PS8-10  Using Logical Reasoning III

Teach this lesson after: 8.2 Unit 6

Goals:
Students will solve problems involving workers who work at different, but constant, rates.

Prior Knowledge Required:
Can add fractions
Can divide whole numbers by a fraction
Can solve worker problems where everyone works at the same rate
Can calculate a percentage of a fraction (for Problem Bank 5)
Can multiply fractions (for Problem Bank 5)
Can convert among percentages, fractions, and decimals (for Problem Banks 5, 6)
Can multiply decimals (for Problem Bank 6)
Can subtract fractions (for Extended Problem)
Can calculate the area of a circle (for Extended Problem)
Can calculate the volume of a cylinder (for Extended Problem)
Can convert between m³ and cm³ (for Extended Problem)

Vocabulary: rate

Materials:
BLM Pool Maintenance (pp. R-62–65, see Extended Problem)

NOTE: Students should complete Lesson PS8-4 before starting this lesson.

Determining the time to finish a job given the fraction completed in a unit time period. Write on the board:

Nora painted a wall. After one hour, she was done \( \frac{1}{3} \) of the wall.

How long would it take her to paint the whole wall?

Read the problem aloud and have a volunteer tell you the answer. (3 hours) Repeat by erasing 1/3 and replacing with 1/2 (2 hours), 1/4 (4 hours), and 1/10 (10 hours). Next, replace the fraction with 2/3 and SAY: This problem is a bit harder, but I think you can still do it. To guide students, write on the board:

After one hour, Nora was done \( \frac{1}{3} \) of the wall.

After one hour, Amy was done \( \frac{2}{3} \) of the wall.

ASK: Who is working faster, Nora or Amy? (Amy) How much faster? (Amy is working twice as fast) SAY: If Amy can get twice as much done in one hour as Nora can, it would take her half as long to finish the wall. ASK: How long does it take Nora to finish the wall? (3 hours) How long does it take Amy? (half as long as 3 hours, so 3/2 hours or 1 1/2 hours)
Exercises: How many hours would it take the person to paint the whole wall, working alone?
a) Lewis can paint \( \frac{1}{5} \) of the wall in one hour.
b) Avril can paint \( \frac{2}{5} \) of the wall in one hour.
c) Marko can paint \( \frac{3}{5} \) of the wall in one hour.

Answers: a) 5 hours, b) 5/2 hours or 2 1/2 hours, c) 3/5 hours or 1 2/3 hours

SAY: You can answer the same type of question given how much is done in any unit of time.

Exercises: How long would it take to complete the task?
a) Jin walks \( \frac{1}{9} \) of the distance from home to school in one minute.
b) Sally walks \( \frac{2}{9} \) of the distance from home to school in one minute.
c) Jin runs \( \frac{1}{15} \) of the race in one minute.
d) Sally runs \( \frac{4}{15} \) of the race in one minute.

Answers: a) 9 minutes, b) 9/2 minutes or 4 1/2 minutes, c) 15 minutes, d) 15/4 minutes or 3 3/4 minutes

To summarize some of these answers, write on the board:

\( \frac{1}{5} \) of the wall in one hour, so five hours to finish the wall
\( \frac{2}{5} \) of the wall in one hour, so \( \frac{5}{2} \) hours to finish the wall
\( \frac{4}{15} \) of the race in one minute, so \( \frac{15}{4} \) minutes to finish the race

ASK: Do you see a shortcut to find the amount of time required to finish the job? (look at the fraction of the job completed in one hour or one minute and turn the fraction upside down)

Exercises: How long would it take to complete the task?
a) Randi completes \( \frac{3}{8} \) of the multiplication questions in one minute.
b) Edmond completes \( \frac{4}{7} \) of his homework in one hour.
c) Randi watched \( \frac{5}{9} \) of the movie in one hour.
d) Edmond walked \( \frac{2}{11} \) of the distance in one minute.

Answers: a) 8/3 minutes or 2 2/3 minutes, b) 7/4 hours or 1 3/4 hours, c) 9/5 hours or 1 4/5 hours, d) 11/2 minutes or 5 1/2 minutes
ASK: When you turn the fraction upside down, what operation does that remind you of? (division, or, more precisely, dividing 1 by the fraction) SAY: To get the amount of time you need to paint the whole wall, you are really dividing 1 by the fraction of the wall that is painted in one hour. Let’s see why this works. Remember that the fraction of the wall painted and the amount of time taken are proportional. Suppose that you can paint half of the wall in one hour. Write on the board:

\[
\begin{align*}
\frac{1}{2} & \text{ of wall painted : } 1 \text{ hour} \\
1 \text{ whole wall painted : } & ?
\end{align*}
\]

SAY: Whatever you multiply 1/2 by to get 1, you have to multiply 1 by to get how much time it takes to paint the whole wall. Show this on the board:

\[
\begin{align*}
\frac{1}{2} & \text{ of wall painted : } 1 \text{ hour} \\
1 \text{ whole wall painted : } & ? \quad \frac{1}{2} \times ? = 1 \text{ so } 1 \div \frac{1}{2} = ?
\end{align*}
\]

SAY: All the question marks in the picture have to stand for the same thing, so you have to divide 1 by 1/2 to get how long it takes to paint 1 whole wall. In this case, 1 divided by 1/2 is 2, so you can paint the whole wall in two hours.

**Problems where workers do not work at the same rate.** Write on the board:

Tess can paint a wall in three hours, working alone.
Ethan can paint the same wall in six hours, working alone.
If they work together, how long will it take them to paint the wall?

SAY: Tess and Ethan don’t work at the same rate. ASK: Will two people take more time or less time than one person working alone? (less time) Draw on the board:

While pointing to their names, SAY: Tess is going to start on this side and Ethan is going to start on that side. ASK: How much of the wall will Tess have done after one hour? (1/3) How do you know? (she takes three hours to do the whole wall, and in one hour, she can do 1/3 as much) How much of the wall will Ethan have done after one hour? (1/6) How do you know? (he takes six hours to do the whole wall, and in one hour, he can do 1/6 as much)
Exercises: How much of the wall does each person paint in one hour?

a) Billy can paint the wall in four hours
b) Lynn can paint the wall in seven hours

Answers: a) 1/4, b) 1/7

SAY: Once you know how much each person paints in one hour, you can find out what fraction of the wall they can paint together in one hour. Colour on the board how much each person will have done after one hour, as shown below:

ASK: What fraction of the wall is painted altogether? (1/2) How did you get that? (added 1/3 and 1/6) SAY: To get the total amount painted, add the amounts they each painted. Write on the board:

\[
\frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \frac{5}{6} = \frac{1}{2}
\]

Exercises: Find the total fraction of the wall painted in one hour.

a) Billy painted \(\frac{3}{10}\) of the wall and Lynn painted \(\frac{1}{5}\) of the wall.

b) Billy painted \(\frac{1}{8}\) of the wall and Lynn painted \(\frac{1}{4}\) of the wall.

c) Billy painted \(\frac{1}{5}\) of the wall and Lynn painted \(\frac{2}{15}\) of the wall.

Answers: a) 1/2, b) 3/8, c) 1/3

SAY: Remember, you can figure out the amount someone can paint in one hour if you know how long it would take them to paint the whole wall by themselves.
Exercises: Find the total fraction of the wall painted in one hour.

a) Cam can paint the wall in two hours, working alone.
   Emma can paint the wall in three hours, working alone.
b) Cam can paint the wall in four hours, working alone.
   Emma can paint the wall in three hours, working alone.
c) Cam can paint the wall in five hours, working alone.
   Emma can paint the wall in two hours, working alone.

Answers: a) 5/6, b) 7/12, c) 7/10

SAY: Once you know the total fraction painted in one hour, you can decide how much time they would need to finish the job. ASK: If two people paint 1/2 of the wall in one hour, how long does it take them to paint the whole wall? (2 hours) What if they painted 1/3 of the wall in one hour, then how long would it take them to paint the whole wall? (3 hours) What if it they painted 2/3 of the wall in one hour, then how long would it take them to paint the whole wall? (3/2 hours or 1 1/2 hours) How did you get that? (it takes half as long to paint the wall as if they painted 1/3 of the wall in 1 hour)

Exercises:
1. Tess can paint a wall in three hours, working alone. Ethan can paint the same wall in five hours, working alone.
   a) How much of the wall will Tess have done after one hour?
   b) How much of the wall will Ethan have done after one hour?
   c) How much of the wall will be done altogether after one hour?
   d) How long will it take them together to paint the wall?

Answers: a) 1/3, b) 1/5, c) 8/15, d) 15/8 hours or 1 7/8 hours

2. Sharon can paint a wall in three hours, working alone, and so can Ren. How long will it take them working together?

Answer: They can each paint 1/3 of the wall in one hour, so working together they can paint 2/3 of the wall in one hour. So, it takes 3/2, or 1 1/2, hours to paint the wall.

3. Look at your answers to Exercises 1 and 2. Does it take Sharon and Ren more or less time than Tess and Ethan? How could you have predicted that?

Answer: less, because Ren is faster than Ethan

SAY: It is always a good idea to go back and check that your answers make sense. ASK: Does it make sense that it takes just under two hours for Tess and Ethan to work together? (yes) Why? (you would expect it to take less than three hours because that is how long it takes Tess alone, but more than 1 1/2 hours because that’s how long it would take if Tess and Ethan both worked at Tess’s pace, but Ethan is slower, so it should take longer)

Exercise: Ella can paint a wall in two hours. Rob can paint the same wall in one hour. How long will it take them, working together, to paint the wall? Check that your answer makes sense.

Answer: In one hour, Ella can paint half the wall and Rob can paint the whole wall by himself, so in one hour, they can do 1 1/2 of the job, or 3/2 of the job. So, they can do the job in 2/3 of an hour. This is less time than it takes Rob by himself, but more time than it would take both of them together, working at Rob’s speed. That makes sense because Ella is slower than Rob.
Problem Bank

1. An engineer can write a program in six days. Another engineer can write the same program in three days. How long would it take if they worked together to write the program?
   **Solution:** The first engineer can write the program in six days. That means he can write 1/6 of the program per day. The second engineer can write the program in three days. That means she can write 1/3 of the program per day. Together they can write 1/6 + 1/3 = 1/2 of the program per day, so it would take two days to complete the program.

2. Mandy can paint a fence by herself in four hours. Kyle can paint the same fence in six hours. They work together for two hours and then Mandy leaves.
   a) How much of the fence will be done after one hour?
   b) How much of the fence will be done after two hours?
   c) How much of the fence will be left for Kyle to do on his own?
   d) How long will it take Kyle to paint the amount of the fence from part c) on his own?
   **Answers:** a) in one hour, Mandy paints 1/4 of the fence and Kyle paints 1/6 of the fence, together they paint 1/4 + 1/6 = 5/12 of the fence; b) in two hours, they paint 10/12, or 5/6, of the fence; c) so when Mandy leaves, Kyle has 1/6 of the fence left to paint; d) it will take him another hour

3. Armand and Jessica work at a factory. Armand is training Jessica to pack bottles. Armand can pack 20 000 bottles in five hours. Jessica can pack 3000 bottles in two hours. How long will it take them to pack 44 000 bottles?
   **Answer:** In one hour, Armand packs 4000 bottles and Jessica packs 1500 bottles, so together they pack 5500 bottles in one hour. So, it will take them 44 000 ÷ 5500 = 8 hours.

4. Tristan can mow the lawn by himself in three hours. Working together, Tristan and Lily mow the lawn in two hours. How long would it take Lily to mow the lawn by herself?
   **Answer:** Tristan can mow 1/3 of the lawn in one hour. In two hours, he completes 2/3 of the lawn. Lily, then, must have done 1/3 of the lawn in two hours. So, she would take six hours to mow the lawn by herself.

5. Jasmin works 20% faster than Arsham. That means that she can do 20% more work than Arsham can do in the same amount of time. Arsham can paint 1/4 of a wall in one hour.
   a) What fraction of the wall is 20% of 1/4 of the wall?
   b) What fraction of the wall can Jasmin paint in one hour?
   c) How long will Arsham and Jasmin take to paint the wall if they work together?
   **Answers:** a) 1/20, b) 3/10, c) 20/11 hours or 1 9/11 hours

6. Zack and Tina are fixing a bike together. Tina works 30% faster than Zack. If they work together, they can do the job in five hours.
   a) How long would it take Zack if he worked alone?
   b) Is your answer more than 10 hours or less than 10 hours? Why does that make sense?
Answer: a) If Zack does $x$ of the job, Tina does $1.3x$, and together they do $2.3x = 1$ whole job. So, in five hours, Zack does $1/2.3 = 10/23$ of the job. In one hour, he does $2/23$ of the job. So, it will take him $23/2$, or $11 \ 1/2$, hours to do the job on his own; b) More than 10 hours, which makes sense because if Zack and Tina worked at the same pace and did the job in five hours, it would take each of them 10 hours working alone. But Zack is slower than Tina, so it should take him longer than 10 hours.

NOTE: In Problem Bank 7, parts a) to e) are guiding questions to help students solve the problem. Some advanced students may be able to solve the problem without guidance.

7. Jayden can paint a fence in 10 hours. Megan can paint the fence in 12 hours. They work for 3 hours and then Alice comes to help. They finish the job 2 hours later. How long would it take Alice to do the job on her own?

a) How much of the job did Megan and Jayden finish in 1 hour?
b) How much of the job did Megan and Jayden finish in 5 hours?
c) How much of the job did Alice do in 2 hours?
d) How much of the job did Alice do in 1 hour?
e) How long would it take Alice to complete the job on her own?

Selected solutions: a) $1/10 + 1/12 = 11/60$, b) $55/60 = 11/12$

Answers: c) $1/12$, d) $1/24$, e) 24 hours
Extended Problem: Pool Maintenance

Materials:
BLM Pool Maintenance (pp. R-62–65)

Extended Problem: Pool Maintenance. This extended problem allows students the opportunity to work with areas of circles and volume of cylinders, converting between units of volume (cm\(^3\) and m\(^3\)), and multiplying decimals. Provide students with BLM Pool Maintenance.

Answers:
1. a) $320, b) $80, c) $62.80, d) $111.64
2. a) 17.584 m\(^3\); b) 100, 1 000 000; c) 17 584 000 cm\(^3\); d) 17 584 L; e) 53.28 days
3. a) 15.7 m\(^2\), b) 1.57 m\(^3\), c) $177.41
Bonus: a) 12 hours, b) 1/4 full, c) 2 1/4 hours or 2 hours 15 minutes
Pool Maintenance (1)

You may use a calculator to complete this task. Use 3.14 for \( \pi \).

1. A store sells pool covers in various sizes and shapes. It decides to make the cost of the cover proportional to the area of the cover. It sells a 4 m by 9 m rectangular cover for $320. How much will the store sell the following covers for? Write your answers to the nearest dollar.
   a) a 6 m by 6 m square cover
   b) a 3 m by 3 m square cover
   c) a circular cover with diameter 3 m
   d) a circular cover with diameter 4 m
Pool Maintenance (2)

2. Simon’s circular pool has a diameter of 4 m and is 1.5 m deep. He fills the pool until the water is 10 cm from the top.
   a) How much water does Simon use in m³?

b) \(1 \text{ m} = \underline{\text{ cm}},\) so \(1 \text{ m}^3 = \underline{\text{ cm}^3}\).

c) How much water does Simon use in cm³?

d) Remember, \(1 \text{ cm}^3 = 1 \text{ mL}\) and \(1 \text{ L} = 1000 \text{ mL}\). How much water does Simon use in L?

e) One estimate suggests that Canadians use an average 330 L of water per day per person. How long would it take for an average Canadian to use as much water as Simon used to fill his pool?
Pool Maintenance (3)

3. Simon wants to put a 1 m wide sidewalk around his circular pool, which has diameter 4 m.
   a) What is the area of the sidewalk in m²?

   b) The sidewalk will be made from concrete 10 cm deep. How much concrete will be needed to make the sidewalk in m³?

   c) Concrete costs $100 per m³, plus a 13% tax. How much will the concrete for the sidewalk cost?
Pool Maintenance (4)

Bonus ► One hose can be used to fill a pool in three hours. Another hose can be used to drain the full pool in four hours.

a) If the hose used to drain the pool is accidentally left on while the owner is trying to fill the pool, how long will it take to fill the pool?

b) After three hours, the owner checks on the pool, expecting it to be full. How full is the pool at the three-hour mark?

c) The owner of the pool turns off the drain hose at the three-hour mark. How much longer will the pool take to fill completely?