Using Proportions to Solve Percentage Problems I

Pages 46–48

Standards: 7.RP.A.3

Goals:
Students will write equivalent statements for proportions by keeping track of the part and the whole, and by solving easy proportions.

Prior Knowledge Required:
Can write equivalent ratios
Can name a ratio from a picture

Vocabulary: comparison fraction, equivalent ratios, multiplier, part-to-whole ratio, percent, proportion, ratio

Materials:
BLM Three Types of Percentage Problems (p. N-79)

Using pictures to review equivalent ratios. Draw on the board:

Have students brainstorm ways of interpreting this picture. SAY: The picture shows four equivalent statements. Write on the board:

\[
\frac{6}{9} \text{ of the circles are shaded.}
\]

\[
\frac{2}{3} \text{ of the circles are shaded.}
\]

\[
6 \text{ is } \frac{2}{3} \text{ of } 9
\]

\[
6 : 9 = 2 : 3
\]

Exercises: Write four equivalent statements for the picture.

a)  

b)  

Answers: a) 6/8 are shaded, 3/4 are shaded, 6 is 3/4 of 8, 6 : 8 = 3 : 4;
b) 8/12 are shaded, 2/3 are shaded, 8 is 2/3 of 12, 8 : 12 = 2 : 3
Writing part-to-whole as a ratio. Tell students that they can write different part-to-whole ratios from the same picture. Draw on the board:

ASK: How many circles are there? (9) How many are shaded? (3) Write on the board:

\[
\text{part} : \text{whole} = 3 : 9
\]

ASK: How many groups are there? (3) How many groups are shaded? (1) Write on the board:

\[
\text{part} : \text{whole} = 1 : 3
\]

Exercises: Write a pair of equivalent ratios for the picture.

a) \[ \text{part} : \text{whole} = 4 : 10 = 2 : 5 \]
b) \[ \text{part} : \text{whole} = 9 : 12 = 3 : 4 \]

Writing part-to-whole as a fraction. Tell students that they can write different fractions of the form \( \frac{\text{part}}{\text{whole}} \) from the same picture. Draw on the board:

ASK: How many circles are there? (12) How many are shaded? (3) Write on the board:

\[
\frac{\text{part}}{\text{whole}} = \frac{3}{12}
\]

Point to the fraction 3/12 and SAY: The comparison fraction is 3/12. ASK: How many groups are there? (4) How many groups are shaded? (1) Write on the board:

\[
\frac{\text{part}}{\text{whole}} = \frac{1}{4}
\]

Point to the fraction 1/4 and SAY: The comparison fraction is 1/4.

Exercises:
1. Write a pair of equivalent fractions for each picture from the previous set of exercises.

Answers: a) part/whole = 4/10 = 2/5, b) part/whole = 9/12 = 3/4
2. Determine the part, the whole, and the comparison fraction. Write an equivalent fraction.

   a) $3$ is $\frac{1}{4}$ of $12$  
   b) $4$ is $\frac{2}{3}$ of $6$  
   c) $6$ is $\frac{3}{5}$ of $10$

**Answers:**

   a) part = $3$, whole = $12$, comparison fraction = $\frac{1}{4}$, part/whole = $3/12 = 1/4$;  
   b) part = $4$, whole = $6$, comparison fraction = $\frac{2}{3}$, part/whole = $4/6 = 2/3$;  
   c) part = $6$, whole = $10$, comparison fraction = $\frac{3}{5}$, part/whole = $6/10 = 3/5$

**Writing ratios with missing parts.** Tell students that in the previous exercises, all four numbers in the equivalent ratios or fractions were given. Usually in questions like those, one number (out of four) is missing. Explain to students that to write a proportion, they have to determine the part, the whole, and what fraction or ratio of the whole the part is, then write them in the correct places. Write on the board:

   $8$ is $\frac{2}{3}$ of what number?

**Exercises:** Determine the part, the whole, and the comparison fraction. Write the proportion, but replace the missing number with a question mark.

   a) $3$ is $\frac{1}{2}$ of what number?  
   b) $4$ is $\frac{1}{3}$ of what number?  
   c) $6$ is $\frac{2}{5}$ of what number?  
   d) What number is $\frac{3}{4}$ of $20$?  
   e) What number is $\frac{4}{5}$ of $20$?  
   f) What number is $\frac{2}{7}$ of $21$?

**Answers:**

   a) part = $3$, whole = $?$, comparison fraction = $1/2$, so $1/2 = 3/?$;  
   b) part = $4$, whole = $?$, comparison fraction = $1/3$, so $1/3 = 4/?$;  
   c) part = $6$, whole = $?$, comparison fraction = $2/5$, so $2/5 = 6/?$;  
   d) part = $?$, whole = $20$, comparison fraction = $3/4$, so $3/4 = ?/20$;  
   e) part = $?$, whole = $20$, comparison fraction = $4/5$, so $4/5 = ?/20$;  
   f) part = $?$, whole = $21$, comparison fraction = $2/7$, so $2/7 = ?/21$

**Changing a verbal proportion problem into a known problem.** Write on the board:

   $12$ is how many fifths of $30$?

Underline “how many fifths” and point out that this is the same as “$?/5$.” SAY: The denominator tells you that the size of the parts is a fifth, and the numerator—the unknown—tells you the
number of fifths. So “12 is how many fifths of 30” is another way of saying “12 is ?/5 of 30.” This is now easy to change to an equivalent ratio. Write on the board:

\[ ? : 5 = 12 : 30 \]

**Exercises:** Write an equivalent ratio for the question. Then write the fraction form.

a) 8 is how many thirds of 12?
b) 21 is how many quarters of 28?
c) 18 is how many tenths of 30?

**Answers:** a) \( ? : 3 = 8 : 12, \frac{?}{3} = 8/12 \); b) \( ? : 4 = 21 : 28, \frac{?}{4} = 21/28 \); c) \( ? : 10 = 18 : 30, \frac{?}{10} = 18/30 \)

**Writing percentage statements in terms of ratios.** Remind students that asking how many hundredths is like asking for “?/100.” **ASK:** What is another name for a fraction with denominator 100? **PROMPT:** What do we use to compare numbers to 100? (a percent) Since students can write fraction statements as equivalent ratios, and a percentage is just a fraction with denominator 100, students can now write percentage statements as equivalent ratios.

**Exercises:** Write the proportion (without solving it). Then write the proportion in terms of fractions. Replace the missing number with a question mark.

a) 19 is how many hundredths of 20?
b) 13 is how many hundredths of 50?
c) 36 is how many hundredths of 60?

**Answers:** a) \( ? : 100 = 19 : 20, \frac{?}{100} = 19/20 \); b) \( ? : 100 = 13 : 50, \frac{?}{100} = 13/50 \); c) \( ? : 100 = 36 : 60, \frac{?}{100} = 36/60 \)

Remind students that a percentage is a hundredth, so asking what is 15% of 40 is asking what is 15 hundredths of 40. If they know how to find a fraction of a whole number, then they know how to find a percentage of a whole number. In questions in which the percentage is unknown, students can write a comparison fraction with denominator 100 and a question mark in the numerator.

**Exercises:** Write the question as a proportion, in ratio form and in fraction form.

a) What is 15% of 40?   b) What is 32% of 50?
c) What is 75% of 48?   d) 24 is 80% of what number?
e) 62 is 25% of what number?   f) 12 is 30% of what number?
g) What percent of 20 is 19?   h) What percent of 24 is 6?

**Answers:** a) \( ? : 40 = 15 : 100, \frac{?}{40} = 15/100 \); b) \( ? : 50 = 32 : 100, \frac{?}{50} = 32/100 \);
c) \( ? : 48 = 75 : 100, \frac{?}{48} = 75/100 \); d) 24 : \( ? = 80 : 100, \frac{24}{?} = 80/100 \); e) 62 : \( ? = 25 : 100, \frac{62}{?} = 25/100 \); f) 12 : \( ? = 30 : 100, \frac{12}{?} = 30/100 \); g) 19 : 20 = \( ? : 100, \frac{19}{20} = \frac{?}{100} \); h) 6 : 24 = \( ? : 100, \frac{6}{24} = \frac{?}{100} \)

Distribute BLM Three Types of Percentage Problems. All three types of questions from the exercises above are summarized on the BLM. Students can use BLM Three Types of Percentage Problems as a reference to help them solve the remaining exercises in this lesson.
Solving proportions. Show students how to solve proportions using equivalent ratios. Use this problem: If 4 bus tickets cost $9, how much would 12 tickets cost?

**Step 1: Make the proportion.** Write a fraction on the board, the top of which is the unknown quantity (in this example, Dollars) and the bottom of which is the other quantity (in this example, Tickets). Write on the board the complete ratio of dollars to bus tickets (9 : 4) in fraction form, then write the incomplete ratio of dollars to bus tickets (? : 12) in fraction form, as shown below:

\[
\frac{\text{Dollars}}{\text{Tickets}} = \frac{9}{4} = \frac{?}{12}
\]

**Step 2: Find the multiplier.** Find the number the first denominator is being multiplied by to get the second denominator (in this example, 3). Write on the board:

\[
\frac{9}{4} \rightarrow \frac{?}{12} \quad \text{Multiplier: } 3
\]

**Step 3: Find the missing number.** Multiply the numerator by that multiplier to find the missing number, as shown below:

\[
\frac{9}{4} \times 3 = \frac{27}{12}
\]

SAY: Since 9/4 = 27/12, 12 tickets cost $27.

Have volunteers complete the first few exercises below, then have students answer the rest on their own.

**Exercises:**

a) If 3 bus tickets cost $4, how much will 15 bus tickets cost?
b) Five bus tickets cost $6. How many can you buy with $30?
c) On a map, 3 cm represents 10 km. How many kilometers do 15 cm represent?
d) Milly gets paid $25 for 3 hours of work. How much would she get paid for working 6 hours?
e) Three centimeters on a map represents 20 km in real life. If a lake is 6 cm long on the map, what is its actual length?
f) There are 2 apples in a bowl for every 3 oranges. If there are 12 oranges, how many apples are there?

**Bonus:** A goalie stopped 18 out of every 19 shots. There were 38 shots. How many goals were scored? Hint: How many did she not stop?

**Answers:** a) $20, b) 25, c) 50 km, d) $50, e) 40 km, f) 8, Bonus: 2

**Extensions**

1. Determine decimals as the value of a percentage.
   a) What percent of 30 is 16.5?
   b) What percent of 18 is 2.7?
   c) What percent of 14 is 2.8?

**Answers:** a) 55%, b) 15%, c) 20%
2. Give word problems involving decimals as the value of a percentage.
   a) A book that costs $18 came to $20.70 after taxes.
      i) How much were the taxes?
      ii) What percent is the tax?
   b) The regular price of a book is $18. The sale price is $12.60.
      i) How much was taken off the regular price?
      ii) What percent was taken off the regular price?
   **Answers:** a) i) $2.70, ii) 15%; b) i) $5.40, ii) 30%

(MP.1, MP.4) 3. Sal buys two T-shirts at full price and gets a 20% discount on the third T-shirt. All three T-shirts have the same original price. After the discount, Sal paid 5% in tax. The total price for all three T-shirts was $35.28. What was the original price of each T-shirt, before taxes?

   **Sample solutions**
   • I worked backwards. The total price after the 5% tax was $35.28, and the tax multiplies the price by 1.05, so before the 5% tax the total price was $35.28 ÷ 1.05 = $33.60. This is 2.8 times the original price of one t-shirt, so I found 33.6 ÷ 2.8 = 12. So the original price of each T-shirt was $12.
   • Let $x$ be the original price of each T-shirt. Before taxes, Sal pays $x + x + x - 0.2x = 2.8x$. After taxes, Sal paid $2.8x + 0.05(2.8x) = 2.8x + 0.14x = 2.94x$. We are told the total price equals $35.28$, so $2.94x = 35.28$, so $x = 35.28 ÷ 2.94 = 12$. So the original price of each T-shirt was $12.

   **NOTE:** Some students might recognize that they can write the price after a 20% discount as $0.8x$ instead of $x - 0.2x$, or the price after a 5% tax by multiplying by 1.05 rather than by adding the 5% and hence write the shorter equation $1.05(2.8x) = 35.28$. These students are looking for and making use of structure (MP.7).

   Redirecting students: Encourage students to start by writing down what they are given and to use a variable for the quantity they are trying to find.

   Whole-class follow-up: Present both solutions above, or have volunteers do so. Emphasize that the numerical solution requires students to notice that the price after a 5% tax is 1.05 times the original price, whereas the algebraic solution can be done without noticing that. The algebraic solution is therefore easier than the numerical solution, as long as you know how to use variables.
RP7-32 Using Proportions to Solve Percentage Problems II

Pages 49–50

Standards: 7.RP.A.3

Goals:
Students will write equivalent statements for proportions by keeping track of the part and the whole, and by solving the proportions.

Prior Knowledge Required:
Can write equivalent ratios
Can solve proportions

Vocabulary: equivalent ratios, lowest terms, multiplier, percent, proportion, ratio

Materials:
calculators
BLM Three Types of Percentage Problems (p. N-79)

NOTE: Students can use BLM Three Types of Percentage Problems as a reference to help them solve the exercises in this lesson.

Review percentage proportions in terms of fractions. Remind students that to write a proportion, they have to determine the part, the whole, and what fraction of the whole is the part. SAY: Suppose that we want to find what percent of 25 is 7. ASK: What is the whole in this question? (25) What is the part? (7) ASK: What is the part-to-whole fraction? (7/25) SAY: There is another way of writing the part-to-whole, which is the missing percent/100 or ?/100, so we can equate the two part-to-whole ratios. Write on the board:

\[
\frac{\text{part}}{\text{whole}} = \frac{7}{25} = \frac{?}{100}
\]

Emphasize that writing each number in the correct place is very important, because writing a number in the wrong place leads to a wrong answer.

Remind students that writing part-to-whole ratios is the first step of solving proportions. In the second step, they have to find the relation between two given numerators or denominators. In this example, students have to find the number the first denominator is being multiplied by to get the second denominator. Write on the board:

\[
\frac{7}{25} \overset{\times 4}{\longrightarrow} \frac{?}{100}
\]
In the third step, students have to multiply the numerator by that multiplier to find the missing number, as shown below:

\[
\begin{align*}
\frac{7}{25} \times 4 & \rightarrow \frac{28}{100} \\
\end{align*}
\]

**Exercises:** Write the proportion in fraction form, then solve the proportion.

a) What percent of 20 is 9?  
  b) What is 50% of 50?  
  c) 9 is what percent of 25?  
  d) 13 is 26% of what number?

**Answers:**

a) \(\frac{9}{20} = \frac{?}{100}\), so 9 is 45% of 20;  
  b) \(\frac{?}{50} = \frac{50}{100}\), so 25 is 50% of 50;  
  c) \(\frac{9}{25} = \frac{?}{100}\), so 9 is 36% of 25;  
  d) \(\frac{13}{?} = \frac{26}{100}\), so 13 is 26% of 50

**MP.1** Solving proportions that need simplifying. Write on the board:

\[
\frac{9}{20} = \frac{?}{100}
\]

Explain to students this proportion is easy to solve because the relationship between the two denominators is obvious. SAY: You can find the second denominator by multiplying the first denominator by 5. You find the multiplier by dividing 100 by 20. Write on the board:

\[
\frac{7}{35} = \frac{?}{100}
\]

SAY: In this proportion, the relation between two denominators is not clear. Ask students to use their calculators to divide 100 by 35 to find the multiplier. ASK: Is the answer a well-known decimal? (no) SAY: Don't give up! Try to reduce 7/35 to lowest terms. Write on the board:

\[
\frac{7}{35} = \frac{1}{5}
\]

SAY: Replace 7/35 by 1/5. Write on the board:

\[
\frac{1}{5} = \frac{?}{100}
\]

ASK: Is this proportion easy to solve? (yes) Ask a volunteer to solve the proportion as shown below:

\[
\begin{align*}
\frac{1}{5} \times 20 & \rightarrow \frac{20}{100} \quad \text{so} \quad \frac{7}{35} = \frac{20}{100}
\end{align*}
\]
Exercises: Find an equivalent ratio to rewrite the proportion. Solve the new proportion.

a) $\frac{?}{100} = \frac{11}{22}$
b) $\frac{6}{24} = \frac{?}{100}$
c) $\frac{12}{?} = \frac{40}{100}$
d) $\frac{?}{100} = \frac{24}{32}$
e) $\frac{75}{48} = \frac{?}{100}$
f) $\frac{12}{?} = \frac{?}{60}$

Answers: a) $\frac{?}{100} = \frac{1}{2}$, so $? = 50$; b) $\frac{1}{4} = \frac{?}{100}$, so $? = 25$; c) $\frac{12}{?} = \frac{2}{5}$, so $? = 30$; d) $\frac{?}{100} = \frac{3}{4}$, so $? = 75$; e) $\frac{?}{48} = \frac{3}{4}$, so $? = 36$; f) $\frac{1}{5} = \frac{?}{100}$, so $? = 20$

Word problem practice.

Exercises: Use your answer to each problem to obtain the answer to the next problem. Discuss the similarities and differences between the problems.

a) 12 is how many fifths of 30?
b) How many fifths of 30 is 12?
c) 12 is what percent of 30?
d) What percent of 30 is 12?
e) A shirt costs $30, and $12 was taken off. What percent was taken off?

Answers: a) 2, b) 2, c) 40, d) 40, e) 40

Selected solutions: a) $\frac{12}{30} = \frac{?}{5}$, so $? = 2$; b) $\frac{?}{5} = \frac{12}{30}$, so $? = 2$; c) $\frac{12}{30} = \frac{?}{100}$, so $? = 40$. The difference between a) and b) is just the order of fractions, but in c) the question asks for percentage so the denominator is 100.

Finding the whole from the part. Write on the board:

$$\frac{2}{3} \text{ of a number is 100. What is the number?}$$

ASK: Is 100 the part or the whole? (the part) What is the whole? (the number that we don’t know) Tell students that this is a part-to-whole ratio. Write on the board:

$$\frac{2}{3} = \frac{100}{?} = \frac{\text{part}}{\text{whole}}$$

Have students solve the proportion. (? = 150)

Exercises: Write the proportion, then find the number.

a) $\frac{3}{4}$ of a number is 9
b) $\frac{4}{9}$ of a number is 24
c) $\frac{7}{13}$ of a number is 21

Answers: a) $\frac{3}{4} = \frac{9}{?}$, so the number is 12; b) $\frac{4}{9} = \frac{24}{?}$, so the number is 54; c) $\frac{7}{13} = \frac{21}{?}$, so the number is 39
More word problem practice. Write on the board:

\[
\frac{2}{3} \text{ of the beads in a box are red.}
\]

100 beads are red.

How many beads are in the box?

**NOTE:** This is the same problem as before, but written a little differently. Use the following sentences and prompts to demonstrate this.

Write on the board:

\[
\frac{2}{3} \text{ of the number of beads in the box is the number of red beads in the box.}
\]

**ASK:** What is the number of red beads in the box? (100) Continue writing on the board:

\[
\frac{2}{3} \text{ of the number of beads in the box is 100.}
\]

Now underline part of that sentence, as shown below:

\[
\frac{2}{3} \text{ of the number of beads in the box is 100.}
\]

Tell students this underlined part is what we want to know. Continue writing on the board:

\[
\frac{2}{3} \text{ of what number is 100?}
\]

This is exactly the problem we solved earlier, so the number is 150.

**Exercises:** A box holds red and blue beads. Find the total number of beads in the box.

- a) \(\frac{2}{3}\) of the beads are red. 8 beads are red.
- b) \(\frac{4}{9}\) of the beads are red. 24 beads are red.
- c) \(\frac{7}{13}\) of the beads are red. 21 beads are red.

**Answers:** a) \(\frac{2}{3} = \frac{8}{?}\), so the total number of beads is 12; b) \(\frac{4}{9} = \frac{24}{?}\), so the total number of beads is 54; c) \(\frac{7}{13} = \frac{21}{?}\), so the total number of beads is 39

Have students compare parts b) and c) of the exercises above with parts b) and c) of the previous set of exercises.
Solving problems with three groups instead of two. Write on the board:

\[
\frac{2}{7} \text{ of the people (boys, girls, and adults) at the park are boys, and } \frac{1}{7} \text{ are adults.}
\]

There are 20 girls at the park. How many people are at the park?

ASK: What fraction of the people are boys or adult? \((2/7 + 1/7 = 3/7)\) ASK: What fraction of the people are girls? \((4/7)\) Challenge students to find a model to help solve the problem. One possible model could be to draw a bar divided into sevenths, then add the given information to it, as shown below:

![Model](image)

Have students use this model, or their own, to solve the problem. SAY: From the picture, it is clear that \(4/7\) of the total number of people is 20. ASK: \(4/7\) of what number is 20? \((35)\) Ask a volunteer to write a proportion and solve it to find the number of people at the park, as shown below.

\[
\frac{4}{7} = \frac{20}{?}, \text{ so } ? = 35
\]

**Exercises:** Draw a model, then solve the problem.

a) 75% of the people at a park are children. There are 24 more children than adults at the park. How many people are at the park? Hint: There are 50% more children than adults at the park.

**Bonus:**

b) There are 5 adults at a park. There are 7 more girls than boys at a park. \(\frac{3}{7}\) of the people at the park are boys. How many people are at the park?

c) There are 18 boys at a park. \(\frac{3}{5}\) of the people (boys, girls, and adults) are girls. There are 6 fewer adults than boys. How many people are at the park?

**Solutions:**

a) 48 people at the park

b) 84 people at the park

c) 75 people at the park

Solving problems with other contexts.

**Exercises:** Draw a model, then solve the problem.

a) Ken and Liz pay for a meal. Ken pays $12. Liz pays \(\frac{3}{5}\) of the total cost. How much did they pay altogether?
b) In a parking lot, \(\frac{3}{4}\) of the vehicles are cars, \(\frac{1}{5}\) are trucks, and the rest are buses. There are 4 buses. How many vehicles are in the parking lot?

c) Jake has a rock collection. He found \(\frac{2}{5}\) of his rocks in Arizona, \(\frac{1}{3}\) in California, and the rest in Alaska. He found 8 more rocks in California than he did in Alaska.
   
i) What fraction of his rocks did he find in Alaska?
   
   ii) What fraction of the total number of rocks do the 8 rocks represent? Hint: 8 is the difference between the number found in California and the number found in Alaska.
   
   iii) How many rocks does Jake have altogether?

d) At a school, \(\frac{3}{7}\) of the people are boys, \(\frac{2}{5}\) are girls, and the rest are adults. There are 6 more students of one gender than the other.
   
i) Are there more boys or more girls?
   
   ii) What fraction of the total number of people do the 6 extra students of one gender represent?
   
   iii) How many people are there in the school altogether?

**Answers:** a) $30; b) 80; c) i) 4/15, ii) 1/15, iii) 15 \times 8 = 120; d) i) 3/7 = 15/35 and 2/5 = 14/35, so there are more boys, ii) 3/7 - 2/5 = 15/35 - 14/35 = 1/35, iii) 6 \times 35 = 210

**Extensions**

(MP.1, MP.7) 1. Find the number.

a) \(\frac{3}{5}\) of \(\frac{4}{5}\) of a number is 60. What is the number?

b) \(\frac{2}{3}\) of \(\frac{3}{4}\) of a number is 36. What is the number?

c) \(\frac{2}{3}\) of \(\frac{3}{4}\) of \(\frac{4}{5}\) of a number is 18. What is the number?

**NOTE:** Some students might find a shortcut way to solve these problems. For example, 2/3 of 3/4 is 2/4 or 1/2, so the question is really asking 1/2 of a number is 36. What is the number?

**Solutions:** a) Solve in two steps: if 3/5 of something is 60, then the something is 100, so 4/5 of a number is 100, this means the number is 125; b) 72; c) Solve in three steps: if 2/3 of (3/4 of 4/5 of a number) is 18, then the part in brackets is 27; 3/4 of 4/5 of a number is 27, so in the bracket is 36, so 4/5 of a number is 36 and the number is 45

2. a) Two thirds of Helen’s age is half of David’s age. David is 10 years older than Helen. How old is Helen?

   b) Tim’s age is two thirds of Sara’s age. Sara’s age is three fifths of Mark’s age. Mark is 9 years older than Tim. How old is Sara?

   **Answers:** a) 30, b) 9

3. Hanna emptied her piggy bank, which only contained pennies. The contents weigh about 1 kg 300 g in total. If each penny weighs \(2\frac{1}{3}\) g, about how much money was in the piggy bank?

   **Answer:** 1,300 \(\div\) 2 1/3 = 1,300 \(\div\) 7/3 = 1,300 \(\times\) 3 \(\div\) 7 \(\approx\) 557 pennies = $5.57, so there is about $5.50

   **NOTE:** To divide by a mixed number, it is essential to change the mixed number to an
improper fraction. We cannot write \( 1,300 \div (2 \frac{1}{3}) = 1,300 \div (2 + \frac{1}{3}) = 1,300 \div 2 + 1,300 \div \frac{1}{3} \), because the distributive law does not apply here. If you try to use the distribute law, you should see right away that doing so gives an answer that can’t be correct: \( 1,300 \div 2 \) is larger than what you started with, \( 1,300 \div (2 \frac{1}{3}) \), because 2 is less than 2 \( \frac{1}{3} \) and dividing by a smaller number gives a larger result.

4. \( \frac{2}{5} \) of the people (boys, girls, adults) at the park are boys. There are 3 more girls than boys. There are 7 adults. How many people are at the park?

**Solution:**

From the model, it is clear that \( \frac{1}{5} \) of the total number of people is 10. \( \frac{1}{5} \) of 50 is 10, so there are 50 people at the park.

**(MP.4)** 5. Tina walks \( \frac{3}{4} \) of a mile in \( \frac{1}{3} \) of an hour. Cam walks \( \frac{5}{4} \) of a mile in \( \frac{1}{2} \) an hour. Who walks faster? How much faster? Use complex fractions to find the answer.

**Answer:**

Tina’s rate of walking is \( \frac{3}{4} \div \frac{1}{3} = \frac{3}{4} \times 3 = \frac{9}{4} = 2 \frac{1}{4} \), so she walks 2 \( \frac{1}{4} \) mph.

Cam’s rate of walking is \( \frac{5}{4} \div \frac{1}{2} = \frac{5}{4} \times 2 = \frac{10}{4} = 2 \frac{1}{2} \), so he walks 2 \( \frac{1}{2} \) mph. Cam walks \( 2 \frac{1}{2} - 2 \frac{1}{4} = \frac{1}{4} \) mph faster than Tina.
RP7-33  Solving Equations (Introduction)

Pages 51–53

Standards: preparation for 7.EE.B.4

Goals:
Students will use the balance model to solve addition and multiplication equations including negative addends and coefficients.

Prior Knowledge Required:
Is familiar with balances
Can solve a simple equation to find an unknown value
Can substitute numbers for unknowns in an expression
Can check whether a number solves an equation

Vocabulary: balance, equation, expression, integer, pan balance, quotient, sides (of an equation), variable

Materials:
pan balance
apples, cubes, or other small objects for demonstration
paper bags, several for display
masking tape
40 connecting cubes for demonstrations

NOTE: You will need a number of identical objects for demonstrations throughout this lesson. The objects you use should be significantly heavier than a paper bag, so that the presence of a paper bag on one of the pans of the balance does not skew the pans. Apples are used in the lesson plan below (to match the pictures in the AP Book), but other objects, like small fruit of equal size, metal spoons, golf balls, tennis balls, or cereal bars, will work well. If a pan balance is not available, refer to a concrete model, such as a seesaw, to explain how a pan balance works, and use pictures or other concrete models during the lesson.

Review pan balances. Show students a pan balance. Place the same number of identical (or nearly identical) apples on both pans, and show that the pans balance. Remind students that when the pans, or scales, are balanced, this means there is the same number of apples on each pan.

Removing the same number of apples from both pans keeps them balanced. Place some apples in a paper bag and place it on one pan, then add some apples beside the bag. Place the same total number of apples on the other pan. ASK: Are the pans balanced? (yes) What does this mean? (the same number of apples are on each pan) Take one apple off each pan. ASK: Are the pans still balanced? Repeat with two apples. Remove the same number of apples from each pan until one pan has only the bag with apples on it. ASK: Are the pans balanced?
Can you tell how many apples are in the bag? Show students the contents of the bag to check their answer. Repeat the exercise with a different number of apples in the bag.

**Solving addition equations given by a balance model.** Make a line on a desk with a piece of masking tape and explain that the parts on either side of the line will be the pans. Ask students to imagine that the pans are balanced. As you did before, place a paper bag with apples in it along with some other apples on one side of the line, and place the same number of apples (altogether) on the other side of the line. Ask students how many apples need to be removed from both sides of the balance to find out how many apples are in the bag. Students can signal their answer. Remove the apples, then ask students to tell how many apples are in the bag. Show the contents of the bag to check the answer. Repeat with a different number of apples.

**Writing equations from scales.** Remind students that we often use variables to represent numbers we do not know. Place a paper bag containing 5 apples and 2 more apples on one side of the line, and place 7 apples on the other side of the line. Ask students to write an expression for the number of apples on the side with the paper bag, as shown below. Explain that an equation is like a pair of balanced pans (or scales), and the equal sign shows that the number of apples in each pan is the same. Remind students that the parts of the equation on either side of the equal sign are called the *sides* of the equation. Each pan of the balance becomes a side in the equation and the “balance” on the desk becomes \( x + 2 = 7 \). Draw on the board:

\[
\begin{array}{c|c}
\text{\includegraphics[width=1cm]{apples}} & \text{\includegraphics[width=1cm]{apples}} \\
\text{\includegraphics[width=1cm]{apples}} & \text{\includegraphics[width=1cm]{apples}} \\
\text{\includegraphics[width=1cm]{apples}} & \text{\includegraphics[width=1cm]{apples}} \\
\text{\includegraphics[width=1cm]{apples}} & \text{\includegraphics[width=1cm]{apples}} \\
\end{array}
\rightarrow \quad x + 2 = 7
\]

The “pans” balance each other. The numbers on both sides are equal.

Create more models and have students write the equation for each one. After you have done a few models that follow this pattern, start placing the bag on different sides of the line, so that students have to write the expressions with unknown numbers on different sides of the equation.

**Solving addition equations using the balance model.** Return to the model that corresponds to the equation \( x + 2 = 7 \). ASK: What do you need to do to find out how many apples are in the bag? (remove two apples from each side) Invite a volunteer to remove the apples, then have students write the old equation and the new equation \( x = 5 \), one below the other. Repeat with a few different examples.

ASK: What mathematical operation describes taking the apples away? (subtraction) Write the subtraction vertically for the equation above, as shown below.

\[
\begin{array}{c}
x + 2 = 7 \\
- 2 - 2
\end{array}
\]
ASK: How many apples are left on the right side of the equation? (5) What letter did we use to represent how many apples are in the bag? (x) Remind students that we write this as “x = 5.”

**Solving addition equations without using the balance model.** Present a few equations without a corresponding model. Have students signal how many apples need to be subtracted from both sides of the equation, then write the vertical subtraction for both sides.

**Exercises:** Solve the equation.

a) \(x + 5 = 9\)  
b) \(n + 17 = 23\)  
c) \(14 + n = 17\)  
d) \(p + 15 = 21\)

**Answers:** a) \(x = 4\), b) \(n = 6\), c) \(n = 3\), d) \(p = 6\)

Students who have trouble deciding how many apples to subtract without the model can complete the following problems.

**Exercises:** Write the missing number. Part a) has been done for you.

a) \(x + 15\)  
b) \(x + 55\)  
c) \(x + 91\)  
Bonus: \(x + 38\)

**Answers:** b) 55, c) 91, Bonus: \(x\)

Finally, give students a few more equations and have them work through the whole process of subtracting the same number from both sides to find the unknown number.

**Exercises:** Solve the equation by subtracting the same number from both sides to find the unknown number.

a) \(x + 5 = 14\)  
b) \(x + 9 = 21\)  
c) \(2 + x = 35\)  
d) \(x + 28 = 54\)

**Sample solution:**

a) \(x + 5 = 14\)

\[
x + 5 \quad -5 \\
\hline
x \quad = 9
\]

**Bonus:** The scale below is balanced. Each bag has the same number of apples in it. How many apples are in the bag? Hint: You can cross out whole bags too!

Answers: b) 12, c) 33, d) 26, Bonus: 2

**Solving multiplication equations given by a model.** Divide a desk into two parts using masking tape and place three bags (with 4 cubes in each) on one side of the line, and 12 separate cubes on the other side of the line. Tell students that the “pans” are balanced.

ASK: What does this say about the number of cubes in both pans? (they are equal) How many cubes are on the pan without the bags? (12) How many cubes are in the bags in
total? (12) How many cubes are in each bag? (4) How do you know? (divide 12 into 3 equal
groups, 12 ÷ 3 = 4) Invite a volunteer to group the 12 cubes into 3 equal groups to check the
answer. Show students the contents of the bags to confirm the answer.

Repeat the exercise with 4 bags and 20 cubes, 5 bags and 10 cubes, 2 bags and 6 cubes.
Students can hold up the right number of fingers to signal the number of cubes in one bag
each time.

**Writing equations from models.** Remind students that the pans of the balance become the
sides of an equation, and that the equal sign in the equation shows that the pans are balanced.
When they have, say, 3 bags with the same number of cubes in each, they write the total
number of cubes in the bags as $3 \times b$. Present a few equations in the form of a model, and have
students write the corresponding equations using the letter $b$ for the unknown number.

**Exercises:** How many are in one bag?

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**Answers:** a) 6, b) 3

Now have students solve the equations by looking at the model. Show them how to write the
solution below the equation and have students record their solutions. Write on the board:

$$2 \times b = 12$$
$$b = 6$$

**Drawing a model to solve the equations.** Tell students that the next task will be the opposite
of what they have been doing: now they will start with an equation and draw a model for it.
Remind students that when we draw pictures in math class, it is important to draw the correct
numbers of objects. Shading, color, and other artistic features or details are not important. Our
drawings in math should be simple and we shouldn’t spend too much time on them.
Demonstrate making a simple drawing of a pan balance, and remind students that they can use
circles or squares or big dots for cubes and boxes for paper bags.

**Exercises:** Draw a model and use it to solve the equation.

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<td>$3 \times b = 15$</td>
<td>$4 \times b = 8$</td>
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<td>$9 \times b = 18$</td>
<td>$8 \times b = 24$</td>
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**Answers:** a) 5, b) 2, c) 2, d) 3

**Using division to find the missing factor.** ASK: Which mathematical operation did you use to
write an equation for each balance? (multiplication) Which mathematical operation did you use
to find the number of apples in each bag? (division) Have students show the division in the
models they have drawn by circling equal groups of dots. For example, in Exercise a) above,
they should circle three equal groups of dots. ASK: What number do you divide by? (the number
of bags)
Multiplying and dividing by the same number does not change the starting number. Write on the board:

\[
\begin{align*}
(5 \times 2) \div 2 & \quad (3 \times 2) \div 2 & \quad (8 \times 2) \div 2 \\
(5 \times 4) \div 4 & \quad (9 \times 3) \div 3 & \quad (10 \times 6) \div 6
\end{align*}
\]

Have students solve each question. (5, 3, 8, 5, 9, 10)

SAY: Look at the questions you solved. ASK: How are they all the same? (they start with a number, multiply by another number, then divide by the same number) Did you get back to the same number you started with? (yes) Does it matter what number you started with? Does it matter what number you multiplied and divided by as long as it was the same number? (no) Have students write their own question of the same type, and check that they get back to the same number they started with.

Explain that you can do the same with unknown numbers. Show one paper bag with some cubes and SAY: I want to multiply this by 3. ASK: What will the answer look like? (3 bags) SAY: I want to divide the result by 3. What will you get? (1 bag again) Write on the board:

\[
\begin{align*}
(\Box \times 3) \div 3 = ___ \\
(b \times 3) \div 3 & \quad (b \times 5) \div 5 & \quad (b \times 6) \div 6 & \quad (b \times 10) \div 10
\end{align*}
\]

Have students solve each question. (3, 5, 6, 10)

**Solving equations by dividing both sides by the same number.** Write on the board:

\[
\begin{align*}
(b \times 7) \div ____ = b & \quad (b \times 2) \div ____ = b & \quad (b \times 4) \div ____ = b \\
(b \times 8) \div ____ = b & \quad (b \times 12) \div ____ = b & \quad (b \times 9) \div ____ = b
\end{align*}
\]

For each equation, have students hold up the correct number of fingers to signal the number they would divide the product by to get back to \(b\). (7, 2, 4, 8, 12, 9)

If a pan balance is available, show students the balance with 3 bags of 5 apples (other objects will work equally well—cereal bars, metal spoons, etc.) on one pan, and 15 apples on the other pan. Invite a volunteer to write the equation for the balance on the board, as shown below:

\[
3 \times 5 = 15
\]

ASK: How many apples are in one bag? (5) Have a volunteer make three groups of five apples on the side without the bags. Point out that there are three equal groups of apples on both sides of the balance. Remove two of the bags from one side, and two of the groups from the other side. SAY: I have replaced three equal groups on each side with only one of these groups.
ASK: What operation have I performed? (division by 3) Are the scales still balanced? (yes) Point out that when you perform the same operation on both sides of the balance, the scales remain balanced. ASK: What does that mean in terms of the equation? Write on the board:

\[ b \times 3 \div 3 = 15 \div 3 \]

Have students calculate the result on both sides. (b on the left side, 5 on the right side) Write on the board (align the equal signs vertically):

\[ b = 5 \]

Demonstrate that the bags indeed contain 5 apples. Repeat with a few more examples. Finally, have students solve equations by dividing both sides of the equation by the same number.

**Exercises:** Solve the equation.

a) \[ b \times 7 = 21 \]  
   b) \[ b \times 2 = 12 \]  
   c) \[ b \times 4 = 20 \]  
   d) \[ b \times 6 = 42 \]  
   e) \[ b \times 3 = 27 \]  
   f) \[ b \times 9 = 72 \]

**Selected solution:**

- a) \[ b \times 7 = 21, b \times \frac{7}{7} = 21 \div 7, b = 3 \]

**Answers:** b) 6, c) 5, d) 7, e) 9, f) 8

**Review adding and subtracting negative numbers.** Remind students that they can subtract a negative number by adding its opposite. Write on the board:

\[ 5 - (-2) = 5 + 2 = 7 \]

SAY: For adding negative numbers, you can subtract the opposite. Write on the board:

\[ 5 + (-2) = 5 - 2 = 3 \]

**Solving equations with integer addends.** Write on the board:

\[ x + (-3) = 5 \]

ASK: If there was no negative sign, how would you solve the equation? (by subtracting both sides by the same number to isolate \( x \)) SAY: Let’s do the same for the equation above and see the result. Write on the board:

\[ x + (-3) - (-3) = 5 - (-3) \]
\[ x = 5 + 3 \]
\[ x = 8 \]

SAY: Let’s check the answer by replacing \( x \) with 8 in the equation. Ask a volunteer to check the answer, as shown below:

\[ x + (-3) = 8 + (-3) = 8 - 3 = 5 \checkmark \]
Say: You can solve equations with negative numbers the same way you solve equations with positive numbers.

**Exercises:** Solve the equation by doing the same thing to both sides of the equation.

a) \( x + (-5) = 7 \)  

b) \( x + 5 = 7 \)  

c) \( x - 5 = 7 \)  

d) \( x - (-5) = 7 \)

e) \( x + 5 = -7 \)  

f) \( x - 5 = -7 \)  

g) \( x - (-5) = -7 \)  

h) \( x + (-5) = -7 \)

**Answers:** a) 12, b) 2, c) 12, d) 2, e) -12, f) -2, g) -12, h) -2

**Review dividing integers.** Remind students that they can divide integers the same way they divide whole numbers. Say: To divide integers, first determine the sign of the quotient, then divide. Write on the board:

\[
\begin{align*}
(+) \div (+) &= + \\
(+) \div (-) &= - \\
(-) \div (+) &= - \\
(-) \div (-) &= +
\end{align*}
\]

**Exercises:** Divide the integers. Determine the sign of the quotient first, then divide as though they are whole numbers.

a) \(-32 \div 4\)  

b) \(25 \div -5\)  

c) \(-24 \div -3\)

**Answers:** a) -8, b) -5, c) 8

**Solving equations with integer coefficients.** Write on the board:

\[3x = -15\]

Ask: If there was no negative sign, how would you solve the equation? (by dividing both sides by the same number, 3) Say: Let’s do the same for the equation above and see the result.

Write on the board:

\[3x \div 3 = -15 \div 3\]

\[x = -5\]

Say: Let’s check the answer by replacing \(x\) with -5 in the equation. Ask: Is 3 times -5 equal to -15? (yes) Students can signal their answer with thumbs up or thumbs down. Say: You can solve equations with negative coefficients the same way you solve equations with positive coefficients. Write on the board:

\[-3x = 15\]

Ask a volunteer to divide both sides by -3 to find the answer. Ask the volunteer to check the answer by replacing \(x\) with -5 in the equation.

**Exercises:** Solve by dividing both sides of the equation by the same number.

a) \(6x = -18\)  

b) \(-6x = 18\)  

c) \(x \times (-6) = 18\)  

d) \(x \times (-6) = -18\)

e) \(-12x = -36\)  

f) \(\frac{1}{2}x = -5\)  

g) \(-\frac{1}{2}x = 5\)  

h) \(-\frac{1}{2}x = -5\)

**Answers:** a) -3, b) -3, c) -3, d) 3, e) 3, f) -10, g) -10, h) 10
Extension

(MP.5) a) Does the model work to solve the equation?
   i) \( x + 3 = 8 \)  
   ii) \( x - 3 = 8 \)  
   iii) \( 3x = 12 \)  
   iv) \( -3x = 12 \)  
   v) \( 0.6x = 1.8 \)

b) Does doing the same thing to both sides work?

Answers: a) i) yes, ii) no, iii) yes, iv) no, v) no; b) i) yes, ii) yes, iii) yes, iv) yes, v) yes
Cross Multiplication (Introduction)

Standards: 7.RP.A.3

Goals:
Students will cross multiply to write an equation for problems involving proportions.

Prior Knowledge Required:
Can convert a fraction \( \frac{a}{b} \) to a decimal by dividing \( a \div b \)
Can find equivalent fractions by multiplying the numerator and denominator by the same number
Can write an equivalent multiplication statement for a given division statement

Vocabulary: canceling, commutative property, complex fraction, cross multiply, equivalent fractions

Materials:
calculators

Review writing a fraction as a division statement. Remind students that we can calculate the value of a fraction such as \( \frac{3}{4} \) by dividing \( 3 \div 4 \). For a quick reminder of why this is true,
SAY: To find 1/4 of something, I would divide it into four equal groups. So to find 1/4 of something, divide it by 4. You can think of 1/4 as 1/4 of 1, so that is 1 ÷ 4. But 3/4 is three times as much as 1/4, so 3/4 is \( 3 \times 1 ÷ 4 = 3 ÷ 4 \).

Exercises: Write as a division statement and use a calculator to find the answer.

\[
a) \quad \frac{3}{5} \quad \text{b) } \frac{5}{8} \quad \text{c) } \frac{7}{20} \quad \text{d) } \frac{3}{10} \quad \text{e) } \frac{8}{25}
\]

Answers: a) \( 3 \div 5 = 0.6 \), b) \( 5 \div 8 = 0.625 \), c) \( 7 \div 20 = 0.35 \), d) \( 3 \div 10 = 0.3 \), e) \( 8 \div 25 = 0.32 \)

Writing fraction statements as equivalent multiplication statements. Remind students that a division statement can be written as a multiplication statement. For example, \( 12 \div 3 = 4 \) can be rewritten as \( 12 = 3 \times 4 \).

Exercises: Change the division statements in the previous set of exercises to multiplication statements.

Answers: a) \( 3 = 5 \times 0.6 \), b) \( 5 = 8 \times 0.625 \), c) \( 7 = 20 \times 0.35 \), d) \( 3 = 10 \times 0.3 \), e) \( 8 = 25 \times 0.32 \)

To guide students in the following exercises, write this template on the board:

\[
\text{_____ ÷ _____ = _____} \\
\text{so} \quad \text{_____ = _____ × _____}
\]
Exercises: Change the fraction statement to a division statement, then to a multiplication statement.

a) \( \frac{7}{8} = 0.875 \)  
   b) \( \frac{1}{4} = 0.25 \)  
   c) \( \frac{17}{20} = 0.85 \)  
   d) \( \frac{4}{5} = 0.8 \)

Answers: a) \( 7 \div 8 = 0.875 \), so \( 7 = 8 \times 0.875 \);  
b) \( 1 \div 4 = 0.25 \), so \( 1 = 4 \times 0.25 \);  
c) \( 17 \div 20 = 0.85 \), so \( 17 = 20 \times 0.85 \);  
d) \( 4 \div 5 = 0.8 \), so \( 4 = 5 \times 0.8 \)

Finding a pattern. Now have students look at their answers to the questions in the previous set of exercises. ASK: If we know the value of a fraction as a decimal, how can we use that to write a multiplication statement? Write on the board:

\[ \underline{\text{_____}} = \underline{\text{_____}} \times \underline{\text{_____}} \]

ASK: In which blank does the numerator—the top number—of the fraction go? (the first blank)  
What number goes in the second blank, the denominator or the value? (it doesn't matter, because multiplication follows the commutative property) Explain that when you know the decimal value of a fraction, the numerator of the fraction can be written as the product of the denominator and the decimal value.

Exercises:
1. Write the equation so that it involves multiplication.
   a) \( \frac{7}{4} = 1.75 \)  
   b) \( \frac{9}{20} = 0.45 \)  
   c) \( \frac{7}{35} = 0.2 \)  
   d) \( \frac{9}{25} = 0.36 \)

Answers: a) \( 7 = 1.75 \times 4 \); b) \( 9 = 0.45 \times 20 \); c) \( 7 = 0.2 \times 35 \); d) \( 9 = 0.36 \times 25 \)

2. Calculate the value of the fraction, then write a multiplication statement.
   a) \( \frac{2}{5} \)  
   b) \( \frac{9}{10} \)  
   c) \( \frac{21}{25} \)  
   d) \( \frac{19}{20} \)

Bonus:
   e) \( \frac{23}{5} \)  
   f) \( \frac{192}{25} \)

Answers: a) \( 2 = 0.4 \times 5 \); b) \( 9 = 0.9 \times 10 \); c) \( 21 = 0.84 \times 25 \); d) \( 19 = 0.95 \times 20 \),  
Bonus: e) \( 23 = 4.6 \times 5 \); f) \( 192 = 7.68 \times 25 \)

Writing fraction statements that involve variables as a product. Write on the board:

\[ \frac{10}{x} = 2 \]

SAY: I don’t know what number \( x \) is, but I know that whatever it is, 2 times \( x \) is equal to 10.  
Write on the board:

\[ 10 + x = 2, \text{ so } 2x = 10 \]
**Exercises:** Rewrite the equation so that it uses multiplication instead of division.

a) \( \frac{24}{x} = 2 \)  

b) \( \frac{24}{x} = 3 \)  

c) \( \frac{24}{x} = 4 \)  

d) \( \frac{x}{3} = 5 \)  

e) \( \frac{x}{6} = 5 \)  

f) \( \frac{x}{7} = 8 \)  

g) \( \frac{8}{2} = x \)  

h) \( \frac{15}{3} = x \)  

i) \( \frac{18}{2} = x \)  

j) \( \frac{18}{x} = 3 \)  

k) \( \frac{33}{3} = x \)  

l) \( \frac{x}{2} = 15 \)

**Answers:**

a) \( 24 = 2 \times x \)  

b) \( 24 = 3 \times x \)  

c) \( 24 = 4 \times x \)  

d) \( x = 5 \times 3 \)  

e) \( x = 6 \times 5 \)  

f) \( x = 8 \times 7 \)  

g) \( 8 = 2 \times x \)  

h) \( 15 = 3 \times x \)  

i) \( 18 = 2 \times x \)  

j) \( 18 = 3 \times x \)  

k) \( 33 = 3 \times x \)  

l) \( x = 15 \times 2 \)

**Changing an equation of equivalent fractions to an equation of multiplication statements.**

Write on the board:

\[
\frac{3}{5} = \frac{12}{20}
\]

SAY: I can write each fraction as a division statement. Write on the board:

\[
3 \div 5 = 12 \div 20
\]

Have students verify this equation by doing long division. (\(3 \div 5 = 0.6\) and \(12 \div 20 = 0.6\)) Tell students that you find it easier to work with multiplication than with division. SAY: I would like to be able to verify this equality by using multiplication instead of division, and I know a trick that lets me change the equation so I can do that. Work through the steps below as a class. Write on the board:

\[
\frac{3}{5} = \frac{12}{20}
\]

SAY: Start by multiplying both sides by \(5 \times 20\) (the product of the denominators). Write on the board:

\[
\frac{3}{5} \times 5 \times 20 = \frac{12}{20} \times 5 \times 20
\]

SAY: Then, cancel common factors and rewrite the equation. The equations should look like this:

\[
\frac{3}{5} \times \frac{5}{5} \times 20 = \frac{12}{20} \times 5 \times 20
\]

\[
3 \times 20 = 12 \times 5
\]

Point out that we have now created an equation of multiplication statements instead of fractions. Have students use multiplication to verify the equation. ASK: Was it easier to use multiplication to verify the equation or was it easier to use division? (multiplication)

Have students use this method to complete the following exercises.
Exercises: Change the equivalent fractions to equivalent multiplication statements.

a) $\frac{2}{3} = \frac{8}{12}$  
   b) $\frac{2}{5} = \frac{6}{15}$  
   c) $\frac{5}{9} = \frac{10}{18}$  
   d) $\frac{3}{8} = \frac{9}{24}$

Answers: a) $2 \times 12 = 8 \times 3$, b) $2 \times 15 = 6 \times 5$, c) $5 \times 18 = 10 \times 9$, d) $3 \times 24 = 9 \times 8$

Multiplying to verify equivalent fractions. Point out that the fractions $\frac{3}{5}$ and $\frac{12}{20}$ are equivalent fractions. ASK: How do I know? PROMPT: What number can we multiply both the numerator and the denominator by in $\frac{3}{5}$ to get $\frac{12}{20}$? (multiply 3 by 4 to get 12 and 5 by 4 to get 20) Write on the board:

$$\frac{3}{5} = \frac{12}{20}$$

ASK: How can we change this to an equation with multiplication instead? (we did it above—it was $3 \times 20 = 12 \times 5$)

Exercises: Change the equivalent fractions to equivalent multiplication statements.

a) $\frac{3}{4} = \frac{15}{20}$  
   b) $\frac{3}{5} = \frac{9}{15}$  
   c) $\frac{7}{9} = \frac{21}{27}$  
   d) $\frac{3}{8} = \frac{15}{40}$

Sample solution:

a) $\frac{3}{4} = \frac{15}{20}$  
   $3 \times 4 \times 20 = 15 \times 4 \times \cancel{4}$  
   $3 \times 20 = 15 \times \cancel{4}$

Answers: b) $3 \times 15 = 9 \times 5$, c) $7 \times 27 = 21 \times 9$, d) $3 \times 40 = 15 \times 8$

(MP.8) Finding a pattern (cross multiplying). Have students look at their answers to the previous set of exercises. ASK: How can you find which numbers to multiply together from the fractions? PROMPT: Do you multiply both numerators together? (no) What do you multiply together? (the numerator of one fraction with the denominator of the other fraction) Go through each one, point to the answer, and verify that this is indeed what students did for each question—join the numerator of each fraction with the denominator of the other fraction to emphasize this point. Tell students that because the products from equivalent fractions can be found by drawing an X, we call this process cross multiplying. Write on the board:

$$\frac{3}{4} = \frac{15}{20}$$

Exercises:

1. Verify that each pair of fractions in the previous two sets of exercises are in fact equivalent by verifying that the products you found are equal.

Answers:

a) $2 \times 12 = 8 \times 3$  
   $24 = 24$

b) $2 \times 15 = 6 \times 5$  
   $30 = 30$

c) $5 \times 18 = 10 \times 9$  
   $90 = 90$

d) $3 \times 24 = 9 \times 8$  
   $72 = 72$

a) $3 \times 20 = 15 \times 4$  
   $60 = 60$

b) $3 \times 15 = 9 \times 5$  
   $45 = 45$

c) $7 \times 27 = 21 \times 9$  
   $189 = 189$

d) $3 \times 40 = 15 \times 8$  
   $120 = 120$
(MP.7) 2. Use cross multiplication to verify that the fractions are equivalent.

a) \( \frac{3}{7} = \frac{6}{14} \)  

b) \( \frac{4}{5} = \frac{12}{15} \)  

c) \( \frac{3}{8} = \frac{15}{40} \)  

Bonus: \( \frac{19}{24} = \frac{133}{168} \)

Answers: a) \( 3 \times 14 = 42 \) and \( 6 \times 7 = 42 \); b) \( 4 \times 15 = 60 \) and \( 12 \times 5 = 60 \); c) \( 3 \times 40 = 120 \) and \( 15 \times 8 = 120 \); Bonus: \( 19 \times 168 = 3,192 \) and \( 133 \times 24 = 3,192 \)

(MP.7) Cross multiply to identify equivalent fractions. For the following exercises, have students decide whether the two fractions are equivalent by multiplying the numerator of each fraction with the denominator of the other fraction and checking whether the two products are equal.

Exercises: Cross multiply to check if the fractions are equivalent.

a) \( \frac{8}{10} \) and \( \frac{64}{100} \)  

b) \( \frac{7}{10} \) and \( \frac{49}{70} \)  

c) \( \frac{3}{5} \) and \( \frac{36}{65} \)  

d) \( \frac{4}{9} \) and \( \frac{36}{90} \)

e) \( \frac{5}{12} \) and \( \frac{45}{96} \)  

f) \( \frac{8}{17} \) and \( \frac{40}{85} \)  

g) \( \frac{7}{9} \) and \( \frac{63}{81} \)  

Bonus: \( \frac{531}{792} \) and \( \frac{59}{98} \)

Answers: a) not equivalent, b) equivalent, c) not equivalent, d) not equivalent, e) not equivalent, f) equivalent, g) equivalent, Bonus: not equivalent

Cross multiplying for complex fractions. SAY: You can cross multiply complex fractions, too. Explain that complex fractions are like other fractions—they just contain fractions in the numerator, or the denominator, or both. Write on the board:

\[
\frac{2}{3} and \frac{5}{4} \quad and \quad \frac{12}{5} \quad \frac{1}{2}
\]

SAY: To verify that they are equivalent, I have to multiply the numerator of the first complex fraction by the denominator of the second complex fraction, then the numerator of the second complex fraction by the denominator of the first complex fraction. Write on the board:

\[
\frac{2}{3} \times \frac{1}{2} = \frac{2 \times 1}{3 \times 2} = \frac{1}{3} \quad \text{and} \quad \frac{5}{12} \times \frac{4}{5} = \frac{5 \times 4}{12 \times 5} = \frac{4}{12}
\]

ASK: Are they equal? (yes) Students can answer by signaling thumbs up. ASK: How do you know? (because 4/12 and 1/3 are equivalent fractions) SAY: So the two complex fractions are equivalent.

Exercises: Cross multiply to check if the complex fractions are equivalent.

a) \( \frac{4}{1} and \frac{4}{2} \)  

b) \( \frac{5}{3} and \frac{3}{7} \)  

Answers: a) equivalent, b) not equivalent
Explain that when the relation between two numerators or two denominators is clear and easy to find, they can find the missing number mentally. Otherwise, it is better to use cross multiplication.

**(MP.5, MP.7) Exercises:** Solve the proportion. Use mental math or cross multiplication. Explain your choice.

- a) \( \frac{3}{5} = \frac{6}{?} \)
- b) \( \frac{5}{6} = \frac{?}{15} \)
- c) \( \frac{80}{100} = \frac{18}{?} \)
- d) \( \frac{55}{100} = \frac{?}{20} \)
- e) \( \frac{10}{11} = \frac{7}{?} \)
- f) \( \frac{50}{65} = \frac{25}{?} \)

**Selected solutions:**

- a) 10, mentally because \( 3 \times 2 = 6 \), so I found \( 5 \times 2 = 10 \);
- b) 12.5, cross multiplying because there isn’t an easy multiplication from 6 to 15 or from 5 to 6

**Answers:**

- c) 22.5, d) 11, e) 7.7, f) 32.5

**Extensions**

1. a) Have students investigate this question: If fractions \( \frac{3}{5} \) and \( \frac{6}{10} \) are equivalent, what fraction is \( \frac{3}{6} \) equivalent to? Write on the board:

\[
\frac{3}{5} = \frac{6}{10} \quad \text{Cross multiply to get } 3 \times 10 = 6 \times 5
\]

Have students decide what fractions they can cross multiply to get \( 3 \times 10 = 5 \times 6 \). Suggest that students look for where the parts of each fraction go in the equation and compare how the equations are different. PROMPT: Which numbers switched positions, and which numbers stayed in the same position?

b) Have students cross multiply to make a new pair of equivalent fractions, then use the commutative property of multiplication for one of the products (not both!) to make a new pair of equivalent fractions.

i) \( \frac{2}{3} = \frac{6}{9} \), so ______ = ______

ii) \( \frac{1}{4} = \frac{5}{20} \), so ______ = ______

iii) \( \frac{3}{5} = \frac{9}{15} \), so ______ = ______

**Answers:**

- a) 3 and 10 are in the same position, but 5 and 6 get switched. So the corresponding fractions are \( \frac{3}{6} = \frac{5}{10} \); b) i) 2/6 = 3/9, ii) 1/5 = 4/20, iii) 3/9 = 5/15. Emphasize to students that to find the second pair of equivalent fractions, they can read the numbers from the first pair across, from left to right.

2. **Mental math and estimation.** Tell students that you know someone who changed the fractions in Extension 1, part b.ii) to \( \frac{1}{20} = \frac{5}{4} \). ASK: How can you tell immediately that this is wrong?

**Answer:** \( 1/20 \) is less than 1, but \( 5/4 \) is more than 1

3. Cross multiply to verify whether the complex fractions are equivalent.

- a) \( \frac{0.2}{0.3} \) and \( \frac{0.4}{0.6} \)
- b) \( \frac{0.03}{0.05} \) and \( \frac{0.4}{0.7} \)
- c) \( \frac{0.02}{0.5} \) and \( \frac{0.1}{0.25} \)

**Sample solution:**

- a) \( 0.2 \times 0.6 = 0.3 \times 0.4, 0.12 = 0.12 \) ✓

**Answers:**

- b) no, c) no
Using Equations to Solve Proportions

Pages 56–57

Standards: 7.RP.A.3

Goals:
Students will cross multiply to solve problems that involve proportions.

Prior Knowledge Required:
Can convert a fraction $a/b$ to a decimal by dividing $a ÷ b$
Can find equivalent fractions by multiplying the numerator and denominator by the same number
Can write a proportion to solve ratio and percentage problems
Can solve multiplicative equations
Can write an equivalent multiplication statement for a given division statement

Vocabulary: cross multiply, equation, equivalent fractions, equivalent ratios, percent, proportion, variable

Materials:
calculators

Using cross multiplying to write equations. Show students how to cross multiply to write an equation when there is a variable in one of the fractions. Write on the board:

$$\frac{10}{x} = \frac{2}{3}$$

SAY: I don’t know what number $x$ is, but I know that no matter what, 2 times $x$ is equal to 10 times 3. Write on the board:

$$10 \times 3 = 2 \times x$$, so $30 = 2x$

Exercises: Cross multiply to write an equation for $x$.

a) $\frac{24}{x} = \frac{2}{5}$  
b) $\frac{24}{x} = \frac{3}{5}$  
c) $\frac{24}{x} = \frac{4}{5}$  
d) $\frac{x}{3} = \frac{12}{9}$

e) $\frac{x}{6} = \frac{5}{3}$  
f) $\frac{x}{7} = \frac{8}{28}$  
g) $\frac{8}{2} = \frac{x}{5}$  
h) $\frac{15}{3} = \frac{x}{4}$

Answers:  
a) $24 \times 5 = 2x$, so $120 = 2x$; b) $120 = 3x$, c) $120 = 4x$, d) $9x = 36$, e) $3x = 30$, f) $28x = 56$, g) $40 = 2x$, h) $60 = 3x$

Using cross multiplying to solve equations. Review multiplicative equations like $2 \times b = 12$. Remind students that to solve this type of equation, they have to divide both sides of the equation by the coefficient of the unknown. For example, in the equation $2b = 12$, the answer is $b = 12 ÷ 2$, so $b = 6$. 
Exercises:
1. Have students solve the equations in their answers to the previous set of exercises.
Sample solution: d) \(9x = 12 \times 3\), \(9x = 36\), \(x = 36 ÷ 9\), \(x = 4\)
Answers: a) \(x = 60\), b) \(x = 40\), c) \(x = 30\), e) \(x = 10\), f) \(x = 2\), g) \(x = 20\), h) \(x = 20\)

2. Rewrite the equation so that it involves multiplication, then solve for \(x\). Check your answer by substitution.

a) \(\frac{20}{x} = 5\)  
b) \(\frac{x}{6} = 7\)  
c) \(\frac{26}{2} = x\)  
d) \(\frac{60}{x} = 15\)  
e) \(\frac{x}{7} = 9\)  
f) \(\frac{48}{8} = x\)

Answers: a) \(20 = 5x\), so \(x = 20 ÷ 5 = 4\); b) \(x = 6 \times 7 = 42\); c) \(2x = 26\), so \(x = 26 ÷ 2 = 13\); d) \(60 = 15x\), so \(x = 60 ÷ 15 = 4\); e) \(x = 7 \times 9 = 63\); f) \(48 = 8x\), so \(x = 48 ÷ 8 = 6\)

Point out that parts c) and f) do not even need to be rewritten, as they can be solved in one step. For example, c) says directly that \(26 ÷ 2 = x\), so we don’t need to first write that \(2x = 26\).

Cross multiplying when the answers are decimal numbers. Tell students to again cross multiply to solve for \(x\), but this time their answers will be decimal numbers. This means that they are comparing equivalent ratios rather than equivalent fractions. Remind students that we can write ratios in fraction form even when both terms are not whole numbers. Review writing fractions as decimal fractions, then as decimals. Write on the board:

\[
\begin{align*}
\frac{1}{2} &= \frac{5}{10} = 0.5 \\
\frac{1}{4} &= \frac{25}{100} = 0.25, \quad \frac{2}{4} = \frac{1}{2} = 0.5, \quad \frac{3}{4} = \frac{75}{100} = 0.75 \\
\frac{1}{5} &= \frac{2}{10} = 0.2, \quad \frac{4}{10} = 0.4, \quad \frac{3}{5} = \frac{6}{10} = 0.6, \quad \frac{4}{5} = \frac{8}{10} = 0.8
\end{align*}
\]

Exercises: Solve for \(x\).

\[
\begin{align*}
a) \quad \frac{10}{3} &= \frac{3}{x} & b) \quad \frac{x}{3} &= \frac{5}{6} & c) \quad \frac{7}{5} &= \frac{x}{6} & d) \quad \frac{7}{x} &= \frac{4}{5} & e) \quad \frac{9}{x} &= \frac{6}{5} & f) \quad \frac{5}{3} &= \frac{11}{x}
\end{align*}
\]

Sample solution: c) \(5x = 42\), so \(x = 42 ÷ 5 = 8.4\)
Answers: a) \(10x = 9\), so \(x = 0.9\); b) \(6x = 15\), so \(x = 2.5\); d) \(4x = 35\), so \(x = 8.75\); e) \(6x = 45\), so \(x = 7.5\); f) \(5x = 33\), so \(x = 6.6\)

Using proportions to solve percentage problems. Review writing a proportion to solve a percentage problem, then demonstrate how using cross multiplication makes the problem easy. Write on the board:

What is 30\% of 8?

SAY: Suppose the answer is \(x\) and we’re going to find \(x\). If 30\% of 8 is equal to \(x\), the ratio of \(x\) to 8 is the same as the ratio of 30 to 100. Write on the board:

\[x : 8 = 30 : 100\]
SAY: To solve this proportion, you can write it as two equivalent fractions. Write on the board:

\[ \frac{x}{8} = \frac{30}{100} \]

Remind students that they already used cross multiplying to solve this type of equation. Ask a volunteer to solve the equation, as shown below.

\[
\frac{x}{8} = \frac{30}{100}, \quad \text{so} \quad 100x = 30 \times 8 \\
100x = 240 \\
x = \frac{240}{100} \\
x = 2.4
\]

Explain to students that writing the proportion is the most important part of the solving process. Write on the board:

12 is 3% of what number?

SAY: Suppose the answer is \( x \) and we’re going to find \( x \). Have students propose different ways of writing the proportion for this problem. SAY: Because there is 3% in the question, one fraction in the proportion is 3/100. Write on the board:

\[
\frac{x}{12} = \frac{3}{100} \quad \text{and} \quad \frac{12}{x} = \frac{3}{100}
\]

ASK: Which proportion gives me the answer, the first or the second? (the second) Students can answer by signaling thumbs up or thumbs down as you point to each proportion. ASK: How do you know? (because 12 is a part of the question and is in the numerator of the second proportion) Ask a volunteer to use cross multiplying to solve the proportion, as shown below.

\[
\frac{12}{x} = \frac{3}{100}, \quad \text{so} \quad 3x = 12 \times 100 \\
3x = 1,200 \\
x = \frac{1,200}{3} \\
x = 400
\]

**Exercises:** Write a proportion in fraction form, then cross multiply and solve.

a) What is 15% of 40?  
    b) What is 32% of 50?  
    c) What is 75% of 48?  
    d) 24 is 80% of what number?  
    e) 62 is 25% of what number?  
    f) 12 is 30% of what number?

**Answers:**  
a) \( \frac{x}{40} = 15/100, \ x = 6 \);  
b) \( \frac{x}{50} = 32/100, \ x = 16 \);  
c) \( \frac{x}{48} = 75/100, \ x = 36 \);  
d) \( \frac{24}{x} = 80/100, \ x = 30 \);  
e) \( \frac{62}{x} = 25/100, \ x = 248 \);  
f) \( \frac{12}{x} = 30/100, \ x = 40 \)
Solving percentage problems mentally. Write on the board:

What is 15% of 20?

Ask a volunteer to write the proportion for the problem in fraction form and use cross multiplication to solve it, as shown below:

\[
\frac{x}{20} = \frac{15}{100}, \text{ so } 100x = 15 \times 20
\]

\[
100x = 3,000
\]

\[
x = 3
\]

Point to the fractions and SAY: Look at the denominators of these equivalent fractions. ASK: What number do I have to multiply 20 by to get 100? (5) What number do I have to multiply \(x\) by to get 15? (the same number: 5) Write on the board:

\[
\frac{x}{20} = \frac{15}{100}
\]

ASK: What is the value of \(x\)? (3)

Exercises: Write the question as a proportion in fraction form, then solve the equation mentally.

a) What is 20% of 25? 
   b) What is 18% of 50?
   c) What is 75% of 10? 
   d) 24 is 48% of what number?
   e) 42 is 21% of what number?

Answers: a) \(x/25 = 20/100, x = 5\); b) \(x/50 = 18/100, x = 9\); c) \(x/10 = 75/100, x = 7.5\);
   d) \(24/x = 48/100, x = 50\); e) \(42/x = 21/100, x = 200\)

Ask students to use cross multiplication to solve the questions in the previous set of exercises. Ask them to find their mistake if they did not get the same answer both ways.

(MP.4) Approximating percentages. Tell students that in some real-world questions, it is not necessary to find the exact percentage. Write on the board:

90 students out of 298 Grade 7 students wear eyeglasses to read. What percent of the Grade 7 students wear eyeglasses?

Ask a volunteer to write the proportion for the question, then use cross multiplication to write the equation, as shown below:

\[
\frac{90}{298} = \frac{x}{100}, \text{ so } 298x = 90 \times 100
\]
Ask students to use a calculator to solve the equation. (30.20134228) ASK: Did you get an
exact number for the question? (no) SAY: Look at the first fraction in the proportion. The
denominator 298 is very close to 300. Write on the board:

\[
\frac{90}{300} = \frac{x}{100}
\]

Ask a volunteer to use cross multiplication to write the equation, then to solve the new
proportion, as shown below:

\[
\frac{90}{300} = \frac{x}{100}, \text{ so } 300x = 90 \times 100, \text{ so } x = \frac{9,000}{300} = 30
\]

Write on the board:

About 30% of students in Grade 7 wear eyeglasses.

**Exercises:** Find the amount.
a) About 10% of students are left-handed. In a school with 731 students, about how many are
left-handed?
b) 28 is 9% of a number.
c) 17 is 23% of a number.

**Answers:** a) about 73, b) 311, c) 74

**(MP.4) More word problem practice.**

**Exercises:** Solve the word problem.
a) A shirt costs $25. After taxes, it costs $30. What percent of the original price are the taxes?
b) A shirt costs $40. After taxes, it costs $46. At what rate was the shirt taxed?
d) A shirt costs $40. It goes on sale for $28. What percent was taken off?

**Bonus:** A shirt costs $20. It goes on sale at 15% off. A 15% tax is then added. What is the final
price?

**Answers:** a) 5/25 = ?/100, so ? = 20; b) 6/40 = ?/100, so ? = 15; c) 12/40 = ?/100, so ? = 30;

Bonus: The sale price is $17, and 15% tax is $2.55, so the final price is $19.55

**Extension**

**(MP.1) NOTE:** This extension emphasizes the importance of checking the answer
by substitution.

a) Can \( \frac{x}{3} = \frac{x}{5} \) be solved? Explain. \[ \text{b) Can } \frac{3}{x} = \frac{5}{x} \text{ be solved? Explain.} \]

**Answers:**
a) Cross multiplying gives us 5x = 3x, 2x = 0, so x = 0, which can be substituted into the original
equation (0/3 = 0/5, 0 = 0), so this equation can be solved.
b) Cross multiplying gives us the same equation and the same result as in part a), x = 0, but
substituting this value into the original equation results in 3/0 = 5/0. These fractions do not make
sense! This equation cannot be solved.
### Three Types of Percentage Problems

#### What percent is it?

<table>
<thead>
<tr>
<th>Given</th>
<th>the whole and the part</th>
</tr>
</thead>
<tbody>
<tr>
<td>To find</td>
<td>What percent of the whole is the part?</td>
</tr>
</tbody>
</table>
| Equation               | \[
| part \quad \frac{?}{100}\cdot \text{whole} \]
| Examples               | 1. What percent of 50 is 10?  
2. 10 is what percent of 50? \[
\text{part} = \frac{10}{50} = \frac{?}{100} \]
3. A shirt costs $50. It is $10 off. What percent off is it?

#### How much is the percentage?

<table>
<thead>
<tr>
<th>Given</th>
<th>the whole and a percent, ( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>To find</td>
<td>How much is this percentage of the whole?</td>
</tr>
</tbody>
</table>
| Equation               | \[
\frac{?}{100}\cdot \text{whole} = P \]
| Examples               | 1. What is 30% of 50?  
2. 30% of 50 is what number? \[
\text{part} = \frac{30}{50} = \frac{?}{100} \]
3. A meal costs $50. Tip and taxes together are 30%. How much are the tip and taxes together?

#### How much is the whole?

<table>
<thead>
<tr>
<th>Given</th>
<th>a part that is ( P% ) of the whole</th>
</tr>
</thead>
<tbody>
<tr>
<td>To find</td>
<td>How much is the whole?</td>
</tr>
</tbody>
</table>
| Equation               | \[
\frac{?}{100} = \frac{P}{\text{whole}} \]
| Examples               | 1. If 5 is 40%, what is the number?  
2. 5 is 40% of what number? \[
\text{part} = \frac{5}{40} = \frac{?}{100} \]
3. A T-shirt was 40% off. The price was reduced by $5. What was the original price?