoning in a variety of meaningful contexts.</td>
</tr>
</tbody>
</table>

Quantity Relationships
By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>-----------</td>
<td>----------------------</td>
</tr>
<tr>
<td>8m11</td>
<td>express repeated multiplication using exponential notation;</td>
</tr>
<tr>
<td>8m12</td>
<td>represent whole numbers in expanded form using powers of ten;</td>
</tr>
<tr>
<td>8m13</td>
<td>represent, compare, and order rational numbers (i.e., positive and negative fractions and decimals to thousandths);</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m14</td>
<td>translate between equivalent forms of a number (i.e., decimals, fractions, percents);</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m15</td>
<td>determine common factors and common multiples using the prime factorization of numbers.</td>
</tr>
</tbody>
</table>
## Operational Sense

By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OCUP Code</strong></td>
<td><strong>Specific Expectation</strong></td>
</tr>
<tr>
<td>8m16</td>
<td>solve multi-step problems arising from real-life contexts and involving whole numbers and decimals, using a variety of tools and strategies;</td>
</tr>
<tr>
<td>8m17</td>
<td>solve problems involving percents expressed to one decimal place and whole-number percents greater than 100;</td>
</tr>
<tr>
<td>8m18</td>
<td>use estimation when solving problems involving operations with whole numbers, decimals, percents, integers, and fractions, to help judge the reasonableness of a solution;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m19</td>
<td>represent the multiplication and division of fractions, using a variety of tools and strategies;</td>
</tr>
<tr>
<td>8m20</td>
<td>solve problems involving addition, subtraction, multiplication, and division with simple fractions;</td>
</tr>
<tr>
<td>8m21</td>
<td>represent the multiplication and division of integers, using a variety of tools;</td>
</tr>
<tr>
<td>8m22</td>
<td>solve problems involving operations with integers, using a variety of tools;</td>
</tr>
<tr>
<td>8m23</td>
<td>evaluate expressions that involve integers, including expressions that contain brackets and exponents, using order of operations;</td>
</tr>
<tr>
<td>8m24</td>
<td>multiply and divide decimal numbers by various powers of ten;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m25</td>
<td>estimate, and verify using a calculator, the positive square roots of whole numbers, and distinguish between whole numbers that have whole-number square roots (i.e., perfect square numbers) and those that do not.</td>
</tr>
</tbody>
</table>
**Proportional Relationships**

By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code Specific Expectation</td>
<td>Part Unit Lesson</td>
</tr>
<tr>
<td>8m26 identify and describe real-life situations involving two quantities that are directly proportional;</td>
<td>2 1:NS 98, 99, 101–103</td>
</tr>
<tr>
<td>8m27 solve problems involving proportions, using concrete materials, drawings, and variables;</td>
<td>1 6:ME 3, 4 2 1:NS 90, 92, 97–99</td>
</tr>
<tr>
<td>8m28 solve problems involving percent that arise from real-life contexts;</td>
<td>2 1:NS 85–89, 92–95</td>
</tr>
<tr>
<td>8m29 solve problems involving rates.</td>
<td>2 1:NS 101–103</td>
</tr>
</tbody>
</table>
**Measurement**

**Overall Expectations**
By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>OCUP Code</th>
<th>Overall Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>8m30</td>
<td>research, describe, and report on applications of volume and capacity measurement;</td>
</tr>
<tr>
<td>8m31</td>
<td>determine the relationships among units and measurable attributes, including the area of a circle and the volume of a cylinder.</td>
</tr>
</tbody>
</table>

**Attributes, Units, and Measurement Sense**
By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>8m32</td>
<td>research, describe, and report on applications of volume and capacity measurement.</td>
</tr>
</tbody>
</table>

**Measurement Relationships**
By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>8m33</td>
<td>solve problems that require conversions involving metric units of area, volume, and capacity (i.e., square centimetres and square metres; cubic centimetres and cubic metres; millilitres and cubic centimetres);</td>
</tr>
<tr>
<td>8m34</td>
<td>measure the circumference, radius, and diameter of circular objects, using concrete materials;</td>
</tr>
<tr>
<td>8m35</td>
<td>determine, through investigation using a variety of tools and strategies, the relationships for calculating the circumference and the area of a circle, and generalize to develop the formulas [i.e., Circumference of a circle = ( \pi \times \text{diameter} ); Area of a circle = ( \pi \times (\text{radius})^2 )];</td>
</tr>
<tr>
<td>8m36</td>
<td>solve problems involving the estimation and calculation of the circumference and the area of a circle;</td>
</tr>
</tbody>
</table>
Measurement Relationships (continued)

By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OCUP Code</strong></td>
<td><strong>Specific Expectation</strong></td>
</tr>
<tr>
<td>8m37</td>
<td>determine, through investigation using a variety of tools and strategies, the relationship between the area of the base and height and the volume of a cylinder, and generalize to develop the formula (i.e., Volume = area of base x height);</td>
</tr>
<tr>
<td>8m38</td>
<td>determine, through investigation using concrete materials, the surface area of a cylinder;</td>
</tr>
<tr>
<td>8m39</td>
<td>solve problems involving the surface area and the volume of cylinders, using a variety of strategies.</td>
</tr>
</tbody>
</table>
Geometry and Spatial Sense

Overall Expectations

By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>OCUP Code</th>
<th>Overall Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>8m40</td>
<td>demonstrate an understanding of the geometric properties of quadrilaterals and circles and the applications of geometric properties in the real world;</td>
</tr>
<tr>
<td>8m41</td>
<td>develop geometric relationships involving lines, triangles, and polyhedra, and solve problems involving lines and triangles;</td>
</tr>
<tr>
<td>8m42</td>
<td>represent transformations using the Cartesian coordinate plane, and make connections between transformations and the real world.</td>
</tr>
</tbody>
</table>

Geometric Properties

By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>8m43</td>
<td>sort and classify quadrilaterals by geometric properties, including those based on diagonals, through investigation using a variety of tools;</td>
</tr>
<tr>
<td>8m44</td>
<td>construct a circle, given its centre and radius, or its centre and a point on the circle, or three points on the circle;</td>
</tr>
<tr>
<td>8m45</td>
<td>investigate and describe applications of geometric properties in the real world.</td>
</tr>
</tbody>
</table>
Geometric Relationships
By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>8m46</td>
<td>determine, through investigation using a variety of tools, relationships among area, perimeter, corresponding side lengths, and corresponding angles of similar shapes;</td>
</tr>
<tr>
<td>8m47</td>
<td>determine, through investigation using a variety of tools and strategies, the angle relationships for intersecting lines and for parallel lines and transversals, and the sum of the angles of a triangle;</td>
</tr>
<tr>
<td>8m48</td>
<td>solve angle-relationship problems involving triangles, intersecting lines, and parallel lines and transversals;</td>
</tr>
<tr>
<td>8m49</td>
<td>determine the Pythagorean relationship, through investigation using a variety of tools and strategies;</td>
</tr>
<tr>
<td>8m50</td>
<td>solve problems involving right triangles geometrically, using the Pythagorean relationship;</td>
</tr>
<tr>
<td>8m51</td>
<td>determine, through investigation using concrete materials, the relationship between the numbers of faces, edges, and vertices of a polyhedron (i.e., number of faces + number of vertices = number of edges + 2).</td>
</tr>
</tbody>
</table>

Location and Movement
By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>8m52</td>
<td>graph the image of a point, or set of points, on the Cartesian coordinate plane after applying a transformation to the original point(s) (i.e., translation; reflection in the x-axis, the y-axis, or the angle bisector of the axes that passes through the first and third quadrants; rotation of 90°, 180°, or 270° about the origin);</td>
</tr>
<tr>
<td>8m53</td>
<td>identify, through investigation, real-world movements that are translations, reflections, and rotations.</td>
</tr>
</tbody>
</table>
## Patterning and Algebra

### Overall Expectations

By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>OCUP Code</th>
<th>Overall Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>8m54</td>
<td>represent linear growing patterns (where the terms are whole numbers) using graphs, algebraic expressions, and equations;</td>
</tr>
<tr>
<td>8m55</td>
<td>model linear relationships graphically and algebraically, and solve and verify algebraic equations, using a variety of strategies, including inspection, guess and check, and using a “balance” model.</td>
</tr>
</tbody>
</table>

### Patterns and Relationships

By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>-----------</td>
<td>---------------------</td>
</tr>
<tr>
<td>8m56</td>
<td>represent, through investigation with concrete materials, the general term of a linear pattern, using one or more algebraic expressions;</td>
</tr>
<tr>
<td>8m57</td>
<td>represent linear patterns graphically (i.e., make a table of values that shows the term number and the term, and plot the coordinates on a graph), using a variety of tools;</td>
</tr>
<tr>
<td>8m58</td>
<td>determine a term, given its term number, in a linear pattern that is represented by a graph or an algebraic equation.</td>
</tr>
</tbody>
</table>
## Variables, Expressions, and Equations

By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>**OCUP Code</td>
<td>Specific Expectation**</td>
</tr>
<tr>
<td>8m59</td>
<td>describe different ways in which algebra can be used in real-life situations;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m60</td>
<td>model linear relationships using tables of values, graphs, and equations, through investigation using a variety of tools;</td>
</tr>
<tr>
<td>8m61</td>
<td>translate statements describing mathematical relationships into algebraic expressions and equations;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m62</td>
<td>evaluate algebraic expressions with up to three terms, by substituting fractions, decimals, and integers for the variables;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m63</td>
<td>make connections between solving equations and determining the term number in a pattern, using the general term;</td>
</tr>
<tr>
<td>8m64</td>
<td>solve and verify linear equations involving a one-variable term and having solutions that are integers, by using inspection, guess and check, and a “balance” model.</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Data Management and Probability

Overall Expectations
By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>OCUP Code</th>
<th>Overall Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>8m65</td>
<td>collect and organize categorical, discrete, or continuous primary data and secondary data and display the data using charts and graphs, including frequency tables with intervals, histograms, and scatter plots;</td>
</tr>
<tr>
<td>8m66</td>
<td>apply a variety of data management tools and strategies to make convincing arguments about data;</td>
</tr>
<tr>
<td>8m67</td>
<td>use probability models to make predictions about real-life events.</td>
</tr>
</tbody>
</table>

Collection and Organization of Data
By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCUP Code</td>
<td>Specific Expectation</td>
</tr>
<tr>
<td>-----------</td>
<td>----------------------</td>
</tr>
<tr>
<td>8m68</td>
<td>collect data by conducting a survey or an experiment to do with themselves, their environment, issues in their school or community, or content from another subject, and record observations or measurements;</td>
</tr>
<tr>
<td>8m69</td>
<td>organize into intervals a set of data that is spread over a broad range;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m70</td>
<td>collect and organize categorical, discrete, or continuous primary data and secondary data, and display the data in charts, tables, and graphs (including histograms and scatter plots) that have appropriate titles, labels, and scales that suit the range and distribution of the data, using a variety of tools;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m71</td>
<td>select an appropriate type of graph to represent a set of data, graph the data using technology, and justify the choice of graph (i.e., from types of graphs already studied, including histograms and scatter plots);</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m72</td>
<td>explain the relationship between a census, a representative sample, sample size, and a population.</td>
</tr>
</tbody>
</table>
## Data Relationships

By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OCUP Code</strong></td>
<td><strong>Specific Expectation</strong></td>
</tr>
<tr>
<td>8m73</td>
<td>read, interpret, and draw conclusions from primary data and from secondary data, presented in charts, tables, and graphs (including frequency tables with intervals, histograms, and scatter plots);</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m74</td>
<td>determine, through investigation, the appropriate measure of central tendency (i.e., mean, median, or mode) needed to compare sets of data;</td>
</tr>
<tr>
<td>8m75</td>
<td>demonstrate an understanding of the appropriate uses of bar graphs and histograms by comparing their characteristics;</td>
</tr>
<tr>
<td>8m76</td>
<td>compare two attributes or characteristics, using a scatter plot, and determine whether or not the scatter plot suggests a relationship;</td>
</tr>
<tr>
<td>8m77</td>
<td>identify and describe trends, based on the rate of change of data from tables and graphs, using informal language;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m78</td>
<td>make inferences and convincing arguments that are based on the analysis of charts, tables, and graphs;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m79</td>
<td>compare two attributes or characteristics, using a variety of data management tools and strategies (i.e., pose a relevant question, then design an experiment or survey, collect and analyse the data, and draw conclusions).</td>
</tr>
</tbody>
</table>
## Probability

By the end of Grade 8, students will:

<table>
<thead>
<tr>
<th>ONTARIO CURRICULUM EXPECTATION</th>
<th>JUMP MATH WORKBOOK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>OCUP Code</strong></td>
<td><strong>Specific Expectation</strong></td>
</tr>
<tr>
<td>8m80</td>
<td>compare, through investigation, the theoretical probability of an event (i.e., the ratio of the number of ways a favourable outcome can occur compared to the total number of possible outcomes) with experimental probability, and explain why they might differ;</td>
</tr>
<tr>
<td>8m81</td>
<td>determine, through investigation, the tendency of experimental probability to approach theoretical probability as the number of trials in an experiment increases, using class-generated data and technology-based simulation models;</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>8m82</td>
<td>identify the complementary event for a given event, and calculate the theoretical probability that a given event will not occur.</td>
</tr>
</tbody>
</table>
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>3</td>
</tr>
<tr>
<td>Patterns and Relations</td>
<td>8</td>
</tr>
<tr>
<td>Shape and Space</td>
<td>10</td>
</tr>
<tr>
<td>Statistics and Probability</td>
<td>14</td>
</tr>
</tbody>
</table>
Notes

To ensure that the curriculum is fully covered, use the worksheets with the lessons plans in the Teacher’s Guide.

Underlined lesson numbers indicate relevant preparatory exercises.

WNCP Abbreviations:

[C] Communication
[CN] Connections
[ME] Mental Mathematics and Estimation
[PS] Problem Solving
[R] Reasoning
[T] Technology
[V] Visualization

JUMP Math workbook units are represented by:

NS  Number Sense
PA  Patterns and Algebra
ME  Measurement
G   Geometry
PDM Probability and Data Management
# Number

## General Outcome
Develop number sense.

## Specific Outcomes
It is expected that students will:

1. **WNCP CURRICULUM**

   **Specific Outcome**
   
   Demonstrate an understanding of perfect square and square root, concretely, pictorially and symbolically (limited to whole numbers). [C, CN, R, V]

   **Achievement Indicators**
   
<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3:NS</td>
<td>58</td>
</tr>
<tr>
<td>1</td>
<td>1:NS</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3:NS</td>
<td>59</td>
</tr>
<tr>
<td>1</td>
<td>1:NS</td>
<td>3, 4</td>
</tr>
<tr>
<td>1</td>
<td>3:NS</td>
<td>59, 61</td>
</tr>
<tr>
<td>1</td>
<td>3:NS</td>
<td>60</td>
</tr>
<tr>
<td>1</td>
<td>3:NS</td>
<td>58</td>
</tr>
</tbody>
</table>

2. **WNCP CURRICULUM**

   **Specific Outcome**
   
   Determine the approximate square root of numbers that are not perfect squares (limited to whole numbers). [C, CN, ME, R, T]

   **Achievement Indicators**
   
<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3:NS</td>
<td>62, 63</td>
</tr>
<tr>
<td>1</td>
<td>3:NS</td>
<td>62, 63</td>
</tr>
<tr>
<td>1</td>
<td>3:NS</td>
<td>62</td>
</tr>
<tr>
<td>1</td>
<td>3:NS</td>
<td>62</td>
</tr>
</tbody>
</table>
### 3. WNCP CURRICULUM

#### Specific Outcome

Demonstrate an understanding of percents greater than or equal to 0%.  
[CN, PS, R, V]

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Provide a context where a percent may be more than 100% or between 0% and 1%.</td>
<td>2</td>
<td>1:NS</td>
<td>93, 94, 95</td>
</tr>
<tr>
<td>Represent a given fractional percent using grid paper.</td>
<td>2</td>
<td>2:PDM</td>
<td>9</td>
</tr>
<tr>
<td>Represent a given percent greater than 100 using grid paper.</td>
<td>2</td>
<td>1:NS</td>
<td>82–84, 94</td>
</tr>
<tr>
<td>Determine the percent represented by a given shaded region on a grid, and record it in decimal, fractional and percent form.</td>
<td>2</td>
<td>1:NS</td>
<td>82–84, 93, 94</td>
</tr>
<tr>
<td>Express a given percent in decimal or fractional form.</td>
<td>2</td>
<td>1:NS</td>
<td>82–87, 96</td>
</tr>
<tr>
<td>Express a given decimal in percent or fractional form.</td>
<td>2</td>
<td>1:NS</td>
<td>82–87, 96</td>
</tr>
<tr>
<td>Express a given fraction in decimal or percent form.</td>
<td>2</td>
<td>1:NS</td>
<td>82–87, 96</td>
</tr>
<tr>
<td>Solve a given problem involving percents.</td>
<td>2</td>
<td>1:NS</td>
<td>88–92, 95, 98, 99</td>
</tr>
<tr>
<td>Solve a given problem involving combined percents, e.g., addition of percents, such as GST + PST.</td>
<td>2</td>
<td>1:NS</td>
<td>83, 98, 99</td>
</tr>
<tr>
<td>Solve a given problem that involves finding the percent of a percent, e.g., A population increased by 10% one year and then increased by 15% the next year. Explain why there was not a 25% increase in population over the two years.</td>
<td>2</td>
<td>1:NS</td>
<td>88–92, 98, 99</td>
</tr>
</tbody>
</table>

### 4. WNCP CURRICULUM

#### Specific Outcome

Demonstrate an understanding of ratio and rate.  [C, CN, V]

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Express a two-term ratio from a given context in the forms 3:5 or 3 to 5.</td>
<td>1</td>
<td>6:ME</td>
<td>1–4</td>
</tr>
<tr>
<td>Express a three-term ratio from a given context in the forms 4:7:3 or 4 to 7 to 3.</td>
<td>2</td>
<td>1:NS</td>
<td>100</td>
</tr>
<tr>
<td>Express a part to part ratio as a part to whole fraction, e.g., frozen juice to water; 1 can concentrate to 4 cans of water can be represented as $\frac{1}{5}$, which is the ratio of concentrate to solution, or $\frac{4}{5}$, which is the ratio of water to solution.</td>
<td>1</td>
<td>6:ME</td>
<td>1–3</td>
</tr>
</tbody>
</table>
4. **Achievement Indicators**

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6:ME</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1:NS</td>
<td>101–103</td>
</tr>
</tbody>
</table>

Identify and describe ratios and rates from real-life examples, and record them symbolically.

Express a given rate using words or symbols, e.g., 20 L per 100 km or 20 L/100 km.

Express a given ratio as a percent and explain why a rate cannot be represented as a percent.

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1:NS</td>
<td>96, 101, 102</td>
</tr>
</tbody>
</table>

5. **WNCP CURRICULUM**

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
</table>

Solve problems that involve rates, ratios and proportional reasoning. [C, CN, PS, R]

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explain the meaning of [\frac{a}{b}] within a given context.</td>
<td>2</td>
<td>1:NS</td>
<td>102</td>
</tr>
</tbody>
</table>

Provide a context in which \[\frac{a}{b}\] represents a:
- fraction
- rate
- ratio
- quotient
- probability.

Solve a given problem involving rate, ratio or percent.

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6:ME</td>
<td>3, 4</td>
</tr>
<tr>
<td>2</td>
<td>1:NS</td>
<td>88–103</td>
</tr>
</tbody>
</table>
### WNCP CURRICULUM

#### Specific Outcome
Demonstrate an understanding of multiplying and dividing positive fractions and mixed numbers, concretely, pictorially and symbolically. [C, CN, ME, PS]

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit 1</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identify the operation required to solve a given problem involving positive fractions.</td>
<td>1</td>
<td>1:NS</td>
<td>31–33</td>
</tr>
<tr>
<td>Provide a context that requires the multiplying of two given positive fractions.</td>
<td>1</td>
<td>1:NS</td>
<td>32</td>
</tr>
<tr>
<td>Provide a context that requires the dividing of two given positive fractions.</td>
<td>1</td>
<td>1:NS</td>
<td>32</td>
</tr>
<tr>
<td>Estimate the product of two given positive proper fractions to determine</td>
<td>1</td>
<td>1:NS</td>
<td>25</td>
</tr>
<tr>
<td>Estimate the quotient of two given positive fractions and compare the</td>
<td>1</td>
<td>1:NS</td>
<td>30</td>
</tr>
<tr>
<td>Express a given positive mixed number as an improper fraction and a given</td>
<td>1</td>
<td>1:NS</td>
<td>9–13</td>
</tr>
<tr>
<td>Model multiplication of a positive fraction by a whole number concretely or</td>
<td>1</td>
<td>1:NS</td>
<td>23, 24</td>
</tr>
<tr>
<td>Model multiplication of a positive fraction by a positive fraction concretely or</td>
<td>1</td>
<td>1:NS</td>
<td>25</td>
</tr>
<tr>
<td>Model division of a positive proper fraction by a whole number concretely or</td>
<td>1</td>
<td>1:NS</td>
<td>26</td>
</tr>
<tr>
<td>Model division of a positive proper fraction by a positive proper fraction pictorially</td>
<td>1</td>
<td>1:NS</td>
<td>27–29</td>
</tr>
<tr>
<td>Generalize and apply rules for multiplying and dividing positive fractions,</td>
<td>1</td>
<td>1:NS</td>
<td>24–29</td>
</tr>
<tr>
<td>Solve a given problem involving positive fractions taking into consideration</td>
<td>1</td>
<td>1:NS</td>
<td>31, 33</td>
</tr>
</tbody>
</table>
### WNCP CURRICULUM

#### Specific Outcome

Demonstrate an understanding of multiplication and division of integers, concretely, pictorially and symbolically. [C, CN, PS, R, V]

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identify the operation required to solve a given problem involving integers.</td>
<td>1</td>
<td>7:NS</td>
<td>71, 74</td>
</tr>
<tr>
<td>Provide a context that requires multiplying two integers.</td>
<td>1</td>
<td>7:NS</td>
<td>74</td>
</tr>
<tr>
<td>Provide a context that requires dividing two integers.</td>
<td>1</td>
<td>7:NS</td>
<td>74</td>
</tr>
<tr>
<td>Model the process of multiplying two integers using concrete materials or pictorial representations and record the process.</td>
<td>1</td>
<td>7:NS</td>
<td>72</td>
</tr>
<tr>
<td>Model the process of dividing an integer by an integer using concrete materials or pictorial representations and record the process.</td>
<td>1</td>
<td>7:NS</td>
<td>73</td>
</tr>
<tr>
<td>Solve a given problem involving the division of integers (2-digit by 1-digit) without the use of technology.</td>
<td>1</td>
<td>7:NS</td>
<td>73</td>
</tr>
<tr>
<td>Solve a given problem involving the division of integers (2-digit by 2-digit) with the use of technology.</td>
<td>1</td>
<td>7:NS</td>
<td>73</td>
</tr>
<tr>
<td>Generalize and apply a rule for determining the sign of the product and quotient of integers.</td>
<td>1</td>
<td>7:NS</td>
<td>72, 73</td>
</tr>
<tr>
<td>Solve a given problem involving integers taking into consideration order of operations.</td>
<td>1</td>
<td>7:NS</td>
<td>73, 74</td>
</tr>
</tbody>
</table>
Patterns and Relations

General Outcomes

• Patterns: Use patterns to describe the world and solve problems.
• Variables and Equations: Represent algebraic expressions in multiple ways.

Patterns

It is expected that students will:

<table>
<thead>
<tr>
<th>1. WNCP CURRICULUM</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific Outcome</td>
<td></td>
</tr>
<tr>
<td>Graph and analyze two-variable linear relations. [C, ME, PS, R, T, V]</td>
<td></td>
</tr>
<tr>
<td><strong>Achievement Indicators</strong></td>
<td><strong>Part</strong></td>
</tr>
<tr>
<td>Determine the missing value in an ordered pair for a given equation.</td>
<td>2</td>
</tr>
<tr>
<td>Create a table of values by substituting values for a variable in the equation of a given linear relation.</td>
<td>2</td>
</tr>
<tr>
<td>Construct a graph from the equation of a given linear relation (limited to discrete data).</td>
<td>2</td>
</tr>
<tr>
<td>Describe the relationship between the variables of a given graph.</td>
<td>2</td>
</tr>
</tbody>
</table>
Variables and Equations
It is expected that students will:

2. WNCP CURRICULUM
   Specific Outcome
   Model and solve problems using linear equations of the form:
   • \( ax = b \)
   • \( \frac{x}{a} = b, a \neq 0 \)
   • \( ax + b = c \)
   • \( \frac{x}{a} + b = c, a \neq 0 \)
   • \( a(x + b) = c \)
   concretely, pictorially and symbolically, where \( a, b \) and \( c \) are integers.
   [C, CN, PS, V]

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model a given problem with a linear equation and solve the equation using concrete</td>
<td>1</td>
<td>2:PA</td>
<td>5, 7, 11</td>
</tr>
<tr>
<td>models, e.g., counters, integer tiles.</td>
<td>2</td>
<td>4:PA</td>
<td>19, 21, 22, 24</td>
</tr>
<tr>
<td>Verify the solution to a given linear equation using a variety of methods, including</td>
<td>1</td>
<td>2:PA</td>
<td>5, 6, 7–9, 11</td>
</tr>
<tr>
<td>concrete materials, diagrams and substitution.</td>
<td>2</td>
<td>4:PA</td>
<td>16, 24</td>
</tr>
<tr>
<td>Draw a visual representation of the steps used to solve a given linear</td>
<td>2</td>
<td>4:PA</td>
<td>19</td>
</tr>
<tr>
<td>equation and record each step symbolically.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solve a given linear equation symbolically.</td>
<td>1</td>
<td>2:PA</td>
<td>8, 9</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4:PA</td>
<td>17, 18</td>
</tr>
<tr>
<td>Identify and correct an error in a given incorrect solution of a linear equation.</td>
<td>1</td>
<td>2:PA</td>
<td>6, 14</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4:PA</td>
<td>19</td>
</tr>
<tr>
<td>Apply the distributive property to solve a given linear equation, e.g., 2(x + 3) = 5;</td>
<td>1</td>
<td>2:PA</td>
<td>13</td>
</tr>
<tr>
<td>2x + 6 = 5; ...</td>
<td>2</td>
<td>4:PA</td>
<td>16, 17, 19, 22</td>
</tr>
<tr>
<td>Solve a given problem using a linear equation and record the process.</td>
<td>1</td>
<td>2:PA</td>
<td>9, 15</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4:PA</td>
<td>19, 22, 24, 25, 29, 30</td>
</tr>
</tbody>
</table>
### Shape and Space

#### General Outcomes
- Measurement: Use direct or indirect measurement to solve problems.
- 3-D Objects and 2-D Shapes: Describe the characteristics of 3-D objects and 2-D shapes, and analyze the relationships among them.
- Transformations: Describe and analyze position and motion of objects and shapes.

#### Measurement

It is expected that students will:

1. **WNCP CURRICULUM**
<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Develop and apply the Pythagorean theorem to solve problems. [CN, PS, R, T, V]</td>
<td></td>
</tr>
</tbody>
</table>

   **Achievement Indicators**
   - Model and explain the Pythagorean theorem concretely, pictorially or using technology.
     - Part 1, Unit 5:G, Lesson 3–5
   - Explain, using examples, that the Pythagorean theorem applies only to right triangles.
     - Part 1, Unit 5:G, Lesson 5
   - Determine whether or not a given triangle is a right triangle by applying the Pythagorean theorem.
     - Part 1, Unit 5:G, Lesson 5
   - Determine the measure of the third side of a right triangle, given the measures of the other two sides, to solve a given problem.
     - Part 1, Unit 5:G, Lesson 4, 5, 6, 7
     - Part 1, Unit 6:ME, Lesson 5, 8
   - Solve a given problem that involves Pythagorean triples, e.g., 3, 4, 5 or 5, 12, 13.
     - Part 1, Unit 5:G, Lesson 7

2. **WNCP CURRICULUM**
<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Draw and construct nets for 3-D objects. [C, CN, PS, V]</td>
<td></td>
</tr>
</tbody>
</table>

   **Achievement Indicators**
   - Match a given net to the 3-D object it represents.
     - Part 2, Unit 6:ME, Lessons 9, 10
   - Construct a 3-D object from a given net.
     - Part 2, Unit 6:ME, Lesson 10
   - Draw nets for a given right circular cylinder, right rectangular prism and right triangular prism, and verify by constructing the 3-D objects from the nets.
     - Part 2, Unit 6:ME, Lessons 10, 17
   - Predict 3-D objects that can be created from a given net and verify the prediction.
     - Part 2, Unit 6:ME, Lesson 10
3. **WNCP CURRICULUM**

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determine the surface area of:</td>
<td></td>
</tr>
<tr>
<td>• right rectangular prisms</td>
<td></td>
</tr>
<tr>
<td>• right triangular prisms</td>
<td></td>
</tr>
<tr>
<td>• right cylinders</td>
<td></td>
</tr>
<tr>
<td>to solve problems. [C, CN, PS, R, V]</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explain, using examples, the relationship between the area of 2-D shapes and the surface area of a given 3-D object.</td>
<td>2</td>
<td>6:ME</td>
<td>16, 17</td>
</tr>
<tr>
<td>Identify all the faces of a given prism, including right rectangular and right triangular prisms.</td>
<td>2</td>
<td>6:ME</td>
<td>10, 16</td>
</tr>
<tr>
<td>Describe and apply strategies for determining the surface area of a given right rectangular or right triangular prism.</td>
<td>2</td>
<td>6:ME</td>
<td>16, 18</td>
</tr>
<tr>
<td>Describe and apply strategies for determining the surface area of a given right cylinder.</td>
<td>1</td>
<td>6:ME</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6:ME</td>
<td>17</td>
</tr>
<tr>
<td>Solve a given problem involving surface area.</td>
<td>2</td>
<td>6:ME</td>
<td>16–18</td>
</tr>
</tbody>
</table>

4. **WNCP CURRICULUM**

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Develop and apply formulas for determining the volume of right prisms and right cylinders. [C, CN, PS, R, V]</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determine the volume of a given right prism, given the area of the base.</td>
<td>2</td>
<td>6:ME</td>
<td>11, 12</td>
</tr>
<tr>
<td>Generalize and apply a rule for determining the volume of right cylinders.</td>
<td>1</td>
<td>6:ME</td>
<td>Z</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6:ME</td>
<td>13</td>
</tr>
<tr>
<td>Explain the connection between the area of the base of a given right 3-D object and the formula for the volume of the object.</td>
<td>2</td>
<td>6:ME</td>
<td>11, 12, 13</td>
</tr>
<tr>
<td>Demonstrate that the orientation of a given 3-D object does not affect its volume.</td>
<td>2</td>
<td>6:ME</td>
<td>11</td>
</tr>
<tr>
<td>Apply a formula to solve a given problem involving the volume of a right cylinder or a right prism.</td>
<td>2</td>
<td>6:ME</td>
<td>11–16, 18</td>
</tr>
</tbody>
</table>
3-D Objects and 2-D Shapes
It is expected that students will:

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Draw and interpret top, front and side views of 3-D objects composed of right rectangular prisms. [C, CN, R, T, V]</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Achievement Indicators</th>
<th>Part</th>
<th>Unit</th>
<th>Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Draw and label the top, front and side views for a given 3-D object on isometric dot paper.</td>
<td>2</td>
<td>6:ME</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>7:G</td>
<td>42, 43</td>
</tr>
<tr>
<td>Compare different views of a given 3-D object to the object.</td>
<td>2</td>
<td>7:G</td>
<td>43–45</td>
</tr>
<tr>
<td>Predict the top, front and side views that will result from a described rotation (limited to multiples of 90 degrees) and verify predictions.</td>
<td>2</td>
<td>7:G</td>
<td>46</td>
</tr>
<tr>
<td>Draw and label the top, front and side views that result from a given rotation (limited to multiples of 90 degrees).</td>
<td>2</td>
<td>7:G</td>
<td>46</td>
</tr>
<tr>
<td>Build a 3-D block object, given the top, front and side views, with or without the use of technology.</td>
<td>2</td>
<td>7:G</td>
<td>42, 45</td>
</tr>
<tr>
<td>Sketch and label the top, front and side views of a 3-D object in the environment with or without the use of technology.</td>
<td>2</td>
<td>7:G</td>
<td>44</td>
</tr>
</tbody>
</table>
Transformations

It is expected that students will:

6. WNCP CURRICULUM

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demonstrate an understanding of tessellation by:</td>
<td></td>
</tr>
<tr>
<td>• explaining the properties of shapes that make tessellating possible</td>
<td></td>
</tr>
<tr>
<td>• creating tessellations</td>
<td></td>
</tr>
<tr>
<td>• identifying tessellations in the environment. [C, CN, PS, T, V]</td>
<td></td>
</tr>
<tr>
<td><strong>Achievement Indicators</strong></td>
<td>Part</td>
</tr>
<tr>
<td>Identify, in a given set of regular polygons, those shapes and combinations of</td>
<td>2</td>
</tr>
<tr>
<td>shapes that will tessellate, and use angle measurements to justify choices, e.g.,</td>
<td></td>
</tr>
<tr>
<td>squares, regular n-gons.</td>
<td></td>
</tr>
<tr>
<td>Identify, in a given set of irregular polygons, those shapes and combinations of</td>
<td>2</td>
</tr>
<tr>
<td>shapes that will tessellate, and use angle measurements to justify choices.</td>
<td></td>
</tr>
<tr>
<td>Identify a translation, reflection or rotation in a given tessellation.</td>
<td>2</td>
</tr>
<tr>
<td>Identify a combination of transformations in a given tessellation.</td>
<td>2</td>
</tr>
<tr>
<td>Create a tessellation using one or more 2-D shapes, and describe the</td>
<td>2</td>
</tr>
<tr>
<td>tessellation in terms of transformations and conservation of area.</td>
<td></td>
</tr>
<tr>
<td>Create a new tessellating shape (polygon or non-polygon) by transforming a</td>
<td>2</td>
</tr>
<tr>
<td>portion of a given tessellating polygon, e.g., one by M. C. Escher, and describe</td>
<td></td>
</tr>
<tr>
<td>the resulting tessellation in terms of transformations and conservation of area.</td>
<td></td>
</tr>
<tr>
<td>Identify and describe tessellations in the environment.</td>
<td>2</td>
</tr>
</tbody>
</table>
Statistics and Probability

General Outcomes
- Data Analysis: Collect, display and analyze data to solve problems.
- Chance and Uncertainty: Use experimental or theoretical probabilities to represent and solve problems involving uncertainty.

Data Analysis
It is expected that students will:

<table>
<thead>
<tr>
<th>Specific Outcome</th>
<th>WNCP CURRICULUM</th>
<th>JUMP MATH LESSONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Critique ways in which data is presented. [C, R, T, V]</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Achievement Indicators</strong></td>
<td><strong>Part</strong></td>
<td><strong>Unit</strong></td>
</tr>
<tr>
<td>Compare the information that is provided for the same data set by a given set of graphs, including circle graphs, line graphs, bar graphs, double bar graphs and pictographs, to determine the strengths and limitations of each graph.</td>
<td>1</td>
<td>4:PDM</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2:PDM</td>
</tr>
<tr>
<td>Identify the advantages and disadvantages of different graphs, including circle graphs, line graphs, bar graphs, double bar graphs and pictographs, in representing a specific given set of data.</td>
<td>1</td>
<td>4:PDM</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2:PDM</td>
</tr>
<tr>
<td>Justify the choice of a graphical representation for a given situation and its corresponding data set.</td>
<td>1</td>
<td>4:PDM</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2:PDM</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8:PDM</td>
</tr>
<tr>
<td>Explain how the format of a given graph, such as the size of the intervals, the width of bars and the visual representation, may lead to misinterpretation of the data.</td>
<td>1</td>
<td>4:PDM</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2:PDM</td>
</tr>
<tr>
<td>Explain how a given formatting choice could misrepresent the data.</td>
<td>1</td>
<td>4:PDM</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2:PDM</td>
</tr>
<tr>
<td>Identify conclusions that are inconsistent with a given data set or graph and explain the misinterpretation.</td>
<td>2</td>
<td>2:PDM</td>
</tr>
</tbody>
</table>
**Chance and Uncertainty**

It is expected that students will:

2. | **WNCP CURRICULUM** | **JUMP MATH LESSONS** |
---|----------------------|-----------------------|
**Specific Outcome** | Solve problems involving the probability of independent events. [C, CN, PS, T] |
**Achievement Indicators** | Part | Unit | Lesson |
---|---|---|---|
Determine the probability of two given independent events and verify the probability using a different strategy. | 2 | 8:PDM | 22, 23 |
Generalize and apply a rule for determining the probability of independent events. | 2 | 8:PDM | 22 |
Solve a given problem that involves determining the probability of independent events. | 2 | 8:PDM | 23, 24 |