Introduction

What is a problem-solving lesson? A JUMP Math problem-solving lesson generally follows the format of a regular JUMP Math lesson, with some important differences:
• There are no AP Book pages that accompany the problem-solving lessons.
• Problem-solving lessons focus on one or more problem-solving strategies rather than focusing on meeting the Common Core State Standards (CCSS). These lessons apply the concepts learned through the standards—often crossing several domains, clusters, or standards—but they are not necessary to complete the standards. Regular lessons, on the other hand, focus on completing the standards and sometimes require problem-solving strategies to do so.
• While regular lessons expose students to all of the problem-solving strategies, the problem-solving lessons provide a way to isolate and focus on the strategies.
• Instead of including extensions, each problem-solving lesson includes an extensive Problem Bank. These questions give students a variety of opportunities to practice the problem-solving strategy from the lesson and to learn new math in the process. Students will need to have mastered the material in the problem-solving lesson (which they do by completing the exercises) in order to tackle the Problem Bank.
• Both the lesson plan and the Problem Bank questions apply the CCSS. All of the standards covered in the lesson are mentioned at the beginning of the lesson plan.
• Some problem-solving lessons include an opportunity for students to complete one or more Performance Tasks. Performance Tasks are multi-part problems intended to determine how well students can apply grade-level CCSS in a new context. While most questions in the task can be done independently of the problem-solving lesson, some questions provide an opportunity to specifically apply the problem-solving strategy. These questions might be challenging for students who have not been taught the problem-solving lesson. Performance Tasks can cover several domains, clusters, or standards at once, all from material covered to date, and so can often be used as a cumulative review.
• While regular lessons cover the standards completely, problem-solving lessons cover clusters of major standards according to the CCSS, and some also cover supporting and additional standards. These lessons provide more challenging independent work while still focusing on the standards.

How do I use problem-solving lessons? Nine problem-solving lessons are provided for Grade 7. The problem-solving lessons can be taught at any point in the grade after the unit indicated in the table that follows. We recommend using as many problem-solving lessons throughout the year as your class time allows, and suggest using them in the order in which they are indicated in the table. However, if required, you can pick and choose based on a careful review of the prior knowledge required for each problem-solving lesson.

We recommend teaching more problem-solving lessons toward the end of the year rather than toward the beginning, as this allows time for students to consolidate their mathematical knowledge and gain confidence before attempting more challenging problems. For this reason,
we recommend using only two of the nine problem-solving lessons during Part 1 of Grade 7. Stronger classes that need fewer bridging lessons for review will have time to finish more of the problem-solving lessons. We recommend that classes needing most of the bridging lessons try at least a few problem-solving lessons.

Some of the Problem Banks include more problems than students can complete in one period. You might wish to use these as extension problems, or have students complete them as problems of the day throughout the year.

**Performance Tasks.** Performance Tasks are included at the end of some problem-solving lessons. Each Performance Task has at least one question that applies the problem-solving strategy covered in the lesson, but most questions can be done independently of the lesson. The Performance Tasks, together with any preparation, require a separate period each. Blackline Masters (BLMs) for each Performance Task are provided at the end of the lesson in which they are cited.

**Problem-solving strategies for Grade 7 and when to use them.** We consider the following problem-solving strategies the most important for this grade level:
- Searching systematically
- Looking for a pattern
- Using logical reasoning
- Guessing, checking, and revising
- Using algebra
- Using structure
- Breaking problems into cases
PS7-5  Using Tape Diagrams and Algebra

Teach this lesson after: 7.2 Unit 3

Standards: 7.RP.A.3, 7.EE.A.1, 7.EE.B.3, 7.EE.B.4

Goals:
Students will use two methods (tape diagrams and algebra) to solve percentage and ratio problems, and they will compare the methods.

Prior Knowledge Required:
Can add and subtract linear expressions
Can solve linear equations
Can draw tape diagrams to represent ratio problems
Can translate between ratios, fractions, and percentages
Can cross multiply to solve proportions
Can add, subtract, multiply, and divide fractions and decimals

Vocabulary: cross multiply, percent, proportion, ratio, tape diagram

Review using tape diagrams to solve “times as many” problems. Write on the board:

Peter made purple paint by mixing red, blue, and white paint.
He used twice as much blue paint as white paint.
He used three times as much red paint as white paint.
If there are 30 cups of purple paint altogether, how much of each color paint did he use?

Have a volunteer demonstrate how to use a tape diagram to show that Peter used twice as much blue paint as white paint, as shown below:

\[
\begin{align*}
\text{white} & \quad \quad \quad \quad \\
\text{blue} & \quad \quad \quad \quad \\
\end{align*}
\]

Leave the tape diagram on the board for students to use as they solve the problem in the following exercise.

Exercise: Complete the problem using a tape diagram.
Answer:

\[
\begin{align*}
\text{white} & \quad \quad \quad \quad \\
\text{blue} & \quad \quad \quad \quad \quad 30 \\
\text{red} & \quad \quad \quad \quad \\
\end{align*}
\]

Peter used 5 cups of white paint, 10 cups of blue paint, and 15 cups of red paint.
Using algebra to solve “times as many” problems. Have a volunteer demonstrate how to use algebra to show that Peter used twice as much blue paint as white paint, as shown below:

Number of cups of white paint = \(x\)
Number of cups of blue paint = 2\(x\)

Leave the volunteer’s work on the board for students to use as they solve the problem in the following exercise.

Exercise: Complete the problem using algebra.

Solution: The number of cups of red paint is 3\(x\), and the total number of cups is 30, so \(x + 2x + 3x = 30, 6x = 30, x = 5\). So Peter used \(x = 5\) cups of white paint, 2\(x = 10\) cups of blue paint, and 3\(x = 15\) cups of red paint.

(MP.1) Comparing using algebra to using tape diagrams to solve “times as many” problems. Ask: What does the \(x\) from the algebra solution represent in the tape diagram solution? (What each block is worth) What else is the same in the two methods? (They both solved “6 times what is 30?”; they got the same answer) To summarize, say: In both solutions, you determined “6 times what is 30?” to figure out what one block is worth, and you then used it to figure out what all the values were.

Review using tape diagrams to solve ratio problems. Write on the board:

There are 10 more boys than girls in a choir. The ratio of girls to boys is 3 : 5. How many girls and how many boys are in the choir?

Have a volunteer draw a tape diagram on the board to represent the situation, as shown below:

```
girls  □□□□□
boys  □□□□□□□□□
```

Ask: What does each block represent? (5) How did you figure that out? (10 \div 2) So, how many girls and how many boys are there? (15 girls and 25 boys)

Using algebra to solve ratio problems. Say: In a tape diagram solution, you can start by figuring out what each block is equal to. You can do that using algebra, too. Write on the board:

If one block is \(x\), then girls = 3\(x\) and boys = 5\(x\)

Say: You can represent any ratio using algebra in this way. If the ratio of girls to boys is 3 to 5, then the number of girls is 3 times something and the number of boys is 5 times the same thing.
**Exercises:** Represent the ratio using variables.

a) The ratio of girls to boys is 4 : 5.
b) The ratio of Yes to No votes is 3 : 7.
c) The ratio of red to yellow to white paint is 3 : 3 : 2.

**Answers:** a) girls: $4x$, boys: $5x$; b) Yes votes: $3x$, No votes: $7x$; c) red: $3x$, yellow: $3x$, white: $2x$

SAY: Let’s go back to the problem with girls and boys that you solved with tape diagrams and solve it again using algebra. If the ratio of girls to boys is 3 to 5, then you can say there are $3x$ girls and $5x$ boys. We know that there are 10 more boys than girls in the choir. ASK: How can you write an equation for that using $x$? ($3x + 10 = 5x$) Have a volunteer solve the equation on the board, as shown below:

\[
3x + 10 = 5x \\
10 = 5x - 3x \\
10 = 2x \\
5 = x
\]

SAY: Now you have to remember to substitute 5 for $x$ to get the number of girls and boys, just like with the tape diagrams. Write on the board:

- girls: $3x = 3 \times 5 = 15$
- boys: $5x = 5 \times 5 = 25$

**Exercises:** Solve the problem first using a tape diagram and then using algebra. Make sure you get the same answer both ways.

a) The ratio of girls to boys in a class is 4 : 7. There are 9 more boys in the class than girls. How many girls and boys are in the class?
b) The ratio of red paint to yellow paint is 3 : 4. There is twice as much yellow paint as white paint. If 54 cups of paint were used altogether, how much of each color of paint was used?
c) Yu was running for class president. When she surveyed her classmates to see who would vote for her, the ratio of Yes to No was 3 : 4. After all the candidates gave a speech to the class explaining what they would do, Yu managed to keep all the Yes voters and change half of the No voters to Yes voters. What is the new ratio of Yes to No votes?
d) The ratio of the number of students in School A to School B was 5 : 8. Then half the students moved from School B to School A. What is the new ratio of students in the two schools?

**Answers:** a) 12 girls and 21 boys; b) 18 cups red, 24 cups yellow, 12 cups white; c) 5 : 2; d) 9 : 4

**Choosing between using tape diagrams and using algebra to solve ratio problems.** Write on the board:

The ratio of boys to girls in a class was 3 : 4. When 9 more boys joined the class, the ratio of boys to girls became 6 : 5. How many girls and boys are in the class now?
Read the problem aloud, then draw on the board:

![Tape Diagram](image)

**(MP.5)** ASK: Which tape diagram is correct? (the first one) What is wrong with the second one? (the number of girls should be the same before and after the boys joined) Erase the second diagram. Point to the remaining tape diagram and tell students that this diagram is tricky to use because the white and gray blocks don’t represent the same amount. ASK: Do the white blocks represent more or do the gray blocks represent more? (the white blocks) How do you know? (they are longer) How can I divide the white blocks and the gray blocks into smaller pieces so that there are the same number of each? (divide the white blocks into 5 pieces each and the gray blocks into 4 pieces each) PROMPT: Since the number of girls is the same before and after the change, 5 gray blocks equal 4 white blocks.

**Exercises:** Use the tape diagram to solve this problem. Use the parts below to guide you.

a) How many gray blocks are there for the boys if there are 20 gray blocks for the girls?
b) How many white blocks are there for the boys if there are 20 white blocks for the girls?
c) How many more gray blocks than white blocks are there for the boys?
d) What does each extra gray block represent?
e) How many girls and boys are now in the class?

**Answers:** a) 24, b) 15, c) 9, d) 1, e) 24 boys and 20 girls

SAY: This method worked but it was a bit tedious. Let’s try using algebra instead. The ratio of boys to girls was 3 : 4. ASK: How can we write that using algebra? (boys: 3x, girls: 4x) Write on the board:

- boys: 3x
- girls: 4x

SAY: There were 3x boys before 9 more joined. ASK: Now how many boys are there? (3x + 9) And how many girls are there? (still 4x) Write on the board:

- new number of boys: 3x + 9
- new number of girls: 4x

ASK: What do we know about the ratio of these new numbers? (the new ratio is 6 : 5) Is that the girls to boys ratio or the boys to girls ratio? (boys to girls) Write on the board:

```
boys : girls
3x + 9 : 4x
```

```
boys : girls
6 : 5
```
SAY: We have two equivalent ratios here. Remember that you can write a ratio using fraction notation. Write on the board:

\[
\frac{3x + 9}{4x} = \frac{6}{5}
\]

ASK: How can you solve this proportion? (cross multiply) Continue writing on the board:

\[(3x + 9)(5) = 6(4x)\]

Have volunteers help you solve for \(x\), as shown below:

\[
15x + 45 = 24x \\
45 = 9x \\
5 = x
\]

SAY: Now that you have \(x\), you have to go back to the original problem and answer the question. Refer students back to the original problem on the board. ASK: What are we being asked to do? (decide how many boys and girls there are now) What is the expression for the new number of boys? (\(3x + 9\)) Write on the board:

\[
3x + 9 = 3(5) + 9 \\
= 15 + 9 \\
= 24
\]

ASK: What is the expression for the new number of girls? (\(4x\)) So how many girls are there now? (20) SAY: So there are 24 boys and 20 girls. ASK: Is the ratio 6 : 5? (yes)

(MP.4) Exercises: Use algebra to solve the problem.

a) The ratio of boys to girls in a class was 3 : 4. When 1 more boy and 4 more girls joined the class, the ratio became 2 : 3. How many girls and boys are in the class now?

b) The ratio of boys to girls in a class was 2 : 3. When 2 more boys joined, the ratio became 5 : 6. How many girls and how many boys were there at first?

c) The ratio of boys to girls in a class was 5 : 4. When 3 more boys and 3 more girls joined, the ratio became 6 : 5. How many girls and boys are in the class now?

d) The ratio of boys to girls in a class is 7 : 8. Half the girls in the class are away for the day to play a basketball game. Now there are 6 more boys in the class than girls. How many boys and girls are normally in the class?

**Answers:** a) 16 boys and 24 girls, b) 8 boys and 12 girls, c) 18 boys and 15 girls, d) 14 boys and 16 girls

SAY: Remember, when you are given the question in terms of fractions or percentages, you can change it to a ratio statement.
**Problem Bank**
1. Solve the problems in two ways, using algebra and using tape diagrams.

   a) The ratio of red paint to blue paint is 3 : 5. There are 10 more cups of blue paint than red paint. How many cups of paint are there altogether?

   b) The ratio of red paint to blue paint is 5 : 6. Cam added red paint so that there was 60% more red paint than there was originally. What is the new ratio of red paint to blue paint?

   c) The ratio of red paint to blue paint is 3 : 4. When Jen poured 6 cups of blue paint into the mix, the ratio became 1 : 2. How much paint is there altogether?

   d) The ratio of students in two Grade 7 classrooms at a school was 3 : 5. Then one sixth of the students in the larger classroom moved to the smaller classroom. What is the new ratio of students in each classroom?

**Solutions:**

   a) Using algebra: $3x + 10 = 5x + 2x$, so $x = 5$. There are $8x = 40$ cups of paint altogether. Using a tape diagram: Draw the diagram so that red paint has 3 blocks and blue paint has 5 blocks; each block in the tape diagram must be worth 5 cups, since the two extra blocks are worth 10 cups. There are 8 blocks, so there are 40 cups altogether.

   b) Using algebra: Red paint is $5x$ cups and blue paint is $6x$ cups. Since red paint increases by 60% and 60% of $5x$ is $3x$, there are now $5x + 3x = 8x$ cups of red paint. There are still $6x$ cups of blue paint, so the new ratio is $8x : 6x = 4 : 3$.

   Using a tape diagram:

   - **red**
     
     [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]

   - **blue**
     
     [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]

   The new ratio is $8 : 6$ or $4 : 3$.

   c) Using algebra: There are $3x$ cups of red paint and $4x$ cups of blue paint; when 6 cups of blue paint are added, there are now $4x + 6$ cups of blue and still $3x$ cups of red. Now, $3x : (4x + 6) = 1 : 2$, so $3x/(4x + 6) = 1/2$. By cross multiplying, $6x = 4x + 6$, so $2x = 6$ and $x = 3$. There are 9 cups of red paint and now 18 cups of blue paint, which is 27 cups altogether. Using a tape diagram:

   - **red**
     
     [ ] [ ] [ ] [ ] [ ] [ ]

   - **blue**
     
     [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]

   Each square represents 3 cups, so there are 9 cups of red paint and 18 cups of blue paint.

   d) Using algebra: The smaller classroom has $3x$ students and the larger classroom has $5x$ students. When one sixth of the $5x$ students move to the smaller classroom, the smaller classroom now has $3x + 5x/6 = 23x/6$ students, and the larger classroom has $5x - 5x/6 = 25x/6$ students, so now the ratio is $23x/6 : 25x/6 = 23 : 25$. 

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(MP.5) **Exercises:** Solve the fraction and percentage problems. Did you use algebra or a tape diagram?

a) There are 25% more boys than girls in a class. When 3 girls are away, there are 40% more boys than girls in the class. How many boys and how many girls are usually in the class?

b) Three fifths of the students in the class are girls. Then 4 more girls join the class. Now two thirds of the students in the class are girls. How many girls and boys are in the class now?

**Answers:** a) 35 boys and 28 girls, algebra; b) 16 girls and 8 boys, algebra
Using a tape diagram: Start with a tape diagram in the ratio 3 : 5; then split each block into sixths:

Moving one sixth of the larger classroom to the smaller classroom moves 5 small pieces, which makes the larger classroom have 25 equal parts and the smaller classroom have 23 equal parts.

(MP.5) 2. For each part in Problem Bank 1, did you prefer using algebra or tape diagrams, or did you have no preference? Explain.

**Answers:** Answers will vary. In parts a), b), and c), both methods are equally easy, but some students might prefer the visual that the tape diagram provides. In part d), students are likely to find the tape diagram method rather tedious.

(MP.1, MP.5) 3. Solve the problems in two ways, using algebra and using tape diagrams. Which method did you like better? Why?

a) Mary has some money saved. She spends \( \frac{2}{5} \) of it on clothes. She spends a third of the rest on movies. She has $24 left. How much money did she have at first?

b) Tony has some money. He spends \( \frac{3}{5} \) of it on clothes. He spends \( \frac{1}{4} \) of the rest on movies. He has $24 left. How much money did he have at first?

c) Hanna has some sports cards. \( \frac{1}{4} \) of them are baseball cards and \( \frac{2}{5} \) of the remainder are hockey cards. The rest are basketball cards. If Hanna has 18 basketball cards, how many sports cards does she have in total? Hint: Use the two different tape diagrams shown below:

\[
\begin{array}{ccc}
\text{total} & \text{the remainder} \\
\text{baseball} & \text{hockey} & \text{basketball}
\end{array}
\]

d) A store has some cans of juice. It sells \( \frac{1}{4} \) of them on Friday, \( \frac{3}{5} \) of the remainder on Saturday, and \( \frac{5}{6} \) of the remainder on Sunday. At the end of the day on Sunday, they still have 30 cans of juice left. How many cans of juice did they have to start?

e) Greg got some money for his birthday. He spent \( \frac{1}{4} \) of it on a T-shirt. He spent \( \frac{2}{5} \) of the rest on a book and \( \frac{3}{10} \) of the rest on music. He has $9.45 left. How much money did he get for his birthday?
Selected solution:
c) Using algebra: Let $x$ be the number of sports cards Hanna has in total. $1/4$ of $x$ is the number of baseball cards and $2/5$ of $3/4$ of $x$ is the number of hockey cards. The number of basketball cards is $x - (1/4 \times x) - (2/5 \times 3/4 \times x) = x(1 - 1/4 - 6/20) = 9x/20$. But the number of basketball cards is 18, so $18 = 9x/20$, $x = 20 \times 18 ÷ 9 = 40$, so she has 40 sports cards in total.

Using tape diagrams: Since there are 18 basketball cards, each block in the second tape diagram is worth 6, which means the remainder, including both hockey and basketball cards, is worth 30. That means that, in the first tape diagram, each block is worth 10, which means the total is 40. So, Hanna has 40 sports cards in total.

Answers: a) $60$, b) $80$, d) 600, e) $30$

NOTE: The problems below are taught using algebra in Grade 8. Challenge your students to do them now using tape diagrams.

(MP.7) 4. Four children and five adults went to a fair. Their tickets cost $26. One child and one adult went to the same fair, and their tickets cost $5.50. Here are tape diagrams to represent each situation:

a) How much would 4 children and 4 adults pay?

b) How much does 1 adult ticket cost? How do you know?
c) How much does 1 child ticket cost? How do you know?

Answers: a) $4 \times 5.50 = 22$; b) The cost of 4 children and 4 adults is $22$. The cost for 4 children and 5 adults is $26$. So the cost for 1 adult is $26 - 22 = 4$. c) $5.50 - 4 = 1.50$.

(MP.1, MP.7) 5. Karen bought 8 pears and 5 apples for $3.40. Carlos bought 3 pears and 2 apples for $1.30. How much does each apple and each pear cost? The tape diagram shows what Karen bought:

a) Draw a picture to show what Carlos bought.
b) Draw a picture that shows how much 6 pears and 4 apples cost.
c) Draw a picture that shows how much 2 pears and 1 apple cost.
d) Draw a picture that shows how much 8 pears and 4 apples cost.
e) How much does 1 apple cost? How do you know?
f) How much does 1 pear cost? How do you know?
Answers:

a) $1.30

\[ \text{Answer: } 1.30 \]

b) $2 \times 1.30 = 2.60$

\[ \text{Answer: } 2.60 \]

c) $3.40 - 2.60 = 0.80$

\[ \text{Answer: } 0.80 \]

d) $4 \times 0.80 = 3.20$

\[ \text{Answer: } 3.20 \]

e) 1 apple costs $3.40 - $3.20 = $0.20; since 8 pears and 4 apples cost $3.20 and 8 pears and 5 apples cost $3.40.

\[ \text{Solution: } 1 \text{ apple costs } $3.40 - $3.20 = $0.20; \text{ since 8 pears and 4 apples cost } $3.20 \text{ and 8 pears and 5 apples cost } $3.40. \]

f) 1 pear costs $0.40; compare the cost of 2 apples, which is $0.40, to the cost of 2 apples and 1 pear, which is $0.80.

\[ \text{Solution: } 1 \text{ pear costs } $0.40; \text{ compare the cost of 2 apples, which is } $0.40, \text{ to the cost of 2 apples and 1 pear, which is } $0.80. \]

(MP.1, MP.7) 6. Tessa bought a box of pears and a box of apples. When she got home, she counted 50 pieces of fruit altogether. Then she realized that \( \frac{1}{5} \) of the pears were bad and \( \frac{1}{6} \) of the apples were bad. If 9 pieces of fruit were bad altogether, how many of each type of fruit were good to eat?

Solution:

\[ \text{bad pears } + \text{ good pears } = 50 \]

\[ \text{bad apples } + \text{ good apples } = 50 \]

From the picture, \( ? = 5 \), so there were 25 good apples and 5 bad apples. There were 4 bad pears, so there were 16 good pears. So, there were 25 apples and 16 pears that were good to eat.