Grade 7

End of Year Teacher Pack

Includes:

- Lesson Plans
- Blackline Masters
- Assessment & Practice Book Answers
- Quizzes and Unit Tests
- Rubrics for Scoring Quizzes and Tests
- Common Core State Standards Correlations
Unit 4  Geometry: Angles and Areas

Introduction
In this unit, students will solve angle problems involving complementary, supplementary, and vertical angles. Students will study areas of rectangles, triangles, parallelograms, trapezoids, composite shapes, and scale drawings. Students are also given the opportunity to solve equations and inequalities in problems involving angles and areas. Students will construct circles and triangles using a compass and verify the triangle inequality learned earlier. Finally, students will study the area and circumference of circles.

Pi. When we give answers, we use 3.14 for pi (unless stated otherwise) and we assume students will do the same. If students use the pi key on their calculator instead, their answer will be slightly different (and slightly more accurate).

Technology: dynamic geometry software. Students are expected to use dynamic geometry software to draw geometric shapes. Most activities in this unit use a program called The Geometer’s Sketchpad®, and some are instructional—they help you teach students how to use the program. If you are not familiar with The Geometer’s Sketchpad®, the built-in Help Centre provides explicit instructions for many constructions. Use phrases such as “How to construct a line segment of given length” to search the Index. NOTE: If you use a different dynamic geometry program to complete these activities, the instructions provided may need to be adjusted.

Grid paper. We recommend that students have grid paper and that you have a background grid on your board. If students do not have grid paper, you will need to have lots of it available (e.g., BLM 1 cm Grid Paper on p. T-1). If you do not have a background grid on your board, you will need to project a transparency of a grid onto the board so that you can write over the grid and erase the board without erasing the grid.

Terminology. In English, “width” means the distance across an object when looking at its front, as in the width of a cereal box, couch, car, or window. But in mathematics, rectangles do not have a front side, so mathematicians use the word “width” very specifically—the width of a rectangle is its shortest side, or the length of the shortest side. The length of a rectangle is its longest side, or the length of the longest side.

“Vertical angles” can also be called “opposite angles.” You can introduce the term “opposite angles” if you wish, and familiarize students with both terms.
G7-11  Complementary Angles and Supplementary Angles

Pages 104–106

Standards: 7.G.B.5, 7.EE.B.4a

Goals:
Students will recognize and evaluate complementary and supplementary angles.
Students will use complementary and supplementary angles to solve for unknown angles.

Prior Knowledge Required:
Can add angles
Knows that a right angle measures 90°
Can solve equations with two operations

Vocabulary: angle, **complementary angles**, right angle, simplify, **straight angles**, supplementary angles

Materials:
BLM Two Ways to Sketch a Problem (p. P-92, see Extension 3)

Review right angles. Remind students that a right angle is a 90° angle. This is the same angle found in a square or in the corner of a sheet of paper.

Introduce complementary angles. Tell students that two angles that add to 90° are called **complementary angles**. Three angles that add to 90° are not called complementary. The two angles that add to 90° are complementary to each other.

Exercises:
1. Fill in the blanks.

   ![Diagram]

   a) \( \angle A \) and \( \angle \) _______ are complementary.  
   b) \( \angle B \) and \( \angle \) _______ are complementary.

   **Answers:** a) \( D \), b) \( C \)

2. \( \angle A \) and \( \angle B \) are complementary. What is the measure of \( \angle B \)?
   a) \( \angle A = 30° \)  
   b) \( \angle A = 70° \)  
   c) \( \angle A = 45° \)  
   d) \( \angle A = 88° \)

   **Answers:** a) 60°, b) 20°, c) 45°, d) 2°

**Bonus:** Is there an angle complementary to \( \angle A = 100° \)? Explain.

**Answer:** No, because 100° can't add to 90° with any other angle.
Identifying complementary angles without the measures. Draw on the board:

ASK: What do ∠1 and ∠2 add to? (90°) SAY: So you can tell that ∠1 and ∠2 are complementary without even knowing what their measures are.

Exercises: Identify the complementary angles in the picture below.

Answers: ∠b and ∠c, ∠d and ∠f, ∠e and ∠g

Introduce straight angles. A straight angle is an angle that makes a straight line. Draw on the board:

ASK: What is the degree measure of a straight angle? (180°) Write on the board:

\[ 180 = 90 + 90 = 2 \times 90 \]

Introduce supplementary angles. Tell students that two angles that add to 180° are called supplementary angles. Three angles that add to 180° are not called supplementary. The two angles that add to 180° are supplementary to each other.

Exercises:
1. Identify all pairs of supplementary angles.

a) 

Answers: a) ∠A and ∠C, ∠B and ∠D; b) ∠a and ∠b, ∠a and ∠d, ∠b and ∠c, ∠c and ∠d
2. ∠A and ∠B are supplementary. What is the measure of ∠B?
   a) ∠A = 30°  b) ∠A = 135°  c) ∠A = 1°  d) ∠A = 100°
   **Bonus:** If ∠B and ∠C are supplementary, what can you say about ∠A and ∠C?
   **Answers:** a) 150°, b) 45°, c) 179°, d) 80°. Bonus: they have the same measure

   **(MP.6) Distinguishing between complementary and supplementary angles.** To remember which word means “add to 90” and which means “add to 180,” students can use the first-letter association between “supplementary” and “straight,” and between “complementary” and “corner.”

   **Exercises:** Identify the angle pairs that are complementary or supplementary.

   a)  
   b)  
   c)  
   d)  

   **Bonus:** In d), are the angles ∠1, ∠2, ∠3, and ∠4 supplementary? Explain.
   **Answers:** a) ∠A and ∠B are supplementary; b) complementary: ∠B and ∠D, ∠C and ∠E, supplementary: ∠A and ∠F; c) complementary: ∠1 and ∠2, supplementary: ∠1 and ∠4, ∠2 and ∠3; d) complementary: ∠1 and ∠2, ∠3 and ∠4, supplementary: ∠5 and ∠6;
   Bonus: No, because only pairs of angles can be supplementary.

   **Finding unknown angle measures using complementary and supplementary angles.**
   SAY: If you know two angles are complementary or supplementary, and you know one of the angles, you can find the other one by solving an equation. Draw on the board:

   ![Diagram](image)

   For each picture, ASK: Are these angles complementary or supplementary? (the first one is complementary, the second is supplementary) Beneath each picture, write the corresponding equation, as shown below:

   \[ x + 30 = 90 \]
   \[ x + 70 = 180 \]

   Demonstrate how to solve the first equation and have a volunteer solve the second one. (see below for solutions)

   \[ x + 30 = 90 \quad x + 70 = 180 \]
   \[ x = 90 - 30 \quad x = 180 - 70 \]
   \[ x = 60 \quad x = 110 \]
(MP.6) SAY: When you are solving the equation, you are just doing rough work, so you don’t need to write the degree symbol but, when you write your answer, you need to write the degree symbol to be precise. Write under each equation:

So $x = 60°$                        So $x = 110°$

Exercises: Use an equation to find $x$.

a) \[ x + 35 = 90, \text{ so } x = 55°; \]

b) \[ x + 140 = 180, \text{ so } x = 40°; \]

c) \[ x + 68 = 180, \text{ so } x = 112° \]

Answers: a) $x + 35 = 90$, so $x = 55°$; b) $x + 140 = 180$, so $x = 40°$; c) $x + 68 = 180$, so $x = 112°$

Equations that require simplifying first. Draw on the board:

\[ x + 2x = 180 \]

ASK: What do these angles add to? (180°) Write on the board:

\[ x + 2x = 180 \]

ASK: Can this equation be simplified before we try to solve it? (yes, $x + 2x$ becomes $3x$) Write on the board the simplified equation and have a volunteer solve it. (see solution below)

\[ 3x = 180 \]

\[ x = 60 \]

(MP.6) SAY: So $x$ is 60°. Don’t forget to write the degree symbol in your answer. Write on the board:

\[ x = 60° \]

Exercises: Use an equation to find $x$. Simplify before solving the equation.

a) \[ 3x \]

b) \[ 4x \]

c) \[ 5x \]

Answers: a) $4x = 90$, so $x = 22.5°$; b) $5x = 180$, so $x = 36°$; c) $9x = 180$, so $x = 20°$; d) $10x = 90$, so $x = 9°$
Draw on the board:

\[ x + 30^\circ - x - 30^\circ \]

Ask one volunteer to write the equation. \((x + 30 + x - 30 = 180)\) Ask another volunteer to simplify the equation. \((2x = 180)\) SAY: Sometimes when you write the equation, the constant terms will cancel. That makes it easier to solve the equation.

**Exercises:** Use an equation to find \(x\). Simplify before solving the equation.

a) \(x + 50^\circ \)

b) \(x - 50^\circ \)

**Answers:** a) \(2x = 180\), so \(x = 90^\circ\); b) \(2x = 90\), so \(x = 45^\circ\)

**NOTE:** In *Question 4, part g*) on AP Book 7.2, p. 2, some students might write the equation \(90 + 57 + x = 180\), then simplify to \(147 + x = 180\), whereas other students might write the equation \(57 + x = 90\) without the need to simplify. Both are acceptable solutions.

**Review two-step equations.** SAY: Some equations you’ll see in this unit require two steps to solve. Write on the board:

\[
3x + 15 = 90 \\
2(x + 40) = 180
\]

Have volunteers solve the equations. \((x = 25^\circ, x = 50^\circ)\) For the second equation, students might solve it directly by dividing both sides by 2, then subtracting 40. Or they might change the equation to \(2x + 80 = 180\) and solve as they would the first equation.

SAY: When you solve equations, you don’t have to put any units in, but if there is a context to the problem, then you have to go back at the end and put the units in that are relevant. In geometry, the units are usually degrees.

**Exercises:** Use an equation to find \(x\).

a) \(30^\circ + 2x + 50^\circ \)

b) \(x + 50^\circ \)

c) \(2x + 70^\circ \)

**Bonus:**

\(45^\circ + x \)

**Answers:** a) \(30 + 2x + 50 = 180\), so \(2x + 80 = 180\), and \(x = 50^\circ\); b) \(x = 65^\circ\); c) \(x = 10^\circ\); Bonus: \(x = 15^\circ\)
(MP.4) Ignoring irrelevant information. Explain that being able to ignore irrelevant information in a problem is a really important skill because most problems that you face in life are not given to you with only the information that you need to solve them. Draw on the board:

ASK: What information do we need to use to find out what $x$ is? ($x + 55 = 90$) What information are we given that we don’t need? (the two unknown angles are both $x$) Ask a volunteer to solve the problem. ($x = 90 - 55$, so $x = 35^\circ$) Then draw another problem on the board:

Ask the same questions as before. (you need $3x + x = 180$, but you don’t need to know about the $90^\circ$ angle)

Exercises: Use an equation to find $x$.

a)

b)

Bonus:

Answers: a) $2x + 3x + 80 = 180$, so $x = 20^\circ$; b) $3x + 16 + 50 = 90$, so $x = 8^\circ$;

Bonus: $3x = 90$, so $x = 30^\circ$ or $100 - x + 2x - 40 = 90$, so $x + 60 = 90$, so $x = 30^\circ$

Extensions

(MP.3, MP.8) 1. $\angle A$ is complementary to $\angle B$ and supplementary to $\angle C$. How are $\angle B$ and $\angle C$ related? Hint: Let $\angle A = x$, then write $\angle B$ and $\angle C$ both in terms of $x$.

Answer: $\angle C$ is $90^\circ$ more than $\angle B$. To see this: Let $\angle A = x$. Then $\angle B = 90^\circ - x$, and $\angle C = 180^\circ - x$, so $\angle C = \angle B + 90^\circ$.

(MP.3) 2. What is wrong with this picture?
Answer: The angle $x - 5$ looks larger than the angle $x + 5$, but the number value of $x + 5$ will always be more than the number value of $x - 5$.

(MP.7) 3. On BLM Two Ways to Sketch a Problem, students learn how to draw the sketch for an angle problem in a way that makes the problem easier to solve.

Answers:
1. a) $2x + 20 = 180$; b) $2x = 180$; c) $x = 80^\circ$, so the angles are $80^\circ$ and $100^\circ$; d) $x = 90^\circ$, so the angles are $80^\circ$ and $100^\circ$; e) yes; f) Sara's; g) because the easiest way to write two numbers with one 20 larger than the other is as $x$ and $x + 20$; h) Because when we write the two numbers as $x + 10$ and $x - 10$ they are still 20 apart, but they make the equation easier to solve.
2. Let the smaller angle be $x - 8$ and the larger angle be $x + 8$, then $2x = 90$, so $x = 45$ and the two angles are $37^\circ$ (because $37 = 45 - 8$) and $53^\circ$ (because $53 = 45 + 8$).

(MP.5) 4. A plant is 15 cm tall and grows 4 cm each week. In how many weeks from now will the plant be 95 cm tall? Use any tool you think will help.

Sample answers:
• I made a T-table.

<table>
<thead>
<tr>
<th>Weeks from Now</th>
<th>Height (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>1</td>
<td>19</td>
</tr>
<tr>
<td>2</td>
<td>23</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>31</td>
</tr>
<tr>
<td>5</td>
<td>35</td>
</tr>
</tbody>
</table>

I noticed that after 5 weeks the plant was 20 cm taller; that makes sense because it grows 4 cm each week. It needs to grow 60 more cm after 5 weeks, so it needs to grow 20 cm three times, which means it needs another 15 weeks. So the plant will be 95 cm tall 20 weeks from now.
• I found $95 - 15 = 80$. That's how many centimetres the plant needs to grow to reach 95 cm. I know that it grows 4 cm each week, so I did $80 ÷ 4 = 20$ to find the number of weeks it takes to grow 80 cm. So the plant will be 95 cm tall 20 weeks from now.
• I used variables. I let $x$ be the number of weeks from now, and I know that the height of the plant for any $x$ is $15 + 4x$. Since I want to know what $x$ is when the height is 95, I need to find $15 + 4x = 95$. I solved the equation in steps: $4x = 95 - 15$, $4x = 80$, $x = 20$. So the plant will be 95 cm tall 20 weeks from now.

Whole-class follow-up: Have several volunteers present their solutions. For each solution, ask the rest of the class to discuss in pairs the following questions: Is the argument correct? Do they understand the argument? Could the argument be improved to make it clearer? Then compare the solutions. For example, to compare the first and third solutions above, ASK: How does the T-table show the equation $15 + 4x = 95$? (the starting height is 15, the gap is 4, and the ending height is 95) For the second and third solutions above, ASK: How are the solutions the same? (they did all the same calculations) Emphasize that when using variables, you don’t need to recognize that the difference is 80 because the equation gets that for you.
### G7-12 Adjacent Angles and Vertical Angles

#### Pages 107–108

#### Standards: 7.G.B.5, 7.EE.B.4a

#### Goals:
Students will recognize vertical and adjacent angles, and will solve for unknown angles.

#### Prior Knowledge Required:
Can identify two angles as supplementary angles that together make a straight angle
Can evaluate the measure of an angle when given the measure of its supplementary angle
Can solve equations with two operations

#### Vocabulary: adjacent angles, intersecting, solve, vertical angles

#### Introduce adjacent angles and vertical angles. Draw on the board:

SAY: Four angles are named when two lines intersect. If two angles are next to each other, they are called **adjacent angles**. If two angles are opposite each other, they are called **vertical angles**. Explain that the word “vertical” here has nothing to do with the word “vertical” in the sense of vertical and horizontal lines. Point to the picture and SAY: \( \angle 1 \) and \( \angle 2 \) are adjacent, but \( \angle 1 \) and \( \angle 3 \) are vertical. ASK: What other angles are adjacent? (\( \angle 2 \) and \( \angle 3 \), \( \angle 3 \) and \( \angle 4 \), \( \angle 4 \) and \( \angle 1 \)) What other pair of angles are vertical? (\( \angle 2 \) and \( \angle 4 \))

#### Exercises: Are the angles adjacent or vertical?

![Diagram](image)

a) \( \angle 1 \) and \( \angle 2 \)  
  b) \( \angle 2 \) and \( \angle 5 \)  
  c) \( \angle 3 \) and \( \angle 6 \)  
  d) \( \angle 1 \) and \( \angle 6 \)  
  e) \( \angle 1 \) and \( \angle 4 \)  
  f) \( \angle 4 \) and \( \angle 5 \)

**Answers:** a) adjacent, b) vertical, c) vertical, d) adjacent, e) vertical, f) adjacent

#### (MP.8) Vertical angles are equal. Draw on the board:

![Diagram](image)
(MP.3) SAY: In these diagrams, x and y are vertical angles. Point to the first picture and ASK: What is x? (60°) How do you know? (it makes 180 with 120) What is y? (60°) How do you know? (it makes 180 with 120)

(MP.8) Repeat for the 150° angle and the 100° angle. ASK: What do you notice? (vertical angles are always equal in measure)

(MP.7) Exercises: Find x and y.

a) \[
x = 120°, \quad y = 50°
\]
b) \[
x = 80°, \quad y = 132°
\]
c) \[
x = 40°, \quad y = 140°
\]

Finding unknown angles using vertical angles. Write on the board:

\[
4x = 120
\]

SAY: The two angles in the picture are equal, so you can write an equation. Have a volunteer solve the equation. \((x = 120 ÷ 4 = 30)\) SAY: You just have to remember to put the degree symbol in your answer. \((x = 30°)\)

Exercises: Use an equation to solve for x.

a) \[
35° + 7x = x - 25° + 120°
\]
b) \[
x - 25 = 120, \quad so \ x = 145°
\]
c) \[
3x + 45 = 90, \quad so \ x = 15°
\]
d) \[
4(x - 10) = 160, \quad so \ x = 50°
\]

Finding vertical angles in more complicated pictures. SAY: In pictures, you are sometimes given information that is distracting. Draw on the board:
SAY: I want to know which angle is vertical to the given angle. There are three intersecting lines in this picture, so part of the picture is irrelevant information. I’m only interested in the two lines that are the arms of the given angle. Thicken those lines on the board and have a volunteer show the vertical angle with an arc. (see below for diagram with thickened lines on the left, and with the vertical angle arc drawn on the right)

Exercises: Find all pairs of vertical angles among the labeled angles.

a) \( \angle 1 \) and \( \angle 4 \), \( \angle 2 \) and \( \angle 5 \), \( \angle 3 \) and \( \angle 6 \); b) \( \angle 1 \) and \( \angle 5 \), \( \angle 2 \) and \( \angle 6 \), \( \angle 3 \) and \( \angle 7 \), \( \angle 4 \) and \( \angle 8 \)

SAY: Now consider a slightly different situation. Draw on the board:

SAY: I want to know which angle is vertical to the given angle. I’m only interested in the lines that show the arms of this angle, so I’m going to highlight them. Do so, by making the lines thicker, then SAY: Now it makes it clear where the vertical angle is. (see below for diagram with thickened lines on the left, and with the vertical angle arc drawn on the right)

Exercises: Which two angles, when added together, make the angle that is vertical to the given angle?

a) \( \angle 2 \) and \( \angle 3 \), b) \( \angle 2 \) and \( \angle 3 \), c) \( \angle 3 \) and \( \angle 4 \)

Answers: a) \( \angle 2 \) and \( \angle 3 \), b) \( \angle 2 \) and \( \angle 3 \), c) \( \angle 3 \) and \( \angle 4 \)
Have volunteers show the angles by adding an arc to each part of the exercises they just did. (see answers below)

a) ![Diagram](image1)

b) ![Diagram](image2)

c) ![Diagram](image3)

Draw on the board:

![Diagram](image4)

ASK: Which angle is vertical to the 150° angle? Have a volunteer show the vertical angles, as shown below:

![Diagram](image5)

SAY: We know the two vertical angles are equal, so we can make an equation. One of the vertical angles is 150° and we know the other angle in terms of \(x\). ASK: What is that angle in terms of \(x\)? (90° + \(x\)) Write the equation on the board:

\[
150 = 90 + x
\]

Have a volunteer solve the equation. (150 − 90 = \(x\), so 60 = \(x\) or \(x = 60°\)) Remind students to include the degree symbol in their answer. (\(x = 60°\))

**Exercises:** Write an equation and solve it for \(x\).

a) ![Diagram](image6)

b) ![Diagram](image7)

c) ![Diagram](image8)

**Answers:**

a) 7\(x\) + 100 = 135, so \(x = 5°\);
b) \(x + 20 + 60 = 120\), so \(x + 80 = 120\), so \(x = 40°\);
c) 4\(x\) + 90 = 142, so \(x = 13°\)
Extensions

(MP.6) 1. Explain how the following proves that vertical angles are equal.

\[
\begin{align*}
180 - (180 - x) &= 180 - 180 + x = x \\
\text{Answer:} & \text{The angle vertical to } x \text{ is also supplementary to } 180 - x. \text{ Therefore, the unknown angle is equal to } 180 - (180 - x). \text{ This equals } x.
\end{align*}
\]

(MP.5) 2. a) Use the sketch to find \(x\).

\[
\begin{align*}
40^\circ + x + 30^\circ &= 180^\circ, \text{ so } x = 55^\circ; \\
b) & \text{ no; c) no}
\end{align*}
\]

(MP.8) 3. If you have \(n\) intersecting lines, name the angles in order, clockwise, from \(\angle 1\) to \(\angle 2n\). What angle is vertical to \(\angle 1\)?

\[\angle (n + 1)\]

(MP.5, MP.6) 4. Solve the equation. Use guess-and-check, mental math, or algebra. Explain your choice using math words like “equivalent,” “equation,” and “solve.”

\[
\begin{align*}
a) 3x + 8 &= 18 + 8 & b) 100x + 500 &= 600 & c) 14x + 38 &= 752 \\
d) 2x + 7 &= 13 & e) 100x + 7 &= 407 & f) 3x + 20 &= 617
\end{align*}
\]

\[
\begin{align*}
a) & \text{ sample explanation: I noticed that the equation is equivalent to } 3x = 18 \text{ and solving that is easy, so I used mental math.} \\
b) & \text{ sample explanation: I used mental math because it’s equivalent to solving } 100x = 100. \\
c) & \text{ sample explanation: I used algebra because the numbers are large, so I didn’t think guess-and-check would work and I couldn’t see a quicker way of doing it.} \\
d) & \text{ sample explanation: I used algebra because then I didn’t have to keep track of what I was doing in my head, or I used guess-and-check because the numbers were small, or I used mental math because I could subtract 7 from both sides and then I just had to solve } 2x = 6. \\
e) & \text{ sample explanation: I used mental math because solving the equation is the same as solving } 100x = 400, \text{ which is easy to solve.} \\
f) & \text{ sample explanation: I used algebra because the numbers are large so I didn’t think guess-and-check or mental math would be as fast.}
\end{align*}
\]
G7-13 Solving Problems Using Angle Properties

Pages 109–111

Standards: 7.G.B.5, 7.EE.B.4a

Goals:
Students will solve for unknown angles using complementary, supplementary, and vertical angles.
Students will name angles using a point on each arm and the vertex.

Prior Knowledge Required:
Can add angles
Can measure angles
Can identify two angles that are complementary or supplementary
Can evaluate the measure of an angle when given the measure of a supplementary or complementary angle
Knows that a right angle measures 90° and a straight angle measures 180°
Can solve equations with two operations
Knows the sum of the angles in a triangle is 180°
Knows that vertical angles are equal

Vocabulary: adjacent angles, angle, arm, complementary angles, right angle, simplify, sketch, supplementary angles, vertex, vertical angles

Materials:
grid paper or BLM 1 cm Grid Paper (p. T-1)
protractors
BLM The Geometer’s Sketchpad® (p. P-93)
blank sheets of paper
BLM Angles and Inequalities (p. P-94, see Extension 4)

(MP.6) Review the four types of angle pairs. ASK: What kinds of angle pairs have we seen so far? (vertical, adjacent, complementary, supplementary) PROMPTS: Two angles can be related by their sum; two angles can be related by where they are in relationship to each other. Write on the board:

1) Vertical
2) Adjacent
3) Complementary
4) Supplementary

Point to each picture and ask students to signal the word that matches the picture by holding up the correct number of fingers. (3, 2, 4, 1)
Diagrams with different types of angle pairs. Draw on the board:

In each picture, ask volunteers to identify pairs of angles that are related and to say how they are related. (In the first picture, the supplementary angle pairs are $\angle 1$ and $\angle 2$, $\angle 2$ and $\angle 3$, $\angle 3$ and $\angle 4$, and $\angle 4$ and $\angle 1$; the vertical angle pairs are $\angle 1$ and $\angle 3$, and $\angle 2$ and $\angle 4$. In the second picture, the supplementary angle pairs are $\angle 1$ and $\angle 5$, and $\angle 4$ and $\angle 5$; the vertical angle pair is $\angle 1$ and $\angle 4$; and the complementary angle pair is $\angle 1$ and $\angle 2$.)

Exercises:
1. Find $x$, $y$, and $z$.
   a) $x$, $y$, $z$
   b) $x$, $y$, $z$
   c) $x$, $y$, $z$

Answers: a) $x = 80^\circ$, $y = 65^\circ$, $z = 35^\circ$; b) $x = 70^\circ$, $y = 125^\circ$, $z = 55^\circ$; c) $x = 55^\circ$, $y = 50^\circ$, $z = 95^\circ$

2. Use the given angle measures to find the measures of all angles in the picture.
   a) $a$, $b$, $c$, $d$, $e$, $f$
   b) $a$, $b$, $c$, $d$
   Bonus: $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $i$

Answers: a) $a = 68^\circ$, $b = 112^\circ$, $c = 68^\circ$, $d = 135^\circ$, $e = 45^\circ$, $f = 135^\circ$; b) $a = 32^\circ$, $b = 58^\circ$, $c = 90^\circ$, $d = 32^\circ$; Bonus: $a = 26^\circ$, $b = 154^\circ$, $c = 26^\circ$, $d = 85^\circ$, $e = 95^\circ$, $f = 85^\circ$, $g = 59^\circ$, $h = 121^\circ$, $i = 59^\circ$

There are 360° in a full turn. Draw on the board:

SAY: There are 90° in a quarter turn, and 180° in a half turn. ASK: How many degrees are in a full turn? (360°)

Finding unknown angles using 360° in a full turn. SAY: If you know there are 360° in a full turn, and you know another way to get the number of degrees in a full turn, then you can make an equation and solve it. Do part a) below together as a class, pointing out that $x + 60$ and 360 are two ways to get the number of degrees in a full turn, so they must be equal.
(MP.2) Exercises: Write an equation and solve it to find $x$.

a) \[ x + 60 = 360, \text{ so } x = 300^\circ; \]
b) \[ 5x + x = 360, \text{ so } x = 60^\circ; \]
c) \[ 2x + 30 + x = 360, \text{ so } x = 110^\circ; \]
d) \[ x + 2x + 2x = 180, \text{ or } x + 2x + 2x + x + 2x + 2x = 360, \text{ so } x = 36^\circ; \]
e) \[ x + 5x + 6x = 180 \text{ or } x + 6x + 5x + x + 6x + 5x = 360, \text{ so } x = 15^\circ\]

The acute angles in a right triangle are complementary. Draw on the board:

\[ x + y = ? \]

Ask students to draw a right triangle on grid paper and measure its two acute angles using a protractor. ASK: What is $x + y$? (90) Have students check with a partner that they both got 90°. If they or their partner did not get 90°, they should help each other to find the mistake.

ASK: What are two angles that add to 90° called? (complementary angles)

Exercises: Use complementary angles to evaluate the variables.

a) \[ x = 30^\circ; \]
b) \[ y = 72^\circ; \]
c) \[ z = 45^\circ; \]
d) \[ x = 40^\circ, y = 50^\circ, z = 40^\circ; \]
e) \[ x = 55^\circ, y = 35^\circ, z = 55^\circ; \]

Bonus: \[ x = 53^\circ, y = 37^\circ, z = 53^\circ \]
Review the sum of the angles in a triangle. Remind students that the sum of the angles in a triangle always make 180°. SAY: That explains in another way why the sum of the two acute angles in a right triangle is 90°. Write on the board:

\[ x + y + 90 = 180 \]  
so \[ x + y = 90 \]

SAY: Because the angle measures in any triangle add to 180, if you know another way to find the sum of the angles in a triangle, you can make an equation and solve it.

(MP.2) Exercises: Evaluate the variables.

**a)**
\[ x = 55° \]

**b)**
\[ x = 18° \]

**c)**
\[ x = 60°, y = 120°, z = 70° \]

**d)**
\[ w = 64°, x = 116°, y = 39°, z = 141° \]

**e)**
\[ x = 38°, y = 115°, z = 65° \]

**f)**
\[ w = 107°, x = 26°, y = 107°, z = 47° \]

Answers: a) \( x = 55° \); b) \( x = 18° \); c) \( x = 60°, y = 120°, z = 70° \); d) \( w = 64°, x = 116°, y = 39°, z = 141° \); e) \( x = 38°, y = 115°, z = 65° \); f) \( w = 107°, x = 26°, y = 107°, z = 47° \)

**Drawing a sketch.** Tell students that if they are given word problems and need to make their own pictures, the angles in the pictures don’t—and often can’t—be accurate, because the measures of the angles aren’t always given. Often, that is what you are trying to find. The picture just has to help you find the angles. Drawing a picture that is a rough estimate instead of being precise is called making a *sketch*.

**NOTE:** The following exercises will be easier to share with students if you copy them onto a transparency.
(MP.2) Exercises: Match each sketch to a problem, then find all the angles in the sketches.

A. \[ x + 90^\circ \]
B. \[ 90^\circ - x \]
C. \[ x \]
D. \[ x \]
E. \[ 45^\circ \]

a) Two equal angles are complementary.
b) Two equal angles are supplementary.
c) Two angles are vertical. One of them is 45°. The other is composed of three equal but smaller angles.
d) Two angles are supplementary. The larger angle is 90° more than the smaller angle.
e) Two angles are vertical and complementary.

Answers: a) C, b) D, c) E, d) A, e) B; see below for the angles in each sketch:

A. \[ 135^\circ \]
B. \[ 45^\circ \]
C. \[ 45^\circ \]
D. \[ \]
E. \[ 135^\circ \]

Naming angles using three points. Tell students that there is another way to name angles. You can name points on the arms of an angle and use those points and the vertex to name the angle. Always write the vertex in the middle. Write on the board:

\[ \angle ADB \text{ or } \angle BDA \]
\[ \angle BDC \text{ or } \angle CDB \]

SAY: It doesn’t matter which arm you start with, but the vertex has to be in the middle. Ask a volunteer to use two ways to name the angle shown by the arc in the third picture. (\(\angle ADC\) and \(\angle CDA\))

(MP.6) Exercises: Name the angle marked with an arc in two ways.

a) \[ A \quad B \quad C \quad D \quad E \]

b) \[ A \quad B \quad C \quad D \quad E \]

c) \[ A \quad B \quad C \quad D \quad E \]

Answers: a) \(\angle CBD\) or \(\angle DBC\), b) \(\angle ABE\) or \(\angle EBA\), c) \(\angle DBE\) or \(\angle EBD\)
You cannot use two points on the same arm to name an angle. Draw on the board:

\[ \angle ACD \quad \angle ADC \quad \angle BDC \quad \angle BDA \]
\[ \angle BAD \quad \angle CDA \quad \angle BCD \]

ASK: Which names match the marked angle? What is wrong with the other ones? (\( \angle ACD \) has vertex \( C \), \( \angle ADC \) is correct, \( \angle BDC \) doesn’t have points on both arms, \( \angle BDA \) is correct, \( \angle BAD \) has vertex \( A \), \( \angle CDA \) is correct, \( \angle BCD \) has vertex \( C \)) Emphasize that the angle name has to have one point from each arm and the vertex in the middle. Two points on the same arm don’t work.

(MP.6) Exercises: Which names are correct for the given angle?

\[ \angle BCD \quad \angle BDC \quad \angle ACB \quad \angle ACD \]
\[ \angle EDC \quad \angle ECD \quad \angle ECB \quad \angle DCE \]

(MP.1) Bonus: Find all the correct ways of naming the angle.

Answers: \( \angle BCD \), \( \angle ACD \), \( \angle ECB \); Bonus: \( \angle ACE \), \( \angle ECA \), \( \angle BCE \), \( \angle ECB \), \( \angle ACD \), \( \angle DCA \), \( \angle BCD \), \( \angle DCE \)

(MP.4) Labeling given angles on a diagram. SAY: If you need to find the measure of a certain angle, it is often convenient to label the measures of any angles that you do know right on the diagram.

Exercises: Show the measures of the angles on the diagram.

a) \( \angle ABD = 70^\circ \) \( \angle CBD = 15^\circ \)
\( \angle BDA = 50^\circ \) \( \angle BAD = 60^\circ \)
\( \angle CDB = 130^\circ \) \( \angle BCD = 35^\circ \)

b) \( \angle MJK = 4x \) \( \angle KLM = 7x \)
\( \angle LKM = 6x \) \( \angle JMK = 7x \)
\( \angle MKJ = 7x \) \( \angle KML = 5x \)
Answers:

Activities 1–2

(MP.5) 1. Provide students with BLM The Geometer’s Sketchpad®. Students will use The Geometer’s Sketchpad® to name angles using the new notation. Students also measure angles and side lengths to verify that vertical angles are equal, that the sum of the angles in a triangle is 180°, and that the side lengths of triangles satisfy the Triangle Inequality. (1. a) yes, c) no, d) yes, yes, e) yes, yes; 2. b) yes, c) no, d) yes, yes; 3. a) yes)

You will need to teach students how to use The Geometer’s Sketchpad®.

(MP.1) 2. Create a right triangle by cutting off a corner from a sheet of paper. Fold the triangle so that the two acute vertices meet at the right angle, as shown below:

How does this verify that the two acute angles in a right triangle are complementary? (The two acute angles, when folded, meet the right angle with no overlaps or gaps.)

(end of activities)

Extensions

1. Use an equation to determine the measure of $\angle ABC$. Use dynamic geometry software to verify your answer.

Solution: The sum $2x + 2y = 180$, so $x + y = 90°$. So $\angle ABC = 90°$.

2. Write an equation and solve for $x$.

a) $2x + 3x = 4x + 15°$, so $5x = 4x + 15°$, so $x = 15°$; b) $3x = x + 60$, so $2x = 60$, so $x = 30°$; c) $x + 90 = 5x$, so $90 = 4x$, so $x = 22.5°$
(MP.3) 3. Find the sum of the angles $\angle A$, $\angle B$, $\angle C$, $\angle D$, $\angle E$, and $\angle F$ as follows.

![Diagram of angles]

a) Use vertical angles and the sum of the angles in a full turn to find the sum $x + y + z$.
b) Use what you know about angles in a triangle to evaluate $\angle A + \angle B + x$, then $\angle C + \angle D + y$, then $\angle E + \angle F + z$.
c) Calculate the sum $\angle A + \angle B + x + \angle C + \angle D + y + \angle E + \angle F + z$ and subtract $x + y + z$.

What is left?

Answers: a) $x + x + y + y + z + z = 360^\circ$, so $x + y + z = 180^\circ$; 
b) $\angle A + \angle B + x = 180^\circ$, and $\angle C + \angle D + y = 180^\circ$, and $\angle E + \angle F + z = 180^\circ$; 
c) $\angle A + \angle B + x + \angle C + \angle D + y + \angle E + \angle F + z = 3 \times 180^\circ = 540^\circ$, so $\angle A + \angle B + x + \angle C + \angle D + y + \angle E + \angle F + z - (x + y + z) = 540^\circ - 180^\circ = 360^\circ$.

4. Have students complete BLM Angles and Inequalities.

Answers: 1. a) $\angle BZC = 60^\circ - x$, $\angle EZD = 60^\circ$; b) $\angle BDC = 90^\circ - 2x$, $\angle ABD = 180^\circ - 2x$, $\angle CDE = 180^\circ - (90^\circ - 2x) = 90^\circ + 2x$; 2. b) $\angle CAB = 90 - x$, so $90 - x < 75$, so $x > 15^\circ$; c) $\angle BCD = 90 - (180 - x) = x - 90 > 70^\circ$, so $x > 160^\circ$; d) $\angle EDA = 90 - (180 - 2x) = 2x - 90$, so $2x - 90 > 60$, so $x > 75^\circ$

Redirecting students: Encourage students to mark all the angles on the diagram once they determine their measures. For the bonus in Question 2, this will involve writing inequality symbols.

(MP.8) 5. Evaluate $x$. How can you get $x$ from $\angle A$ and $\angle B$? Explain.

![Images of triangles]

Answers:

a) $130^\circ$
b) $90^\circ$
c) $x$ is supplementary to $110^\circ$, so $x = 70^\circ$; in all cases, the angle supplementary to $x$ is $180 - (\angle A + \angle B)$, so $x$ is $180 - (180 - (\angle A + \angle B)) = 180 - 180 + (\angle A + \angle B) = \angle A + \angle B$
G7-14 Area of Rectangles and Triangles

Pages 112–114

Standards: 7.EE.B.4a

Goals:
Students will solve equations using the area of triangles.

Prior Knowledge Required:
Can find the area of a rectangle
Recognizes the use of the distributive property
Can substitute numbers for variables in a formula
Can solve equations with two operations
Can perform operations with decimals in the standard order

Vocabulary: base, height, perpendicular, right triangle

Review the area of a rectangle. Draw on the board:

Remind students that the area of the rectangle, measured using grid squares as units, is the number of squares down times the number of squares across. So, in this case, it is $4 \times 5 = 20$ square units. Tell students that the length of a rectangle is the length of the longest side and the width of the rectangle is the length of the shortest side.

Exercises: Use the formula $A = \ell \times w$ to find the area of the rectangle.

a) b) c)

Answers: a) $A = 15 \text{ cm}^2$, b) $A = 32 \text{ cm}^2$, c) $A = 7 \text{ cm}^2$
The area of a right triangle. Draw on the board:

SAY: A right triangle is half of a rectangle. Ask volunteers to draw rectangles around each of the other right triangles to show that each one is half of a rectangle. Then point to the first triangle, on the grid, and ASK: What is the area of this shaded triangle? (10 square units) How do you know? (it is half the area of the rectangle, which is 20 square units)

Exercises: Find the area of the triangle by first finding the area of the rectangle.

a)  
![Diagram of a right triangle with sides 2 cm and 7 cm]  
2 cm  
7 cm  

b)  
![Diagram of a right triangle with sides 6 cm and 3 cm]  
6 cm  
3 cm  

c)  
![Diagram of a right triangle with sides 4 cm and 4 cm]  
4 cm  
4 cm  

d)  
![Diagram of a right triangle with sides 3 cm and 5 cm]  
3 cm  
5 cm  

Answers: a) rectangle: $A = 14 \text{ cm}^2$, triangle: $A = 7 \text{ cm}^2$; b) rectangle: $A = 18 \text{ cm}^2$, triangle: $A = 9 \text{ cm}^2$; c) rectangle: $A = 16 \text{ cm}^2$, triangle: $A = 8 \text{ cm}^2$; d) rectangle: $A = 15 \text{ cm}^2$, triangle: $A = 7.5 \text{ cm}^2$

Splitting a triangle into two right triangles to find its area. SAY: Any triangle can be decomposed into two right triangles. Draw on the board:

Ask volunteers to come up to the board and divide each triangle into two right triangles. (see answers below)

SAY: So if you can find the area of two right triangles, then you can find the area of any triangle.
Exercises: Find the area of the triangle (in grid squares) by adding the areas of the two right triangles.

a) ![Diagram of triangle A]  

Selected solution: a) The right triangle on the left has area 2 grid squares and the right triangle on the right has area 10 grid squares. So the total area is $2 + 10 = 12$ grid squares.

Answers: b) $4 + 8 = 12$, c) $6 + 6 = 12$

ASK: What do you notice about your answers? (they are all the same) Using part a) above, draw on the board:

![Diagram of triangle B]

SAY: When you split the triangle into two right triangles, you also split the rectangle into two smaller rectangles. ASK: What fraction of the area of the big rectangle is the area of the big triangle? (half) How do you know? (each right triangle is half of a rectangle, and the two rectangles together make the big rectangle) Point out that this is a general property of numbers; point at the picture above as an example, and write on the board:

$$\frac{1}{2} \times 4 + \frac{1}{2} \times 20 = \frac{1}{2} \times (4 + 20) = \frac{1}{2} \times 24$$

$$\frac{4}{2} + \frac{20}{2} = \frac{4 + 20}{2} = 24 \div 2$$

ASK: What property is that? (the distributive property) SAY: All the triangles from the exercises were half of the same rectangle, so they have the same area.

Introduce base and height in triangles. Point out that when students split the triangles into two right triangles, they drew a line from one of the vertices to the opposite side. The line they drew made a right angle with the side. This line (and its length) is called the height of the triangle and the side they drew it to is called the base.

SAY: A base is something that an object can stand on. If you think of a triangle as standing on its base, you can ask how high it is. The height of the triangle is how high it is when standing on its base, so it has to be measured perpendicular to, or at a right angle to, the base.

The formula for the area of a triangle. Ask students to identify and label the base and the height in each of the triangles they split into right triangles earlier. ASK: Which side of the
rectangle is the base of the triangle? (the length) Which side of the rectangle is the height of the triangle? (the width) Write on the board:

\[
\text{Area of rectangle} = \text{length} \times \text{width} \\
\text{Area of triangle} = \frac{\text{length} \times \text{width}}{2} = \frac{\text{base} \times \text{height}}{2}
\]

SAY: The area of a rectangle is length \(\times\) width and the area of the triangle within it is half that. But the base of the triangle is the length of the rectangle and the height of the triangle is the width of the rectangle. So now you have a formula for the area of a triangle.

**Exercises:** Use the base \((b)\) and the height \((h)\) of the triangle to find the area \(A = b \times h \div 2\).

\[
\begin{align*}
a) \quad & \quad \text{4 cm} \\
& \quad \text{9 cm} \\
b) \quad & \quad \text{12 cm} \\
& \quad \text{6 cm} \\
c) \quad & \quad \text{5.5 cm} \\
& \quad \text{8 cm}
\end{align*}
\]

**Answers:** a) 18 cm\(^2\), b) 36 cm\(^2\), c) 22 cm\(^2\)

**Drawing the height to any side as the base.** Tell students that, in all the examples above, the base of the triangle was horizontal and the height was inside the triangle, but they don’t have to be. The base can be any side of the triangle. The perpendicular distance to the opposite side is still called the height. Draw on the board three copies of the same triangle and thicken each side in turn. Ask volunteers to use a dot to show the vertex opposite the base. Demonstrate how to draw the height for the first base. SAY: To draw the height of the triangle, the base always has to meet the opposite vertex at a right angle. You might need to extend the base so you can make a right angle. Have volunteers show the height for the other two bases. (see sample pictures below; the height in each is shown with an arrow)

In the exercises below, students who are struggling can start by drawing a dot at the vertex opposite the base.
**Exercises:** The thick side is the base. Draw the height. Hint: You may need to extend the base.

![Diagram](image)

**Answers:**

![Diagram](image)

**The formula for the area works with any base.** Draw on the board:

![Diagram](image)

SAY: You can find the area of the shaded triangle by subtracting the areas of the two right triangles. Have students do this. Then have a volunteer show their work on the board:

\[
\begin{align*}
\text{Area} &= 10 \times 4 \div 2 - 3 \times 4 \div 2 \\
&= 40 \div 2 - 12 \div 2 \\
&= 20 - 6 \\
&= 14
\end{align*}
\]

Students might also choose to show their work using fraction notation instead of division, as shown below:

\[
\begin{align*}
\frac{10 \times 4}{2} - \frac{3 \times 4}{2} \\
= \frac{40}{2} - \frac{12}{2} \\
= 20 - 6 \\
= 14
\end{align*}
\]

This notation should also be shown and used throughout the unit.

ASK: What if you used the formula base times height divided by 2? Would that get the same answer? Write on the board:

\[7 \times 4 \div 2\]
Ask a volunteer to do the calculation to check. \((28 \div 2 = 14)\) Circle the 10 and the 3 in the first volunteer’s work, and SAY: you can just subtract 10 − 3 and multiply by 4 divided by 2. That will also get the same answer by the distributive property. So even when the height is outside the triangle, you can still use the formula \(b \times h \div 2\) to find the area.

**Exercises:**

1. Find the area of the triangle.
   a) \[ A = b \times h \div 2 = 5 \times 4 \div 2 = 10 \text{ cm}^2, \]
   b) \[ A = 11 \times 6 \div 2 = 33 \text{ cm}^2, \]
   c) \[ A = 5 \times 12 \div 2 = 30 \text{ in}^2, \]
   d) \[ A = 2 \times 4.5 \div 2 = 4.5 \text{ m}^2 \]

2. Use the formula to find the area of the triangle.
   a) base 10 in, height 3 in  
   b) base \(2 \frac{1}{2}\) ft, height \(5 \frac{3}{5}\) ft  
   c) base 4 ft, height 3.25 ft

   **Answers:** a) 15 in\(^2\), b) 7 ft\(^2\), c) 6.5 ft\(^2\)

SAY: You have to make sure you’re using the correct base and height to find the area.

**(MP.6) Exercises:** Can the two given values be the base and height used to calculate the area of the triangle? You do not actually have to find the area.

a) \[ 5 \text{ cm} \]
   b) \[ 8 \text{ in} \]
   c) \[ 7.4 \text{ in} \]
   d) \[ 5 \text{ cm} \]
   e) \[ 7.4 \text{ in} \]

**Answers:** a) yes, b) no, c) no, d) no, e) yes

SAY: Sometimes the different bases and heights are shown together on the same picture. This can get confusing, so you have to make sure you’re not multiplying two bases or two heights to get the area. Remind students that the height is the height when standing on the base, so the base and height are always perpendicular.
(MP.6) Exercises:
1. Can the two marked lengths be the base and height used to calculate the area of the triangle?
   a) ![Diagram](a)
   b) ![Diagram](b)
   c) ![Diagram](c)
   d) ![Diagram](d)
   e) ![Diagram](e)

   **Answers:** a) no, b) no, c) yes, d) yes, e) no

(MP.1, MP.7) 2. Find the area two ways. Make sure you get the same answer both ways.

   a) ![Diagram](a)
   b) ![Diagram](b)
   c) ![Diagram](c)
   d) ![Diagram](d)

   **Answers:**
   a) \( A = 3 \times 4 \div 2 = 6 \text{ cm}^2 \) or \( A = 5 \times 2.4 \div 2 = 6 \text{ cm}^2 \),
   b) \( A = 8 \times 4.5 \div 2 = 18 \text{ cm}^2 \) or \( A = 12 \times 3 \div 2 = 18 \text{ cm}^2 \),
   c) \( A = 10 \times 4.5 \div 2 = 22.5 \text{ in}^2 \) or \( A = 5 \times 9 \div 2 = 22.5 \text{ in}^2 \),
   d) \( A = 11 \times 3 \div 2 = 16.5 \text{ in}^2 \) or \( A = 5 \times 6.6 \div 2 = 16.5 \text{ in}^2 \)

   **SAY:** When you can find the area two ways (because you are given two bases and two heights),
you can write an equation. That means that if there is one measurement that you don’t know,
but you have the other three, then you can find the one that you don’t know.

(MP.7) Exercises: Find the area two ways to make an equation. Then solve for \( x \).

   a) ![Diagram](a)
   b) ![Diagram](b)
   c) ![Diagram](c)

   **Selected solution:**
   c) \( 6(2x + 2.5) \div 2 = 6.3(10) \div 2 \)
      \( 6(2x + 2.5) = 6.3(10) \)
      \( 12x + 15 = 63 \)
      \( 12x = 48 \)
      \( x = 4 \text{ m} \)

   **Answers:**
   a) \( 10(4.25) \div 2 = 5x + 2 \) (or \( 10(4.25) = 5x \)), so \( x = 8.5 \text{ m} \); b) \( 27.5x + 2 = 20(5.5) \div 2 \),
   so \( x = 4 \text{ m} \)
SAY: Sometimes, you’re given the area numerically and one way to get the area from a variable. Then you can solve for the variable.

**Exercises:** Solve for $x$.

a) Area = 14 m$^2$

![Diagram of a triangle with base 4 m and height $x$.

b) Area = 24 ft$^2$

![Diagram of a triangle with base 8 ft and height $x$.

c) Area = 20 m$^2$

![Diagram of a triangle with base 2x + 1 and height 4 m.

**Answers:**

a) $4x + 2 = 14$, so $4x = 28$, so $x = 7$ m; b) $x = 6$ ft; c) $x = 3$ m

**Extensions**

(MP.1) 1. Which rectangle is larger, 2 inches by 3 centimeters or 3 inches by 2 centimeters?

Hint: How many 1 in by 1 cm rectangles fit inside each rectangle?

**Answer:** They are the same size—six 1 in by 1 cm rectangles fit inside each rectangle.

2. Find the areas of the shaded triangles A and B using the formula for the area of a triangle. What fraction of the area of the rectangle is the area of A? The area of B?

![Diagram showing two shaded triangles A and B within a rectangle.

**Answer:** Area of A = $6 \times 5 \div 2 = 15$, Area of B = $10 \times 3 \div 2 = 15$, Area of A = Area of B, so the rectangle in the picture is divided into four parts of equal area. The area of A and the area of B are each $1/4$ of the area of the rectangle.

3. All squares are rectangles. Use this to make a formula for the area of a square from the formula for the area of a rectangle. Hint: What special case is a square?

**Solution:** A square is a rectangle with length = width. So, let $s$ be the side length of the square. Then the area of the square is $\ell \times w = s \times s = s^2$.

(MP.1, MP.4) 4. The purity of gold is measured in karats. For example, a 24-karat gold necklace is 100% pure gold, and an 18-karat gold necklace is 75% pure, because $\frac{18}{24} = 75\%$. This means that 75% of the mass of the necklace comes from gold.

A jeweler buys gold and then makes gold jewelry. She buys pure gold at the price of $40$ per gram and the other materials for $5$ per gram. She then marks up the price by 60% and sells the jewelry to a store. At what price will she sell a 21-karat gold necklace that weighs 16 g?

**Solution:** I solved the proportion $21/24 = x/16$, which gives $x = 14$, so 14 of the 16 grams are pure gold. She buys the gold for $40 \times 14 = 560$ and she buys the rest of the material for $5 \times 2 = 10$. Her total cost for materials is $570$. She marks up the price by 60% of $570$, which is $342$, so she charges $912$ for the necklace.
G7-15 Area of Parallelograms

Pages 115–116

Standards: 7.EE.B.4a

Goals:
Students will use the area of parallelograms to solve equations.

Prior Knowledge Required:
Can find the area of a rectangle
Can substitute numbers for variables in a formula
Can solve equations with two operations
Can perform operations with decimals in the standard order

Vocabulary: base, height, parallelogram, perpendicular, trapezoid

Materials:
rulers
a picture of a parallelogram to tape to the board

Introduce parallel lines. Tell students that parallel lines are straight lines that are always the same distance apart. Draw on the board:

Point to the first pair of lines and point out the lines are getting closer together in the one direction and farther apart in the other direction. The second pair of lines are not getting closer together or farther apart in any direction, so they are parallel. SAY: If you measure the distances from any point on one line to the other, you have to measure the perpendicular distance. Draw the lines on the board, as shown below:

Ask volunteers to measure the two distances on each picture. Are the two distances the same? (no for the first picture, yes for the second picture) SAY: Because parallel lines stay the same distance apart, they can never meet, but lines that get closer together in one direction will meet
when extended far enough. You can tell that lines are parallel when they go in exactly the same direction. In the exercise below, ensure that students are measuring the distances at a right angle.

**Exercise:** Trace the two sides of a ruler to create parallel lines. Then check that they are the same distance apart no matter where you measure the distance from.

**Introduce parallelograms and trapezoids.** Tell students that a *parallelogram* is a quadrilateral with two pairs of parallel sides, and a *trapezoid* is a quadrilateral with only one pair of parallel sides. **NOTE:** Some mathematicians define trapezoid to mean “a quadrilateral with at least one pair of parallel sides.” By this definition, all parallelograms are trapezoids. By our definition, also a common definition, no parallelograms are trapezoids. Tell students that this lesson will focus on parallelograms, and the next lesson will focus on trapezoids.

**MP.7 Exercises:** Find the area of the parallelogram by making a rectangle with the same answer. Include the units in your answer.

**Bonus:** Finish the picture yourself.

**Answers:** $4 \times 5 = 20$, so $A = 20 \text{ cm}^2$; Bonus: $3 \times 7 = 21$, so $A = 21 \text{ cm}^2$

**MP.6 Introduce base and height of a parallelogram.** Explain that the side of the parallelogram that they drew the perpendicular line to is the base of the parallelogram. The height of the parallelogram is the height when standing on the base, so the height is always measured perpendicular to the base.

Point out that “base” is also short for “the length of the base” and "height" refers to both the distance between the base and the opposite side, and the perpendicular line it is measured along.

**Exercises:** Find the base and the height of the parallelogram.

**Answers:** a) base = 4 and height = 4, b) base = 4 and height = 3, c) base = 5 and height = 3
The formula for the area of a parallelogram. Write on the board:

\[
\text{Area of parallelogram} = \text{area of rectangle} = \text{length} \times \text{width} = \text{base} \times \text{height} = b \times h
\]

SAY: The area of the parallelogram is the same as the area of the rectangle, which is the length of the rectangle times the width of the rectangle. But the length of the rectangle is the base of the parallelogram, and the width of the rectangle is the height of the parallelogram.

**Exercises:** Find the area of the parallelogram. All measurements are in inches.

a) 
\[
\begin{array}{c}
2\\5
\end{array}
\]

b) 
\[
\begin{array}{c}
3\\7
\end{array}
\]

c) 
\[
\begin{array}{c}
3
\end{array}
\]

**Answers:** a) 10 in\(^2\), b) 21 in\(^2\), c) 21 in\(^2\)

**(MP.2) When measurements are given in different units.** Tell students that, when finding the area using a formula, it’s important to check first that all the measurements are given using the same units. Draw on the board:

For each picture, ASK: Are the units the same? (yes for the first picture, no for the second) Tell students that when writing equations, they don’t need to write the units, but they have to make sure to write the units in their final answer and, if the units are different, they have to convert all the measurements to the same units first. Demonstrate by writing the equations for each parallelogram on the board:

\[
A = b \times h = 2 \times 3 = 6 \quad A = 30 \times 8 = 240
\]

So \(A = 6\) cm\(^2\)

SAY: I know the units for the first picture are cm\(^2\) because the lengths are all in centimeters. ASK: What units did I choose for the second picture, centimeters or millimeters? (millimeters)
How do you know? (because you kept 8 the same and changed 3 to 30) So what are the units for area? (mm²) Write on the board under $A = 30 \times 8 = 240$:

$$A = 240 \text{ mm}^2$$

SAY: It doesn’t matter whether you use millimeters or centimeters, as long as you use the same units for both.

**(MP.2, MP.6) Exercises:** Find the area of the parallelogram. Make sure to check your units.

- **a)**
  - $6 \text{ cm}$
  - $6 \text{ mm}$

- **b)**
  - $3.5 \text{ cm}$
  - $40 \text{ mm}$

- **c)**
  - $1.2 \text{ cm}$
  - $90 \text{ mm}$

**Bonus:**

- $20 \text{ in}$
- $1 \text{ ft}$

**Answers:**

- a) $A = 60 \times 6 = 360 \text{ mm}^2$ or $6 \times 0.6 = 3.6 \text{ cm}^2$
- b) $35 \times 40 = 1,400 \text{ mm}^2$ or $3.5 \times 4 = 14 \text{ cm}^2$
- c) $9 \times 1.2 = 10.8 \text{ cm}^2$ or $90 \times 12 = 1,080 \text{ mm}^2$
- Bonus: $15 \times 20 = 300 \text{ in}^2$ or $1 \frac{1}{4} \times 1 \frac{2}{3} = 25/12 \text{ ft}^2$ or $2 \frac{1}{12} \text{ ft}^2$

Point to the parallelogram in part a) of the exercises above. SAY: It doesn’t matter whether you use the top side or the bottom side as the base because they are equal. Opposite sides in a parallelogram are always equal.

**(MP.3) Exercises:** A parallelogram has height 80 cm and base 1.5 m.

- a) Tom says: The area is $80 \text{ cm} \times 1.5 \text{ m} = 120 \text{ cm}^2$. Is he correct? Explain.
- b) Convert both measurements to centimeters and find the area.

**Answers:**

- a) no, he used two different units—the units must be the same;
- b) height = 80 cm, base = 150 cm, area = 12,000 cm²

Any side of a parallelogram can be a base. Tape a picture of a parallelogram on the board so that no side is horizontal. SAY: The bases of the parallelogram don’t have to be horizontal, but you could make them horizontal by rotating the parallelogram. Demonstrate rotating the parallelogram so that one pair of sides is horizontal. SAY: The height is now the vertical distance between the bases. Draw this distance on the parallelogram, then rotate the parallelogram to the way it was and SAY: The height is always the perpendicular distance between the base and the opposite side, even if the base is not horizontal.

**Drawing the height of a parallelogram when the height is inside the shape.** Draw some parallelograms on the board (restrict to parallelograms in which the height is inside the
parallelogram) and thicken one side to show that it is the base. Have volunteers draw the height (using a dashed line). Encourage students to use a small square to show that the height and base are perpendicular. (see sample drawings below)

Point out that it doesn’t matter where students draw the perpendicular because the distance between the lines is always the same: the lines are parallel, so they are always the same distance apart. Tell students that the convention is to start the height at a vertex.

**Drawing the height of a parallelogram when the height is outside the shape.** SAY: Even when the height isn’t inside the parallelogram, you can still extend the base to draw a perpendicular height from the base to the opposite side. Draw on the board:

Have volunteers draw the height on the shapes above. You may need to demonstrate the first one. (see completed pictures below)

**Use the same formula for the area of parallelograms with height falling outside the parallelogram.** Draw on the board:

12 × 2 − 7 × 2 = 5 × 2

SAY: The large rectangle is 12 by 2 and the two right triangles, when put together, make a 7 by 2 rectangle, so when you subtract their areas, you get 12 × 2 − 7 × 2 = 5 × 2, by the distributive property. But that’s exactly the base of the parallelogram times the height of the parallelogram, so the formula still works.
Finding the area of parallelograms using different sides as bases. Draw on the board:

SAY: I drew the same parallelogram twice, so I know that they have the same area. Have students calculate the area for each base to make sure that they get the same answer both ways. (36 cm²)

(MP.1) Exercises: Use different sides as a base to find the area in two different ways. Make sure you get the same answer both ways.

a)

\[ A = 4 \times 6 = 24 \text{ cm}^2 \] and \[ 8 \times 3 = 24 \text{ cm}^2 \];

b)

\[ A = 3.6 \times 2 = 7.2 \text{ in}^2 \] and \[ 2.4 \times 3 = 7.2 \text{ in}^2 \];

c)

\[ A = 2.5 \times 2.5 = 6.25 \text{ ft}^2 \] and \[ 5 \times 1.25 = 6.25 \text{ ft}^2 \]

SAY: Sometimes two bases and two heights are given in the same picture. You have to be able to determine which base goes with which height.

(MP.7) Exercises: Find the area in two different ways to make an equation, then solve for \( x \).

a)

\[ 6.4(2.5) = 4x, \ 16 = 4x, \ x = 4 \text{ cm} \];

b)

\[ 6x = 8(1.5), \ 6x = 12, \ x = 2 \text{ in} \];

Bonus: convert 3 ft = 36 in and 1.5 ft = 18 in, \[ 36x = 18(20), \ x = 360/36 = 10 \text{ in} \]
Extensions

1. Find the area of the same parallelogram two different ways, using the pictures below.

Answers: $4 \times 4 = 16$ or $8 \times 2 = 16$.

(MP.7) 2. All rectangles are parallelograms. How can you get the formula for the area of a rectangle from the formula for the area of a parallelogram?

Solution: A rectangle is a parallelogram in which the sides are perpendicular. That means that both the base and the height are sides of the rectangle (in a parallelogram, only the base is a side). So the formula $A = b \times h$ becomes multiplying the side lengths, which is the formula for the area of a rectangle.

(MP.1) 3. Draw two parallelograms with the same side lengths but different areas.

Sample answer:

(MP.4) 4. An e-book costs $16 before taxes and $16.48 after taxes. A can of soda costs $1.60 before taxes and $1.68 after taxes. Which item is taxed at a higher rate?

Sample solutions:

- I found $16.48 - 16 = 0.48$, so the tax on the e-book is $0.48. I found what percentage of 16 is 0.48 by solving $0.48/16 = x/100$ and got $x = 3$, so the e-book has a 3% tax. I found $1.68 - 1.6 = 0.08$, so the tax on the can of soda is $0.08$. I found what percentage of 1.60 is 0.08 by solving $0.08/1.6 = x/100$ and got $x = 5$, so the can of soda has a 5% tax. The can of soda is taxed at a higher rate.
- I found $16.48 \div 16 = 1.03$, so the tax on the e-book is 3%, and $1.68 \div 1.6 = 1.05$, so the tax on the can of soda is 5%. The can of soda is taxed at a higher rate.
- I found $16.48 - 16 = 0.48$ and $1.68 - 1.6 = 0.08$, so the tax on the e-book is 6 times as much as the tax on the can of soda, but the e-book cost more than six times as much as the can of soda, so the can of soda must be taxed at a higher rate.

Whole-class follow-up: Brainstorm reasons why some products might be taxed at a higher rate than others. Allow students to share reasons with a partner and then have volunteers suggest reasons to the class. (sample answer: the government might want to discourage people from drinking sugary drinks so they might want to tax that more than an e-book)
G7-16  Area of Trapezoids

Pages 117–118

Standards: 7.EE.B.4a, 7.EE.B.4b

Goals:
Students will solve equations and inequalities using the area of a trapezoid.

Prior Knowledge Required:
Can find the area of a rectangle, triangle, and parallelogram
Can substitute numbers for variables in a formula
Can solve equations with two operations
Can solve inequalities with two operations
Can perform operations with decimals using the standard order of operations

Vocabulary: base, height, parallelogram, perpendicular, trapezoid

Materials:
a loose sheet of grid paper
transparency of grid paper or BLM 1 cm Grid Paper (p. T-1)

(MP.7) Developing a formula for the area of a trapezoid. Demonstrate the following to students: Fold a sheet of grid paper in two along a grid line, draw a trapezoid that is not symmetrical and has no right angles, and cut the trapezoid through both halves of the sheet to obtain two copies of the same trapezoid. Then arrange the two trapezoids so that, together, they make a parallelogram (turn one of the trapezoids upside down and place it beside the other one), as shown below:

![Diagram of trapezoid and parallelogram]

ASK: How does the area of the trapezoid compare to the area of the parallelogram? (the area of the trapezoid is half of the area of the parallelogram) Tell students you want to make a formula for the area of the trapezoid. Write on the board:

Area of trapezoid  =  (area of parallelogram) ÷ 2

ASK: What is the formula for the area of a parallelogram? (base × height) Write on the board:

= (base of parallelogram) × (height of parallelogram) ÷ 2
ASK: Which sides of the trapezoid form the base of the parallelogram—the parallel sides or the non-parallel sides? (the parallel sides) SAY: Let’s call the parallel sides of the trapezoid the bases of the trapezoid. So a trapezoid has two bases. Write on the board:

![Diagram of a trapezoid with bases labeled]

ASK: How can you get the height of the parallelogram from the sides of the trapezoid? (it is the distance between the two bases) SAY: The perpendicular distance between the bases is called the height of the trapezoid, and is equal to the height of the parallelogram. Write on the board:

\[
= (\text{base } 1 + \text{base } 2) \times (\text{height}) \div 2 \\
= (b_1 + b_2) \times h \div 2
\]

**Exercises:** Use the formula to find the area of the trapezoid.

a) ![Diagram of a trapezoid with measurements]

b) ![Diagram of a trapezoid with measurements]

c) ![Diagram of a trapezoid with measurements]

**Answers:** a) 35 cm², b) 34 cm², c) 35 cm²

**MP.4 Ignoring irrelevant information.** SAY: Sometimes there is more information than you need to solve the problem, and you have to decide which information to use.

**Exercises:** Ignore the information you don’t need to find the area of the trapezoid.

a) ![Diagram of a trapezoid with measurements]

b) ![Diagram of a trapezoid with measurements]

c) ![Diagram of a trapezoid with measurements]

d) ![Diagram of a trapezoid with measurements]
Answers: a) $3(5 + 7) + 2 = 18\text{ in}^2$, b) $5(3 + 4.5) + 2 = 18.75\text{ in}^2$, c) $8(19 + 7) + 2 = 104\text{ in}^2$, d) $1.5(1.75 + 3.25) + 2 = 3.75\text{ in}^2$, e) $2.5(2.6 + 5) + 2 = 9.5\text{ in}^2$, f) $3(8 + 16) + 2 = 36\text{ in}^2$

Writing expressions for the area when some measurements are unknown. Draw on the board:

Have volunteers dictate how to substitute the correct values into the formula. PROMPTS: What is $b_1$? What is $b_2$? What is $h$? Show this on the board:

$$\frac{(b_1 + b_2) \times h}{2} = \frac{(x + 2(x - 1))(5)}{2} = \frac{5(x + 2x - 2)}{2} = \frac{5(3x - 2)}{2}$$

Ask volunteers to explain why each step makes sense.

Exercises: Write an expression for the area of the trapezoid. Simplify.

Answers: a) $7(x + 8) + 2$, b) $(5 + x)(12 + 18) + 2 = 30(5 + x) + 2 = 15(5 + x)$, c) $4(2x + 5) + 2 = 2(2x + 5)$, d) $10(3x + x) + 2 = 10(4x) + 2 = 20x$

(MP.7) Finding the area two ways to write an equation. SAY: If you know the area as a number and you have an expression for the area, then you can find the unknown measurement.
Refer to the example above, and SAY: If we know the area is 40 ft\(^2\), then we can write an equation. Write on the board:

\[
\frac{5(3x - 2)}{2} = 40
\]

SAY: Here we have two ways of finding the area. One, we know the area is 40 ft\(^2\) because we are given that. Two, we have an expression in terms of \(x\) that tells us how to find the area. So now we can solve the equation we get by putting area = area. Write on the board:

\[
5(3x - 2) = 80
\]

SAY: You can cross multiply to get \(5(3x - 2) = 80\) or just use logic: If a number divided by 2 is 40, that number is 80. Have a volunteer finish solving the equation. (The volunteer can expand the expression, then solve \(15x - 10 = 80\), or they can divide both sides by 5 and solve \(3x - 2 = 16\). Either way, the answer is \(x = 6\).)

SAY: So now we know the two bases on the trapezoid. Update the labels in the diagram already on the board, as shown below:

![Diagram of a trapezoid with labels](image)

(MP.6) Remind students that they don’t need to write the units when solving the equation, but they need to put the units in at the end. You might point out that, in real life, not saying what units you are using can be disastrous. There was a case of a NASA Mars orbiter that was lost because the two teams working on it were using different units and forgot to mention to each other what units they were using!

**Exercises:** The area of the trapezoid is 18 cm\(^2\). Write an equation and solve for \(x\).

- a) ![Diagram](image)
- b) ![Diagram](image)
- c) ![Diagram](image)
- d) ![Diagram](image)
- e) ![Diagram](image)
- f) ![Diagram](image)
Sample solution: a) \( \frac{x(5+7)}{2} = 18 \) so \( 12x = 36 \), so \( x = 3 \text{ cm} \)

Answers: b) \( x = 1.5 \text{ cm} \), c) \( x = 1 \text{ cm} \), d) \( x = 3 \text{ cm} \), e) \( x = 4 \text{ cm} \), f) \( x = 2 \text{ cm} \)

(MP.7) Using conditions on the area to determine conditions on the unknown measurement. Refer back to the example before the previous exercises. Suppose that instead of being told that the area is exactly \( 40 \text{ ft}^2 \), we are told that it is at least \( 40 \text{ ft}^2 \). ASK: Then what would we be able to say about the unknown sides? (they are at least 6 and 10) SAY: Our intuition tells us that if the area has to be at least \( 40 \text{ ft}^2 \), then the sides should be at least as long as what we need to make the area exactly \( 40 \text{ ft}^2 \). But let’s try to see that algebraically. Write on the board:

\[
\frac{5(3x-2)}{2} \geq 40
\]

Remind students that they can solve inequalities the same way they solve equations. Have a volunteer show the same steps for solving the inequality that were used for solving the equation, as shown below:

\[
\frac{5(3x-2)}{2} \geq 40
\]

\[
5(3x-2) \geq 80
\]

\[
3x - 2 \geq 16
\]

\[
x \geq 6
\]

SAY: It makes sense that if \( x = 6 \) makes the expression equal 40, then \( x \) being at least 6 will make the expression at least 40, because as \( x \) gets bigger, so does the expression. This is always true when the coefficient of \( x \) is positive.

Exercises: The area of the trapezoid must be at least \( 60 \text{ ft}^2 \). What is \( x \)?

a) \[
\begin{align*}
8 \text{ ft} & \\
12 \text{ ft} & \\
\end{align*}
\]

b) \[
\begin{align*}
2 \text{ ft} + x & \\
12.5 \text{ ft} & \\
\end{align*}
\]

Answers: a) \( x(8 + 12) + 2 \geq 60 \), so \( x(8 + 12) \geq 120 \), so \( 20x \geq 120 \), so \( x \geq 6 \text{ ft} \);

b) \( 12.5(6.4 + 2 + x) \geq 120 \), so \( x \geq 1.2 \text{ ft} \);

c) \( 5(6 + 2x + x) \geq 120 \), so \( x \geq 6 \text{ ft} \);

d) \( 7.5(1 + 2x + 4x) \geq 120 \), so \( x \geq 2.5 \text{ ft} \)
Draw on the board:

![Diagram of a trapezoid with dimensions labeled: 7 ft, 8 ft - x, and 8 ft]

Ask a volunteer to write an expression for the area of the trapezoid. \(7(16 - x) + 2\) SAY: The area has to be at least 35 ft\(^2\). Have a volunteer add that information to the expression, then have volunteers dictate the next two lines of the solution as shown below. PROMPTS: If a number divided by 2 is at least 35, what does it say about that number? (the number is at least 70) If 7 times a number is at least 70, what does it say about that number? (the number is at least 10)

\[
\frac{7(16 - x)}{2} \geq 35
\]

\[
7(16 - x) \geq 70
\]

\[
16 - x \geq 10
\]

Then remind students that they want to make the coefficient of \(x\) positive, so they have to move it over to the other side. Write on the board:

\[
16 \geq 10 + x
\]

Then have a volunteer finish the solution as shown below:

\[
16 - 10 \geq x
\]

\[
6 \geq x
\]

\[
x \leq 6
\]

**Exercises:** The area of the trapezoid must be at least 100 ft\(^2\). What is the measure of \(x\)?

a) ![Image of trapezoid with dimensions 8 ft, 25 ft - x, and 10 ft]

b) ![Image of trapezoid with dimensions 12.5 ft, 10 ft - x, and 10 ft]

c) ![Image of trapezoid with dimensions 8 ft, 15 ft - x, and 15 ft]

**Answers:** a) \(8(10 + 25 - x) + 2 \geq 100\), so \(x \leq 10\) ft; b) \(12.5(20 - x) + 2 \geq 100\), so \(x \leq 4\) ft; c) \(8(30 - x) + 2 \geq 100\), so \(x \leq 5\) ft

**MP.8** A triangle is like a special case of a trapezoid. After students do Question 5 on AP Book 7.2, p. 15, point out that a triangle is a special case of a trapezoid with one of the bases being 0.
Extensions
1. Tom and Beth are finding the area of the trapezoid:

Tom and Beth both make parallelograms twice as big:

![Parallelograms](image)

a) Did they make the same parallelogram?
b) What did they do differently?
c) Do their parallelograms have the same area?
d) Do they get the same answer for the area of the trapezoid?

**Answers:** a) no, b) Tom flipped the trapezoid upside down and put it to the right of the original trapezoid. Beth also flipped the trapezoid upside down, but she put it to the left of the original trapezoid. c) yes, d) yes

(MP.7) 2. You discovered in Question 5 on AP Book 7.2, p. 15 that a triangle is like a trapezoid with $b_1 = 0$.

a) What shape is like a trapezoid with $b_1 = b_2$?
b) Find a formula for the area of that shape by using $b_1 = b_2$ in the formula for the area of a trapezoid. Do you get the correct formula?

**Answers:** a) parallelogram, b) $(b_1 + b_2)h = \frac{(b_1 + b_2)h}{2} = \frac{2bh}{2} = b_1h$

(MP.6, MP.7) 3. Mindy solved $83x = 1,328$ and got $x = 16$.

a) Is Mindy correct?
b) Use Mindy’s solution to solve the equation. Explain how you came up with your answer.

i) $5(83x) = 5(1,328)$
ii) $83x + 2 = 1,330$
iii) $83x = −1,328$
iv) $166x = 1,328$
vi) $83(x + 5) = 1,328$

**Answers:**
a) Mindy is correct because $83 \times 16 = 1,328$.
b) i) $x = 16$ because multiplying both sides by 5 doesn’t change the value of $x$ that makes both sides equal; ii) $x = 16$ because adding 2 to both sides doesn’t change the value of $x$ that makes both sides equal; iii) $x = −16$ because $83 \times (−16) = −(83 \times 16) = −1,328$; iv) $x = 8$ because $166$ is double $83$, so in order to get the same product on the right side, I need to multiply by half of $16$, which is $8$; v) $x + 5$ must be $16$ because Mindy found $83 \times 16 = 1,328$ and that means $x = 11$; vi) I changed the left side to $83(x + 1)$, and $x + 1$ must equal $16$ because Mindy found $83 \times 16 = 1,328$, and that means $x = 15$
G7-17  Areas of Composite Shapes

Pages 119–120

Standards: 7.G.B.6

Goals:
Students will solve problems involving the area of two-dimensional objects composed of triangles, quadrilaterals, and polygons, including shapes that overlap.

Prior Knowledge Required:
Can find the area of rectangles, triangles, trapezoids, and parallelograms
Can find missing side lengths when there is enough information to do so

Vocabulary: parallelogram, trapezoid

Areas of shapes made of rectangles. Draw on the board:

Tell students that the shapes are identical, but they are broken down into rectangles differently. Refer students to the first picture. ASK: What are the side lengths of the white rectangle? (4 cm and 6 cm) What are the sides lengths of the shaded rectangle? (13 cm and 5 cm) Write on the board:

\[
\text{Area} = 4 \times 6 + 13 \times 5 \\
= 24 + 65 = 89
\]

Exercise: Find the areas of the white rectangle and the shaded rectangle in the second picture. Make sure they add to 89.
Answer: \[6 \times 9 + 5 \times 7 = 54 + 35 = 89\]
SAY: All these lengths are in centimeters, but not all sides are given. Have a volunteer outline the two missing lengths. Point to the bottom side and ASK: How can you find this length? (add the two top sides) Write under the bottom side:

\[ 4 + 7 = 11 \]

Repeat for the right side. (this time, \( 1 + ? = 5 \), so the unknown side is \( 5 - 1 = 4 \)) Write next to the right side:

\[ 5 - 1 = 4 \]

Ask students to find the area of the rectangle by dividing it into two smaller rectangles. 
\((4 \times 1 + 4 \times 11 = 4 + 44 = 48 \) or \(5 \times 4 + 7 \times 4 = 20 + 28 = 48 \)\) Then ask students to look at the larger rectangle (extend the top and right sides to see it), and to find the area by subtracting the small rectangle that emerges in the top right from the larger rectangle. \((11 \times 5 - 1 \times 7 = 55 - 7 = 48 \)\)

Exercises: a) Find the area by adding the areas of two smaller rectangles.

\[ \begin{array}{ccc}
\text{i)} & 12 \text{ m} & 23 \text{ m} \\
\text{ii)} & 4 \text{ ft} & 3 \text{ ft} \\
\text{iii)} & 2 \text{ cm} & 3.2 \text{ cm} \\
\end{array} \]

b) Find the area of the shapes in part a) by subtracting the area of a small rectangle from the area of a large rectangle

Answers: a) i) \( A = 180 \text{ m}^2 + 276 \text{ m}^2 = 456 \text{ m}^2 \), ii) \( A = 12 \text{ ft}^2 + 45 \text{ ft}^2 = 57 \text{ ft}^2 \), iii) \( 2 \text{ cm}^2 + 2.4 \text{ cm}^2 = 4.4 \text{ cm}^2 \); b) i) \( A = 621 \text{ m}^2 - 165 \text{ m}^2 = 456 \text{ m}^2 \), ii) \( A = 72 \text{ ft}^2 - 15 \text{ ft}^2 = 57 \text{ ft}^2 \), iii) \( 6.4 \text{ cm}^2 - 2 \text{ cm}^2 = 4.4 \text{ cm}^2 \)

Combining units. SAY: The lengths in a shape might not always be given in terms of the same unit. In that case, you have to change them to be in the same unit. SAY: It doesn’t matter which unit you use, as long as you use the same unit throughout.

(MP.2) Exercises: Find the area by changing all the measurements to one unit. Then find the area again by changing all the measurements to the other unit.

a)  

\[ \begin{array}{ccc}
32 \text{ mm} & 4 \text{ cm} & 1 \text{ cm} \\
36 \text{ mm} & & \\
\end{array} \]

\[ A = 21.2 \text{ cm}^2 = 2,120 \text{ mm}^2 \]

b)  

\[ \begin{array}{ccc}
1.15 \text{ m} & 150 \text{ cm} & 2.75 \text{ m} \\
285 \text{ cm} & & \\
\end{array} \]

\[ A = 62,850 \text{ cm}^2 = 6.285 \text{ m}^2 \]
Tell students that their answers should be the same, but in different units. Write on the board:

1 cm

1 cm = ______ mm

So 1 cm² = ______ mm²

Ask a volunteer to fill in the blanks. (1 cm = 10 mm, so 1 cm² = 100 mm²) Have students determine the missing value in 1 m² = ______ cm². (1 m = 100 cm, so 1 m² = 10,000 cm²) Together, verify that the answers to the exercises above are indeed the same, regardless of the units.

(MP.1, MP.6) Exercises: Change all the measurements to one unit of measurement and find the area. Then change them all to the other unit, and find the area again. Check that you got the same answer both ways.

a)  b)

Answers: a) A = 46 cm² = 4,600 mm² (1 cm² = 100 mm²); b) A = 5.6 m² = 56,000 cm² (1 m² = 10,000 cm²)

Areas of shapes composed of rectangles, triangles, trapezoids, and parallelograms. Tell students that they have already found areas of composite shapes when they found the area of parallelograms and trapezoids. Draw on the board:

SAY: If you can find the area of each part of a shape, then you can find the total area.

(MP.7) Exercises:
1. Find the area of the composite shapes. All measurements are in feet.

a)  b)

Answers: a) 26 ft², b) 11 ft²
**Bonus:** Draw a pentagon and divide it into triangles. Measure the base and height of each triangle in centimeters. Find each area, then find the total area of the polygon.

**Sample answer:**

So area = \( \frac{1.2 \times 1.9}{2} + \frac{1.5 \times 2.3}{2} + \frac{1.2 \times 1.9}{2} = 4.005 \text{ cm}^2 \)

**Solving equations to find an unknown measurement.** Draw on the board:

![Diagram](image)

Ask a volunteer to write an expression for the area of the shaded part. (32 – 3x) SAY: The area of the shaded part is 14 cm², and I want to know the unknown measurement. Write on the board:

\[ 32 - 3x = 14 \]

ASK: What’s the first thing we have to do to solve the equation? PROMPT: When we find the solution, the coefficient of \( x \) is 1, and \( x \) is by itself. (make the coefficient of \( x \) positive by moving the variable term to the other side) Write on the board:

\[ 32 = 14 + 3x \]

Ask a volunteer to finish solving the equation. (see solution below)

\[ 32 = 14 + 3x \]
\[ 32 - 14 = 3x \]
\[ 18 = 3x \]
\[ x = 6 \]

**(MP.7) Exercises:** All measurements shown are in meters. The area of the shaded figure is 50 m². Evaluate \( x \).

a) ![Diagram](image)

b) ![Diagram](image)

**Bonus:**

![Diagram](image)
**Answers:**
a) \(10x - 2.5(8) = 50\), so \(x = 7\) m; 
b) \(5(12) - 2.5x = 50\), so \(x = 4\) m; 
Bonus: \((10 + 2x)(5) + 2 - (x + 1)(2.5) + 2 = 50\), so \(x = 7\) m

**(MP.7) Areas of overlapping shapes.** Draw on the board:

ASK: When you put these rectangles end to end, what is the area? \((5 + 9 = 14)\) Now what if we draw the rectangles so that they overlap one square—what is the new area? Draw on the board:

SAY: If we add the areas of the two rectangles \((5 + 9 = 14)\) we count the overlapping part twice, but we only need to count it once, so we have to subtract it. SAY: When you count the squares across, there are only 13 because one is overlapping. Write on the board:

\[
\text{Area} = 5 + 9 - 1 = 13
\]

Repeat with an overlap of 2 squares and 3 squares, but have volunteers write the expression for the area as shown below:

Area = 5 + 9 − 2 = 12  
Area = 5 + 9 − 3 = 11

SAY: You can always calculate the area of any two overlapping shapes by adding the two areas and subtracting the area of the overlapping part, because you counted the overlapping part twice. Draw on the board:
Exercises: Find the total area. All measurements are in feet.

a)  
\[ A = 3(2) + 4(2) - 1(1.5) = 12.5 \text{ ft}^2 \]

b)  
\[ A = 20 + 28 - 4 = 44 \text{ ft}^2 \]

c)  
\[ A = 30 + 16 - 1 = 45 \text{ ft}^2 \]

d)  
\[ A = 81 + 52.25 - 5.5 = 127.75 \text{ ft}^2 \]

e)  
\[ A = 84 + 84 - 37.72 = 130.28 \text{ ft}^2 \]

Answers: a) \[ A = 3(2) + 4(2) - 1(1.5) = 12.5 \text{ ft}^2 \], b) \[ A = 20 + 28 - 4 = 44 \text{ ft}^2 \], c) \[ A = 30 + 16 - 1 = 45 \text{ ft}^2 \], d) \[ A = 81 + 52.25 - 5.5 = 127.75 \text{ ft}^2 \], e) \[ A = 84 + 84 - 37.72 = 130.28 \text{ ft}^2 \]

Extensions

1. The area and perimeter of the same rectangle were calculated two ways:

\[
\begin{array}{c|c|c}
\text{Length} & \text{Width} & \text{Area} & \text{Perimeter} \\
\hline
1 \text{ cm} & 2 \text{ cm} & 2 \text{ cm}^2 & 6 \text{ cm} \\
10 \text{ mm} & 20 \text{ mm} & 200 \text{ mm}^2 & 60 \text{ mm} \\
\end{array}
\]

John says the perimeter is greater than the area because 6 > 2. Mona says the area is greater than the perimeter because 200 > 60. Who is correct?

Answer: Neither is correct. You cannot compare area and perimeter because there is no way of putting them in the same units.

(MP.3) 2. The lengths of two sides of a triangle are each \( s \) cm. What can you say about the third side? Be as specific as you can. Explain how you know.

Answer: Let \( x \) be the length of the third side. If \( s \) is the longest side, then \( x \leq s \) and \( x + s > s \), so \( 0 < x \leq s \). If \( x \) is the longest side, then \( x < s + s \). So \( s \leq x < 2s \). In any case, \( 0 < x < 2s \).
G7-18 Areas of Scale Drawings

Pages 121–123

Standards: 7.G.A.1, 7.RP.A.2

Goals:
Students will compute actual areas from a scale drawing.

Prior Knowledge Required:
Can compute actual lengths from a scale drawing
Can make a scale drawing given the original drawing and the scale
Can compute the area of a rectangle, triangle, parallelogram, and trapezoid

Vocabulary: base, height, exponent, parallelogram, scale, square of a number, trapezoid

Materials:
grid paper or BLM 1 cm Grid Paper (p. T-1)

Comparing areas of scale drawings.

(MP.8) Exercises: Copy the shape onto grid paper and draw a scale drawing with all sides twice as long. Then find the area of both shapes.

Answers: a) original: $A = 8$, scale drawing: $A = 32$; b) original: $A = 12$, scale drawing: $A = 48$; c) original: $A = 16$, scale drawing: $A = 64$; d) original: $A = 15$, scale drawing: $A = 60$

ASK: What do you multiply the smaller area by to get the larger area? (4) SAY: When all sides multiply by 2, the area multiplies by 4. Demonstrate the special case of a square. Draw on the board:

ASK: If all sides multiply by 3, what do you predict the area will multiply by? (some students might say 6; some students might say 9) ASK: How could you check your prediction? PROMPT: What is the easiest shape to find the area of? (a square) SAY: Draw a 1 by 1 square
and multiply each side length by 3. ASK: What is the new area? (3 × 3 = 9) SAY: So when you multiply the side lengths by 3, the area multiplies by 9.

Write on the board:

\[
\text{Area} = (b_1 + b_2) \times h \div 2 = \frac{(b_1 + b_2)h}{2}
\]

**Exercises:**

a) Find the area of the trapezoid.

![Trapezoid diagram](image)

b) The original trapezoid is shown in part a). Find the bases and the height of a trapezoid drawn to the given scale.

i) (original) : (new) = 1 : 2
ii) (original) : (new) = 1 : 3
iii) (original) : (new) = 1 : 4

c) Find the area of the trapezoid using the new bases and heights from part b).

d) For each trapezoid in part b), how many times the original area is the area?

**Answers:**
a) 5 cm²; b) i) \(b_1 = 12, b_2 = 8, h = 2\), ii) \(b_1 = 18, b_2 = 12, h = 3\), iii) \(b_1 = 24, b_2 = 16, h = 4\);

ii) \(A = 20 cm^2\), ii) \(A = 45 cm^2\), iii) \(A = 80 cm^2\); d) i) 4 times, ii) 9 times, iii) 16 times

**MP.3** Finding the area of the scale drawing from the actual area and the scale. ASK: How can you get the area of a scale drawing if you know the scale and the area of the original drawing? (multiply the original area by the scale times the scale) Write on the board:

The area of a scale drawing with (original) : (new) = 1 : \(s\) is

\[
(\text{original area}) \times s \times s = (\text{original area}) \times s^2
\]

SAY: Remember that when you multiply a number by itself, you are taking the number to the exponent 2, and you can show that by a raised 2.

**MP.7** Exercises: The area of an actual shape is given, along with the ratio of a scale drawing. What is the area of the scale drawing?

a) The area of a square is 16 cm². The scale drawing has ratio (original) : (new) = 1 : 3.
b) The area of a parallelogram is 6 cm². The scale drawing has ratio (original) : (new) = 1 : 5.
c) The area of a hexagon is 8.5 cm². The scale drawing has ratio (original) : (new) = 1 : 2.
d) The area of a star is 11 cm². The scale drawing has ratio (original) : (new) = 1 : 3.

**Answers:** a) \(A = 16 cm^2 \times 9 = 144 cm^2\), b) \(A = 6 cm^2 \times 25 = 150 cm^2\),
c) \(A = 8.5 cm^2 \times 4 = 34 cm^2\), d) \(A = 11 cm^2 \times 9 = 99 cm^2\)
Choosing between formulas. Draw on the board:

![Shapes](image)

Ask volunteers to write the formula they would use to find the area of each shape. \((b \times h + 2, ℓ \times w, (b_1 + b_2) \times h + 2, b \times h)\)

Draw and label the measurements for the first shape on the board, as shown below:

![Labelled Shape](image)

Have students copy the remaining three shapes to use in the following exercise.

**Exercise:** Draw and label the measurements used in the formulas for each shape.

**Answers:**

![Shapes](image)

**Fractional scales.** SAY: When you find areas of scale drawings, the ratio (original) : (new) can be a fraction, too.

**Exercises:**

a) Write the dimensions of the shape drawn to the scale (original) : (new) = \(1 : \frac{3}{4}\).

i) \(ℓ = 15, w = 6\); ii) \(b = 12, h = 9\); iii) \(b_1 = 15, b_2 = 9, h = 6\);

b) Find the area of the original shapes and the scale drawings from part a) by using the area formula for each shape.

**Answers:**

a) i) \(A = 160\ \text{in}^2\), scale drawing: \(A = 90\ \text{in}^2\), ii) \(A = 192\ \text{in}^2\), scale drawing: \(A = 108\ \text{in}^2\), iii) \(A = 128\ \text{in}^2\), scale drawing: \(A = 72\ \text{in}^2\); c) i) \(90\ \text{in}^2\), ii) \(108\ \text{in}^2\), iii) \(72\ \text{in}^2\), yes
Finding the area of the actual figure from the area of the scale drawing and the scale.

**(MP.4) Exercises:** Use the scale to find the area of each room. Convert the measurements first.

- **a)** Scale: 1 cm = 2 ft
- **b)** Scale: 1 cm = 1 yd
- **c)** Scale: 1 cm = 5 ft

**Answers:**

- **a)** Dining Room: \[ A = 15 \times 8 = 120 \text{ ft}^2 \], Kitchen: \[ A = 6 \times 13 = 78 \text{ ft}^2 \]  
- **b)** Banquet Hall: \[ A = 9 \times 14 = 126 \text{ yd}^2 \], Foyer: \[ A = 7 \times 4 = 28 \text{ yd}^2 \]  
- **c)** Bedroom: \[ A = 12 \times 10 = 120 \text{ ft}^2 \], Hall: \[ A = 4 \times 30 = 120 \text{ ft}^2 \], Closet: \[ A = 5 \times 2 = 10 \text{ ft}^2 \]

Write on the board:

\[
(\text{original length}) : (\text{scaled length}) = 1 : s
\]

\[
(\text{scaled area}) = (\text{original area}) \times s^2
\]

**SAY:** The \( s \) is what you multiply the original length by to get the scaled length, but I could start with the scaled length and ask what do I multiply the scaled length by to get the original length. Write on the board:

\[
(\text{original length}) : (\text{scaled length}) = t : 1
\]

**SAY:** Sometimes the unit ratio tells us what to multiply the scaled length by to get the original length. Write on the board:

\[
(\text{original area}) = (\text{scaled area}) \times t^2
\]

This tells us how to get the area of the original figure from the area of the scaled drawing.

**(MP.1) Exercise:** Verify your answers to part a) of the previous exercises by finding the area of the scaled drawing first, then finding the area of the actual rooms.

**Solution:**

\[ (\text{original}) : (\text{scaled}) = 2 \text{ ft} : 1 \text{ cm}, \text{ so } t = 2, \text{ so multiply the scaled area in cm}^2 \text{ by 4 to get the actual area in ft}^2 \text{. The scaled area for the dining room is 30 cm}^2, \text{ so the actual area is } 120 \text{ ft}^2, \text{ the scaled area for the kitchen in 19.5 cm}^2, \text{ so the actual area for the kitchen in 78 ft}^2 \]
Comparing areas. SAY: I want to compare the areas of the rooms to know which one is larger. I can tell easily when both rooms are measured in square feet, but when one room is measured in square yards, how can I tell which is larger? PROMPTS: How many feet are in a yard? (3) How many square feet are in a square yard? (9) Draw on the board:

So 1 yd$^2 = 9$ ft$^2$

(MP.4) Exercise: Which room in the previous exercise is largest? Which is smallest? Answers: The largest room is the banquet hall. Its area is 126 yd$^2 = 1,134$ ft$^2$. The smallest room is the closet. Its area is 10 ft$^2$.

Exercises: Use the floor plan and scale to find the area of each room. Which room is largest?

<table>
<thead>
<tr>
<th>A. Scale: 1 cm = 2 ft</th>
<th>B. Scale: 1 cm = 3 yd</th>
<th>C. Scale: 2 cm = 1 ft</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 cm</td>
<td>1.5 cm</td>
<td>24 cm</td>
</tr>
<tr>
<td>4.5 cm</td>
<td>3 cm</td>
<td>30 cm</td>
</tr>
</tbody>
</table>

Answers: A. $A = 16 \times 9 = 144$ ft$^2$ or $8 \times 4.5 = 36$ cm$^2$ and $36 \times 4 = 144$ ft$^2$, so $A = 144$ ft$^2$; B. $A = 4.5 \times 9 = 40.5$ yd$^2$; C. $A = 15 \times 12 = 180$ ft$^2$; so Room B is the largest. Its area is 40.5 yd$^2 = 364.5$ ft$^2$.

Finding the area from a scale drawing two ways. Tell students that Wyoming is roughly shaped like a rectangle. Draw a rough map of Wyoming on the board:

```
73 mm
```

Casper
```
55 mm
```
Cheyenne

Scale: 1 mm = 5 mi

ASK: What is the area of the map? $(73 \text{ mm} \times 55 \text{ mm} = 4,015 \text{ mm}^2)$ How can you use the area of the map to get the area of Wyoming? (multiply by the square of the scale) Write on the board:

$4,015 \times 25 = 100,375$ mi$^2$
Point to 4,015 and SAY: This tells you how many square millimeters are in the drawing. Point to the 25 and SAY: There are 5 mi for each millimeter of drawing, so there are 25 mi² for each square millimeter in the drawing. So the total area is 100,375 mi².

ASK: How can you get the length and the width of Wyoming from the map? (multiply them by 5 to get the number of miles across) How can you get the area from the length and the width? (multiply the length and the width) Have students determine the area this way. Tell students to make sure they get the same area this way.

**MP.1** Exercises: The picture is a scale drawing of a regulation soccer field with the scale 1 cm = 12 ft.

![Diagram of soccer field scale drawing]

Find the area of the field in two ways:
a) Find the area of the scale drawing, then use it to find the area of the actual field.
b) Find the actual length and width, then use them to find the area of the field.
Did you get the same answer both ways? If not, find your mistake.

**Answers:**

a) Scale drawing $A = 600 \text{ cm}^2$, Field $A = 600 \times 12 \times 12 = 86,400 \text{ ft}^2$;
b) Field: $l = 30 \times 12 = 360 \text{ ft}$, $w = 20 \times 12 = 240 \text{ ft}$, $A = l \times w = 360 \times 240 = 86,400 \text{ ft}^2$

**Scales in which neither term of the ratio is 1.** Write on the board:

![Diagram of room scale drawing]

$2 \text{ cm} = 5 \text{ ft}$

The area of the scale drawing is 80 cm².

Tell students that we want to know how large this room is, but neither term of the ratio is 1. SAY: I know the area of the scale drawing in square centimeters, and I want the number of square feet in the actual room. Write on the board:

$______ \text{ ft}^2 = 1 \text{ cm}^2$
ASK: If you knew how many square feet were represented by 1 cm², then could you tell me how many square feet were represented by 80 cm²? (yes) How? (multiply by 80) Write on the board:

5 ft = 2 cm

____ ft = 1 cm

____ ft² = 1 cm²

Have volunteers fill in the blanks. (5/2 or 2.5 and 25/4 or 6.25) So what is the area of the room in square feet? (80 × 25/4 = 500 ft²) Write on the board:

\[
\frac{\text{scaled area}}{\text{original area}} \times \left(\frac{\text{original}}{\text{scaled}}\right)^2 = \text{original area}
\]

\[
80 \times \frac{25}{4} = 500 \text{ ft}^2
\]

SAY: In the exercises below, sometimes you will be given the original area and sometimes you will be given the scaled area.

(MP.4) Exercises:
a) The area of an Olympic ice rink is 1,800 m². A school builds an ice rink to the scale (Olympic rink) : (school rink) = 5 : 4. What is the area of the school rink?
b) A picture used 5 mL of paint. Another picture is drawn to the scale (original) : (new) = 5 : 8. How much paint will the new picture need?

Bonus: A picture drawn on 2 cm grid paper has area 400 cm². Len copies the same picture on a wall, using grids 3 ft long. His picture will be half red and 1 can of paint covers 350 ft² of wall. How many cans of red paint will Len need?

Answers: a) 1,800 × (4/5)² = 1,152 m² or (school rink) × (5/4)² = 1,800 m²; b) 12.8 mL;

Bonus: The area of the painting is 400 × (1.5)² = 900 ft², so 450 ft² is red, so Len will need 2 cans of red paint.

Extensions
1. A scale drawing of a room is shown. The area of the room is 480 ft². What is the scale of the drawing?

\[
\begin{array}{c}
15 \text{ cm} \\
8 \text{ cm}
\end{array}
\]

Solution: The area of the rectangle is 120 cm², so 1 cm² = 4 ft², so 1 cm = 2 ft is the scale of the drawing.
2. For Question 11 on AP Book 7.2, p. 20, how can you get the area of the school soccer field in square feet without doing any calculations? Explain. **Answer:** Since the soccer field is 8,514 yd², the school field is 8,514 ft², because 1 foot is one third of 1 yd.

3. Which room is larger? A. B. 

**Solution:** A. $A = (3 \text{ yd} \times 2 \text{ yd}) + (1 \text{ yd} \times 4 \text{ yd}) = 10 \text{ yd}^2 = 90 \text{ ft}^2$; B. $A = (11 \text{ ft} \times 5 \text{ ft}) + (2 \text{ ft} \times 4 \text{ ft} + 2) = 59 \text{ ft}^2$, so room A is bigger.

4. The side length of a square increases by 10%. By what percentage will the area of the square increase? Explain. **Sample solutions:** • The new side length becomes $x + 0.1x = 1.1x$, and the area increases from $x^2$ to $1.21x^2$, because $1.1x \times 1.1x = 1.21x^2$. That is $x^2 + 21/100x^2$, so the area increased by 21%. • $1.1x$ is the new side length because that is another way of writing 10% more than $x$. So the area becomes $1.1x \times 1.1x = 1.21x^2$. The original area was $x^2$, so this is an increase of 21%.

Whole-class follow-up: Have volunteers present both solutions above. ASK: How did writing the side length as $1.1x$ instead of $x + 0.1x$ make finding the area easier? (you are multiplying a single term by itself instead of two terms) How can you tell that $1.21x^2$ is a 21% increase on the original $x^2$? (you’re adding $1x^2$ to $0.21x^2$, so you’re adding 21% of $x^2$ to $x^2$)

5. A store offers you a choice between two options. Which would you choose? a) 3 pairs of socks for the price of 2 or 30% off all pairs of socks b) 3 pairs of skis for the price of 2 or 30% off all pairs of skis c) Discuss with a partner: What is the same for both questions? What is different? What might make your answers different? **Selected sample answers:** a) if all pairs of socks are the same price, the first option gives a discount of $1/3 = 33.3\% > 30\%$, so the first option saves more money when you buy 3 pairs of socks (or multiples of 3 pairs of socks), so I would choose the first option; b) if you are buying skis for 3 people, and all the skis are the same price, then buying 3 for the price of 2 is the better buy, but if you only need one pair, or two pairs, or even if you need four pairs, it would be better to get 30% off all pairs.
G7-19 Circles

Pages 124–126

Standards: 7.G.A.2

Goals:
Students will construct circles with given radius and diameter.

Prior Knowledge Required:
Knows the triangle inequality

Vocabulary: center, circle, congruent, diameter, non-congruent, radius

Materials:
transparency of BLM Circle (p. P-95)
large compass to use on the board (or a shoelace)
compasses
centimeter rulers
circular object, such as a can or lid, for each student
BLM Congruent Triangles and The Geometer's Sketchpad® (p. P-96)

A circle is the set of points all the same distance from its center. Project BLM Circle on the board. Ask students if anyone knows the mathematical definition of a circle. Tell students that a circle is the set of points that are the same distance from a given point, and that point is the center of the circle. Have volunteers find the distance from the center to various points on the circle to verify that the distance is the same.

Using a compass to draw a circle. Demonstrate how to use a compass to draw a circle by using a large compass on the board:
Step 1: Put the point of the compass at the center of the circle.
Step 2: Put the pencil of the compass at any point.
Step 3: Keeping the point at the center, and without changing the width of the compass, draw the circle.

If you don't have a large compass, you can use a shoelace. You can place the chalk depending on how large you want the circle. Provide each student with a compass for the following exercises.

Exercises:
a) On a blank sheet of paper, draw several circles with the same center.
b) Draw a circle that takes up less than half the page. Then use any point on the circle as the center and draw another circle the same size (i.e., don't change the width of the compass).
c) Draw a fairly large circle. Then draw a smaller circle with a different center that lies completely inside the larger circle.
**Introduce the term “radius.”** Remind students that the distance from the center of the circle to any point on the circle is the same for all points on the circle. SAY: That distance is called the *radius* of the circle. Provide each student with a centimeter ruler and a compass.

**Exercises:** Set the width of the compass to the given radius, then draw a circle.
- a) radius = 3 cm
- b) radius = 4 cm
- c) radius = 35 mm

**NOTE:** If your students only have inch rulers, replace parts a) to c) with 1 in, 2 in, and 2 1/2 in.

**Introduce the term “diameter.”** Draw on the board:

A. ![](circle_diameter_a.png)  B. ![](circle_diameter_b.png)  C. ![](circle_diameter_c.png)

Tell students that a *diameter* of a circle is any line that joins two points on the circle and passes through the center of the circle. ASK: Which of the circles shows a diameter? (Circle A) Why aren’t the other two diameters? (the line in B doesn’t join two points on the circle and the line in C doesn’t pass through the center)

Use a large compass to draw another circle on the board. Have volunteers draw several diameters. Point out how each one of them is twice as long as the radius. Then SAY: The radius is halfway across the circle just to the center, and the diameter is the whole way across the circle. You can use the word “diameter” to mean the length of the line or the line itself.

**Exercises:** What is the radius of a circle with the following diameter?
- a) 6 in
- b) 4 cm
- c) 10 ft
- d) 7 in
- e) 9 mm

**Answers:**
- a) 3 in
- b) 2 cm
- c) 5 ft
- d) 3.5 in
- e) 4.5 mm

ASK: How would you draw a circle with diameter 4 in? (put the radius at 2 in, then draw the circle) SAY: You need to know the radius when drawing a circle. If you are given the diameter, you have to divide by 2.

**Exercises:** Draw a circle with the given diameter.
- a) 3 in
- b) 7 cm
- c) 0.5 ft
SAY: Sometimes you will be given the diameter and sometimes you will be given the radius. Draw on the board:

- 19 cm
- 26 cm

Have volunteers tell you the radius and the diameter of each circle. Make sure they include the units in their answer. (radius 19 cm and diameter 38 cm; radius 13 cm and diameter 26 cm)

**Exercises:** Write the radius ($r$) and diameter ($d$) of the circle.

- a) $r = 22$ mm, $d = 44$ mm;  
- b) $r = 12$ mm, $d = 24$ mm;  
- c) $r = 5$ mm, $d = 10$ mm;  
- d) $r = 15.5$ mm, $d = 31$ mm

Draw a circle on the board and have volunteers measure the distance across the circle at various heights:

ASK: Where is the circle the widest? (in the middle) SAY: The distance between two points on a circle is greatest when the line joining them passes through the center of the circle. That means the diameter of a circle is the largest width across the circle.

**Activity 1**

Give students something circular such as a can or lid, and have them trace the circle. Then have students use a ruler to find the longest way across the circle horizontally. Repeat for the longest vertical way across the circle. Students should ensure that they get the same distance for both diameters. Have students use their work to locate the center of the circle. (the center is where the two lines meet because both diameters pass through the center)

(end of activity)

**Drawing triangles with given side lengths.** Tell students that they can use circles to draw a triangle with given side lengths. Leaving plenty of space around the line, draw on the board:

- $18$ in
- $18$ in
- $10$, $12$, $18$ in
SAY: I drew the longest side to start with, but I could have started with any side. I drew a horizontal line because that is easiest to work with. Have a volunteer draw a circle with radius 10 inches centered at point $A$ and another volunteer draw a circle with radius 18 inches centered at point $B$. Students can use a large compass, if available. (see completed picture below)

![Completed picture showing two circles with centers $A$ and $B$.](imaginary)

ASK: How can I find the third vertex, $C$? (it must be at the intersection of the two circles) SAY: It doesn’t matter which of the two intersection points you use. Both work to make a triangle with the given conditions. Using one of the two points, draw the triangle in red. Using the other point, draw a blue triangle. SAY: These are two different triangles, but they are congruent. ASK: What does it mean for two triangles to be congruent? (same size and shape; I can fit one exactly onto the other without overlaps or gaps) How can you fit the top triangle onto the bottom triangle? (flip it) SAY: You could make two more triangles by centering the larger circle at $A$ and the smaller circle at $B$, but all four circles would be congruent.

**Exercises:** Draw a horizontal line 5 cm long. Then construct four different triangles with side lengths 5 cm, 2 cm, and 4 cm. Are all four triangles congruent? If not, find your mistake.

**(MP.8) Using a shortcut to draw triangles.** Tell students that they don’t need to draw the whole circle to find the third point. SAY: You only need to draw a quarter of each circle. Demonstrate on the board as shown below:

![Shortcut to draw triangles](imaginary)

**Exercises:** Use the shortcut to draw a triangle with the given measurements.
- a) 5 cm, 8 cm, 9 cm
- b) 4 cm, 5 cm, 5 cm
- c) 3 cm, 4 cm, 5 cm

SAY: Remember that not all three given side lengths can be the sides of triangles. ASK: What is the condition for three side lengths to make a triangle? (the smaller two have to add to more than the largest)

**Exercises:** Do the three sides make a triangle? Use a compass to check your answer by trying to draw the triangle.
- a) 2 cm, 5 cm, 6 cm
- b) 2 cm, 5 cm, 8 cm
- c) 2 cm, 5 cm, 7 cm

**Answer:** only a) makes a triangle
Discuss what goes wrong in b) and c) when they try to draw the triangle. (in part b), the two circles don't intersect, so there is no point that is 2 cm from one point and 5 cm from the other; in part c), the third point is on the 7 cm line, so it doesn't make a triangle) Draw a picture of each case on the board:

a) ![Diagram](image1)
b) ![Diagram](image2)
c) ![Diagram](image3)

**Activity 2**
Have students complete **BLM Congruent Triangles and The Geometer's Sketchpad®**. Students will use The Geometer's Sketchpad® to verify that three side lengths are enough to determine a unique triangle. (3. \(AB = DE, BC = EF, AC = FD\); 4. rotations (by moving e) or translations (by moving D or F), but not reflections; 5. yes, the triangles are congruent; 6. a) yes, I can change the side lengths. I couldn't change them when moving \(\Delta DEF\) because the side lengths were fixed from \(\Delta ABC\); b) \(\Delta DEF\) is modified the same way as I modify \(\Delta ABC\), keeping all side lengths equal; 7. yes)

(end of activity)

**Extensions**
1. Construct a circle with three given side lengths by starting with a shorter side instead of the longest side.
   a) Draw a line 4 cm long, and find all the points that are 2 cm from one of the vertices and all the points that are 5 cm from the other point.
   b) Find three points that make a triangle with side lengths 2 cm, 4 cm, and 5 cm.
   c) Cut out the triangle you made in part b) and place it on top of one of the other triangles you made earlier to check that they are congruent.

**Bonus:** Start with a line 2 cm long and draw a triangle with side lengths 2 cm, 4 cm, and 5 cm.

**Answers:**
Start with the 5 cm side: ![Diagram](image4)
Start with the 4 cm side: ![Diagram](image5)

Note that the triangles are no longer a flip of one another. Now you need to rotate and flip.
(MP.5, MP.8) 2. a) Use a ruler and compass or dynamic geometry software to try to draw two non-congruent polygons with the given side lengths.
   i) 2 cm, 2 cm, 3 cm   ii) 2 cm, 3 cm, 3 cm
   iii) 2 cm, 2 cm, 3 cm, 3 cm   iv) 1 cm, 2 cm, 3 cm, 4 cm
   v) 3 cm, 4 cm, 5 cm   vi) 3 cm, 4 cm, 5 cm, 6 cm
   vii) 3 cm, 3 cm, 3 cm, 3 cm   viii) 2 cm, 2 cm, 3 cm, 3 cm, 4 cm
   ix) 2 cm, 2 cm, 2 cm, 2 cm, 2 cm, 2 cm   x) 2 cm, 2 cm, 2 cm, 3 cm
b) When can you draw two non-congruent polygons with the same side lengths? When can you not do this?
c) Check your answers to part b) with two other examples, one where you think it will work and one where you think it will not.
Selected answer: b) you can draw these polygons when the number of sides is more than 3, you can't when the polygons you draw are triangles

(MP.3, MP.6) 3. a) Andy wants to draw a scale drawing of his apartment, including pieces of furniture. He has a circular kitchen table. Will the scale drawing also be a circle? Explain.
b) In pairs, discuss your answers to part a). Do you agree with each other? Discuss why or why not.
Selected answers: a) Yes. All the distances on the scale drawings are proportional to the distances in real life, and the distances of all the points on the circle to the center of the circle are all the same in real life, so they will be on the scale drawing too. That is exactly what it means for the picture on the scale drawing to be a circle.

NOTE: In part b), encourage partners to ask questions to understand and challenge each other’s thinking (MP.3)—see p. A-49 for sample sentence and question stems.

(MP.3, MP.5) 4. a) How many triangles have side lengths 3 cm and 4 cm? Explain how you know. Use any tools you think will help.
b) In pairs, explain your thinking to part a). Do you agree with each other? Discuss why or why not.
Sample answers:
• I used a ruler to draw a line segment 3 cm long. The ends of this segment will be two of the vertices of the triangle. I used a compass to draw a circle with radius 4 cm at one of the endpoints of the line segment. I saw there are only two points on the circle that cannot be the third point of the triangle, so I can make infinitely many triangles this way.
• I did the problem mentally. I looked for lengths that can be the length of the third side. I need the length of the two shortest sides to be greater than the longest side. So, the third side has to be greater than 1 but less than 7. Any length in between 1 cm and 7 cm can be the length of the third side. There are infinitely many numbers between 1 and 7 because the numbers can be decimals too, so there are infinitely many triangles that have side lengths 3 cm and 4 cm.

NOTE: In part b), encourage partners to ask questions to understand and challenge each other’s thinking (MP.3)—see p. A-49 for sample sentence and question stems.
G7-20  Circumference

Pages 127–129

Standards: 7.G.B.4

Goals:
Students will discover and apply the relationship between the radius of a circle and the circumference of the circle.

Prior Knowledge Required:
Knows the diameter is the largest distance across a circle
Can recognize the radius and diameter of a circle
Can multiply a fraction by a whole number

Vocabulary: center, circle, circumference, diameter, pi (π), radius

Materials:
circular objects of different sizes (one for each student) with diameter of at most 3.5 inches
materials for measuring circumference, such as pipe cleaners, string, small tokens, etc.
strip of paper 11 inches long for each student
rulers

Introduce the word “circumference.” ASK: What is the word for the distance around a polygon? (perimeter) SAY: There is a separate word for the distance around a circle. It is called the circumference of a circle.

Finding the circumference of a circle informally.

Activity
Provide students with circular objects such as a can or lid, with diameter of at most 3.5 inches. Brainstorm ways to find the distance around the circle. (take string or pipe cleaner, carefully place it along the outside of the circle, then straighten it and measure the distance; see how many small tokens fit along the outside of the circle, then put them in a straight line and measure how long the line is; etc.) Have students estimate the circumference of their circle using whichever method they choose.

(end of activity)

Take a strip of paper 11” long and show how you can curl it to form a cylinder with a circle on top and bottom. Have students put an 11” strip of paper around their circular object and mark where the overlap occurs (a circle 3.5 inches wide will have no overlap).
SAY: The place where the overlap starts shows how long the distance around the circle is. (see sample below)

![Diagram of circle and overlap]

**Comparing the circumference to the diameter.** Have students find the diameter of their circle by using a ruler to find the largest distance across. Have students record their results in chart format and compile a list of at least three other students’ results in their chart with headings “Student’s Name,” “Circumference (C),” and “Diameter (d).” Tell students to leave room for two more columns. Tell students you think there might be a rule for how to get the circumference from the diameter. ASK: Do you think adding the same number will always work? What about multiplying the same number? How can you check? (to check if adding works, use \( C - d \); to check if multiplying words, use \( C \div d \)) Have students add two more columns, \( C - d \) and \( C \div d \), then complete their chart. ASK: What do you notice? (\( C \div d \) is always a little more than 3)

Tell students that, for any circle, the quotient of the circumference divided by diameter is always the same. SAY: This number is called \( \pi \). Show the symbol for it on the board:

\[
C \div d = \pi
\]

SAY: Pi is different from all the numbers you’ve seen before because it has an infinite number of digits after the decimal point. You can decide to be as accurate as you want. Write on the board:

\[
\pi \approx 3.14159265
\]

SAY: This is pi rounded to eight decimal places. Before providing the exercises below, you may need to remind some students of the rules for rounding: if the next digit is 5 or higher, round up; otherwise, round down.

**Exercises:** Write \( \pi \) to the nearest …

a) thousandth  

b) ten thousandth  

c) millionth  

d) hundredth  

**Answers:** a) 3.142, b) 3.1416, c) 3.141593, d) 3.14

**Finding the circumference from the diameter.** Write on the board:

\[
\frac{C}{d} = \pi \quad \text{so} \quad \frac{C}{d} = \frac{\pi}{1}
\]
SAY: You can cross multiply. Write on the board:

\[
\frac{C}{d} = \frac{\pi}{1} \quad C = \pi \times d \quad \text{or} \quad C = \pi d
\]

SAY: That makes sense because \( C \div d = \pi \) and \( C = \pi \times d \) are in the same fact family. Then remind students that when there is a variable, they don’t need to write the times sign. When they’re using the formula, they can use a bracket instead of the times sign. Write on the board:

\[
C = \pi d = \pi (5)
\]

SAY: You don’t usually need to be accurate to more than two decimal places, so let’s use 3.14 for pi. Write on the board:

\[
C = \pi d = \pi (5) \approx (3.14)(5) = 15.7 \text{ in}
\]

SAY: I used the “approximately equal to” sign because pi is only approximately 3.14.

**Exercises:** Find the circumference of the circle with given diameter. Include the units in your answer.

a) \(23 \text{ mm}\)  
   \[C \approx 3.14 \times 23, \text{ so } C \approx 72.22 \text{ mm}\]

b) \(3 \text{ ft}\)  
   \[C \approx 3.14 \times 3, \text{ so } C \approx 9.42 \text{ ft}\]

c) \(9 \text{ cm}\)  
   \[C \approx 3.14 \times 9, \text{ so } C \approx 28.26 \text{ cm}\]

d) \(5.5 \text{ in}\)  
   \[C \approx 3.14 \times 5.5, \text{ so } C \approx 17.27 \text{ in}\]

**Solutions:** a) \(C \approx 3.14 \times 23, \text{ so } C \approx 72.22 \text{ mm}\); b) \(C \approx 3.14 \times 3, \text{ so } C \approx 9.42 \text{ ft}\); c) \(C \approx 3.14 \times 9, \text{ so } C \approx 28.26 \text{ cm}\); d) \(C \approx 3.14 \times 5.5, \text{ so } C \approx 17.27 \text{ in}\)

**Finding the circumference from the radius.** SAY: If you know the radius of a circle, you can find the diameter then find the circumference. Write on the board:

\[
d = 2r = 2(8) = 16 \text{ yd}
\]
\[
C = \pi d = \pi (16) \approx (3.14)(16) = 50.24 \text{ yd}
\]

**Exercises:** Find the diameter, then find the circumference.

a) \(5 \text{ mm}\)  
   \[d = 2r = 2(8) = 16 \text{ yd}\]
   \[C = \pi d = \pi (16) \approx (3.14)(16) = 50.24 \text{ yd}\]

b) \(6 \text{ ft}\)  
   \[d = 2r = 2(8) = 16 \text{ yd}\]
   \[C = \pi d = \pi (16) \approx (3.14)(16) = 50.24 \text{ yd}\]

c) \(9 \text{ mi}\)  
   \[d = 2r = 2(8) = 16 \text{ yd}\]
   \[C = \pi d = \pi (16) \approx (3.14)(16) = 50.24 \text{ yd}\]

d) \(7 \text{ yd}\)  
   \[d = 2r = 2(8) = 16 \text{ yd}\]
   \[C = \pi d = \pi (16) \approx (3.14)(16) = 50.24 \text{ yd}\]
Answers: a) $d = 10 \text{ mm}, C \approx 31.4 \text{ mm}$; b) $d = 12 \text{ ft}, C \approx 37.68 \text{ ft}$; c) $d = 18 \text{ mi}, C \approx 56.52 \text{ mi}$; d) $d = 14 \text{ yd}, C \approx 43.96 \text{ yd}$

SAY: You can find the circumference directly from the radius, too. Write on the board:

\[
C = \pi d = \pi (2r) \\
= \pi \times 2 \times r \\
= 2 \pi r \\
\approx 2(3.14)r
\]

SAY: This is another formula for the circumference of a circle, in terms of the radius instead of the diameter. Write on the board:

\[
C = 2\pi r
\]

Exercises: Find the circumference of the circle with given radius. Include the units.

a) $C \approx 2(3.14)(7)$, so $C \approx 43.96 \text{ mm}$; b) $C \approx 2(3.14)(11)$, so $C \approx 69.08 \text{ ft}$; c) $C \approx 2(3.14)(14)$, so $C \approx 87.92 \text{ mi}$; d) $C \approx 2(3.14)(4.5)$, so $C \approx 28.26 \text{ yd}$

Deciding which formula to use. Draw on the board:

\[
\begin{align*}
\text{a)} & \quad \text{b)} & \quad \text{c)} & \quad \text{d)} \\
\qquad 7 \text{ mm} & \quad \qquad 11 \text{ ft} & \quad \qquad 14 \text{ mi} & \quad \qquad 4.5 \text{ yd} \\
\qquad 8 \text{ cm} & \quad \qquad 5 \text{ in} & \quad \qquad 6 \text{ cm} & \quad \qquad 9 \text{ mm} \\
\qquad 12 \text{ cm}
\end{align*}
\]

(MP.1) Point to each circle in turn and ASK: Which formula would you use, $2\pi r$ or $\pi d$? ($\pi d$, $\pi d$, $2\pi r$, $2\pi r$, $\pi d$) Write the formulas on the board as students say them. Then have students compute the circumference of each circle, using 3.14 for $\pi$. Ask volunteers to share their answers. (25.12 cm, 15.7 in, 37.68 cm, 56.52 mm, 37.68 cm)

ASK: Which two answers are the same? (the circle with radius 6 cm and the circle with diameter 12 cm) Why are the answers the same? (the circles are the same; having radius 6 cm is the same thing as having diameter 12 cm)
Exercises:
1. Write the formula you would use \((2 \pi r\) or \(\pi d)\) to find the circumference of the circle.
   a) \(8\text{ in}\)  
   b) \(12\text{ ft}\)  
   c) \(6.5\text{ yd}\)  
   d) \(9\text{ mi}\)

   **Answers:** a) \(C = 2\pi r\), b) \(C = \pi d\), c) \(C = \pi d\), d) \(C = 2\pi r\)

2. Find the circumference. Include the units.
   a) \(15\text{ cm}\)  
   b) \(16\text{ ft}\)  
   c) \(8\text{ in}\)  
   d) \(19\text{ mi}\)

   **Answers:** a) \(C \approx 47.1\text{ cm}\), b) \(C \approx 100.48\text{ ft}\), c) \(C \approx 50.24\text{ in}\), d) \(C \approx 59.66\text{ mi}\)

**Finding a fraction of the circumference.** Draw on the board:

Pointing to each circle in turn, ASK: What fraction of the circumference is covered by the bold line? \((1/2, 1/4, 3/4)\) Since a full turn is \(360^\circ\), what degree of turn is made going from \(A\) to \(B\)? \((180^\circ, 90^\circ, 270^\circ)\) SAY: Half a full turn is half of \(360^\circ\), so that’s \(180^\circ\). A quarter of a turn is a quarter of \(360^\circ\), so that’s \(90^\circ\). You can always find the fraction of the turn if you know the angle. Draw on the board:

ASK: What fraction of \(360^\circ\) is \(120^\circ\)? \((1/3)\) Write on the board:

\[
\frac{120}{360} = \frac{1}{3}
\]

SAY: So \(1/3\) of the circumference is covered and, once you know the fraction of the circumference, you can find the distance covered just from knowing the circumference.
Write on the board:

\[
\text{Circumference} = \pi d \approx (3.14)(10) = 31.4 \text{ cm}
\]

ASK: Where did I get the 10 from? (the radius is 5, so the diameter is 10) Write on the board:

\[
\text{Length} = \frac{1}{3} (31.4 \text{ cm}) = 10.47 \text{ cm}
\]

**Exercises:** The circle has circumference 24 cm. Find the length of the bolded part.

- **a)**
- **b)**
- **c)**
- **d)**
- **Bonus:**

**Answers:** a) 6 cm, b) 12 cm, c) 8 cm, d) 16 cm, Bonus: 19.2 cm

Adding sides to find the distance around a shape. Tell students that they can find the distance around shapes by adding all the side lengths, even when the side lengths are circular.

**Exercises:** Find the distance around the shape.

- **a)**
- **b)**
- **c)**
- **d)**
- **Bonus:**

**Answers:** a) 16 cm, b) 13 cm, c) 10.28 cm, d) 20.14 in, Bonus: 12.85 cm or 128.5 mm

Do part a) below together as a class. (the half-circle has length 
\[
\frac{1}{2} \pi d \approx \frac{1}{2} (3.14)(2) = 3.14 ,
\]
so the perimeter is \(3.14 + 2 = 5.14\) cm)

**Exercises:** Find the distance \((D)\) around the shape.

- **a)**
- **b)**
- **c)**
- **d)**

**Answers:** a) \(D \approx 5.14\) cm, b) \(D \approx 14.28\) ft, c) \(D \approx 40.26\) m, d) \(D \approx 16.28\) mi
Extensions

(MP.1, MP.7) 1. The radii (or radiuses) of the two small half-circles in the picture below are $x$ and $3x$.

![Diagram of two small half-circles with radii $x$ and $3x$.]

a) What is the radius of the large half-circle?
b) Solve for $x$ if the distance around the shape is $24\pi$ cm.

Sample answers:
a) The diameter of the large circle is $3x + 3x + x + x = 8x$, so the radius is $4x$;
b) $\frac{1}{2}(2\pi x) + \frac{1}{2}(2\pi)(3x) + \frac{1}{2}(2\pi)(4x) = \frac{1}{2}(2\pi)(x + 3x + 4x) = 8\pi x$,
so $8\pi x = 24\pi$ cm, and $x = 3$ cm

(MP.3) 2. The radius of one circle is twice as big as the radius of another.
Will the circumference be twice as big?
Answer: Yes. Let $C_1$ be the circumference of the smaller circle and $r$ be its radius. Let $C_2$ be the circumference of the larger circle, so its radius is $2r$. Then $C_1 = 2\pi r$ and $C_2 = 2\pi (2r) = 4\pi r$ and the $4\pi r$ is twice $2\pi r$.

3. Which circle has greater circumference? Explain.
a) $d = 1$ yd or $r = 18$ in  
b) $d = 1$ m or $r = 50$ cm

Answers:
a) Neither. If $r = 18$ in, then $d = 36$ in = 1 yd (same diameters = same circumferences)  
b) Neither. If $r = 50$ cm, then $d = 100$ cm = 1 m (same diameters = same circumferences)

(MP.7) 4. a) Calculate $22/7$ and round to two decimal places. Is your answer close to $\pi$?
b) Find the circumference. Use $\frac{22}{7}$ or 3.14 for $\pi$, whichever will make the calculations easier.

i) $d = 28$ cm  
ii) $r = 17$ in

iii) $d = 35$ mm  
iv) $r = 24.5$ mi

Bonus: Use $22/7$ for $\pi$ and calculate the circumference using mental math.
c) $d = 3.5$ cm (Hint: Convert the measurement to mm.)  
d) $d = 1.75$ in (Hint: Change the decimal to an improper fraction.)

NOTE: Encourage students to attempt the bonus questions without the hints.

Answers: a) yes, the answer, rounded to two decimal places, is 3.14; 
b) i) $C \approx 88$ cm, ii) $C \approx 106.76$ in, iii) $110$ mm, iv) 154 mi;
Bonus: c) $C \approx 110$ mm = 11 cm, d) $C \approx 5.5$ in
5. People have been trying to approximate pi for thousands of years. Here are some of the earliest approximations of pi:

<table>
<thead>
<tr>
<th>Approximate year</th>
<th>Region</th>
<th>Fraction used to approximate $\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000 BCE</td>
<td>Babylon</td>
<td>$3\frac{1}{8} = 3.125$</td>
</tr>
<tr>
<td>2000 BCE</td>
<td>Egypt</td>
<td>$\frac{256}{81} = 3.185$</td>
</tr>
<tr>
<td>250 BCE</td>
<td>Greece</td>
<td>$3\frac{10}{71} &lt; \pi &lt; 3\frac{1}{7}$ so $\pi$ is between these two numbers</td>
</tr>
<tr>
<td>150 CE</td>
<td>Egypt</td>
<td>$\frac{377}{120} = 3.141666$</td>
</tr>
<tr>
<td>450 CE</td>
<td>China</td>
<td>$\frac{355}{113} = 3.141593$</td>
</tr>
<tr>
<td>530 CE</td>
<td>India</td>
<td>$\frac{3927}{1250} = 3.141600$</td>
</tr>
</tbody>
</table>

- a) Complete the chart by using a calculator to turn the fractions into decimals by dividing. Round your answers to six decimal places.
- b) Use your calculator to write $\pi$ correctly to as many decimal places as possible. Press the $\pi$ key and write all the decimals you see.
- c) To how many decimal places was Egypt's 2000 BCE estimate of $\pi$ accurate? What about Egypt's estimate in 150 CE?
- d) Which of the regions above knew $\pi$ to the most correct decimal places? The least correct?
- e) Put the regions and dates in order from most accurate to least accurate. Do later dates tend to be more accurate?
- f) Which region, Egypt (150 CE) or Greece, had the more accurate approximation?

**Answers:**

a) Egypt: $3.160494$, Greece: $3.140845 < \pi < 3.142857$, Egypt: $3.141667$, China: $3.141593$, India: $3.1416$ or $3.141600$;
b) $\pi \approx 3.14159265359$;
c) Egypt 2000 BCE estimate ($256/81 = 3.16$) is one decimal place accurate; Egypt 150 CE estimate ($377/120 = 3.141666$) is 3 decimal places accurate;
d) China was most accurate (6 decimal places) while Egypt in 2000 BCE was the least accurate;
e) China $450$ CE $355/113 = 3.141593$
   India $530$ CE $3927/1250 = 3.1416$
   Egypt $150$ CE $377/120 = 3.141666$
   Greece $250$ BCE $3\frac{10}{71} = 3.140845$
   Babylon $2000$ BCE $3\frac{1}{8} = 3.125$
   Egypt $2000$ BCE $256/81 = 3.16049$

Yes, later dates tend to be more accurate.
f) Greece had a more accurate calculation of $\pi$ than Egypt (150 CE)
(MP.8) 6. Teach students how mathematicians have approximated the circumference of a circle by using polygons. Draw on the board:

Point out that the shortest way to get from \( A \) to \( B \) is along the triangle, not along the circle, (a straight line is always the shortest way to get from one point to another, and people use that fact every time they take a shortcut to get somewhere). So the perimeter of the triangle is a lower bound for the circumference of the circle. In fact, as the number of sides of the polygon gets bigger, the perimeter is getting closer and closer to the circumference of the circle, but is still smaller than the circumference. So, instead of finding the circumference of a circle, they found closer and closer estimates by finding the circumference of polygons. Inside a circle with diameter 4 cm, a regular hexagon has side length 2 cm, a regular octagon has side length 1.53 cm, and a regular decagon has side length 1.24 cm.

Have students calculate the perimeter of each polygon and verify that they are getting closer to the actual circumference. (the perimeters are 12 cm, 12.24 cm, 12.4 cm; the circumference of the circle is 12.56 cm) In 1424, the Persian astronomer Jamshid al-Kashi calculated \( \pi \) to 16 digits. To do that, al-Kashi used a polygon with \( 3 \times 2^{28} \) sides. His 16-digit calculation held the world record for about 180 years.

(MP.3, MP.6) 7. a) Alexa says that a circle with radius \( r \) is a scale drawing of a circle with radius 1. Do you agree with Alexa? Why or why not? Use math words.

b) Cameron says that the area of a circle is proportional to the square of its radius. Do you agree with Cameron? Why or why not? Use math words.

c) In pairs, discuss your answers to part a). Do you agree with each other? Discuss why or why not.

Selected answers:

a) I agree with Alexa. In a circle with radius 1, all the distances from the center to the circle are 1. If you scale those distances by a factor of \( r \), then you get a circle with radius \( r \), so the circle with radius \( r \) is a scale drawing of the circle with radius 1 with a scale factor of \( r \).

b) I agree with Cameron. I know that \( \text{area of scale drawing} = (\text{scale factor})^2 \times \text{(area of original)} \). So I let \( A \) be the area of the circle with radius 1, and \( r \) is the scale factor, so the area of the circle with radius \( r \) is \( r^2 \times A \). For any circle of any radius, its area is always \( A \) multiplied by the square of its radius. So the area of a circle is proportional to the square of its radius, with constant of proportionality \( A \).

NOTE: In part c), encourage partners to ask questions to understand and challenge each other’s thinking (MP.3) and use of vocabulary (MP.6)—see p. A-49 for sample sentence and question stems.
G7-21 Area of Circles

Pages 130–131

Standards: 7.G.B.4

Goals:
Students will discover and apply the relationship between the radius of a circle and the area of the circle.
Students will discover and apply the relationship between the circumference of a circle and the area of the circle.

Prior Knowledge Required:
Knows how to find the circumference of a circle given the area
Can find the area of composite shapes
Knows 3.14 as an approximation for pi
Can multiply a fraction by a whole number

Vocabulary: area, circle, circumference, diameter, formula, pi (\(\pi\)), radius

Materials:
BLM Area Formula for Circles (p. P-97)
calculator
blank sheets of paper
compasses
scissors for each student
BLM Approximating the Area of a Circle (p. P-98)

(MP.8) Discovering the formula for the area of a circle.

Exercise: Complete BLM Area Formula for Circles.
Answers: 1. a) the areas (in grid squares as units) are: 12, 9, 5, 5, 9, 12, and 81;
b) 133; c) 532; d) 13; e) 3.15; f) The shaded part is really close to being the full circle—the shaded parts outside the circle look as though they are almost the same area as the unshaded parts inside the circle.

When students have finished the exercise above, ASK: Is your answer to part e) close to pi? Tell students that mathematicians have shown that the area of a circle is always exactly equal to pi times the square of the radius. Write on the board:

\[ A = \pi r^2 \]
Exercises: Find the area of the circle. Use 3.14 for \( \pi \) and round to two decimal places.

a) \[ A \approx (3.14)(4)^2, \text{ so } A \approx 50.24 \text{ yd}; \]
b) \[ A \approx (3.14)(9)^2, \text{ so } A \approx 254.34 \text{ ft}^2; \]
c) \[ A \approx (3.14)(13)^2, \text{ so } A \approx 530.66 \text{ mm}^2; \]
d) \[ A \approx (3.14)(2.75)^2, \text{ so } A \approx 23.75 \text{ in}^2 \]

Be sure students know how to use their calculator—in particular, the \( x^2 \) button.

SAY: Sometimes you are given the diameter instead of the radius. But the formula is only in terms of the radius. ASK: How can you get the radius if you know the diameter? (divide by 2)

(MP.6, MP.7) Exercises: Find the radius of the circle. Then find the area and round to two decimal places.

a) \[ r = 14 \text{ mm}, \quad A = 615.44 \text{ mm}^2; \]
b) \[ r = 5.5 \text{ ft}, \quad A = 94.99 \text{ ft}^2; \]
c) \[ r = 3.7 \text{ cm}, \quad A = 42.99 \text{ cm}^2; \]
d) \[ r = 11 \text{ in}, \quad A = 379.94 \text{ in}^2 \]

SAY: In the same way we can calculate the distance around part of a circle, we can also calculate the area of part of a circle, as long as we know what fraction of the area we are finding.

Exercises: A circle has area 48 cm\(^2\). What fraction of the circle is shaded? What is the area of the shaded part?

a) \[ 1/2 \text{ is shaded}, \quad A = 24 \text{ cm}^2; \]
b) \[ 3/4 \text{ is shaded}, \quad A = 36 \text{ cm}^2; \]
c) \[ 1/3 \text{ is shaded}, \quad A = 16 \text{ cm}^2; \]
d) \[ 2/3 \text{ is shaded}, \quad A = 32 \text{ cm}^2 \]

SAY: When shapes are made of different parts, you can add their areas to find the total area of the shape.
(MP.1) **Exercises:** Find the area of the shape made from circle parts and/or rectangles. Use 3.14 for $\pi$. Round to two decimal places.

(a) [Diagram of a shape with dimensions 1 m, 3 m, and 2 m]

(b) [Diagram of a shape with dimensions 3 m, 1 m, and 1 m]

(c) [Diagram of a shape with dimensions 7 yd and 5 yd]

**Bonus:** You will need to subtract.

(bonus diagram with dimensions 7 in, 5 in, 5 in, and 10 in)

**Answers:**

(a) $4 + 1 + \frac{1}{4}(3.14)(1^2)$, so 5.79 m$^2$; 
(b) $\frac{1}{2} \pi (0.5)^2 + \frac{1}{2} \pi (2)^2$, so 6.68 m$^2$;

(c) $\frac{1}{2} \pi (2.5)^2 + \frac{1}{2} \pi (2.5)^2 + 35$, so 54.63 yd$^2$; 
Bonus: $12(15) - \frac{1}{4}(3.14)(5^2)$, so 160.38 in$^2$

(MP.1) **Understanding the formula for the area of a circle.** Complete **Question 5** on AP Book 7.2, p. 131 together as a class. Students will need a blank sheet of paper, a compass, and a pair of scissors. Demonstrate each step, then have students do it on their own. Verify that everyone has done the step correctly before proceeding. When doing part g), you may need to trace your finger along the outside of the circle then along the bottom part of the shape you made, to help students see that they are actually the same.

When you are finished, provide students with BLM **Approximating the Area of a Circle**. Emphasize how close the length of the rectangle is to half the circumference when you use 16 pieces, and how it’s not as close when you use four pieces. Emphasize also how the area of the rectangle gets closer to the area of the circle as you use more pieces.

**NOTE:** The area of a rectangle is the upper bound for the area of the circle. The area of the rectangle is larger than the area of the circle because the rectangle goes around the circle, but as the number of sides increases, the areas get closer.

**Relating the area to the circumference.** SAY: If you know the circumference, you can figure out the radius. Write on the board:

$$C = 2\pi r \quad \text{so} \quad r = \frac{C}{2\pi}$$
Exercises: The circumference $C$ is given. Find the radius.

a) $C = 12\text{ in}$  
   b) $C = 19\text{ cm}$  
   c) $C = 23\text{ mm}$  
   d) $C = 29\text{ mi}$

**Answers:**
a) $r \approx 1.91\text{ in}$,  
b) $r \approx 3.03\text{ cm}$,  
c) $r \approx 3.66\text{ mm}$,  
d) $r \approx 4.62\text{ mi}$

SAY: If you know the circumference, you can figure out the radius, and if you know the radius, you can figure out the area.

Exercises: The circumference is given. Find the radius, then the area of the circle.

a) $C = 25\text{ mm}$  
   b) $C = 18\text{ yd}$  
   c) $C = 30\text{ ft}$  
   d) $C = 17\text{ cm}$

**Answers:**
a) $r \approx 3.98\text{ mm}$,  
   $A \approx 49.76\text{ mm}^2$;  
   b) $r = 2.87\text{ yd}$,  
   $A \approx 25.86\text{ yd}^2$;  
   c) $r \approx 4.78\text{ ft}$,  
   $A \approx 71.74\text{ ft}^2$;  
   d) $r = 2.71\text{ cm}$,  
   $A = 23.06\text{ cm}^2$

Tell students that there is a way to find the area directly if you know the circumference, without even determining the radius. Point out that the radius is squared in the formula for the area of a circle. So if you want a formula for the area of a circle in terms of the circumference only, you should square the circumference.

**(MP.8) Exercises:** Complete the chart.

<table>
<thead>
<tr>
<th></th>
<th>$r = \text{Radius}$</th>
<th>$A = \text{Area}$</th>
<th>$C = \text{Circumference}$</th>
<th>$C^2$</th>
<th>$C^2 \div A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>5 m</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b)</td>
<td>6 mi</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c)</td>
<td>7 mm</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d)</td>
<td>8 yd</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>e)</td>
<td>9 cm</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Answers:**

<table>
<thead>
<tr>
<th></th>
<th>$r = \text{Radius}$</th>
<th>$A = \text{Area}$</th>
<th>$C = \text{Circumference}$</th>
<th>$C^2$</th>
<th>$C^2 \div A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>5 m</td>
<td>78.5 $m^2$</td>
<td>31.4 $m$</td>
<td>985.96</td>
<td>12.56</td>
</tr>
<tr>
<td>b)</td>
<td>6 mi</td>
<td>113.04 $mi^2$</td>
<td>37.68 $mi$</td>
<td>1,419.78</td>
<td>12.56</td>
</tr>
<tr>
<td>c)</td>
<td>7 mm</td>
<td>153.86 $mm^2$</td>
<td>43.96 $mm$</td>
<td>1,932.48</td>
<td>12.56</td>
</tr>
<tr>
<td>d)</td>
<td>8 yd</td>
<td>200.96 $yd^2$</td>
<td>50.24 $yd$</td>
<td>2,524.06</td>
<td>12.56</td>
</tr>
<tr>
<td>e)</td>
<td>9 cm</td>
<td>254.34 $cm^2$</td>
<td>56.52 $cm$</td>
<td>3,194.51</td>
<td>12.56</td>
</tr>
</tbody>
</table>

Challenge students to predict $C^2 \div A$ in terms of $\pi$. If students need help, have them divide their last column by 3.14 on a calculator. (4 is the answer for all rows) Write on the board:

$$C^2 \div A = 4\pi$$

so \hspace{5mm} $$A = C^2 \div (4\pi)$$
ASK: How did I get the second equation from the first? (switched the divisor and the quotient)

Write on the board:

\[ 20 \div 5 = 4 \text{ so } 20 \div 4 = 5 \text{ or } 5 = 20 \div 4 \]

SAY: The expressions in each place of the division equation just stand for numbers, so you can switch them around the same way you can with numbers.

(MP.1, MP.7) Exercises: Find the area of the circles you did in the exercise before the chart, without finding the radius. Did you get the same answer both times? If not, find your mistake.

Bonus: Use \( C = 2\pi r \) and \( A = \pi r^2 \) to verify the formula \( A = C^2 \div (4\pi) \).

Solution: \( C^2 + (4\pi) = (2\pi r)^2 + (4\pi) = 4\pi^2 r^2 + 4 + \pi = \pi r^2 \)

Extensions

(MP.1, MP.7) 1. Find the area of the shaded figure. Round to two decimal places.

\[ a) \quad \frac{1}{4}(3.14)(5^2) = 108.88 \text{ cm}^2 \\
 b) \quad (9 + 5) \times 5 + 2 \times (3.14)(5^2) = 93.88 \text{ cm}^2 \\
 c) \quad \frac{1}{2} (3.14)(5^2) + \frac{1}{12} (3.14)(3^2) + \frac{1}{6} (3.14)(5^2) + \frac{1}{4} (3.14)(3^2) = 61.75 \text{ cm}^2 \\
 d) \quad (3.14)(4^2) + \frac{1}{2} (7 \times 7) - \frac{1}{4} (3.14)(4^2) = 62.18 \text{ cm}^2 \\

(MP.3) 2. The radius of a circle is twice as big as the radius of another. Will the area also be twice as big?

Answer: No. For example, a circle with radius 1 cm has area \( \pi \text{ cm}^2 \), but a circle with radius 2 cm has area \( 4\pi \text{ cm}^2 \), which is 4 times as big.
Goals:
Students will draw a sketch to solve area and angle problems.
Students will recognize when two cases are required to solve a problem.

Prior Knowledge Required:
Knows that the sum of the angles in a triangle is 180°
Can find the area of rectangles, trapezoids, parallelograms, and triangles
Can solve equations with two operations

Vocabulary: base, height, parallelogram, sketch, trapezoid

Labeling a sketch to solve an area problem. Point out to students that when they are given a problem, they are not always given a picture, but making a picture themselves can make the problem easier to solve. Write on the board:

A parallelogram has height 4 cm and base 7 cm.

ASK: What type of shape do I need to draw? (a parallelogram) Draw on the board:

Ask a volunteer to label the measurements that are given, as shown below:

Exercises: Copy the shape and label the measurements. (Hint: You may not be given one of the measurements, but you can still figure it out.)

a) A triangle has base 3 cm and height 5 cm.

b) A triangle has height 3 cm and base twice as long.

c) A triangle has base 15 inches and height one third as long.
Using a sketch to solve an area problem. SAY: Now that you have a sketch, it is easier to find the area.

Exercises: Find the area of each shape in the previous exercises.
Answers: a) $A = 7.5 \text{ cm}^2$, b) $A = 9 \text{ cm}^2$, c) $A = 37.5 \text{ cm}^2$

Drawing a sketch to solve an area problem. Tell students that the shapes do not have to be drawn exactly to scale, but they should be approximate. The triangle will usually be easier to work with if you draw the base horizontal. Draw on the board:

ASK: If the height is three times the base, which triangle is the better sketch? (the first one)

Exercises: Sketch a shape that matches the description. Draw the base horizontal.
a) The base of a triangle is twice the height.
b) The base of a parallelogram is half of the height.
c) One base of a trapezoid is half as long as the other base. The height is equal to the larger base.
Sample answers:

SAY: Now you will sketch the shape and label all the information you need to find the area of the shape. Write on the board:

The base of a triangle is twice the height. The base is 8 cm. Find the area.

Have a volunteer sketch the triangle according to the first sentence. Have another volunteer label the base. ASK: What is the height of this triangle? (4 cm) How do you know? (because 8 cm is twice the height) Label the height, as well. The final picture should look like this:

SAY: Then find the area. \((8 \times 4) \div 2 = 16 \text{ cm}^2\)
**MP.5) Exercises:** Draw a sketch to show the information. Then find the area.

a) The base of a parallelogram is four times as long as the height. The height is 3 cm.
b) A right-angled trapezoid is made from a square and a right triangle. The base and height of the right triangle, and the side length of the square, are all 2 inches long.
c) One base of a trapezoid is three times as long as the other base. The height is half the length of the shortest base. The longest base is 6 inches long.

**Answers:** a) 36 cm², b) 6 in², c) 4 cm²

**Sample sketches:**

a) ![Sketch](image)

b) ![Sketch](image)

c) ![Sketch](image)

**Sketches of angle diagrams.** Write on the board:

The two smallest angles in a triangle are equal.

A. ![Diagram](image)

B. ![Diagram](image)

ASK: Which is a better sketch? (B) PROMPT: In one of the triangles, the larger angles are equal. SAY: In triangle A, the largest angles are equal. In triangle B, the smaller angles are equal, so that is the better diagram. Now erase the incorrect sketch, triangle A, and add the information below to the description:

The two smallest angles in a triangle are equal.
The largest angle is 30° more than the smallest.

SAY: Let’s call the smallest angle \(x\). Have a volunteer label where \(x\) goes in the diagram. Have another volunteer label the largest angle in terms of \(x\), as shown below:

![Diagram](image)

ASK: Do you have enough information to solve for \(x\) now? (yes) PROMPT: What do all three angles add to? (180°) Write on the board:

\[ x + x + x + 30 = 180 \]

Ask a volunteer to simplify the expression on the left:

\[ 3x + 30 = 180 \]
Ask another volunteer to solve the equation:

\[ 3x = 150 \]
\[ x = 50° \]

Ask other volunteers to explain each step. (subtract 30 from both sides, then divide both sides by 3) Then ask a volunteer to label each angle of the triangle with the actual degrees instead of expressions. Remind them to use the degree symbol. (see labeled triangle below)

![Labeled Triangle](image)

**(MP.6)** When students have solved the exercises below, encourage them to clearly state what the variable represents.

**Exercises:** The two largest angles in a triangle are equal. Draw a sketch and solve the problem.

a) The largest angle is 30° more than the smallest. What are the angles?
b) The largest angle is twice the smallest angle. What are the angles?

**Answers:** a) 70°, 70°, 40°; b) 36°, 72°, 72°

**Sample sketches:**

a) ![Sketch A](image)

b) ![Sketch B](image)

**(MP.3)** Dividing a problem into cases. SAY: You might be told that two angles are equal but not told whether they are the larger angle or the smaller angle. Then you have to do two cases.

Write on the board:

Two angles in a triangle are equal.
The largest angle is 7 times the smallest angle.
What are the angles in the triangle?

ASK: What are the two cases? (the largest angles are equal and the smallest angles are equal)

**Exercise:** Do both cases.

**Answer:** If the largest angles are equal, the sum of the angles is 15\(x = 180\), so \(x = 12°\), and the angles are 12°, 74°, 74°; if the smallest angles are equal, the sum of the angles is 9\(x = 180\), so \(x = 20°\), and the angles are 20°, 20°, and 140°)
SAY: A triangle has two equal angles. One of the angles measures 40°. ASK: What are the two cases here? (the 40 degree angle is one of the equal angles or it is not) Write on the board:

Case 1: 40° is one of the equal angles.  
Case 2: 40° is not one of the equal angles.

Have volunteers draw a sketch and label the angles according to the two cases, using \( x \) for the unknown angle. (see sample sketches below)

Students will use these cases in Exercise 1, below.

**Exercises:**
1. Find the angles in the triangle in each case. Label the cases.
   **Answer:** Case 1: 40° is one of the equal angles, so \( x + 80 = 180 \), \( x = 100° \), and so the angles are 40°, 40°, and 100°; Case 2: 40° is not one of the equal angles, \( 2x + 40 = 180 \), \( 2x = 140 \), \( x = 70° \), and so the angles are 40°, 70°, and 70°.

2. A triangle has two equal angles. What are all the angles if …
   a) one (or more) of the angles measures 50°  
   b) one (or more) of the angles measures 60°  
   c) one (or more) of the angles measures 70°
   **Answers:** a) 50°, 65°, 65° or 50°, 50°, 80°; b) 60°, 60°, 60°; c) 70°, 55°, 55° or 70°, 70°, 40°

**MP.3** SAY: One of the angles measures 90°. Can the 90° angle be one of the two equal angles? (no) Why not? (because a triangle cannot have two 90° angles) SAY: So in this example, there is only one case because the other case is impossible.

**Exercises:** A triangle has two equal angles. What are all the angles if …
   a) one of the angles measures 120°  
   b) one of the angles measures 140°  
   c) one of the angles measures 156°
   **Answers:** a) 120°, 30°, 30°; b) 140°, 20°, 20°; c) 156°, 12°, 12°

**Extensions**
**MP.3, MP.7** 1. a) Divide a quadrilateral into two triangles. What is the sum of the angles in the quadrilateral? Will the answer be the same for any quadrilateral? Explain.
   b) A quadrilateral has two pairs of equal angles. One of the angles is 40°. What are all the angles in the quadrilateral? Sketch different possible arrangements of the angles.
   c) A trapezoid has two right angles. The other two angles are such that one is double the other. What are the angles? Sketch the trapezoid.
d) A quadrilateral has three equal angles. One (or more) of the angles is 45°. What are the angles in the quadrilateral? Draw a sketch for each possible set of angles.

**Answers:**

a) To find the sum of the angles in a quadrilateral, split the quadrilateral into two triangles. For example:

![Diagram of a quadrilateral split into two triangles]

The sum of the four angles in the quadrilateral is: \( y + (z + a) + b + (c + x) = x + y + z + a + b + c \).

But the sum of the angles in a triangle is 180°, so the sum of the four angles in the quadrilateral is \( 180° + 180° = 360° \). This will be the same for any quadrilateral.

b) 40°, 40°, 140°, 140°

c) 90°, 90°, 60°, 120°

d) 45°, 45°, 45°, 225° or 45°, 105°, 105°, 105°; possible sketches:

![Sketches of quadrilaterals]

2. Describe the values of \( x \) that make the statement true. You will need to use one of these symbols: \(<\), \(>\), \(\geq\), \(\leq\), or \(\neq\).

a) \( x + x = 1 \) when \( \neq \).

b) \(-7x + 14 \geq 0 \) when \( > \).

c) Let \( C \) be the cost of a taxi ride. The taxi company charges a $2 flat fee plus $3 per mile. Let \( x \) be the number of miles traveled. The cost of the taxi ride is \( C = 3x + 2 \) when \( \leq \).

d) Let \( C \) be the cost of renting a bike. The bike company offers the first hour free, then charges $5 per hour after that. For example, renting the bike for 90 minutes costs $2.50. Let \( x \) be the number of hours the bike is rented for. The cost of the bike rental is \( C = 5(x - 1) \) when \( > \).

**Answers:** a) \( x \neq 0 \), b) \( x > 0 \), c) \( x \leq 2 \), d) \( x > 1 \)

**(MP.6)** Whole-class follow-up: Have partners take turns explaining the symbols they used in parts a) and c). (sample answers: \( \neq \) means “not equal to”, \( \leq \) means “less than or equal to,” “no more than,” or “at most”)
G7-23 Using a Formula

Pages 134–136


Goals:
Students will use equations to solve problems involving areas and circumferences.

Prior Knowledge Required:
Knows that the sum of the angles in a triangle is 180°
Can find the area of rectangles, triangles, trapezoids, parallelograms, and circles
Can find the circumference of a circle
Can solve equations with two operations
Can draw a sketch
Knows how area is affected by scale drawings
Can convert between mm and cm, m and cm, ft and in, and ft and yd

Vocabulary: area, base, circle, circumference, diameter, height, formula, parallelogram, radius, sketch, trapezoid

Materials:
BLM Formula Flash Cards (p. P-99–100), a double-sided copy for each pair of students
grid paper or BLM 1 cm Grid Paper (p. T-1)

Review the formulas seen so far in the unit. Tell students that they have seen a lot of formulas so far in this unit, and you want to review what they have done so far.

Activity
(MP.1) Provide each pair of students with BLM Formula Flash Cards. Have students give each other the flash cards until they can confidently name what the formula is for. Then go in the other direction. Start with the side of the cards that show the shape and whether the area (by shading) or circumference is to be found and have students say or write the formula for each.
(end of activity)

Exercises: Write the formula for the area or circumference. Copy the shape and label the variables in the formula.

a)  

b)  

c)  

d)  

e)  

f)  

Answers: a) \(A = l \times w\), b) \(A = (b \times h) \div 2\), c) \(A = (b_1 + b_2) \times h \div 2\), d) \(C = \pi d\) or \(2\pi r\), e) \(A = \pi r^2\), f) \(A = b \times h\)

If students label the variables incorrectly, show the correct way then provide more of that shape for them to label. You may need to start by labeling one of the variables. **NOTE:** Some students may confuse \(\pi\) for a variable, but it is in fact a number.

**Using a formula to solve a problem.**

(MP.5) Exercises:
1. Determine what you are given and what you need to find, then decide what formula you would use. Write your answers in the table below.
   a) A circle has diameter 7 mm. What is its circumference?
   b) Find the area of a triangle with height 3 cm and base 7 cm.
   c) A circle has radius 9 m. What is its area?
   d) A trapezoid has bases 2.5 cm and 5.5 cm, and height 9 cm. What is its area?

<table>
<thead>
<tr>
<th>Shape</th>
<th>Given</th>
<th>Find</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle</td>
<td>Diameter = 7 mm</td>
<td>Circumference</td>
<td>(C = \pi d)</td>
</tr>
<tr>
<td>Triangle</td>
<td>Height = 3 cm</td>
<td>Area</td>
<td>(A = b \times h + 2)</td>
</tr>
<tr>
<td>Circle</td>
<td>Radius = 9 m</td>
<td>Area</td>
<td>(A = \pi r^2)</td>
</tr>
<tr>
<td>Trapezoid</td>
<td>Base 1 = 2.5 cm</td>
<td>Base 2 = 5.5 cm</td>
<td>Area</td>
</tr>
</tbody>
</table>

2. Solve the problems from the previous exercises.
   **Answers:** a) Circumference \(\approx 21.98\) mm\(^2\), b) Area = 10.5 cm\(^2\), c) Area \(\approx 254.34\) m\(^2\), d) Area = 36 cm\(^2\)

**Finding dimensions given the area.** Write on the board:

The base of a parallelogram is 5 cm.
Its area is 20 cm\(^2\). What is its height?
SAY: Here, you are given the area. ASK: What is the formula for the area of a parallelogram? (base × height) Write on the board:

\[ \text{Area} = \text{base} \times \text{height} \]

ASK: What are we given? (the base and the area) What do we need to find? (the height) Write on the board:

\[ 20 = 5 \times h \]

SAY: In this case, the formula isn’t for what you are trying to find, but it can still help you to find the missing measurement. Write on the board:

\[ h = 20 \div 5 = 4 \text{ cm} \]

(MP.5) Exercises:
1. Determine what you are given and what you need to find, then decide, what formula you would use. Write your answers in the table below.
   a) Find the base of a parallelogram with height 6 cm and area 27 cm\(^2\).
   b) A triangle with base 6 m has area 12 m\(^2\). What is its height?
   c) A circle has circumference 62.8 mm. What is its radius?
   d) A trapezoid has bases 6 cm and 10 cm, and area 60 cm\(^2\). What is its height?

<table>
<thead>
<tr>
<th>Shape</th>
<th>Given</th>
<th>Find</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Answers:

<table>
<thead>
<tr>
<th>Shape</th>
<th>Given</th>
<th>Find</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Parallelogram</td>
<td>Height = 6 cm</td>
<td>Base</td>
<td>( A = b \times h )</td>
</tr>
<tr>
<td></td>
<td>Area = 27 cm(^2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) Triangle</td>
<td>Base = 6 m</td>
<td>Height</td>
<td>( A = b \times h \div 2 )</td>
</tr>
<tr>
<td></td>
<td>Area = 12 m(^2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c) Circle</td>
<td>Circumference = 62.8 mm</td>
<td>Radius</td>
<td>( C = 2 \pi r )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d) Trapezoid</td>
<td>Base 1 = 6 cm</td>
<td>Height</td>
<td>( A = (b_1 + b_2) \times h \div 2 )</td>
</tr>
<tr>
<td></td>
<td>Base 2 = 10 cm</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Area = 60 cm(^2)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Solve the problems in Exercise 1.
Answers: a) base = 4.5 cm, b) height = 4 m, c) radius = 10 mm, d) height = 7.5 cm
(MP.2) Converting between units when necessary to solve problems. Write on the board:

A triangle has area 18 cm² and base 45 mm. What is its height?

Ask a volunteer to write on the board the formula for the area of a triangle:

$$A = b \times h \div 2$$

Write on the board:

$$18 = 45 \times h \div 2$$

(MP.6) SAY: I’m given that the base is 45 and the area is 18, so I just plugged in those numbers. ASK: Is it right? (no) Why not? PROMPT: Look at the units. (the units are not the same) Tell students that the first thing they have to do is always check that the units are the same. If they are not, they have to change the units to be the same. SAY: If length is given in mm, then area has to be given in mm², or if area is given in cm², then length has to be in cm. ASK: Which is easier to change, the 18 cm² to mm² or the 45 mm to cm? (the 45 mm to cm) SAY: Length units are usually easier to convert than area measurements. ASK: What is 45 mm in cm? (4.5 cm) Write on the board:

A triangle has area 18 cm² and base 45 mm. What is its height? 4.5 cm

SAY: Now that the units are consistent and we are using cm and cm², we can solve the problem. Write the formula on the board and have a volunteer fill in the correct numbers:

$$A = b \times h \div 2$$

$$18 = 4.5 \times h \div 2$$

SAY: The 18 is in cm² and the 4.5 is in cm, so what are the units for height? (cm) SAY: Once we find the number part from the formula, we know the units already, so we will have solved the problem. Have a volunteer solve the equation. (see sample solution below)

$$18 = 4.5 \times h \div 2$$

$$18 \div 4.5 \times 2 = h$$

$$h = 8 \text{ cm}$$
Exercises: Convert the measurement. Replace the parts of the formula with the given information. Then solve the equation for the unknown. Include the units in your final answer.

a) \(b = 75 \text{ mm} = \underline{______} \text{ cm}\)  
\(A = 45 \text{ cm}^2\)  
Find: height \((h)\) of parallelogram  
Formula: \(A = b \times h\)

b) \(b = 9 \text{ ft} = \underline{______} \text{ yd}\)  
\(A = 6 \text{ yd}^2\)  
Find: height \((h)\) of triangle  
Formula: \(A = b \times h + 2\)

c) Base 1 = 45 mm = \underline{______} cm  
Base 2 = 35 mm = \underline{______} cm  
Area = 28 cm\(^2\)  
Find: height \(h\) of trapezoid  
Formula: area = \((b_1 + b_2) \times h + 2\)

d) Base 1 = 42 in = \underline{______} ft  
Height = 18 in = \underline{______} ft  
Area = 9 ft\(^2\)  
Find: base 2 of trapezoid  
Formula: area = \((b_1 + b_2) \times h + 2\)

Answers: a) \(h = 6 \text{ cm}\), b) \(h = 4 \text{ yd}\), c) \(h = 7 \text{ cm}\), d) \(b_2 = 8.5 \text{ ft}\)

(MP.2) Scale drawings of circles. Have students draw a circle on grid paper with radius 2 grid squares. Have students find the area of the circle in grid squares, using 3.14 for pi.  
\((3.14 \times 2^2 = 12.56)\)

Exercises: Draw scale drawings of the circle on grid paper with the given scale. Find the area of each circle. What do you multiply the area of the original circle by to get the area of the new circle?

a) (original) : (new) = 1 : 2  
b) (original) : (new) = 1 : 3  
c) (original) : (new) = 1 : 4  
Bonus: (original) : (new) = 1 : 5

Answers: a) \(A \approx 50.24\) (multiply by 4), b) \(A \approx 113.04\) (multiply by 9), c) \(A \approx 200.96\) (multiply by 16), Bonus: \(A \approx 314\) (multiply by 25)

(MP.4) Word problems practice. In the exercises below, round your answers to two decimal places.

Exercises:

(MP.7) 1. What area of lawn does the sprinkler cover?

Answer: \(A = \frac{1}{2}\) of a square + \(\frac{3}{4}\) of a circle \(\approx \frac{1}{2}(15)(15) + \frac{3}{4}(3.14)(15)(15) = 642.38 \text{ ft}^2\)
2. A large, circular mural is being painted, as shown below:

Each of the two smaller circles has a diameter of 4 m. The two smaller circles are being painted yellow, while the rest of the area inside the larger circle is being painted green.

a) What is the total area of the two small circles?
b) What is the area of the entire mural?
c) What area will be painted green?
d) If yellow paint costs $4.50 per square meter and green paint costs $5.50 per square meter, how much will it cost to paint the entire mural?

Answers: a) \( A \approx 25.12 \text{ m}^2 \), b) \( A \approx 50.24 \text{ m}^2 \), c) \( A \approx 25.12 \text{ m}^2 \), d) cost = $251.20

(MP.6) 3. A rectangular backyard with an area of 84 m\(^2\) has two quarter-circle shaped ponds separated by grass. The grass needs to be replaced by concrete.

If concrete costs $6 per square meter, what will it cost to replace the grass with concrete? Use one of the symbols = or \( \approx \) in your answer. Explain what the symbol means and why you used it.

Answer: The width of the backyard is \( w = 6 \text{ m} \), because \( 84 \div 14 = 6 \). The two quarter circles together make a half circle with radius 6 m and area \( A = \pi(6)^2/2 \approx (3.14)(18) = 56.52 \text{ m}^2 \). Using \( G \) for the area of the grass in square meters, \( G = 84 - 56.52 = 27.48 \text{ m}^2 \). Concrete costs $6 per square meter, so using \( C \) for the cost of replacing the grass with concrete, \( C = 27.48 \times 6 = $164.88 \). I used “=” when the amount was exactly equal, but I used “≈” to say the amounts were only approximately equal.
4. A bicycle wheel has spokes that connect the center of the wheel with the rim. Each spoke is 15 cm long. The width of the tire is 5 cm.

![Diagram of bicycle wheel and tire]

a) i) What is the diameter of the rim (in cm)?
    ii) What is the circumference of the rim (in cm)?
    iii) If the spokes are 4.71 cm apart, how many spokes are there?

b) i) What is the diameter of the wheel (in cm)?
    ii) What is the circumference of the tire (in cm and m)?
    iii) How many revolutions of the tire will it take to cover 376.8 m?
    iv) If the cyclist pedals at two revolutions per second, how long will it take to cover this distance?

**Answers:**

a) i) diameter of rim = 30 cm, ii) Circumference of rim ≈ 94.2 cm, iii) 20 spokes;
    b) i) diameter of wheel = 40 cm, ii) Circumference of tire = 125.6 cm or 1.256 m,
    iii) 300 revolutions, iv) 150 seconds or 2.5 minutes or 2 minutes, 30 seconds

**Extensions**

(MP.4) 1. Tasha has a 14 ft wide circular pool in her backyard. She wants to put a 2-foot wide cement path around it. If cement costs $3 per square foot, how much will the path cost?

**Solution:** Area of path = \[ \pi(9)^2 - \pi(7)^2 \approx 100.48 \text{ ft}^2 \], so the path will cost $301.44.

(MP.4) 2. a) In the running track below, each lane is 1 m wide. How much of a head start should the outside lane runner get? Assume the runner runs along the most inside part of the lane.

![Diagram of running track]
b) Project idea: Interested students can look up how the amount of head start from the next lane is chosen for the Olympic 400 m running track and verify the fairness in terms of distance run using the math learned in this unit. Are there other factors involved in fairness (such as distance on a straight or curved path or ability to see other runners)?

**Selected solution:** a) Assuming the runner runs along the most inside part of their lane, the distance the inside lane runner has to run, starting from point $A$, is: $100 + 40\pi \approx 225.6$ m. The outside lane runner, if starting from the same position, would have to run $100 + 42\pi \approx 231.9$ m, so the outside runner should get a head start of 6.3 m.

(MP.3) 3. Vicky bought 12 bus tickets for $7.50. She calculated how much 30 bus tickets cost as follows:

$$\frac{12}{7.50} = \frac{x}{30}, \text{ and } 4 \times 7.50 = 30, \text{ so } x = 4 \times 12 = 48, \text{ so 30 bus tickets cost } $48$. 

a) Do you agree with Vicky’s answer? Why or why not?
b) What did Vicky do correctly? What did she do incorrectly?
c) How much would 30 bus tickets cost?

**Answers:** a) no, because the cost of 12 tickets is less than $12, so the cost of 30 tickets should be less than $30; b) she solved the proportion correctly and interpreted what $x$ means correctly, but she should have written the proportion as $12/7.50 = 30/x$; c) $30/12 = 2.5$, so $x = 2.5 \times 7.50 = 18.75$. So 30 tickets will cost $18.75$.

Whole-class follow-up: Discuss the solution, then ASK: What could Vicky have done to make sure she doesn’t get the tickets and the cost mixed up? (use a ratio table with clear titles or write the ratio in words) Show students both representations:

<table>
<thead>
<tr>
<th>Number of Tickets</th>
<th>Cost of Tickets ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>7.50</td>
</tr>
<tr>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

SAY: By making sure to clearly label which is which, Vicky can make sure she doesn’t get the tickets and the cost mixed up.
Two Ways to Sketch a Problem

1. Look at the following problem.

Two angles add to 180°. The larger angle is 20° more than the smaller angle. What are the two angles?

Mark and Sara both make sketches to solve the problem.

Mark's sketch: \[ x \quad x + 20° \]

Sara's sketch: \[ x - 10° \quad x + 10° \]

a) What equation does Mark get? ________________

b) What equation does Sara get? ________________

c) Solve Mark's equation.

What angles does he find? _______ and _______

d) Solve Sara's equation.

What angles does she find? _______ and _______

e) Do Mark and Sara both get the same answer? ________________

f) Whose equation was easier to solve? ________________

g) Explain why you think Mark drew the sketch the way he did.

h) Explain why you think Sara drew the sketch the way she did.

2. Use Sara’s method to solve the following problem.

Two angles are complementary. One angle is 16° more than the other. What are the two angles?
The Geometer’s Sketchpad®

Use The Geometer’s Sketchpad® to carry out each step.

1.  a) Start a new file. Draw a line. Label it \( m \). Mark a point \( A \) on the line \( m \). Move the line around. Does the point \( A \) stay on line \( m \)?

b) Draw another line through point \( A \). Call the new line \( n \). Mark a point \( B \) on line \( m \) that is not on line \( n \). Mark a point \( C \) on line \( n \) that is not on line \( m \).

c) Move line \( m \) by moving point \( B \). Does line \( n \) move?

d) Move line \( m \) by moving point \( A \). Does point \( A \) stay on both lines?

   Does line \( n \) move?

e) Move line \( m \) by moving it at a point that is not marked. Does line \( n \) move?

   Does point \( A \) stay on both lines?

2.  a) Start a new file. Create three points \( A, B, \) and \( C \). Create the line segments \( AB \) and \( AC \).

b) Measure \( \angle BAC \) and \( \angle CAB \). Are your answers the same?

c) Rotate \( \angle BAC \) various degrees. Does rotating the angle change the size?

d) Delete everything except the triangle made by \( A, B, \) and \( C \). Draw a copy of \( \angle BAC \).

   Place the copy directly on top of \( \angle BAC \), and rotate the copy 180°. Place the rotated angle so that its vertex is at \( A \).

   Is your rotated copy vertical to \( \angle BAC \)?

   Does your rotated copy have the same measure as \( \angle BAC \)?

3.  a) Start a new file. Use the perpendicular lines command from the menu to draw perpendicular lines. Do these lines stay perpendicular, even if points are moved around?

b) Use the perpendicular lines and line segment commands to draw a right triangle.

c) Measure the two acute angles in the right triangle you created. Check that the sum remains 90°, even when the triangle is modified.

d) Draw another triangle and verify that the sum of the three angles remains 180°, even when you modify it.

e) Measure the lengths of the sides of your triangle. Check that the length of the longest side is shorter than the sum of the two shorter side lengths. Check that this remains true even after modifying the triangle.
Angles and Inequalities

1. Show the measures of the given angles on the diagram. Then write an expression for the measures of the other angles.
   a) \( \angle AZB = x \) \( \angle AZC = 60^\circ \)
   \( \angle BZC = \) __________
   \( \angle EZD = \) __________
   b) \( \angle DBC = 2x \) \( \angle DCB = 90^\circ \)
   \( \angle BDC = \) __________
   \( \angle ABD = \) __________
   \( \angle CDE = \) __________

2. What possible measures can \( \angle ABC \) be?
   a) \( \angle CBD > 150^\circ \)
      Let \( \angle ABC = x \)
      Then \( \angle CBD = 180^\circ - x \)
      So \( 180 - x \) ______ > 150
      \( 180 - x - 150 > 0 \)
      \( 180 - 150 > x \)
      \( 30 > x \)
      So \( \angle ABC < 30^\circ \)
   b) \( \angle CAB < 75^\circ \)
      Let \( \angle ABC = x \)
      Then \( \angle CAB = \) __________
      So __________ < 75°
      __________
      __________
      __________
      So __________
   c) \( \angle BCD > 70^\circ \)
   Bonus \( \angle ABC = \angle ACB, \angle EDA > 60^\circ \)
Circle
Congruent Triangles and The Geometer’s Sketchpad®

1. Using the polygon tool, construct a triangle with vertices \(A, B,\) and \(C.\) Name it \(\triangle ABC\) (read as “triangle \(ABC\)”). Measure the sides and the angles of your triangle.

2. a) Construct a point \(D,\) Using a command for constructing circles and the length of \(AB\) as the radius, construct a circle with centre \(D\). Construct line segment \(DE = AB\). Hide the circle.

   b) Using a command for constructing circles and the length of \(BC\) as the radius, construct a circle with centre \(E\).

   c) Using a command for constructing circles and the length of \(AC\) as the radius, construct a circle with centre \(D\).

   d) Construct a point that is on both circles. Label it \(F\). Use the polygon tool to construct triangle \(\triangle DEF\). Hide the circles.

3. Which sides are equal in \(\triangle ABC\) and \(\triangle DEF\)?

   \(AB = \underline{\quad}\) \(BC = \underline{\quad}\) \(AC = \underline{\quad}\)

4. Try to move the vertices of \(\triangle DEF\). How does your triangle change?
   Which transformations can you make: rotations, reflections, translations?

5. Move \(\triangle DEF\) onto \(\triangle ABC\). You many need to use a command for reflection.
   Does it fit without overlaps or gaps? \(\underline{\quad}\) Are \(\triangle ABC\) and \(\triangle DEF\) congruent? \(\underline{\quad}\)

6. Move \(\triangle DEF\) away from \(\triangle ABC\). Try to move the vertices of \(\triangle ABC\).
   a) Are the changes you can make in this triangle different from the changes you could make to \(\triangle DEF\)? Why?

   b) What happens to \(\triangle DEF\) when you modify \(\triangle ABC\)? What can you say about the triangles \(ABC\) and \(DEF\)?

7. Are three side lengths enough to determine a unique triangle? \(\underline{\quad}\)
Area Formula for Circles

1. Estimate the area of the circle by finding the area of the shaded part.
   a) Find the area of each shaded shape in the top right quarter of the circle.
   b) Add the areas in part a) to estimate the area of one quarter of the circle.
   c) Find the area of the entire shaded part of the grid using your answer from part b). \( A = \) ______
   d) The radius of the circle is \( r = \) ______.
   e) Use a calculator to calculate \( A \div r^2 \) to two decimal places. ______
   f) Explain why the area \( A \) that you found in part c) is a good estimate for the area of the circle.
Approximating the Area of a Circle

As you cut the circle into more pieces, the resulting shape looks more like a rectangle.

Area of circle $\approx$ Area of rectangle $= l \times w \approx \frac{C}{2} \times r = \pi r \times r = \pi r^2$

4 pieces

8 pieces

16 pieces
Formula Flash Cards (1)

\[ C = \pi d \]
\[ A = \pi r^2 \]

\[ A = \frac{b \times h}{2} \]
\[ A = b \times h \]

\[ A = l \times w \]
\[ A = \frac{\left(b_1 + b_2\right) \times h}{2} \]
Formula Flash Cards (2)

- Circle: $r$
- Circle: $d$
- Trapezoid: $b_1$, $b_2$, $h$
- Triangle: $b$, $h$
- Parallelogram: $b$, $h$
- Rectangle: $l$, $w$
Unit 5  The Number System: Repeating Decimals and Terminating Decimals

This Unit in Context
In Grade 6, students fluently added, subtracted, multiplied, and divided multi-digit decimals using the standard algorithm for each operation (6.NS.B.3). In 7.1 Unit 4, students reviewed how to convert decimal fractions to decimals, compare fractions to decimals, and add and subtract multi-digit decimals.

In this unit, students will learn how to convert a rational number to a decimal using long division. They will learn that the decimal form of a rational number terminates in zero or eventually repeats (7.NS.A.2d). In Grade 8, students will learn that some decimals neither terminate nor repeat and that these numbers are called irrational numbers. Students will learn that real numbers consist of the union of rational and irrational numbers. In high school, students will encounter complex numbers, which do not form part of the real number system.

Mathematical Practices in This Unit
In this unit, you will have the opportunity to assess MP.1 to MP.8. The MP labels in this unit flag both opportunities to develop the mathematical practice standards and opportunities to assess them. Below is a list of where we recommend assessing each standard, as well as some examples of how students can show that they have met a standard.

MP.1: NS7-46 Extension 3, NS7-48 Extension 2

MP.2: NS7-46 Extension 3, NS7-47 Extension 3
In NS7-46 Extension 3, students reason abstractly and quantitatively when they recognize what the results of each computation mean in context. For example, students recognize that, when they find $2,880 \div 355$ they get either $8 \text{ R } 40$ or approximately $8.1$, someone needs to buy $9$ whole cans of ginger ale.

MP.3: NS7-49 Extension 3, NS7-50 Extension 5

MP.4: NS7-47 Extension 3, NS7-48 Extension 2
In NS7-47 Extension 3, students model mathematically a real-world situation when they use equations to determine how many adult tickets and child tickets were sold at a zoo, given specific conditions.

MP.5: NS7-46 Extension 3, NS7-52 Extension 5

MP.6: NS7-47 Extension 3, NS7-51 Extension 5

MP.7: NS7-48 Extension 2, NS7-51 Extension 4, NS7-52 Extension 5
**MP.8:** NS7-50 Extension 5, NS7-52 Question 7 on AP Book 7.2 p. 151

In NS7-52 Question 7 on AP Book 7.2 p. 151, students look for regularity when they multiply 0.2 × 3, 0.22 × 3, and 0.222 × 3, and they express the regularity they notice when they multiply $0. \overline{2} \times 3$. 
Unit 5  The Number System: Repeating Decimals and Terminating Decimals

Introduction
In this unit, students will review the long-division algorithm and learn how to apply it to two-digit divisors, decimal numbers, and negative numbers.

Students will use long division to convert fractions to decimals. They will identify the decimals as terminating or repeating, and use bar notation for repeating decimals. They will position terminating decimals on number lines and see that repeating decimals have approximate positions on these number lines. Students will determine whether a fraction’s decimal expansion terminates or repeats by looking at the denominator of the fraction in reduced form.

Students will round, compare, add, subtract, and multiply positive and negative terminating and repeating decimals.

Fraction notation. We show fractions in two ways in our lesson plans:

Stacked: \( \frac{1}{2} \)  
Not stacked: 1/2

If you show students the non-stacked form, remember to introduce it as new notation.

Materials. We recommend that students always work on grid paper. Paper with 1/4-inch grids works well in most lessons. In this unit, grid paper will help students align the numbers correctly when doing long division and working with decimal numbers. If students do not use grid paper in general, you will need to have lots of it on hand throughout the unit (e.g., from BLM 1 cm Grid Paper on p. T-1). If students who have difficulties in visual organization will be working without grid paper, they should be taught to draw a grid before starting to work on a problem.
Goals:
Students will use rounding to estimate the quotient when dividing by two-digit numbers including cases in which the estimate yields the correct quotient.

Prior Knowledge Required:
Can round two-digit numbers to the nearest ten
Can use long division to divide multi-digit numbers by one-digit numbers
Can divide decimals by whole numbers
Can divide positive and negative numbers

Vocabulary: absolute value, dividend, divisor, estimate, quotient, remainder, rounding

Materials:
transparency of BLM 2-Digit Division (p. Q-42, optional)

Introduce rounding to estimate the quotient. Go through the steps to divide 25 by 3 on the board, as shown below:

\[
\begin{array}{c}
8 \\
3 \overline{)25} \\
-24 \\
1 \\
\end{array}
\]
So, \(25 \div 3 = 8 \text{ R } 1\).

SAY: Let’s say \(25 \div 3\) represents dividing 25 cents equally among 3 people. Each person would get 8 cents with 1 cent left over. Now let’s divide 253 cents by 30 people. We are sharing about 10 times as much among 10 times as many people, so we expect a similar answer—about 8 cents each. Go through the steps to calculate \(253 \div 30\) on the board:

\[
\begin{array}{c}
8 \\
30 \overline{)253} \\
-240 \\
13 \\
\end{array}
\]
So, \(253 \div 30 = 8 \text{ R } 13\).

SAY: Notice that these two divisions have the same quotients but different remainders. You can always look for an easier division (25 ÷ 3) to help you do a harder division (253 ÷ 30). We know that the quotient of the harder division is 8 because 3 goes into 25 eight times. Once you know the quotient, you can finish the long division.
Write on the board:

465 ÷ 70

ASK: What easier division can we do to find the quotient? (46 ÷ 7) What is the quotient? (6)

Finish the long division on the board:

\[
\begin{array}{c|cc}
70 & 465 \\
\hline
420 & \\
45 & \\
\end{array}
\]

So, 465 ÷ 70 = 6 R 45.

**Exercises:** Divide.

a) 20)177  

b) 30)143  

c) 50)384  

d) 40)352  

**Bonus:**

e) 80)715  

f) 60)473  

g) 90)608  

h) 70)571  

**Answers:** a) 8 R 17, b) 4 R 23, c) 7 R 34, d) 8 R 32, Bonus: e) 8 R 75, f) 7 R 53, g) 6 R 68, h) 8 R 11

**NOTE:** We will use the following example and set of exercises to go through each step of long division. You could project a transparency of BLM 2-Digit Division to help you organize the lesson. You could also write the example and exercises on the board, leaving enough space to show all the steps.

**Rounding the divisor to estimate the quotient.** Project the example from BLM 2-Digit Division or write on the board:

49)167

ASK: What makes this problem harder than the ones we’ve done already? (49 is not a multiple of 10) Is it close to a multiple of 10? (yes) Which one? (50) Write “50” in a circle beside 49, as shown below:

\[
\begin{array}{c}
50 \\
49)167
\end{array}
\]

SAY: 49 is close to 50, so 49 goes into 167 about the same number of times as 50 does.  

ASK: How many times does 50 go into 167? (3) How did you get that? (5 goes into 16 three times) Write “3” above the 7, as shown below:

\[
\begin{array}{c}
50 \\
3 \\
49)167
\end{array}
\]

SAY: We don’t know for sure that 3 is the right quotient. We’re just guessing because 49 is close to 50.
Exercises: Round the divisor to the nearest 10. Then estimate the quotient. Write the estimated quotient above the ones digit of the dividend.

a) 19)164  
b) 38)251  
c) 41)351  
d) 81)342  

Bonus:  
e) 79)581  
f) 62)502  
g) 91)557  
h) 78)724  

Answers: a) 20, 8; b) 40, 6; c) 40, 8; d) 80, 4; Bonus: e) 80, 7; f) 60, 8; g) 90, 6; h) 80, 9

Multiplying the divisor by the estimated quotient. Refer back to the example on BLM 2-Digit Division or that you left on the board. ASK: Now that we have a guess for what the quotient is, what’s the next step? (multiply the quotient by the divisor) Emphasize that you multiply by the actual divisor, not the rounded divisor. Ask a volunteer to multiply 49 × 3 on the board and show where to put the answer in the division, as shown below:

\[
\begin{array}{c}
50 \\
\underline{49)167} \\
147 \\
\underline{\times 3} \\
147 \\
\end{array}
\]

Exercises: Continue with the previous exercises. Multiply the divisor by the estimated quotient. Write the answer below the dividend.

Answers: a) 19\|164 , 6  
b) 38\|251 , 6  
c) 41\|351 , 8  
d) 81\|342 , 4  
Bonus: e) 79\|581 , 7  
f) 62\|502 , 8  
g) 91\|557 , 9  
h) 78\|724 , 9

Finishing the long division. Remind students that the final step of long division is to determine the remainder through subtraction. Finish the example on BLM 2-Digit Division or on the board, as shown below:

\[
\begin{array}{c}
50 \\
\underline{49)167} \\
147 \\
\underline{\text{−} 147} \\
20 \\
\end{array}
\]

SAY: We are sharing 167 objects among 49 groups. ASK: How many objects are in each group? (3) How many objects have been shared so far? (147) How many have not been shared? (20) Are we finished with the division? (yes) How do you know? (the remainder is less than the number of groups) Remind students that when they solve long division and get a remainder, they can write the answer in an equation format. Write “So, 167 ÷ 49 = 3 R 20.” on the board below the division.

Exercises: Finish the previous exercises by finding the remainder.

Answers: a) 8 R 12, b) 6 R 23, c) 8 R 23, d) 4 R 18, Bonus: e) 7 R 28, f) 8 R 6, g) 6 R 11, h) 9 R 22
Review the steps of long division. SAY: Let’s review the four steps of long division. Point to the example on BLM 2-Digit Division or on the board, and ask volunteers to explain each step. (Step 1: Round the divisor to the nearest 10. Step 2: Estimate the quotient. Step 3: Multiply the divisor by the estimated quotient. Step 4: Subtract to find the remainder.)

Exercises: Complete the long division.

a) 327 ÷ 51  b) 184 ÷ 28  c) 148 ÷ 31  d) 211 ÷ 47
Bonus:

e) 583 ÷ 62  f) 642 ÷ 91

Answers: a) 6 R 21, b) 6 R 16, c) 4 R 24, d) 4 R 23, Bonus: e) 9 R 25, f) 7 R 5

Identifying the part of the dividend that is at least as large as the divisor. Write on the board:

18|472  42|217  11|586  15|1,076

SAY: Now we are going to introduce some division problems with quotients greater than 10. To start a division problem, we have to find the first part of the dividend that is at least as large as the divisor. ASK: In the first division, what is the first part of the dividend that is at least 18? PROMPT: Is 1 at least 18? (no) Is 14? (no) Is 147? (yes) Circle 147. Repeat with the other divisions. (217, 58, 107)

Exercises: Circle the first part of the dividend that is at least as big as the divisor.

a) 34|426  b) 37|1,196  c) 23|2,347
d) 15|1,439  e) 15|167

Answers: a) 42, b) 119, c) 23, d) 143, e) 16

Write on the board:

11|455

ASK: What is the first part of 455 that is at least as big as 11? (45) Circle 45. SAY: Estimate the quotient by dividing 10 into 45. (4) Tell students to write “4” above the ones digit of the circled number. Have a volunteer complete the long division on the board, as shown below:

41
11|455
- 44   1
15
- 11
4

So, 455 ÷ 11 = 41 R 4.
Exercises: Divide. Hint: Circle the first part of the dividend that is at least as big as the divisor. Write the first digit of the quotient above the ones digit of the circled number.

a) 215 ÷ 27  
b) 852 ÷ 34  
c) 521 ÷ 45  
d) 478 ÷ 52

Bonus:
e) 4,358 ÷ 81  
f) 8,327 ÷ 79

Answers: a) 7 R 26, b) 25 R 2, c) 11 R 26, d) 9 R 10, Bonus: e) 53 R 65, f) 105 R 32

How many digits will the whole number quotient have? Write on the board:

479 ÷ 52  
627 ÷ 52

ASK: Will the whole number quotient of the first division have 1 digit or 2 digits? (1) How do you know? (we circle all 3 digits so the quotient will be in the ones column) Will the whole number quotient of the second problem have 1 digit or 2 digits? (2) How do you know? (we circle the first 2 digits so the quotient’s first digit will be in the tens column)

(MP.7) Exercise: Circle the divisions that have two-digit quotients.

275 ÷ 42  
723 ÷ 21  
367 ÷ 5  
432 ÷ 54  
2,594 ÷ 324  
3,482 ÷ 213

Answers: 723 ÷ 21, 367 ÷ 5, and 3,482 ÷ 213

(MP.3) Review dividing integers. Remind students that as long as they can divide positive numbers by positive numbers, they can also divide negative numbers. Write on the board:

(+ ÷ (+) = (+)  
(− ÷ (−) = (+)  
(+ ÷ (−) = (−)  
(− ÷ (+) = (−)

−624 ÷ 78 =

ASK: What sign does the answer to this division have? (negative) PROMPT: Which rule for dividing integers applies? ((− ÷ (+) = (−))) SAY: Now let’s use the absolute values to divide. Remind students that the absolute value of a number is its distance from zero: |−624| = 624. SAY: So we’ll divide 624 by 78. Have volunteers go through the steps on the board. The final picture should look like this:

8
78 | 624
−624
0

ASK: So what is the answer to −624 ÷ 78? (−8)

Exercises: Divide.
a) 147 ÷ (−21)  
b) −938 ÷ (−67)

Bonus:
c) −8,833 ÷ 73  
d) −6,460 ÷ (−85)

Answers: a) −7, b) 14, Bonus: c) −121, d) 76
**Review dividing decimals.** Remind students that as long as they can divide whole numbers by whole numbers, they can divide decimals by whole numbers, too. Write on the board:

\[
\begin{array}{c}
\begin{array}{r}
34 \\
\hline
20.4
\end{array}
\end{array}
\]

SAY: Let's work this out as if we were dividing 204 by 34. Have volunteers go through the steps on the board, as shown below:

\[
\begin{array}{r}
6 \\
\hline
34 \big) 20.4 \\
- 204 \\
\hline
0
\end{array}
\]

ASK: How do you know where to put the decimal point in the quotient? (it goes in the same place where it is in the dividend) Have a volunteer place the decimal point. (0.6)

**Exercises:** Divide.

a) \(81.9 \div (-39)\)  
b) \(-4.84 \div 11\)

**Bonus:**

c) \(90.19 \div 29\)  
d) \(-9.963 \div (-81)\)

**Answers:** a) -2.1, b) -0.44, Bonus: c) 3.11, d) 0.123

Tell students they may have to add zeros to get the correct place value of the quotient. Write on the board:

\[
\begin{array}{r}
3 \\
\hline
51 \big) 1.53 \\
- 153 \\
\hline
0
\end{array}
\]

Show that we need to add a zero for the tenths to get the answer 0.03.

**Exercises:** Divide.

a) \(-16.4 \div (-41)\)  
b) \(-2.52 \div 42\)

**Bonus:**

c) \(0.219 \div (-73)\)  
d) \(1.428 \div 68\)

**Answers:** a) 0.4, b) -0.06, Bonus: c) -0.003, d) 0.021

**Extensions**

**(MP.7)** 1. You can divide by three-digit numbers the same way you divide by two-digit numbers. Circle the first part of the dividend that is at least as big as the divisor. Round the divisor to the nearest hundred, and estimate the quotient. Complete the long division.

a) \(2,136 \div 512\)  
b) \(31,073 \div 428\)  
c) \(256,581 \div (-387)\)  
d) \(-56,235 \div (-489)\)

**Answers:** a) 4 R 88, b) 72 R 257, c) -663, d) 115
2. Fill in the missing numbers.

   a) \[ \begin{array}{c}
   42) \underline{22} \\
   - \underline{84} \\
   \underline{13}
   \end{array} \]

   b) \[ \begin{array}{c}
   87) \underline{287} \\
   - \underline{174} \\
   - \underline{435} \\
   \underline{84}
   \end{array} \]

   Answers:
   a) 229784
   b) 252259

(MP.1, MP.2, MP.5) 3. Bev made a grape drink by mixing \( \frac{1}{3} \) cup of ginger ale with \( \frac{1}{2} \) cup of grape juice. She used all her ginger ale, but she still has lots of grape juice. She wants to make 30 cups of the grape drink for a party. How many 355 mL cans of ginger ale does she need to buy? Use 1 cup = 240 mL and any tool you think will help. Say what each step means in the situation.

Sample answers:
- I made a ratio table.

<table>
<thead>
<tr>
<th>Number of Cups of Ginger Ale</th>
<th>Total Cups of Ginger Ale and Grape Juice</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} + \frac{1}{2} = \frac{5}{6} )</td>
</tr>
</tbody>
</table>

I solved \( \frac{5}{6}x = 30 \) by finding \( 30 ÷ \frac{5}{6} = 30 \times \frac{6}{5} = 36 \). Then I multiplied \( 1/3 \times 36 = 12 \) to get the number of cups of ginger ale needed for 30 cups of grape juice. I found \( 12 \times 240 = 2,880 \) mL of ginger ale. I found \( 2,880 ÷ 355 = 8 \text{ R } 40 \), so she needs to buy 9 cans of ginger ale.

- I used paper and pencil. I found what fraction of the drink is ginger ale. Since \( 2/6 \) of a cup is ginger ale and \( 3/6 \) of a cup is grape juice, the ratio of ginger ale to grape juice is \( 2 : 3 \), so \( 2/5 \) is ginger ale. For the party, Bev needs \( 2/5 \) of 30 cups, or 12 cups of ginger ale. As above, Bev needs 2,880 mL of ginger ale, so she needs to buy 9 cans of ginger ale.
**NS7-47  2-Digit Division—Guess and Check**

Pages 140–142

Standards: 7.NS.A.2c

**Goals:**
Students will use rounding to estimate the quotient when dividing by two-digit numbers, including cases in which the estimate needs to be adjusted.

**Prior Knowledge Required:**
Can use long division to divide multi-digit numbers by two-digit numbers
Can divide decimals
Can divide positive numbers and negative numbers

**Vocabulary:** decimal, decimal point, dividend, divisor, estimate, “guess and check,” quotient, remainder, rounding

**Applying the guess, check, and revise strategy to division.** Tell students that different people guessed the quotient for different divisions. Challenge students to decide whether the quotient guessed is too high or too low. Write on the board:

\[
\begin{array}{cccc}
3 & 19 & 3 & 25 \\
3 & 12 & 2 & 17 \\
9 & 15 & 9 & 18 \\
5 & 27 & 9 & 12 \\
4 & 25 & 2 & 17 & 2
\end{array}
\]

Point to the first division, and SAY: If we divide 19 into 3 groups of 5 each, we have 4 left over. ASK: Can we put one more in each group? (yes) Is the estimate of 5 in each group too low or too high? (too low) Point to the second division, and SAY: With 9 objects in each group, we would place 27 objects, but we only have 25 objects. ASK: Is 9 too low or too high? (too high) Point to the third division, and ASK: Is 9 too low or too high? (too high) How do you know? (because 18 objects is too many to place—we only have 17) Repeat for the fourth division. (6 is too low because we have 2 left over, so we can put one more in each group)

Point out that looking at the remainder tells us if the estimate is good or not. If the remainder is negative, the estimate is too high. As soon as you see that the remainder will be a negative number, stop the division. If the remainder is positive and less than the divisor, the estimate is good. If the remainder is greater than or equal to the divisor, the estimate is too low. Write on the board:

\[
\begin{array}{ccccccc}
23 & 149 & 17 & 123 & 19 & 171 & 31 & 284 & 33 & 255 \\
7 & 26 & 8 & 9 & 8 \\
-161 & -102 & -152 & -279 & -264 \\
27 & 19 & 5
\end{array}
\]
Point to each division in turn, and ASK: Is the quotient too high, too low, or just right? Have students signal thumbs up for too high, thumbs down for too low, or flat hand for just right. (high, low, low, just right, high)

(MP.3) Exercises: Multiply the estimated quotient by the divisor. Determine the remainder. Is the estimate too high, too low, or just right?

a) \( \frac{7}{36} 298 \)
b) \( \frac{6}{27} 183 \)
c) \( \frac{6}{73} 435 \)
d) \( \frac{9}{24} 198 \)

**Bonus:**
e) \( \frac{7}{821} 5,735 \)
f) \( \frac{6}{683} 4,557 \)
g) \( \frac{7}{867} 6,936 \)
h) \( \frac{6}{714} 4,228 \)

**Answers:** a) \( R = 46 \), too low; b) \( R = 21 \), just right; c) \( R < 0 \), too high; d) \( R < 0 \), too high; Bonus: e) \( R < 0 \), too high; f) \( R = 459 \), just right, g) \( R = 867 \), too low, h) \( R < 0 \), too high

Revising the estimate. Write on the board:

\[
\begin{array}{c}
7 \\
23 \overline{156} \\
\hline
-161
\end{array}
\]

SAY: The first estimate of 7 gives a negative remainder, so it's too high. ASK: Which is a better next guess, 6 or 8? (6) How do you know? (7 was too high, so we should choose a smaller number) Cross out the “8” guess. Have a volunteer multiply 23 \( \times \) 6 on the board. (138) Have another volunteer finish the division. (6 R 18)

**Exercises:**

1. Use the first estimate to make a better estimate. Then divide.

a) \( \frac{3}{19} 78 \)
b) \( \frac{4}{42} 164 \)

\[
\begin{array}{c}
3 \\
-57 \\
\hline
21
\end{array}
\]

21 > 19, negative number

\[
\begin{array}{c}
21 \\
-168 \\
\hline
4\text{ is too high.}
\end{array}
\]

**Bonus:**
c) \( \frac{7}{353} 2956 \)
d) \( \frac{5}{534} 2627 \)

\[
\begin{array}{c}
7 \\
-2471 \\
\hline
485
\end{array}
\]

7 is too _______. 5 is too _______.

**Answers:** a) 4 R 2; b) 3 R 38; Bonus: c) low, 8 R 132; d) high, 4 R 491

2. Divide.

a) \( 231 \div 16 \)
b) \( 632 \div 53 \)
c) \( 243 \div 31 \)

**Bonus:**
d) \( 6,943 \div 851 \)
e) \( 2,657 \div 322 \)

**Answers:** a) 14 R 7, b) 11 R 49, c) 7 R 26, Bonus: d) 8 R 135, e) 8 R 81
(MP.1) Encourage students to use multiplication and addition to check their answers. Emphasize that doing so is very important when dividing multi-digit numbers because there are many opportunities for making mistakes. Write on the board:

\[ 231 ÷ 16 = 14 \text{ R } 7 \quad 14 \times 16 + 7 = 224 + 7 = 231 \]

SAY: There are 14 groups with 16 objects in each group (point to “14 × 16”), and 7 left over (point to “+ 7”). So, there are 231 objects in total.

Review dividing decimals. Remind students that as long as they can divide whole numbers by whole numbers, they can divide decimals by decimals, too. Write on the board:

\[ 3.4 \overline{17.65} \]

SAY: Although you are not dividing by a whole number here, you can change it to a problem in which you are. ASK: How can you do that? (multiply both the divisor and dividend by 10) Show where to place the decimal point in the quotient, as shown below:

\[ 3.4 \overline{176.5} \rightarrow 34 \overline{1765} \rightarrow 34 \overline{176.5} \]

Point out that you sometimes have to add zeros to the dividend when you multiply by 10. Write on the board:

\[ 0.0034 \overline{176.528} \]

Have a volunteer demonstrate on the board how to change this problem so that the divisor is a whole number. (34)\overline{1765,280}

Exercises: Create an equivalent problem that has a whole number divisor. Then divide.

a) 5.7)\overline{76.38}  
b) 0.0027)\overline{37.8}  
c) 8.4)\overline{981.12}  

Answers: a) 13.4, b) 14,000, c) 116.8

(MP.4) Word problems practice.

Exercises: Use long division to solve the problem.

a) Rani has 3.6 pounds of cheese. She needs 0.08 pounds of cheese for each sandwich. How many sandwiches can she make?
b) Jim has $12.50. How many $0.25 pencil crayons can he buy?
c) A shelf is 41.4 inches long. How many books 2.3 inches thick can the shelf hold?
d) The temperature changes by −11.76°F over 73.5 minutes. How much does it change each minute on average?
e) The altitude of an airplane changes by −1,250 feet per minute. How long would it take a plane to make a change in altitude of −26,250 feet to land? NOTE: A negative number can represent a decrease in height. In this case, the plane is landing and therefore is losing altitude.

Bonus: A quarter is about 0.18 cm thick. What is the value of a stack of quarters 6.3 cm high?
Answers: a) 45; b) 50; c) 18; d) \(-0.16\)°F; e) 21 minutes; Bonus: \(6.3 \div 0.18 = 35\) and 
\(35 \times 0.25 = 8.75\), so the value of the stack of quarters is $8.75

Extensions

(MP.2) 1. Investigate if the estimate is more likely to be correct when the divisor is closer to the rounded number you used to make your estimate. For example, when the divisor is 31 rounded to 30, is your estimate more likely to be correct than when the divisor is 34 rounded to 30? Try these examples:

<table>
<thead>
<tr>
<th>Divisor</th>
<th>Dividend</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>243</td>
</tr>
<tr>
<td>31</td>
<td>249</td>
</tr>
<tr>
<td>31</td>
<td>257</td>
</tr>
<tr>
<td>31</td>
<td>265</td>
</tr>
<tr>
<td>31</td>
<td>274</td>
</tr>
<tr>
<td>34</td>
<td>243</td>
</tr>
<tr>
<td>34</td>
<td>249</td>
</tr>
<tr>
<td>34</td>
<td>257</td>
</tr>
<tr>
<td>34</td>
<td>265</td>
</tr>
<tr>
<td>34</td>
<td>274</td>
</tr>
</tbody>
</table>

Answer: Rounding 31 to 30 gives the right answer in all but the first and last cases. Rounding 34 to 30 doesn't give the right answer in any case. So rounding 31 to 30 gives the correct answer more often than rounding 34 to 30.

(MP.3) 2. Explain what is wrong with the answer. Find the correct answer.

a) \(\frac{9}{34}\) 
\[306 \div 34 \rightarrow \frac{9}{274} - 306 \quad \frac{352}{965} \rightarrow \frac{4}{4485} - 32 \quad \frac{378}{1716.75} = 0.6\]

Answers: a) estimate is too high, 8 R 2; b) estimate is too low, 50 R 85; 
c) The 6 should be above the 8, not the 7, because 63 goes into 378, not 37, 6 times. Then, 
zero should be added to keep the place value, 0.06.

(MP.2, MP.4, MP.6) 3. A zoo sells adult tickets for $8.95 each and child tickets for $3.50 each. 
In one day, the zoo sells 600 tickets for a total of $3,816.75. How many adult tickets and how 
many child tickets did they sell? Write an equation with a variable to solve the problem and state 
what the variable represents. Write your answer as a full sentence.

Answer: Let \(x\) be the number of adult tickets sold. Then the number of child tickets is \(600 - x\). 
The total amount of money made is, in dollars, \(8.95x + 3.5(600 - x)\). The total amount of money 
made is $3,816.75, so I made the equation \(8.95x + 3.5(600 - x) = 3,816.75\). After simplifying, I 
got \(8.95x + 2,100 - 3.5x = 3,816.75\), or \(5.45x = 1,716.75\). Then \(x = 1,716.75 + 5.45 = 315\). 
So the number of adult tickets sold is 315. They sold 600 tickets altogether, so the number of 
child tickets sold is 600 – 315 = 285.

Whole-class follow-up: Write the sequence of equations on the board without the explanations 
and have volunteers explain each step. ASK: Is 315 the final answer to the question? (no) What 
else is the question asking for? (the number of child tickets) So what is the final answer? (they 
sold 315 adult tickets and 285 child tickets)
Writing Fractions as Decimals

Pages 143–144

Standards: 7.NS.A.2d

Goals:
Students will use long division to convert fractions to decimals.

Prior Knowledge Required:
Can use long division to divide multi-digit numbers by multi-digit numbers

Vocabulary: dividend, quotient, remainder

NOTE: In this lesson, students will see that fractions can be written as one of two types of decimals: decimals that go on forever and decimals that stop. We will introduce the terms “repeating decimals” and “terminating decimals” in the next lesson.

Adding zeros to the dividend to get a remainder of zero. ASK: If I divide $6 among 2 people, how much does each person get? ($3) What bills or coins do I need to be able to do this? (six $1 bills so each person gets three $1 bills) Write on the board:

\[
\begin{array}{c}
3 \\
2 \overline{)6} \\
-6 \\
0
\end{array}
\]

SAY: Now I want to divide $6 among 5 people. I can’t do this with six $1 bills, but I could do it with 60 dimes, since 60 ÷ 5 = 12. Each person gets 12 dimes, or $1.20. Write on the board:

\[
\begin{array}{c}
1.2 \\
5 \overline{)6.0} \\
-5 \downarrow \\
10 \\
-10 \\
0
\end{array}
\]

Point out that just like $6 = 60 dimes, 6 = 60 tenths, which can be written as 6.0, because you use one place after the decimal when you write tenths. SAY: To divide $6 among 8 people, I need 600 pennies, since 600 ÷ 8 = 75. Each person gets 75 pennies. Write on the board:

\[
\begin{array}{c}
0.75 \\
8 \overline{)6.00} \\
-56 \downarrow \\
40 \\
-40 \\
0
\end{array}
\]
Point out that 600 pennies is 600 hundredths of a dollar, and we write 600 hundredths as 6.00. SAY: Because 6 = 6.0 = 6.00, we were able to get an answer with no remainder to these division problems.

**Writing fractions as terminating decimals.** Remind students that they can use long division to write a fraction as a decimal. Write on the board:

\[
\frac{7}{20} = 7 \div 20
\]

SAY: 7 can't be divided by 20 but, if we write 7 as 7.0, we can use long division to work out 7.0 ÷ 20. Ask a volunteer to solve the long division on the board, as shown below:

\[
\begin{align*}
& \hspace{1cm} 0.3 \\
& \hspace{1cm} 20 \overline{)7.0} \\
& \hspace{1cm} -60 \\
& \hspace{1cm} 10
\end{align*}
\]

SAY: There is still a remainder, so let's add another zero to 7.0 and see if we get a remainder of zero. Ask a volunteer to continue the division on the board, as shown below:

\[
\begin{align*}
& \hspace{1cm} 0.35 \\
& \hspace{1cm} 20 \overline{)7.0} \\
& \hspace{1cm} -60 \\
& \hspace{1cm} 100 \\
& \hspace{1cm} -100 \\
& \hspace{1cm} 0
\end{align*}
\]

SAY: So 7/20 = 0.35. Point out that by solving the long division this way, we are thinking of 7 as 7 and 0 tenths and 0 hundredths. If we needed to, we could have added 0 thousandths. In fact, we can add as many zeros as we need to get an even division with no remainder.

**Exercises:** Use long division to write the fraction as a decimal. Keep dividing until the remainder is zero.

a) \( \frac{3}{5} \)  

b) \( \frac{17}{20} \)  

c) \( \frac{5}{8} \)  

d) \( \frac{3}{40} \)  

**Answers:** a) 0.6, b) 0.85, c) 0.625, d) 0.075, Bonus: 0.2625

**Introduce the possibility of never getting a remainder of zero.** SAY: Now let's divide $6 by 9 people. Six $1 bills can't be divided by 9. ASK: Can 6 and 0 tenths, which is 60 dimes, be divided by 9? Have a volunteer solve the division on the board, as shown below:

\[
\begin{align*}
& \hspace{1cm} 0.6 \\
& \hspace{1cm} 9 \overline{)6.0} \\
& \hspace{1cm} -54 \\
& \hspace{1cm} 6
\end{align*}
\]
Say: We have a remainder. Let's try dividing 6 and 0 tenths and 0 hundredths, which is 600 pennies. Have a volunteer add a zero and continue the division on the board, as shown below:

\[
\begin{array}{c}
9 \overline{)6.00} \\
-54 \downarrow \\
60 \\
-54 \\
6
\end{array}
\]

(MP.8) Say: There is still a remainder! Ask: Do you think if we added 0 thousandths to make 6.000, that we would get a remainder of 0? (No) How do you know? (The remainder is always 6) Point out that every subtraction is 60 − 54, so every remainder is 6. Say: Our method of adding zeros didn’t work the way we thought it would, but it did still give us something useful: a pattern to predict how the division will continue.

Writing fractions as repeating decimals. Ask a volunteer to use long division to write 1/9 as a decimal, as shown below:

\[
\begin{array}{c}
9 \overline{)1.000} \\
-9 \downarrow \\
10 \\
-9 \downarrow \\
10
\end{array}
\]

Say: We still have a remainder. Ask: Should we keep going? (No) Why? (We will never get a remainder of zero) Why not? (The remainder will always be 1) Point out that all the divisions are 10 ÷ 9, so we’re always getting the same quotient (1) and the same remainder (10 − 9 = 1). Say: Because the remainders are repeating, you always do the same division, so you always get the same quotient. Ask: If you continue the division, what will the next digit be? (1) And the digit after that? (1) And the digit after that? (1) Say: When you divide some numbers, you will never get a remainder of zero and you could continue the division forever! But you can stop dividing when you see a repeating pattern in the quotient. To show that the quotient in the example continues with only 1s, write on the board:

\[
\frac{1}{9} = 0.111...
\]

(MP.8) Exercises: Write the fraction as a decimal. Stop adding zeros to the dividend when you see a pattern in the quotient.

a) \(\frac{2}{9}\)  
b) \(\frac{1}{3}\)  
c) \(\frac{44}{99}\)  
d) \(\frac{5}{15}\)

Answers: a) 0.222…, b) 0.333…, c) 0.444…, d) 0.333…
Writing fractions as decimals. SAY: When you are writing a fraction as a decimal, some decimals will end and some will go on forever. For example, $\frac{7}{20} = 0.35$ ends because we got a remainder of zero in the long division, and $\frac{1}{9} = 0.111\ldots$ goes on forever because we always had a remainder in the long division. Just keep adding zeros to the dividend until you get a remainder of zero, or see a repeating pattern.

(MP.8) Exercises: Use long division to write the fraction as a decimal. Stop adding zeros to the dividend when the remainder is zero, or when you see a pattern.

a) $\frac{1}{5}$ b) $\frac{6}{18}$

c) $\frac{17}{50}$ d) $\frac{12}{36}$

Answers: a) 0.2, b) 0.333..., c) 0.34, d) 0.333...

(MP.3) Writing negative fractions as decimals. Remind students that they can write negative fractions as decimals the same way they can write fractions as decimals. First, use these rules to determine the sign of the decimal. Write on the board:

\[ (+) ÷ (+) = (+) \quad (-) ÷ (-) = (+) \quad (+) ÷ (-) = (-) \quad (-) ÷ (+) = (-) \]

Then, divide as though both numbers are positive. Go through the following example with students on the board:

\[-\frac{7}{9} \text{ is a negative decimal} \]

\[
\begin{array}{c}
0.77 \\
9 \) 7.00 \\
-63 \\
70 \\
So -\frac{7}{9} = -0.777... \\
\end{array}
\]

Point out that the remainder will always be 7, and we can stop the division as soon as we see that pattern.

Exercises: Use long division to write the fraction as a decimal.

a) $-\frac{7}{10}$ b) $-\frac{11}{33}$

c) $\frac{16}{-18}$ d) $-\frac{9}{-60}$

Answers: a) −0.7, b) −0.333..., c) −0.888..., d) 0.15
Extensions
NOTE: It’s not always the first digit after the decimal that repeats. Sometimes the repetition starts with the second digit (e.g., $5/6 = 0.8333\ldots$) or a group of numbers repeats (e.g., $1/11 = 0.090909\ldots$). Let students know about these possibilities before they begin the Extensions.

1. a) Use long division to write the fraction as a decimal. Stop adding zeros to the dividend when the remainder is zero, or when you see a pattern.

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 4 & 5 & 6 & 8 & 9 & 10 \\
\end{array}
\]

b) Which fractions from part a) have a remainder of zero?

**Answers:** a) $1/2 = 0.5$, $1/3 = 0.333\ldots$, $1/4 = 0.25$, $1/5 = 0.2$, $1/6 = 0.166\ldots$, $1/8 = 0.125$, $1/9 = 0.111\ldots$, $1/10 = 0.1$, $1/11 = 0.0909\ldots$; b) $1/2$, $1/4$, $1/5$, $1/8$, $1/10$

(MP.1, MP.4, MP.7) 2. Marco and Tessa decide to go for a walk together. Marco walked towards Tessa’s house and Tessa walked towards Marco’s house. They both left at 4 p.m. and met each other at 4:20 p.m. Their houses are 3.6 miles apart. If Marco walked at a speed of 4 mph, what was Tessa’s speed? Find a fast way to solve the problem.

**Sample solutions:**
- Marco walks 4 mph, so in 20 minutes he walks $4/3 = 1 \frac{1}{3}$ miles. Their houses are 3.6 miles apart, so Tessa walked the rest of the way in the 20 minutes. She walked $3.6 - 1 \frac{1}{3} = 3 \frac{3}{5} - 1 \frac{1}{3} = 18/5 - 4/3 = 54/15 - 20/15 = 34/15 = 2 \frac{4}{15}$ miles. At that rate, in one hour she would have walked three times that distance, so I found $3 \times 2 \frac{4}{15} = 6 \frac{12}{15} = 6.8$. So Tessa’s speed was 6.8 mph.
- Marco and Tessa get closer together at a speed of $3 \times 3.6 = 10.8$ mph. Since Marco walked at a speed of 4 mph, Tessa must have walked at a speed of $10.8 - 4 = 6.8$ mph.
- I used variables. Let $s$ be Tessa’s speed in mph. The distance Tessa walked in 20 minutes is $s/3$. The distance Marco walked in 20 minutes is $4/3$. The total distance is $3.6 = s/3 + 4/3$. So, $s/3 = 3.6 - 4/3 = 3 \frac{3}{5} - 4/3 = 18/5 - 4/3 = 54/15 - 20/15 = 34/15$. So $s = 34/5 = 6 \frac{4}{5} = 6.8$. Tessa’s speed was 6.8 mph.
- I used variables. $3.6 = s/3 + 4/3$, so $3.6 = (s + 4)/3$, so then I multiplied both sides by 3 and I got $3 \times 3.6 = s + 4$, so $10.8 = s + 4$, so $s = 6.8$. Tessa’s speed was 6.8 mph.

Whole-class follow-up: Present, or have volunteers present, the four solutions. Compare the solutions. **ASK:** Which algebraic solution is like the first numerical solution? (the first one) Which algebraic solution is like the second numerical solution? (the second one)
Standards: 7.NS.A.2d

Goals:
Students will distinguish between repeating decimals and terminating decimals.
Students will use bar notation for repeating decimals.

Prior Knowledge Required:
Can use long division to write positive and negative fractions as decimals

Vocabulary: bar notation, repeating core, repeating decimal, terminating decimal

Introduce repeating decimals and terminating decimals. Write on the board:

0.5  0.555…

ASK: What is the difference between these two numbers? (one is exactly 0.5, the other is a number that is a bit bigger than 0.5) Why do you think the second number has a “…” after it? (the 5s continue on forever) SAY: We call 0.5 a terminating decimal, because its digits stop. After the 5, there will only be zeros (0.5000000…). The number 0.555… is not a terminating decimal because its digits will continue on and on forever, and will never become zero. More importantly, the digits form a pattern that repeats on and on forever. So we call the decimal a repeating decimal. Repeating decimals have a repeating core—a digit or group of digits that repeats.

Identifying the repeating core. Write on the board:

0.777…  0.444…

ASK: What is the repeating core? (7, 4) Point out that these decimals have a single digit that repeats. Write on the board:

0.1666…  0.8333…

ASK: What is the repeating core? (6, 3) Point out that the repeating digit doesn’t have to be the first digit after the decimal point. Write on the board:

0.363636…  0.2818181…

ASK: What is the repeating core? (36, 81) Point out that sometimes there is more than one digit in the repeating core.
Write on the board:

\[ 0.12233444... \quad \quad 0.121121121... \]

Point out that although these decimals have a pattern, there is no repeating core. SAY: This is a third type of decimal—a decimal that does not terminate or repeat, but we will not be working with these in this unit.

**Exercises:** Identify the repeating core of the decimal.

a) 0.55555...  
   b) 0.27777...  
   c) 0.1363636...  
   d) 0.636363...  
   e) 0.423423423...  
   f) 0.41666...  

**Answers:** a) 5, b) 7, c) 36, d) 63, e) 423, f) 6

**Identifying repeating decimals and terminating decimals.** Write a variety of decimals on the board and have students signal a terminating decimal with thumbs up and a repeating decimal with thumbs down. Examples:

0.3 0.777... 0.45454545... 0.67 0.22 0.01666... 0.500...

(terminating, repeating, repeating, terminating, terminating, repeating, terminating)

**Exercise:** Circle the repeating decimals. Hint: Can you identify a repeating core?

0.22 0.343434... 0.234 0.2333... 0.2343434... 0.34

**Answers:** 0.343434..., 0.2333..., and 0.2343434...

**Introduce bar notation for repeating decimals.** SAY: Mathematicians have a way of writing repeating decimals exactly, without rounding and without having to write many digits. Mathematicians put a bar over the digit or group of digits that forms the repeating core. This is called **bar notation**. Write on the board:

\[ \bar{0.7} \]

SAY: The bar over the 7 means that the repeating core is 7. Write on the board:

\[ 0.\bar{7} \approx 0.77777777 \]

Write on the board:

\[ 0.0\bar{2} \quad 0.\bar{25} \quad 0.00\bar{5} \quad 0.4\bar{56} \quad 0.1\bar{452} \quad 0.1\bar{846} \]

Ask students to identify the repeating core. PROMPT: What is under the bar? (2, 25, 5, 456, 52, 1846) Have volunteers come up to the board to write the first 8 digits of each decimal. (0.02222222, 0.25252525, 0.00555555, 0.45645645, 0.14525252, 0.18461846) Point out that we have written these numbers to eight decimal places, or 8 place value positions after the decimal point.
**Exercises:** Write the decimal to 8 decimal places.

- a) 0.8
- b) 0.08
- c) 0.008
- d) 0.17
- e) 0.17
- f) 0.517
- g) 0.517
- h) 0.651

**Answers:**
- a) 0.88888888
- b) 0.08888888
- c) 0.00888888
- d) 0.17171717
- e) 0.17777777
- f) 0.51717171
- g) 0.51751751
- h) 0.65151515

Write on the board:

0.555…

**ASK:** What is the repeating core? (5) Continue writing on the board:

0.555... = 0.5

Write on the board:

0.1666…

**ASK:** What is the repeating core? (6) Continue writing on the board:

0.1666... = 0.16

**MP.6** Point out that only the 6 is part of the repeating core and that the 1 is not, so be sure to only put the bar over the 6. To write 0.16 would be incorrect.

Write on the board:

0.123012301230…

**ASK:** What is the repeating core? (1230) Continue writing on the board:

0.123012301230... = 0.1230

Point out that we only put a bar over numbers after the decimal point. To write 0.123 would be incorrect. Write on the board:

16.6666666666…

Have a volunteer show how to use bar notation to write the decimal. (16.6) Point out that even though there is a 6 before the decimal, the bar must be placed above the 6 after the decimal.

**Exercises:** Use bar notation to write the repeating decimal.

- a) 0.7777777777…
- b) 0.8333333333…
- c) 0.2666666666…
- d) 7.2727272727…
- e) 0.3232323232…
- f) 0.126126126126…
- g) 0.045454545454…
- h) 8.0769230769230769…
- i) 12.142857142857142857…

**Answers:**
- a) 0.7, b) 0.8\̅, c) 0.2\̅, d) 7.2\̅, e) 0.3\̅2, f) 0.126, g) 0.0\̅45, h) 8.07692\̅3, i) 12.1\̅42857
Writing fractions as repeating decimals. Remind students that we can use long division to write fractions as decimals. Write on the board:

\[
\frac{6}{11} = 6 \div 11 = 11 \overline{.54}
\]

Have a volunteer solve the long division on the board, adding zeros to the dividend as needed. The final picture should look like this:

\[
\begin{array}{c|c}
\text{0.545} \hline
11 & 6.000 \\
-55 & \\
50 & \\
-44 & \\
60 & \\
55 & \\
5 & \\
\end{array}
\]

ASK: How do we know when we can stop the division? (when we see that the digits 5 and 4 repeat in the quotient since we get a remainder of 5, then 6, then 5 again) What pattern would the quotient follow if we kept dividing? (0.54545454…) How can we use bar notation to write 0.54545454…? (0.\overline{54})

**Exercises:** Use long division to write the fraction as a decimal. Use bar notation.

a) \(\frac{2}{3}\)  

b) \(\frac{5}{6}\)  

c) \(\frac{4}{11}\)  

d) \(\frac{8}{99}\)

**Answers:** a) 0.\overline{6}, b) 0.8\overline{3}, c) 0.3\overline{6}, d) 0.0\overline{8}

Write on the board:

\[
-\frac{6}{11}
\]

ASK: What is the sign of the decimal? (negative) SAY: When we divided 6 by 11, the answer was 0.54. ASK: So what is \(-\frac{6}{11}\) ? (\(-0.\overline{54}\))

**Exercises:** Use long division to write the positive fraction as a decimal. Then use bar notation to write the decimal with the correct sign.

a) \(-\frac{14}{21}\)  

b) \(\frac{12}{27}\)  

c) \(-\frac{5}{11}\)  

d) \(-\frac{1}{6}\)

**Answers:** a) \(-0.\overline{6}\), b) \(-0.4\), c) 0.4\overline{5}, d) 0.1\overline{6}
Extensions

(MP.1) 1. Four friends want to share a pizza that costs $10.
   a) If the four friends split the cost evenly, how much should each person pay?
   b) If one friend decides to not have pizza, then how much should each person pay? Round your answer to the nearest cent.
   c) When the three friends round to the nearest cent, do they pay the full $10? If not, what should they do?
   **Answers:** a) $10 ÷ 4 = $2.50 each; b) $10 ÷ 3 ≈ $3.33 each; c) $3.33 × 3 = $9.99, they don’t pay the full $10 and one person needs to pay $3.34

(MP.7) 2. a) Use long division to write $\frac{1}{5}$, $\frac{2}{9}$, $\frac{3}{9}$, and $\frac{4}{9}$ as decimals.
   b) Extend the pattern to find $\frac{5}{9}$, $\frac{6}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$.
   c) Use long division to write $\frac{12}{99}$, $\frac{25}{99}$, $\frac{37}{99}$, and $\frac{48}{99}$ as decimals.
   d) Extend the pattern to find $\frac{52}{99}$, $\frac{61}{99}$, $\frac{75}{99}$, $\frac{89}{99}$, and $\frac{95}{99}$.
   e) Predict what $\frac{123}{999}$ is as a decimal.
   f) Use long division to check your answer to part e).
   **Bonus:** Write the repeating decimal as a fraction.
   g) $\frac{0.7}{9}$ h) $\frac{0.23}{99}$ i) $-\frac{0.05}{99}$ j) $\frac{0.441}{999}$
   k) $-\frac{0.652}{999}$ l) $0.98$ m) $-0.5$ n) $0.461$
   o) $0.3\overline{8}$ p) $-0.06\overline{1}$
   **Answers:** a) $0.1$, $0.2$, $0.3$, $0.4$; b) $0.5$, $0.6$, $0.7$, $0.8$; c) $0.1\overline{2}$, $0.2\overline{5}$, $0.3\overline{7}$, $0.4\overline{8}$;
   d) $0.5\overline{2}$, $0.6\overline{1}$, $0.7\overline{5}$, $0.8\overline{9}$, $0.9\overline{5}$; e) $0.12\overline{3}$, f) $123 ÷ 999 = 0.\overline{123}$; Bonus: g) $\frac{7}{9}$; h) $\frac{23}{99}$; i) $-\frac{5}{99}$;
   j) $\frac{441}{999}$; k) $-\frac{652}{999}$; l) $\frac{98}{99}$; m) $-\frac{5}{9}$; n) $\frac{461}{999}$; o) $\frac{38}{99}$; p) $-\frac{61}{999}$

(MP.3) 3. Eddy painted a square wall. Randi is painting a square wall that is twice as wide as the wall that Eddy painted. Eddy used 1.3 gallons of paint. Randi says she will need 2.6 gallons of paint because that is twice as much as Eddy needed and the wall she is painting is twice as big. Do you agree with Randi? Why or why not?
   **Answer:** No. Both walls are square walls, so if Randi’s wall is twice as wide, it is also twice as tall, so it is really four times as big. Randi will need four times as much paint as Eddy.
Repeating Decimals and Terminating Decimals on Number Lines

Pages 146–147

Standards: 7.NS.A.2d

Goals:
Students will position terminating decimals and repeating decimals on number lines.

Prior Knowledge Required:
Can position a terminating decimal on a number line
Can count by tenths, hundredths, and thousandths
Understands bar notation for repeating decimals

Vocabulary: repeating decimal, terminating decimal

Materials:
transparency of BLM Positioning Decimals on Number Lines (p. Q-43, optional)

Identifying hundredths and thousandths on number lines. Project a number line from BLM Positioning Decimals on Number Lines and label it as shown below, or draw on the board:

SAY: We are going to identify the hundredths marked with X's. Point out that our starting point, 0.5 or 5 tenths, is the same as 5 tenths and 0 hundredths, or 50 hundredths. Write “0.50” in the first blank. ASK: What is our ending point? (0.60) Write “0.60” in the last blank. Explain that the markings on this number line show all the hundredths between 50 hundredths and 60 hundredths. ASK: What hundredths are marked with an X? (0.52, 0.55, 0.59)

Exercise: Write the hundredths that are marked with an X.

Answers: 0.40, 0.42, 0.44, 0.47, 0.50
Project a number line from BLM Positioning Decimals on Number Lines and label it as shown below, or draw on the board:

```
0.19                                     0.20
          *                               *
          |                               |
0.19      0.20                          0.20
          |                               |
          *                               *
          |                               |
          |                               |
          |                               |
```

SAY: This number line covers exactly one hundredth, from 19 hundredths to 20 hundredths, so the number line counts by thousandths. ASK: How many thousandths is 19 hundredths? (190) Write “0.190” in the first blank. ASK: How many thousandths is 20 hundredths? (200) Write “0.200” in the last blank. ASK: What are the other thousandths marked with an X? (0.192, 0.195, 0.199)

**Exercise:** Write the thousandths that are marked with an X.

0.74                                     0.75
          *                               *
          |                               |
          |                               |
          |                               |
```
0.74      0.75                          0.75
          |                               |
          |                               |
          |                               |
```

**Answers:** 0.740, 0.743, 0.746, 0.748, 0.750

**Positioning terminating decimals on number lines.** SAY: If you have a number line divided into place value units, such as tenths, hundredths, or thousandths, then it is easy to place a terminating decimal on it. Remind students that if we had a number line divided into hundreds, each section of hundred could be divided into 10 tens, and each section of ten could be divided into 10 ones, and each section of one could be divided into 10 tenths, and so on, counting by smaller and smaller amounts each time.

Project a number line from BLM Positioning Decimals on Number Lines and label it as shown below, or draw on the board:

```
0   0.1   0.2   0.3   0.4   0.5   0.6   0.7   0.8   0.9   1

0   0.1   0.2   0.3   0.4   0.5   0.6   0.7   0.8   0.9   1
```

SAY: We are going to position 0.294 on the number line. This number line counts by tenths because the interval from 0 to 1 is divided into tenths. ASK: Which tenths is 0.289 between? (0.2 and 0.3) Circle the interval from 0.2 to 0.3. Point out that this gives us an approximate position of 0.289. SAY: Now, we are going to zoom in on this interval to find a more exact position. Project another number line from BLM Positioning Decimals on Number Lines and label it, as shown below, or continue drawing on the board:

```
0   0.2   0.21  0.22  0.23  0.24  0.25  0.26  0.27  0.28  0.29  0.3

0   0.2   0.21  0.22  0.23  0.24  0.25  0.26  0.27  0.28  0.29  0.3
```
Point out that the second number line counts by hundredths because the tenth from 0.2 to 0.3 is divided into hundredths. ASK: Which hundredths is 0.289 between? (0.28 and 0.29) Circle that interval and explain that we are going to zoom in on it. Project another number line from BLM Positioning Decimals on Number Lines and label it, as shown below, or continue drawing on the board:

![Number Line Diagram]

SAY: This third number line shows the thousandths from 0.28 to 0.29. ASK: Where do we place 0.289? (on the tick) Circle 0.289 on the number line. SAY: A terminating decimal always has an exact position on a number line. We sometimes need to zoom in on the line or divide it into very small pieces to see it. The final digit tells you how far in you have to zoom. The number we positioned, 0.289, has an exact position on the number line counting in thousandths because the last digit has that place value.

**Exercise:** Circle the position of 0.912 on the number lines.

![Number Line Diagram]

**Bonus:** Use the exact position of 0.912 on the third number line to more accurately estimate its position on the second number line, then on the first number line.  
**Answers:** between 0.9 and 1 on the first line, between 0.91 and 0.92 on the second line, on the tick at 0.912 on the third line; Bonus: closer to 0.91, closer to 0.9

**Estimating the position of terminating decimals by place value.** Write on the board:

0.168

Underline the zero, and SAY: This number is bigger than 0, but smaller than 1, so if you’re counting by ones, 0.168 is between 0 and 1. Underline the 1. SAY: This number is bigger than 0.1 but less than 0.2. ASK: If you count by tenths, which numbers is it between? (0.1 and 0.2)
Underline 16. ASK: If you count by hundredths, which numbers is 0.168 between? (0.16 and 0.17) SAY: And if you count by thousandths, 0.168 has an exact position.

**Exercises:** Which two numbers is the given number between if you count by ones, tenths, and hundredths?

<table>
<thead>
<tr>
<th></th>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 0.592</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) 0.138</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c) 6.476</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bonus: 0.079</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Answers:**

<table>
<thead>
<tr>
<th></th>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 0.592</td>
<td>0 and 1</td>
<td>0.5 and 0.6</td>
<td>0.59 and 0.60</td>
</tr>
<tr>
<td>b) 0.138</td>
<td>0 and 1</td>
<td>0.1 and 0.2</td>
<td>0.13 and 0.14</td>
</tr>
<tr>
<td>c) 6.476</td>
<td>6 and 7</td>
<td>6.4 and 6.5</td>
<td>6.47 and 6.48</td>
</tr>
<tr>
<td>Bonus: 0.079</td>
<td>0 and 1</td>
<td>0 and 0.1</td>
<td>0.07 and 0.08</td>
</tr>
</tbody>
</table>

(MP.3) **Positioning repeating decimals on number lines.** Project three number lines from BLM Positioning Decimals on Number Lines and label them, as shown below, or draw on the board:

```
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
```

```
<table>
<thead>
<tr>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.4</th>
<th>0.4</th>
<th>0.4</th>
<th>0.4</th>
<th>0.4</th>
<th>0.4</th>
<th>0.4</th>
</tr>
</thead>
</table>
```

```
<table>
<thead>
<tr>
<th>0.4</th>
<th>0.41</th>
<th>0.42</th>
<th>0.43</th>
<th>0.44</th>
<th>0.45</th>
<th>0.46</th>
<th>0.47</th>
<th>0.48</th>
<th>0.49</th>
<th>0.5</th>
</tr>
</thead>
</table>
```

```
<table>
<thead>
<tr>
<th>0.44</th>
<th>0.441</th>
<th>0.442</th>
<th>0.443</th>
<th>0.444</th>
<th>0.445</th>
<th>0.446</th>
<th>0.447</th>
<th>0.448</th>
<th>0.449</th>
<th>0.45</th>
</tr>
</thead>
</table>
```

SAY: Now let's try to position a repeating decimal on a number line. ASK: What are the first eight decimal places of \( \frac{4}{9} = 0.\overline{4} \) ? (0.44444444) Write on the board:

\[
\frac{4}{9} = 0.\overline{4} \approx 0.44444444
\]

Cover up everything but 0.44, and ASK: What number is \( 0.\overline{4} \) between on the first number line that counts by tenths? (0.4 and 0.5) Circle that interval and draw arrows to show that the second number line is a magnification of that interval. Cover up everything but 0.444, and ASK: What number is \( 0.\overline{4} \) between on the second number line that counts by hundredths? (0.44 and 0.45)
Circle the interval and draw arrows to show that the third number line is a magnification of that interval. Repeat these steps by covering up everything but 0.4444. ASK: Does 0.4\̅ have an exact position on the third number line? (no) If you drew number lines that measured smaller and smaller intervals of ten thousandths, hundred thousandths, etc., would you find an exact place to put this decimal? (no, it will always be between two markings no matter what place value you count by, because there is no final digit) SAY: Because repeating decimals go on forever and have no final digit, we can’t determine their exact position on a number line divided into tenths, hundredths, etc.

**Exercise:** Circle the position of 0.1\̅ on the number lines.

```
0  0.1  0.2  0.3  0.4  0.5  0.6  0.7  0.8  0.9  1
```

```
0  0.1  0.11  0.12  0.13  0.14  0.15  0.16  0.17  0.18  0.19  2
```

```
0  0.1  0.11  0.111  0.112  0.113  0.114  0.115  0.116  0.117  0.118  0.119  0.12
```

**Answers:** between 0.1 and 0.2 on the first number line, between 0.11 and 0.12 on the second number line, between 0.111 and 0.112 on the third number line

**Estimating the position of repeating decimals by place value.** Write on the board:

0.4\̅

ASK: What is this decimal to 4 places? (0.4555) On a number line divided into tenths, where would you position this number? (between 0.4 and 0.5) PROMPT: Underline the first decimal place to show that 0.4\̅ is between 0.4 and 0.5. On a number line divided into hundredths, where would you position this number? (between 0.45 and 0.46) On a number line divided into thousandths, where would you position this number? (between 0.455 and 0.456)

**Exercises:** What two numbers is the given number between if you count by tenths, hundredths, and thousandths?

<table>
<thead>
<tr>
<th></th>
<th>Tenths</th>
<th>Hundredths</th>
<th>Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>0.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b)</td>
<td>0.3\̅</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c)</td>
<td>0.18\̅</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bonus:</td>
<td>0.15\̅</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Answers:

<table>
<thead>
<tr>
<th></th>
<th>Tenths</th>
<th>Hundredths</th>
<th>Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 0.6 = 0.6666…</td>
<td>0.6 and 0.7</td>
<td>0.66 and 0.67</td>
<td>0.666 and 0.667</td>
</tr>
<tr>
<td>b) 0.3 = 0.3333…</td>
<td>0.3 and 0.4</td>
<td>0.33 and 0.34</td>
<td>0.333 and 0.334</td>
</tr>
<tr>
<td>c) 0.16 = 0.1888…</td>
<td>0.1 and 0.2</td>
<td>0.18 and 0.19</td>
<td>0.188 and 0.189</td>
</tr>
<tr>
<td>Bonus: 0.15 = 0.1515…</td>
<td>0.1 and 0.2</td>
<td>0.15 and 0.16</td>
<td>0.151 and 0.152</td>
</tr>
</tbody>
</table>

Extensions

1. Fill in the blank.
   a) 0.5 has an exact position when you count by ________________.
   b) 0.269 has an exact position when you count by ________________.
   c) 47 has an exact position when you count by ________________.
   d) 0.1325 has an exact position when you count by ________________.
   e) 0.21 has an exact position when you count by ________________.

   Bonus: 7.00 has an exact position when you count by ________________.

   Answers: a) tenths, b) thousandths, c) ones, d) ten thousandths, e) hundredths, Bonus: ones

2. a) \( \frac{1}{3} = 0.\bar{3} \). Explain why 0.\( \bar{3} \) does not have an exact position on a number line divided into tenths, hundredths, thousandths, etc.

   b) Label this number line so that \( \frac{1}{3} \) has an exact position.

   \[ \text{Answer:} \quad 0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1 \]

   (MP.1) 3. The position of \( \frac{2}{3} \) on a number line from 0 to 1 is the same as the position of \( \frac{2}{3} \) of 10
   on a number line from 0 to 10.

   \[ \text{Answer:} \quad 0 \quad \frac{2}{3} \quad 1 \]

   a) What is \( \frac{2}{3} \) of 10? Write your answer as a mixed number.
b) Show where $\frac{2}{3}$ of 10 is on the number line. Hint: What fraction of the distance from 6 to 7 is it?

\[ \begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array} \]

c) Use your answer to part b) to place $\frac{2}{3}$ on the number line. Hint: What fraction of the distance from 0.6 to 0.7 is it?

\[ \begin{array}{cccccccccccc}
0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 1 \\
\end{array} \]

**Answers:**
a) $20/3 = 6 \frac{2}{3}$, b) it is $2/3$ of the way from 6 to 7, c) it is $2/3$ of the way from 0.6 to 0.7

(MP.3) 4. You can use long division to do Extension 3. To find $2 \div 3$, you can think of 2 as 20 tenths and share 20 tenths among 3 people. Since $20 \div 3 = 6 R 2$, each person would get $6 \frac{2}{3}$ tenths. So $\frac{2}{3}$ is between 6 tenths and 7 tenths, and is $\frac{2}{3}$ of the way from 6 tenths to 7 tenths. Think of 2 as 200 hundredths and use 200 hundredths $\div 3$ to place $\frac{2}{3}$ on the number line. Explain your reasoning.

\[ \begin{array}{cccccccccccc}
0.60 & 0.61 & 0.62 & 0.63 & 0.64 & 0.65 & 0.66 & 0.67 & 0.68 & 0.69 & 0.70 \\
\end{array} \]

**Answer:** 200 hundredths $\div 3 = 66 \frac{2}{3}$ hundredths, so $2/3$ is $2/3$ of the way from 66 hundredths to 67 hundredths. So it is $2/3$ of the way from 0.66 to 0.67.

(MP.3, MP.8) 5. To find $\frac{7}{15}$, Kathy does long division:

\[
\begin{array}{c|cccc}
\multicolumn{1}{r}{15}& 7.000 \\
\cline{2-5}
\multicolumn{1}{r}{} & -6.0 \\
\multicolumn{1}{r}{} & 100 \\
\multicolumn{1}{r}{} & -90 \\
\multicolumn{1}{r}{} & 100 \\
\multicolumn{1}{r}{} & -90 \\
\multicolumn{1}{r}{} & 10 \\
\end{array}
\]

She stops dividing and writes $\frac{7}{15} = 0.4\overline{6}$. How does she know that the 6 will keep repeating?

**Answer:** The 6 will always repeat because $15 \times 6$ will always be 90, and when you subtract 90 from 100, you always get 10, and when you bring down the 0, you always get 100.
NS7-51  Is the Fraction a Repeating Decimal or a Terminating Decimal?

Pages 148–149

Standards: 7.NS.A.2d

Goals:
Students will look at the denominator of a fraction in reduced form to determine whether the fraction’s decimal equivalent will terminate or repeat.

Prior Knowledge Required:
Can convert a decimal fraction to a decimal
Can identify whether a decimal terminates or repeats
Can use long division to convert a fraction to a decimal
Can reduce a fraction to lowest terms

Vocabulary: bar notation, decimal fraction, lowest terms, repeating decimal, terminating decimal

Materials:
cutout fractions from BLM Terminating or Repeating? (pp. Q-44–45), one for each student

Review terminating and repeating decimals. ASK: Can you name some fractions that you know are written as terminating decimals? (1/2, 1/4, 1/5, 1/10, 1/100, etc.) Can you name some fractions that you know are written as repeating decimals? (1/3, 1/6, 1/9, etc.) SAY: After today’s lesson, you will be able to tell that 91/325 can be written as a terminating decimal and that 80/96 can be written as a repeating decimal, without having to do the long division!

Review converting decimal fractions to decimals. Remind students that in a decimal fraction, the denominator is a power of 10. Write on the board:

\[
\frac{7}{100}
\]

Point to the fraction, and ASK: How do we say this number? (7 hundredths) Don’t accept “7 over 100.” ASK: How do we write 7/100 as a decimal? (0.07) Write on the board:

\[
\frac{17}{10}
\]

ASK: What is the decimal form of this fraction? (1.7) Point out that improper fractions are greater than 1 so their decimal is greater than one, too.
Exercises: Write the decimal fraction as a decimal.

\[
\begin{align*}
a) \quad & \frac{36}{100} \\
b) \quad & \frac{5}{1000} \\
c) \quad & \frac{2341}{1000} \\
d) \quad & \frac{23456}{100000} \\
e) \quad & \frac{7}{10000} \\
f) \quad & \frac{79}{10}
\end{align*}
\]

Answers: a) 0.36, b) 0.005, c) 2.341, d) 0.23456, e) 0.0007, f) 7.9

ASK: Are your answers terminating decimals or repeating decimals? (terminating) Can every decimal fraction be written as a terminating decimal? (yes) Why? (because the denominator indicates the place value, so the decimal always ends somewhere) SAY: The final place value of the decimal is determined by the denominator of the decimal fraction. For example, if the decimal fraction has denominator ten thousand, then the final place value of the decimal is ten thousandths.

Activity (MP.4) Give each student one cutout fraction from BLM Terminating or Repeating?. Students use long division to write the fraction as a decimal. Divide the board into two sections, labeled “Terminating Decimals” and “Repeating Decimals.” Students affix their fraction in the appropriate section. (Terminating Decimals: 1/2 = 0.5, 1/4 = 0.25, 2/4 = 0.5, 3/4 = 0.75, 1/5 = 0.2, 2/5 = 0.4, 3/5 = 0.6, 4/5 = 0.8, 3/6 = 0.5, 3/8 = 0.375, 5/8 = 0.625, 7/20 = 0.35, 21/40 = 0.525, 3/50 = 0.06; Repeating Decimals: 1/3 = 0.\overline{3}, 2/3 = 0.\overline{6}, 1/6 = 0.1\overline{6}, 2/6 = 0.\overline{3}, 4/6 = 0.\overline{6}, 5/6 = 0.8\overline{3}, 1/7 = 0.14285\overline{7}, 3/7 = 0.42857\overline{1}, 4/9 = 0.\overline{4}, 8/9 = 0.8\overline{3}, 3/11 = 0.2\overline{7}, 7/11 = 0.\overline{63}, 1/12 = 0.0\overline{83}, 5/12 = 0.41\overline{6}, 1/13 = 0.07692\overline{3}, 4/15 = 0.2\overline{6}, 11/30 = 0.3\overline{6}, 31/75 = 0.41\overline{3})

(end of activity)

If a fraction has an equivalent decimal fraction, it can be written as a terminating decimal.

Choose five fractions from the Terminating Decimals section in the activity above and have students write them as equivalent decimal fractions. (see examples below)

\[
\begin{align*}
\frac{1}{2} &= \frac{5}{10} \\
\frac{1}{4} &= \frac{25}{100} \\
\frac{2}{4} &= \frac{50}{100} \\
\frac{3}{4} &= \frac{75}{100} \\
\frac{1}{5} &= \frac{2}{10}
\end{align*}
\]

SAY: A fraction can be written as a terminating decimal if it has an equivalent decimal fraction.

Choose five fractions from the Repeating Decimals section in the activity above, and ASK: Is it possible to find an equivalent decimal fraction for these fractions? (no) SAY: Let’s think about why 1/3 can’t be written as a decimal fraction. To write an equivalent fraction, we multiply the numerator and denominator by the same number. Write on the board:

\[
\begin{align*}
\frac{1\times?}{3\times?} &= \frac{10}{100} \\
\frac{1\times?}{3\times?} &= \frac{100}{1000} \\
\frac{1\times?}{3\times?} &= \frac{1000}{10000}
\end{align*}
\]

ASK: Can we find a number to multiply by to make 1/3 equivalent to a decimal fraction? (no) Show students that 10 ÷ 3 = 3 R 1, 100 ÷ 3 = 33 R 1, etc. SAY: There will always be a remainder of 1, so there is no number than can multiply 3 to make a power of 10. A fraction that is written as a repeating decimal can’t be written as an equivalent decimal fraction.
Exercises:
1. Find a decimal fraction equivalent to the fraction, if possible.
   a) \( \frac{3}{5} \)  
   b) \( \frac{2}{3} \)  
   c) \( \frac{3}{20} \)  
   d) \( \frac{5}{6} \)  
   e) \( \frac{7}{9} \)  
   f) \( \frac{7}{40} \)  
   Answers: a) 6/10, b) impossible, c) 15/100, d) impossible, e) impossible, f) 175/1,000

2. Use each fraction from Exercise 1 that could be written as a decimal fraction, and write it as a decimal.
   Answers: \( \frac{3}{5} = 0.6 \), \( \frac{3}{20} = 0.15 \), \( \frac{7}{40} = 0.175 \)

3. Use long division to confirm that the fractions from Exercise 1 that did not have an equivalent decimal fraction are repeating decimals.
   Answers: \( \frac{2}{3} = 0.\overline{6} \), \( \frac{5}{6} = 0.8\overline{3} \), \( \frac{7}{9} = 0.\overline{7} \)

(MP.8) Using the denominator to determine if the decimal will terminate or repeat.
SAY: We know that decimal fractions can be written as terminating decimals. ASK: How can we tell if a fraction has an equivalent decimal fraction? (look at the denominator and see if it can be multiplied to get a power of 10). Let’s find an easy way to do this. Write on the board:

\[
10 = 2 \times 5
\]

SAY: We know that 10 = 2 × 5. ASK: What about the next two powers of 10? Write on the board:

\[
10^2 = 100 = \\
10^3 = 1,000 =
\]

ASK: What is 100 as a product of only 2s and/or 5s? \( (2 \times 5 \times 2 \times 5) \) Write the product on the board. ASK: What is 1,000 as a product of only 2s and/or 5s? \( (2 \times 5 \times 2 \times 5 \times 2 \times 5) \) Write the product on the board.

Write on the board:

\[
\frac{1}{4} = \frac{25}{100}
\]

SAY: \( 4 = 2 \times 2 \). We multiplied 4 by 25 to get 100. And \( 25 = 5 \times 5 \). That’s how we get to 100—\( 4 \times 25 \), or \( 2 \times 2 \times 5 \times 5 \).

Write on the board:

\[
\frac{1}{9}
\]

SAY: \( 9 = 3 \times 3 \). Powers of 10 are written as products of only 2s and 5s. There is no way to get a 3 in there, so \( 1/9 \) can’t be written as a decimal fraction.
Exercises:
1. Choose four fractions from the Terminating Decimals section in the preceding activity and write their denominators as products of only 2s and/or 5s.

**Selected answers:** 3/8, 8 = 2 × 2 × 2; 7/20, 20 = 2 × 2 × 5; 21/40, 40 = 2 × 2 × 2 × 5; 3/50, 50 = 2 × 5 × 5

2. Choose four fractions from the Repeating Decimals section in the preceding activity and show that the denominator is written as a product of a number other than 2 or 5.

**Selected answers:** 4/9, 9 = 3 × 3; 1/12, 12 = 2 × 2 × 3; 4/15, 15 = 3 × 5; 11/30, 30 = 2 × 3 × 5

Point to 3/6 and SAY: The denominator is not a product of 2s and/or 5s (6 = 2 × 3) but the decimal is terminating. ASK: Why? (it’s equivalent to 1/2) SAY: We should reduce the fraction before looking at its denominator. Write on the board:

\[
\frac{45}{72}
\]

ASK: What is this fraction in lowest terms? PROMPT: Divide the numerator and denominator by a common factor. (5/8) ASK: Can 8 be written as a product of 2s and/or 5s? (yes, 8 = 2 × 2 × 2) So does the decimal equivalent of 45/72 terminate? (yes)

Write on the board:

\[
\frac{80}{96}
\]

ASK: What is this fraction in lowest terms? (5/6) Can 6 be written as a product of 2s and 5s? (no, 6 = 2 × 3) So does the decimal equivalent of 80/96 terminate? (no)

Exercises:
1. a) Write out all the fifteenths from 1/15 to 14/15 in lowest terms.
b) Use the denominators to decide which fractions terminate as decimals.


2. Write the fraction in lowest terms and decide if the decimal equivalent terminates.

\[
\begin{array}{cccccc}
\text{a)} & \frac{27}{75} & \text{b)} & \frac{34}{40} & \text{c)} & \frac{9}{150} \\
\text{d)} & \frac{10}{14} & \text{e)} & \frac{6}{21} & \text{f)} & \frac{44}{110} \\
\text{g)} & \frac{91}{325}
\end{array}
\]

**Answers:** a) 9/25, terminates; b) 17/20, terminates; c) 3/50, terminates; d) 5/7, repeats; e) 2/7, repeats; f) 2/5, terminates; g) 7/25, terminates

Extensions
1. a) Create a fraction that can be written as a terminating decimal. The fraction must have ...
   i) a one-digit denominator    ii) a two-digit denominator
   iii) a four-digit denominator iv) a six-digit denominator

   b) Explain how you chose the denominators.
Sample answers: a) i) $\frac{1}{2}$, ii) $\frac{1}{50}$, iii) $\frac{11}{1,000}$, iv) $\frac{111}{100,000}$; b) the denominators are products of only 2s and/or 5s

2. a) Create a fraction that can be written as a repeating decimal. The fraction must have ...
   i) a one-digit denominator
   ii) a two-digit denominator
   iii) a three-digit denominator
   iv) a five-digit denominator
b) Explain how you chose the denominators.
Sample answers: a) i) $\frac{1}{3}$, ii) $\frac{1}{33}$, iii) $\frac{1}{333}$, iv) $\frac{1}{33,333}$; b) the denominators are products of numbers that are not only 2s and/or 5s

3. Roy rolls a regular die. Write the probability of the event in lowest terms and as a repeating decimal with bar notation.
a) A multiple of 3 is rolled.      b) An odd number is rolled.
c) A 6 is rolled.                d) A number greater than 4 is rolled.
e) A number less than 5 is rolled. f) A factor of 12 is rolled.
Answers: a) $\frac{1}{3} = 0.\overline{3}$, b) $\frac{1}{2} = 0.5$, c) $\frac{1}{6} = 0.1\overline{6}$, d) $\frac{1}{3} = 0.\overline{3}$, e) $\frac{2}{3} = 0.\overline{6}$, f) $\frac{5}{6} = 0.8\overline{3}$

(MP.7) 4. a) Use $\frac{1}{7} = 0.142857$ to write $\frac{10}{7}$ as a repeating decimal.
b) How can you use your answer to part a) to write $\frac{3}{7}$ as a repeating decimal?
Answers: a) $\frac{1}{7} \times 10 = \frac{10}{7}$, and $0.142857 \times 10 = 1.428571$, so $\frac{10}{7} = 1.\overline{428571}$; b) $3/7 = 10/7 - 7/7$, so since $10/7 = 1.428571$ and $7/7 = 1$, then $3/7 = 1.428571 - 1 = 0.428571$

(MP.6) 5. May and Alex both write $\frac{23}{90}$ as a repeating decimal. May writes 0.2$\overline{5}$ and Alex writes 0.2$\overline{5}$. Who is correct? What mistake did the other person make? Explain using what the bar means.
Answer: I found $23 \div 90$ and I stopped when the remainders started to repeat. My answer was 0.25555 …. Only the 5 repeats, so only the 5 should be under the bar. Alex is correct. May didn’t realize that you only put the bar above the repeating part, or maybe she wasn’t being careful in how she wrote her answer.
NS7-52  Operations with Repeating Decimals (Advanced)

Pages 150–151

Standards: 7.NS.A.1d, 7.NS.A.2c, 7.NS.A.2d

Goals:
Students will round, compare, add, subtract, and multiply repeating decimals and terminating decimals.

Prior Knowledge Required:
Can round a number to a given place value
Can use < and > to compare decimal numbers
Can write a fraction as a decimal
Can use bar notation for repeating decimals
Can add, subtract, and multiply terminating decimals

Vocabulary: bar notation, repeating decimal, terminating decimal

Materials:
calculators

(MP.5) Using a calculator to convert decimals to fractions. Show students how to use a calculator to convert decimals to fractions. Write on the board:

\[ \frac{2}{3} \]

ASK: How do you use a calculator to calculate the decimal equivalent to \( \frac{2}{3} \)? \((2 ÷ 3)\) Point out that some calculators round the last digit. For example, when calculating \( 2 ÷ 3 \), one calculator may display 0.66666666 while another displays 0.666666667. Tell students that both displays give the answer \( 0.\overline{6} \).

Rounding repeating decimals. SAY: Repeating decimals go on and on forever, but sometimes we don’t need all that information and we are happy to use the decimal in a rounded form. Write on the board:

\[ \frac{9}{11} = 0.\overline{81} = 0.818181... \]

ASK: What is this number rounded to the nearest tenth? \((0.8)\) PROMPT: Underline 1 in the hundredth position, and ASK: Is this number higher or lower than 5? (lower) Do we round 8 up or down? (down) What is the number to the nearest hundredth? \((0.82)\) Thousandth? \((0.818)\)
Exercises: Use a calculator to find the decimal equivalent of the fraction. Round the decimal to the nearest tenth, hundredth, and thousandth.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal Equivalent (to 5 digits)</th>
<th>Nearest Tenth</th>
<th>Nearest Hundredth</th>
<th>Nearest Thousandth</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) (\frac{3}{7})</td>
<td>≈ 0.42857</td>
<td>0.4</td>
<td>0.43</td>
<td>0.429</td>
</tr>
<tr>
<td>b) (\frac{2}{3})</td>
<td>≈ 0.66666</td>
<td>0.7</td>
<td>0.67</td>
<td>0.667</td>
</tr>
<tr>
<td>c) (\frac{13}{99})</td>
<td>≈ 0.13131</td>
<td>0.1</td>
<td>0.13</td>
<td>0.131</td>
</tr>
<tr>
<td>d) (\frac{15}{21})</td>
<td>≈ 0.71428</td>
<td>0.7</td>
<td>0.71</td>
<td>0.714</td>
</tr>
</tbody>
</table>

Review how to compare terminating decimals. Remind students that to compare terminating decimals, they look for the largest place value where the decimals have different digits. Write on the board:

0.3746 □ 0.3728 □ 0.4589 □ 0.4723 □ 0.35 □ 0.32141

Have volunteers use < or > to compare the decimals. (>, <, >) Use the third example to emphasize that the number of digits after the decimal point tells you nothing about how large a number is; 0.35 has fewer digits after the decimal point than 0.32141, but it is larger.

NOTE: If students struggle with this, they can write the numbers vertically on grid paper, lining up the place values to help them find the largest place value where the numbers are different.

Exercises: Use < or > to compare the numbers.

a) 0.4896 □ 0.4899 □ 1.2 □ 1.02
b) 0.23 □ 0.2127 □ 0.671 □ 0.681

Answers: a) <, b) >, c) >, d) <

(MP.3) Comparing repeating decimals. Write on the board:

0.358 □ 0.358

Tell students that it’s easier to see which decimal is greater if we write the first six digits of the two numbers. Tell them that they may want to add zeros to terminating decimals. Writing the
numbers in a grid will keep the place values aligned. An easy way to do this is to line up the decimal points. Have volunteers tell you the first six digits and write on the board:

\[
\begin{array}{c}
0.358 \\
0 3 5 8 0 0 0 \\
0 3 5 8 3 5 8
\end{array}
\]

ASK: What are the first digits where the decimals differ? (fourth digits) Circle the fourth digits, as shown below:

\[
\begin{array}{c}
0 3 5 8 0 0 0 \\
0 3 5 8 3 5 8
\end{array}
\]

ASK: So which number is greater? \(0.358\) Write the < sign in the box in the original statement.

**Exercises:** Write the first few digits of each decimal and use > or < to compare them.

a) 0.254     0.25  

b ) 0.754     0.75  

c ) 0.285     0.285  

d ) 0.729     0.72  

e) 5.548     5.548  

f ) 0.382     0.382  

g ) 0.312     0.312  

h ) 0.318     0.318

**Answers:** a) >, b) <, c) <, d) >, e) <, f) <, g) >, h) <

**Using decimals to compare fractions.** Write on the board:

\[
\begin{array}{c}
6 \\
13
\end{array}
\quad \begin{array}{c}
11 \\
26
\end{array}
\]

SAY: To compare these fractions, we could write them with common denominators. ASK: Which denominator should we use? (26) What is 6/13 equal to? (12/26) So which fraction is greater? (6/13) Using a calculator, what is the decimal equivalent of 6/13? \(0.4615\ldots\) 11/26? \(0.4230\ldots\) Which decimal is greater? \(0.4615\ldots\) SAY: This also shows that 6/13 is greater. Point out that when you compare fractions, you usually have to find a common denominator or write the fractions as decimals. In contrast, you can compare decimal numbers just by looking at them. This is an advantage to working with decimal numbers.

**(MP.1) Exercises:**

1. Compare the fractions by writing them with common denominators.

a) \(\frac{7}{15}\) \(\square\) \(\frac{19}{45}\)  

b) \(\frac{3}{8}\) \(\square\) \(\frac{7}{16}\)  

c) \(\frac{5}{6}\) \(\square\) \(\frac{3}{5}\)

**Answers:** a) 21/45 > 19/45, b) 6/16 < 7/16, c) 25/40 > 24/40

2. Compare the fractions in Exercise 1 by writing them as decimals.

**Answers:** a) 0.4\(\bar{6}\) > 0.42, b) 0.375 < 0.4375, c) 0.625 > 0.6

3. Check that you got the same answers in Exercises 1 and 2. If you didn’t, find your mistake.
Review adding and subtracting terminating decimals to the thousandths. Write on the board:

\[3.45 + 2.764\]

Remind students that when adding and subtracting decimals, it is important to line up the place values, to ensure they are always adding the same place values together. Rewrite the problem on the board vertically, as shown below:

\[
\begin{array}{c}
3.45 \\
+ 2.764 \\
\hline
6.214
\end{array}
\]

SAY: Add the digits from right to left. Hint: You may want to add zeros to terminating decimals.

ASK: What is the sum? (6.214)

**Exercises:** Add the decimals by lining up the decimal places.

- a) 8.45 + 3.582
- b) 13.45 + 1.345
- c) 43.218 + 7.469
- d) 8.9452 + 3.479
- **Bonus:** 9.415 + 23.74 + 12.678

**Answers:**
- a) 12.032
- b) 14.795
- c) 50.687
- d) 12.424
- Bonus: 45.833

**(MP.3) Adding a terminating decimal to a repeating decimal with no regrouping.** Write on the board:

\[
\begin{array}{c}
0.32 \\
+ 0.44444444 \\
\hline
0.76444444
\end{array}
\]

Have students solve the problems. Write their answers on the board. (0.76, 0.764, 0.7644, 0.7644444) ASK: How do we use bar notation to express the pattern of the number that are we adding to 0.32? (0.\overline{4}) Using bar notation, what is the sum of 0.32 + 0.\overline{4}? (0.76\overline{4}) Explain that the sum has 4 in every place value from thousandths on because, in those place values, we are always adding 0 (from 0.3200000….) and 4 (from 0.44444444…).

SAY: To add a repeating decimal to a terminating decimal, line up the first few decimal places. Point out that this is how we compared repeating decimals—we pretended that the decimal didn’t go on forever and that was enough to know which was greater. Similarly, adding the first few decimal places might be enough to see the repeating pattern in the sum. Suggest that students write at least eight decimal places.

**Exercises:** Line up the first eight decimal places and add the decimals.

- a) 0.42 + 0.\overline{5}
- b) 0.\overline{45} + 0.3
- c) 0.3\overline{2} + 0.475
- d) 0.123 + 0.6\overline{51}

**Answers:**
- a) 0.9\overline{75}
- b) 0.7\overline{54}
- c) 0.797\overline{2}
- d) 0.774\overline{651}
(MP.3) Adding repeating decimals with no regrouping. Write on the board:

\[ 0.\overline{523} + 0.\overline{16} \]

Write the decimals vertically to eight decimal places, as shown below:

| 0 | 5 | 2 | 3 | 2 | 3 | 2 | 3 | 2 | ...
---|---|---|---|---|---|---|---|---|---
| + | 0 | 1 | 6 | 1 | 6 | 1 | 6 | 1 | 6 | ...

SAY: We can’t start adding from the right since there is no right-most digit. You can add from the left as long as no regrouping is required. ASK: What is the sum? (0.\overline{684}) How do we write this in bar notation? (0.6\overline{84})

**Exercises:** Line up the first eight decimal places and add the decimals.

a) \(0.27 + 0.31\)  
b) \(4.13 + 0.31\)  
c) \(0.413 + 0.2\overline{31}\)  
d) \(4.\overline{13} + 0.2\overline{31}\)  
e) \(0.345 + 0.2\overline{3}\)  
f) \(0.345 + 0.1\overline{32}\)  
g) \(0.345 + 0.4\overline{32}\)  
h) \(0.241 + 0.3\overline{155}\)

**Answers:** a) 0.58, b) 4.44, c) 0.6\overline{4}, d) 4.3\overline{62}, e) 0.5\overline{78}, f) 0.4\overline{77}, g) 0.7, h) 0.55\overline{67}

(MP.3) Subtracting repeating decimals with no regrouping. Tell students that subtracting repeating decimals is just like adding them: line up the first few decimal places and subtract from left to right (as long as there is no regrouping) until you see a pattern in the difference.

Write on the board:

\[ 0.\overline{74} - 0.34 \]

Write the decimals vertically to eight decimal places, as shown below:

| 0 | 7 | 4 | 7 | 4 | 7 | 4 | 7 | 4 | ...
---|---|---|---|---|---|---|---|---|---
| - | 0 | 3 | 4 | 3 | 4 | 3 | 4 | 3 | 3 | ...

ASK: What is the answer? (0.40\overline{40}4040\ldots) What is the answer in bar notation? (0.\overline{40})

**Exercises:** Line up the first eight decimal places and subtract the decimals.

a) \(0.\overline{67} - 0.4\overline{3}\)  
b) \(0.874 - 0.2\overline{34}\)  
c) \(0.8\overline{74} - 0.3\overline{4}\)  
d) \(0.874 - 0.0\overline{32}\)

**Answers:** a) 0.24, b) 0.6\overline{40}, c) 0.5\overline{31440}, d) 0.8\overline{425516}
(MP.3) Multiplying repeating decimals by whole numbers with no regrouping. Write on the board:

\[
\begin{array}{c}
0.4 \\
\times 2 \\
\hline
0.8
\end{array}
\quad \begin{array}{c}
0.44 \\
\times 2 \\
\hline
0.88
\end{array}
\quad \begin{array}{c}
0.444 \\
\times 2 \\
\hline
0.888
\end{array}
\quad \begin{array}{c}
0.4444 \\
\times 2 \\
\hline
0.8888
\end{array}
\]

Have students solve the problems. (0.8, 0.88, 0.888, 0.8888) Write the answers on the board.

ASK: How do we use bar notation to express the pattern of the number that we are multiplying? (0.4) What is the product \(0.4 \times 2\)? (0.8) Write on the board:

\[
\begin{array}{c}
0.07 \\
\times 5 \\
\hline
0.35
\end{array}
\quad \begin{array}{c}
0.0707 \\
\times 5 \\
\hline
0.3535
\end{array}
\quad \begin{array}{c}
0.070707 \\
\times 5 \\
\hline
0.353535
\end{array}
\]

Have students solve the problems. Write their answers on the board. (0.35, 0.3535, 0.353535)

ASK: What is \(0.0\overline{7} \times 5\)? (0.35)

**Exercises:** Multiply.

a) \(0.4 \times 2\) \hspace{1cm} b) \(0.12 \times 3\) \hspace{1cm} c) \(0.08 \times 6\) \hspace{1cm} d) \(0.1\overline{2}3 \times 3\)

**Answers:** a) 0.8, b) 0.36, c) 0.48, d) 0.369

**Extensions**

(MP.1) 1. a) Add \(0.2\overline{5} + 0.5\overline{2}\). Identify the repeating core.

b) Find two decimals with a two-digit repeating core that add to \(0.6\).

c) Find three decimals with a two-digit repeating core that add to \(0.7\).

d) Add \(0.345 + 0.432\). Identify the repeating core.

e) Find two decimals with a three-digit repeating core that add to \(0.8\).

f) Find two decimals with a four-digit repeating core that add to \(0.78\).

**Answers:** a) 0.7, repeating core 7; d) 0.7, repeating core 7

**Sample answers:** b) 0.24 + 0.42, c) 0.12 + 0.23 + 0.42, e) 0.345 + 0.543, f) 0.4123 + 0.3755

2. You can divide repeating decimals by terminating decimals. To divide \(0.\overline{8} + 0.6\), change the question to divide by a whole number \((8.\overline{8} \div 6\). Solve the long division until you see a pattern:

\[
\begin{array}{c}
1.481... \\
6)8.8\overline{8} \\
-6 \\
28 \\
-24 \\
48 \\
-48 \\
08
\end{array}
\]

So, \(0.\overline{8} \div 0.6 = 1.481\).
Divide. Use bar notation to write your answer.

a) \(0.8 \div 0.4\)  
b) \(0.1\overline{2} \div 0.06\)  
c) \(0.5\overline{2} \div 0.15\)  
d) \(0.1\overline{00} \div 0.25\)

**Answers:** a) \(2.\overline{2}\), b) \(2.\overline{02}\), c) \(3.4\overline{81}\), d) \(0.4\overline{00}\)

3. Write the numbers in order from least to greatest. Hint: Write the first few digits of the decimals to compare them.

a) \(0.5, 0.5\overline{1}, 0.51, 0.51\)  
b) \(0.28, 0.\overline{2}, 0.2\overline{8}, 0.2\overline{8}\)  
c) \(0.1\overline{83}, 0.1\overline{83}, 0.1\overline{83}, 0.1\overline{83}\)

**Answers:** a) \(0.5, 0.5\overline{1}, 0.51, 0.51\); b) \(0.2, 0.28, 0.2\overline{8}, 0.2\overline{8}\); c) \(0.1\overline{83}, 0.1\overline{83}, 0.1\overline{83}, 0.1\overline{83}\)

4. Use long division to write the fraction as a decimal. Round the answer to two decimal places. Write the approximate percentage.

Example: \(\frac{5}{12} = 0.41\overline{6} \approx 0.42 = 42\%\)

a) \(\frac{1}{3}\)  
b) \(\frac{6}{7}\)  
c) \(\frac{17}{99}\)  
d) \(\frac{11}{40}\)

**Answers:** a) \(33\%\), b) \(86\%\), c) \(17\%\), d) \(28\%\)

**MP.5, MP.7** 5. Are \(\frac{0.2}{0.3}\) and \(\frac{0.45}{0.675}\) equivalent complex fractions? Use any tool except a calculator. Show your work.

**Sample answers:**
- I found what I needed to multiply 0.2 by to get 0.45: \(0.45 \div 0.2 = 4.5 \div 2 = 2.25\). Then I multiplied 0.3 \(\times 2.25\) and got 0.675, so the complex fractions are equivalent.
- I found \(0.2 \div 0.3 = 2 \div 3 = 0.\overline{6}\), and I found \(0.45 \div 0.675 = 450 \div 675 = 0.6\), so the complex fractions are equivalent.
- I wrote 0.45/0.675 as 450/675 and then I divided the numerator and denominator by 45: \(450 \div 45 = 10\) and \(675 \div 45 = 15\) (by long division), so \(450/675 = 10/15 = 2/3 = 0.2/0.3\).
- I found 675 \(\div 450\) using mental math. I noticed that 675 = 450 + 225, and 225 is half of 450, so 225 is \(0.5 \times 450\), and 675 is \(1.5 \times 450\). So \(675 \div 450 = 1.5\), and \(0.3 + 0.2 = 1.5\), so 675/450 and 0.3/0.2 are equivalent, and hence \(0.45/0.675\) and 0.2/0.3 are equivalent.
- I cross multiplied. I found \(0.2 \times 0.675 = 0.135\) and \(0.3 \times 0.45 = 0.135\), so they are equivalent complex fractions; I used the standard algorithm.
- I did the cross multiplication mentally. I split 675 into 600 + 75 and found 675 \(\times 2 = 1,200\) + 150 = 1,350, so \(0.675 \times 0.2 = 0.135\). I know that 50 \(\times 3\) is 150 and 45 \(\times 3\) is 15 less than that, so 45 \(\times 3 = 135\), so \(0.3 \times 0.45 = 0.135\).

Whole-class follow-up: Have several volunteers present solutions. Have the class discuss the following questions in pairs: Is the argument correct? Is there a part of the argument you do not understand? Could the argument be improved to make it clearer? Compare the solutions.

**ASK:** Which solution was the fastest? (cross multiplying or mental math) Will it work for any numbers? (cross multiplying will but mental math will not)
2-Digit Division

Example:

\[
\begin{array}{c}
49 \\
\hline
167
\end{array}
\]

Exercises

a) \[
\begin{array}{c}
19 \\
\hline
164
\end{array}
\]

b) \[
\begin{array}{c}
38 \\
\hline
251
\end{array}
\]

c) \[
\begin{array}{c}
41 \\
\hline
351
\end{array}
\]

d) \[
\begin{array}{c}
81 \\
\hline
342
\end{array}
\]

Bonus ►

e) \[
\begin{array}{c}
79 \\
\hline
581
\end{array}
\]

f) \[
\begin{array}{c}
62 \\
\hline
502
\end{array}
\]

g) \[
\begin{array}{c}
91 \\
\hline
557
\end{array}
\]

h) \[
\begin{array}{c}
78 \\
\hline
724
\end{array}
\]
Positioning Decimals on Number Lines
## Terminating or Repeating? (1)

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Terminating?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{1}{3} )</td>
<td>No</td>
</tr>
<tr>
<td>( \frac{2}{3} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{1}{4} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{2}{4} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{3}{4} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{1}{5} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{2}{5} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{3}{5} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{4}{5} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{1}{6} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{2}{6} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{3}{6} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{4}{6} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{5}{6} )</td>
<td>Yes</td>
</tr>
<tr>
<td>( \frac{1}{7} )</td>
<td>No</td>
</tr>
</tbody>
</table>
### Terminating or Repeating? (2)

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Simplified Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{7}$</td>
<td>$\frac{3}{8}$</td>
</tr>
<tr>
<td>$\frac{5}{8}$</td>
<td>$\frac{4}{9}$</td>
</tr>
<tr>
<td>$\frac{8}{9}$</td>
<td>$\frac{3}{11}$</td>
</tr>
<tr>
<td>$\frac{7}{11}$</td>
<td>$\frac{1}{12}$</td>
</tr>
<tr>
<td>$\frac{5}{12}$</td>
<td>$\frac{1}{13}$</td>
</tr>
<tr>
<td>$\frac{4}{15}$</td>
<td>$\frac{7}{20}$</td>
</tr>
<tr>
<td>$\frac{11}{30}$</td>
<td>$\frac{21}{40}$</td>
</tr>
<tr>
<td>$\frac{3}{50}$</td>
<td>$\frac{31}{75}$</td>
</tr>
</tbody>
</table>
Unit 6  Geometry: Volume, Surface Area, and Cross Sections

This Unit in Context
In Grade 6, students found the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and showed that the volume is the same as would be found by multiplying the edge lengths of the prism. They applied the formulas \( V = \ell \cdot w \cdot h \) and \( V = b \cdot h \) to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real world and mathematical problems.

In this unit, students will continue their study of three-dimensional shapes that began in Grade 6. They will solve real world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms (7.G.B.6). In Grade 8, their study of three-dimensional shapes will expand to include formulas for the volumes of cones, cylinders, and spheres, which students will use to solve real world and mathematical problems.

Mathematical Practices in This Unit
In this unit, you will have the opportunity to assess MP.1 to MP.8. The MP labels in this unit flag both opportunities to develop the mathematical practice standards and opportunities to assess them. Below is a list of where we recommend assessing each standard, as well as some examples of how students can show that they have met a standard.

MP.1: G7-27 Extension 4
In G7-27 Extension 4, students make sense of, and persevere in solving, a non-routine problem when they determine how many pieces they need to cut a cake into to ensure that the amount of sugar in each piece is no more than the recommended daily limit.

MP.2: G7-30 Extension 6, G7-31 Extension 5

At the bottom of p. R-55 in G7-30, students construct an argument when they explain why any cross section of a rectangular pyramid must have at most five sides and so cannot be a hexagon.

In G7-29 Question 8 on AP Book 7.2 p. 167, students model a real-world situation when they calculate the volume of an A-frame cottage to determine how many people can safely stay in the cottage given specific fire regulations.

MP.5: G7-24 Extension 4, G7-29 Extension 4

MP.6: G7-27 Extensions 3 and 4, G7-28 Extension 1, G7-31 Extension 5
**MP.7:** G7-31 Extensions 3 and 4

**MP.8:** G7-26 Extension 3, G7-28 Extension 1
Unit 6  Geometry: Volume, Surface Area, and Cross Sections

Introduction
In this unit, students will deepen their understanding of volume and surface area. As they find the volume of right prisms and the surface area of right prisms and right pyramids, students will add, subtract, multiply, and divide whole numbers, fractions, and decimals. They will use algebra to solve inequalities. They will also work with 3-D shapes composed of two or more right polygonal prisms. Students will use nets, develop and use formulas, and use structures to solve real-world and mathematical problems involving volume and surface area. Finally, students will describe the shapes of the cross sections that result from slicing rectangular prisms and rectangular pyramids with planes.

Conventions for the use of length, width, and height. In two-dimensional shapes, we follow the convention that length is used for the longest side of the rectangle. In three-dimensional shapes, we use length and width interchangeably. Label diagrams clearly and let students know that length will not always be the longest measurement in 3-D geometry. We use height consistently for how tall a prism is if it’s standing on its base, or for the perpendicular distance between the two bases of the prism.

Materials. For this unit, students will benefit from seeing a variety of 3-D shapes. If you do not have a commercial set, collect and use various boxes. For example, cereal and tissue boxes are rectangular prisms, and some candy bar boxes are triangular prisms. You can turn a standard milk carton into a pentagonal prism by cutting off the strip on the top and covering the triangular openings at the top with paper, as shown below:

![Milk Carton](image)

You can also create 3-D shapes using BLM Nets of 3-D Shapes (pp. R-68–78). Photocopy the BLMs on cardstock to make sturdier shapes. Nets are available for the following shapes:

Right prisms with the following bases:
1. Rectangle
2. Square
3. Isosceles right triangle
4. Scalene right triangle
5. Parallelogram
6. Trapezoid
7. Pentagon
8. Hexagon

Right pyramids with the following bases:
9. Square
10. Rectangle
11. Equilateral triangle
Materials for slices of 3-D shapes. BLM Nets for Cuts of Cubes and Rectangular Pyramids (pp. R-84–113) contains pairs of nets for 3-D shapes that together make a cube or rectangular pyramid. All pairs of shapes from this BLM show a plane cut of the cube or rectangular pyramid. BLM Fake Net (pp. R-118–119) also has one such pair, except the cut is not through a plane. Photocopy the BLMs on cardstock to make sturdier shapes. Although we recommend providing students with the opportunity to make the 3-D shapes themselves for the activity in Lesson G7-30, you might wish to prepare some ahead of time to ensure that the shapes will be made correctly, especially those shapes that will be needed as demonstrations in class (A4, C10, G1, I5, J7, K15, L9, M12, N13, and O3).

Fraction Notation. We show fractions in two ways in our lesson plans:

Stacked: \( \frac{1}{2} \)  
Not stacked: 1/2

If you show students the non-stacked form, remember to introduce it as new notation.
Goals:
Students will find the volume of rectangular prisms and 3-D shapes composed of two or more rectangular prisms with whole-unit, decimal, and fractional edge lengths. Students will solve inequalities to find a missing measurement that satisfies a given constraint on the volume.

Prior Knowledge Required:
Knows different units of measurement for length, area, and volume
Can find the area of a rectangle
Can multiply whole numbers, fractions, and decimal numbers
Can substitute numbers for variables in a formula
Can find missing side lengths when there is enough information to do so
Can solve inequalities

Vocabulary: cubic units, dimension, formula, inequality, one-dimensional (1-D), rectangular prism, three-dimensional (3-D), two-dimensional (2-D), volume

Materials:
24 connecting cubes for demonstration
7 different colors of chalk

(MP.6) Review units of length, area, and volume. Draw on the board:

Remind students that one-dimensional objects, such as a string or a line, have length, and are measured in cm, m, in, ft, and yd. Two-dimensional objects, such as the top of a desk or the surface of a wall, have length and width, and are measured in square units such as cm², m², in², ft², and yd². Three-dimensional objects, such as a box or a closet, have length, width, and height, and are measured in cubic units such as cm³, m³, in³, ft³, and yd³. We call the space taken up by a 2-D object “area,” and we call the space taken up by a 3-D object “volume.”
SAY: To get the area of a 1-inch square, we multiply 1 in × 1 in. Point at the “2” in “in²” on the 2-D shape on the board and SAY: Two measurements are multiplied to get the area, so area = 1 in². Point at the “3” in “in³” on the 3-D shape on the board and ASK: Why is volume measured in cm³? (three measurements are multiplied to get the volume: length, width, and height)

**Counting cubes in a stack using horizontal layers.** Build a 2 × 4 rectangle using connecting cubes. ASK: How many cubes are in this rectangle? (8) How do you know? (I counted the cubes; I calculated 4 × 2; I multiplied length by width) Write on the board:

\[
\begin{array}{ccc}
4 & \times & 2 \\
\text{length} & \times & \text{width} \\
cubes in one layer
\end{array}
\]

Add another 2 × 4 rectangle to create a second layer. ASK: How many cubes are in the new stack? (16) How can you use the number of cubes in one layer to figure this out? (8 cubes × 2 layers) Add a third layer to the stack and repeat the questions. (24 cubes, 8 cubes × 3 layers) Write on the board:

\[
\begin{array}{ccc}
8 & \times & 3 \\
cubes in one layer & \times & \text{number of layers} \\
cubes in stack
\end{array}
\]

(MP.4) **Determine the number of cubes in a stack as a product of length, width, and height.** ASK: How can you quickly work out that there are 24 cubes in the stack? (4 × 2 × 3) Write on the board:

\[
4 \times 2 \times 3 = 24
\]

SAY: Since a stack is a 3-D shape, it has length, width, and height. So when we multiply 4 × 2 × 3, we’re multiplying length × width × height. Add this information to the multiplication statement on the board, as shown below:

\[
\begin{array}{ccc}
4 & \times & 2 \\
\text{length} & \times & \text{width} \\
cubes in stack
\end{array}
\]

\[
\begin{array}{ccc}
2 & \times & 3 \\
\text{height} & \times & \text{number of layers} \\
cubes in stack
\end{array}
\]

SAY: Multiplying length by width gives the number of cubes in one layer, height is the number of layers, and volume is the number of cubes in the stack. Mathematicians call stacks, or rectangular boxes, rectangular prisms. Write on the board:

\[
\text{Volume} = \text{length} \times \text{width} \times \text{height} \quad \text{or} \quad V = l \times w \times h
\]

SAY: We can use this formula to calculate the volume of a rectangular prism. Remind students that a formula is a short way to write the instructions for how to calculate something. Draw on the board:
ASK: What is the length of this prism? (5 in) The width? (2 in) The height? (3 in) Write on the board:

\[ \text{Volume} = 5 \text{ in} \times 2 \text{ in} \times 3 \text{ in} \]

ASK: Are all the units the same? (yes) SAY: Since all the units are the same, we can just plug in the numbers to calculate the volume. We’ll do the calculations without the units because it’s tedious to have to write the units all the time. But the first step of writing the volume with all the units is really important because then we can see if all the units are the same or not. Write on the board:

\[ \text{Calculation: } 5 \times 2 \times 3 = \]

Have a volunteer do the calculation. (30) ASK: What unit do we write for the answer? (in\(^3\)) Write on the board:

\[ \text{So, } V = 30 \text{ in}^3. \]

(MP.2) Exercises: Find the volume of the rectangular prism.

\begin{align*}
\text{a) } & \quad \text{Volume} = \underline{\quad} \quad \text{Calculation: } \underline{\quad} \\
\text{b) } & \quad \text{Volume} = \underline{\quad} \quad \text{Calculation: } \underline{\quad} \\
\text{c) } & \quad \text{length} = 5 \text{ in}, \text{ width} = 3 \text{ in}, \text{ height} = 9 \text{ in} \\
\text{d) } & \quad l = 2 \text{ m}, w = 2 \text{ m}, h = 6 \text{ m}
\end{align*}

So, \( V = \underline{\quad} \)

Answers:

a) \( \text{Volume} = 4 \text{ cm} \times 1 \text{ cm} \times 8 \text{ cm}, \text{ Calculation: } 4 \times 1 \times 8 = 32, \text{ So, } V = 32 \text{ cm}^3 \)
b) \( \text{Volume} = 12 \text{ ft} \times 2 \text{ ft} \times 5 \text{ ft}, \text{ Calculation: } 12 \times 2 \times 5 = 120, \text{ So, } V = 120 \text{ ft}^3 \)
c) \( \text{Volume} = 5 \text{ in} \times 3 \text{ in} \times 9 \text{ in}, \text{ Calculation: } 5 \times 3 \times 9 = 135, \text{ So, } V = 135 \text{ in}^3 \)
d) \( \text{Volume} = 2 \text{ m} \times 2 \text{ m} \times 6 \text{ m}, \text{ Calculation: } 2 \times 2 \times 6 = 24, \text{ So, } V = 24 \text{ m}^3 \)

(MP.6) SAY: Here’s an example that shows the importance of checking the units before calculating volume. Draw on the board:

\[ \text{Volume} = \frac{80 \text{ cm} \times 1 \text{ m} \times 150 \text{ cm}}{} \]
ASK: What does the volume of this prism equal? (150 cm × 1 m × 80 cm) Write this on the board. ASK: Are all the units the same? (no) SAY: We can’t calculate the volume with these numbers because they are not given in the same units. But if we write 1 m as 100 cm, then we can calculate the volume. Write on the board:

Calculation: 150 × 100 × 80 = 1,200,000 cm³  
So, \( V = 1,200,000 \text{ cm}^3 \).

**(MP.1, MP.3)** Show that volume does not depend on orientation. Draw on the board:

Have three volunteers come to the board and each calculate the volume of one prism.  
(3 \( \times \) 2 \( \times \) 4 \( \text{ in} \) = 24 \( \text{ in}^3 \), 4 \( \times \) 3 \( \times \) 2 \( \text{ in} \) = 24 \( \text{ in}^3 \), 2 \( \times \) 4 \( \times \) 3 \( \text{ in} \) = 24 \( \text{ in}^3 \)) Point to the length of each prism (3 in, 4 in, 2 in) and ASK: Do the three prisms have the same length? (no) Do they have the same width? (no) Do they have the same height? (no) So why do the three prisms have the same volume? (the order of the dimensions changed, not the dimensions themselves; it’s the same prism each time) Remind students that order does not matter in multiplication, so length \( \times \) width \( \times \) height can be rewritten in any order. In addition, the dimensions can be assigned to the rectangular prism in any order; in other words, you can assign any side to be the length, width, or height. Make sure students see that these pictures actually show the same prism—it has just been rotated.

**(MP.5)** Exercises:

a) Find the volume of the prism in three different ways.

b) Which way is the easiest to calculate mentally?

**Solutions:** a) 13 \( \times \) 2 \( \times \) 5 = 26 \( \times \) 5 = 130, so \( V = 130 \text{ cm}^3 \), 13 \( \times \) 5 \( \times \) 2 = 65 \( \times \) 2 = 130, so \( V = 130 \text{ cm}^3 \), 5 \( \times \) 2 \( \times \) 13 = 10 \( \times \) 13 = 130, so \( V = 130 \text{ cm}^3 \); b) 5 cm \( \times \) 2 cm \( \times \) 13 cm is the easiest to calculate because 5 \( \times \) 2 = 10 and it is easy to multiply by 10

**(MP.2)** Finding the volume of prisms with fractional side lengths. Draw on the board:

Have a volunteer write out the product used to find the volume, including the units, on the board. (Volume = 2 1/4 \( \text{ in} \) \( \times \) 1/2 \( \text{ in} \) \( \times \) 3/4 \( \text{ in} \)) ASK: Are all the units the same? (yes) SAY: So we can just
plug in the numbers and multiply to get the volume. Remind students that to multiply fractions, they need to convert any mixed numbers to improper fractions. ASK: How many quarters does 2 1/4 equal? (9) How do you know? (2 × 4 + 1 = 9) SAY: To multiply the fractions, multiply the numerators and multiply the denominators. Write on the board:

\[
\text{Calculation: } \frac{9}{4} \times \frac{1}{2} \times \frac{3}{4} = \frac{9 \times 1 \times 3}{4 \times 2 \times 4} = \frac{27}{32}
\]

SAY: Since all the units are in inches, volume is measured in in³. So, the volume is 27/32 in³.

**Exercises:** Find the volume of the rectangular prism.

a) length = \( \frac{1}{2} \) in, width = \( \frac{1}{2} \) in, height = \( \frac{1}{2} \) in

b) \( \ell = 3 \frac{1}{2} \) ft, \( w = 3 \frac{1}{4} \) ft, \( h = 2 \) ft

**Answers:** a) 15/8 in³ or 1 7/8 in³, b) 182/8 = 91/4 ft³ or 22 3/4 ft³

**Finding the missing side lengths of a 2-D shape.** Draw on the board:

With colored chalk, trace the three horizontal sides. Point to the bottom of the shape and ASK: What is this length? (13 ft) How do you know? (6 + 7 = 13) With a different color of chalk, trace the three vertical sides. Point to the vertical side with no measurement and ASK: What is this length? (3 ft) How do you know? (5 − 2 = 3) Write the two lengths on the diagram. Leave the diagram on the board.

**Exercises:** Find the missing side lengths. All measurements are in centimeters.

a) \[
\begin{array}{c}
3 \\
12 \\
x \\
\end{array}
\]

b) \[
\begin{array}{c}
7 \\
22 \\
x \\
\end{array}
\]

**Bonus:**

\[
\begin{array}{c}
4.5 \\
5 \frac{2}{9} \\
x \\
\end{array}
\]

**Answers:** a) \( x = 4 \) cm, \( y = 18 \) cm; b) \( x = 39 \) cm, \( y = 6 \) cm; Bonus: \( x = 8 \frac{5}{9} \) cm, \( y = 4.25 \) cm

**Finding the volume of composite 3-D shapes.** Add diagonal lines to the example on the board to make the 2-D shape 3-D, as shown below:
Point out that not all the edges are labeled because the drawing would be too messy if they were. Have a volunteer use colored chalk to trace the two visible edges that have a length of 6 ft, a different color to trace the five visible edges that have a length of 9 ft, and so on until all the visible edges are colored. SAY: To find the volume of this shape, we could divide it into two rectangular prisms, because we know how to find the volume of rectangular prisms. Have a student show how the shape could be divided. Label the prisms $V_1$ and $V_2$. (the two possible divisions are shown below)

![Diagram](image)

Write on the board:

$$V_1 = \quad V_2 = \quad \text{Total volume} =$$

Have students tell you what calculations to use to find the volumes. They may use their calculators to multiply. (see solutions below)

$$V_1 = 6 \text{ ft} \times 9 \text{ ft} \times 3 \text{ ft} = 162 \text{ ft}^3 \quad \text{or} \quad V_1 = 6 \text{ ft} \times 9 \text{ ft} \times 5 \text{ ft} = 270 \text{ ft}^3$$

$$V_2 = 13 \text{ ft} \times 9 \text{ ft} \times 2 \text{ ft} = 234 \text{ ft}^3 \quad V_2 = 7 \text{ ft} \times 9 \text{ ft} \times 2 \text{ ft} = 126 \text{ ft}^3$$

ASK: What is the total volume? (396 ft$^3$)

(MP.1) Have a volunteer come to the board to show a different way to divide the shape and calculate the volume. (see above) ASK: Why doesn’t it matter which way we divide the shape? (it doesn’t matter which parts you add to make the whole shape) Point out that calculating the volume in a different way allows us to check our answer.

**Exercises:** Divide the whole shape into rectangular prisms. Find the volume of each prism to find the total volume of the shape.

**a)**

![Diagram](image)

<table>
<thead>
<tr>
<th>6 cm</th>
<th>16 cm</th>
<th>9 cm</th>
<th>4.5 cm</th>
</tr>
</thead>
</table>

**b)**

![Diagram](image)

<table>
<thead>
<tr>
<th>2 ft</th>
<th>4 ft</th>
<th>5 ft</th>
</tr>
</thead>
</table>

**(MP.1)** Bonus: Find the volumes of the 3-D shapes in parts a) and b) in two ways. Check that you got the same answer both ways.

**Answers:** a) 495 cm$^3$, b) 190 ft$^3$

Determining which dimensions satisfy a given volume. Write on the board:

3 in, 4 in, 1 in  
2 in, 2 in, 3 in  
1 in, 2 in, 5 in  
3 in, 3 in, 4 in
SAY: I need a box to have a volume of at least 12 in$^3$. Point to the first set of dimensions and ASK: Could these be the dimensions of my box? (yes) Why? (the volume is 12 in$^3$) Repeat with the next three sets of dimensions. (yes, no, yes)

**Exercises:** Which boxes give a volume of no more than 100 m$^3$?
A. 7 m, 7 m, 2 m   B. 10 m, 10 m, 10 m   C. 5 m, 5 m, 5 m   D. 5 m, 4 m, 5 m  
**Bonus:** E. 1.5 m, 2.5 m, 24 m   F. 15 m, 15 m, $\frac{1}{2}$ m

**Answers:** A, D; Bonus: E

**(MP.1, MP.4) Solving inequalities to find a missing dimension.** Draw on the board:

![Volume Inequality Diagram]

SAY: We don’t know the width of the box, but we know that the volume must be at least 60 in$^3$. We can solve an inequality to find the unknown width. ASK: What formula do we use to find the volume of a prism? ($V = \ell \times w \times h$) Write on the board:

$$V \geq 60 \text{ in}^3$$

$$\ell \times w \times h \geq 60 \text{ in}^3$$

ASK: What dimensions do we know? ($\ell = 10 \text{ in}$ and $h = 3 \text{ in}$) Remind students that they can leave the units out when they do the calculations, but they have to remember to write the answer with units. SAY: Let’s substitute the dimensions we know in the inequality on the board. We will write $x$ for width, the unknown dimension. Continue writing on the board:

$$10(x)(3) \geq 60$$

SAY: Now we have an inequality that we can solve. Have students explain step by step how to solve the inequality. Write each step on the board, as shown below:

$$30x \geq 60$$

$$x \geq \frac{60}{30}$$

$$x \geq 2$$

SAY: So the width must be at least 2 inches for the volume to be at least 60 in$^3$.

**Exercises:** Write and solve an inequality to find the unknown dimension.

a) $V \geq 150 \text{ in}^3$, $\ell = 10 \text{ in}$, $w = 3 \text{ in}$   b) $V \leq 72 \text{ ft}^3$, $\ell = 3 \text{ ft}$, $h = 6 \text{ ft}$

c) $V$ is at least 6 cm$^3$, $w = 5 \text{ cm}$, $h = 3 \text{ cm}$  
**Bonus:** $V$ is at most 6 m$^3$, $w = \frac{1}{2}$ m, $h = \frac{4}{5}$ m

**Answers:** a) $h \geq 5$ in, b) $w \leq 4$ ft, c) $\ell \geq 0.4$ cm, Bonus: $\ell \leq 5$ m
(MP.4) Word problems practice.

Exercises:
1. A sandbox measures 4 ft by $3\frac{3}{4}$ ft. Micky wants the sand to have a depth of $\frac{1}{2}$ ft. If each 50 lb bag of sand fills $\frac{1}{2}$ ft$^3$, how many bags would Micky need to fill the sandbox?

**Answer:** the volume of sand needed is $7\frac{1}{2}$ ft$^3$, and $7\frac{1}{2}$ ft$^3 \div \frac{1}{2}$ ft$^3 = 15$, so Micky needs 15 bags of sand

2. a) The cargo hold of a delivery truck measures 9 ft long, 6 ft wide, and 6 ft high. What is the volume of the cargo hold?
   b) A box is 1.5 ft long, 1.5 ft wide, and 2 ft high. What is the volume of the box?
   c) How many boxes can fit in the cargo hold by volume?
   d) Check your answer to part c) by sketching the cargo hold and showing how the boxes fit. Does it matter which way they are packed?

**Answers:** a) $324$ ft$^3$; b) $4.5$ ft$^3$; c) $324 \div 4.5 = 72$ boxes; d) 72 boxes fit if they are packed with height of $2$ ft to fit $6$ long, $4$ wide, and $3$ high, or if you pack them with width of $2$ ft to fit $6$ long, $3$ wide, and $4$ high. But if you pack them with length of $2$ ft, only $64$ boxes will fit ($4$ long, $4$ wide, and $4$ high).

3. a) Box A measures 9 in by 9 in by 6 in, and Box B measures 10 in by 8 in by 7 in. Predict which box will have a greater volume. Explain.
   b) Calculate the volume of each box. Was your prediction in part a) correct?

**Answers:** a) Box B has $2$ dimensions that are greater than Box A so I think Box B will have a greater volume; b) Box A has $V = 486$ in$^3$ and Box B has $V = 560$ in$^3$, so my prediction was correct

Extensions
(MP.2) 1. A cube is a rectangular prism with all edges of the same length.
   a) Find the volume of the cube with the given edge length.
      i) 12 in  ii) 3 ft  iii) 100 cm
   b) Use your answers from part a) to convert the cubic unit.
      i) $1$ ft$^3 = \underline{\ }$ in$^3$  ii) $1$ yd$^3 = \underline{\ }$ ft$^3$  iii) $1$ m$^3 = \underline{\ }$ cm$^3$
   c) Use your answers from part b) to convert the unit.
      i) $5$ yd$^3 = \underline{\ }$ ft$^3$  ii) $2$ m$^3 = \underline{\ }$ cm$^3$  iii) $0.5$ ft$^3 = \underline{\ }$ in$^3$
      iv) $500,000$ cm$^3 = \underline{\ }$ m$^3$  v) $9$ ft$^3 = \underline{\ }$ yd$^3$  vi) $1$ yd$^3 = \underline{\ }$ in$^3$

**Answers:** a) i) $1,728$ in$^3$, ii) $27$ ft$^3$, iii) $1,000,000$ cm$^3$; b) i) $1,728$ in$^3$, ii) $27$ ft$^3$, iii) $1,000,000$ cm$^3$; c) i) $135$ ft$^3$, ii) $2,000,000$ cm$^3$, iii) $864$ in$^3$, iv) $1/2$ m$^3$, v) $1/3$ yd$^3$, vi) $46,656$ in$^3$

(MP.1) 2. The volume of a rectangular prism is $24$ cm$^3$ and its height is $2$ cm. What could be the dimensions of the base of the prism?

**Answer:** The base of the prism has area $24 \div 2 = 12$ cm$^2$, so the dimensions of the base could be $1$ cm $\times$ $12$ cm, $2$ cm $\times$ $6$ cm, $3$ cm $\times$ $4$ cm, $2.4$ cm $\times$ $5$ cm, and so on.
3. a) What is the effect on the volume when you double...
   i) one dimension?
   ii) two dimensions?
   iii) all three dimensions?

b) What is the effect on the volume when you triple all three dimensions?

c) The original volume of a prism has been multiplied by 1,000. What changes could have been made to the dimensions? Find at least five different possibilities.

**Bonus:** The original volume of a prism has been divided by four. What changes could have been made to the dimensions?

**Answers:**
  a) i) the volume doubles, ii) the volume is multiplied by 4, iii) the volume is multiplied by 8; b) the volume is multiplied by 27; c) Sample answers: all three dimensions were multiplied by 10; one dimension was multiplied by 1,000; the dimensions were multiplied by 2, 5, and 100 or 4, 50, and 50 or 4, 10, and 25; Bonus: Sample answers: two dimensions were multiplied by 1/2, one dimension was multiplied by 1/4

4. A cereal company puts one of four animal cards—a cat, a dog, a bird, or a fish—in each cereal box, each with equal likelihood. If a customer collects all four animals, the customer wins a free box of cereal.

a) Sandy buys 4 boxes, Tony buys 5 boxes, and Emma buys 6 boxes. Estimate the probability that each person gets a free box of cereal. Use any tool you think will help.

b) Does the probability of winning increase as a customer buys more boxes?

**Selected sample answers:**

a) • I simulated picking one of the animal cards in the box by spinning a spinner with four equal regions. I did this four times for Sandy, five times for Tony, and six times for Emma, since each spin represents buying a box. I recorded the results of each spin and looked for whether each region was spun at least once. That was one experiment; I need to repeat this many times in order to estimate the probability. (Students might combine their results with those of several classmates to get a more accurate estimate.)

• I used four-sided dice. I rolled four dice to represent Sandy’s four boxes, five dice for Tony, and six dice for Emma. I recorded the results of each roll and looked for whether each number was rolled at least once. That was one experiment; I did 40 experiments to estimate the probability.

• I used a random number generator. To estimate the probability that Sandy wins, I picked 400 numbers from 1 to 4 for Sandy and asked the computer to put them in 4 columns. Each row represented an experiment and I checked how many rows out of 100 included all four numbers. This gave me the experimental probability that Sandy wins. I repeated this for Tony with 500 numbers from 1 to 4 in 5 columns, and for Emma with 600 numbers from 1 to 4 in 6 columns.

**NOTE:** The theoretical probabilities are 3/32 for four boxes, 15/64 for five boxes, and 195/512 for six boxes. Students should find that the probability increases as the customer buys more boxes.
G7-25 Volume of Polygonal Prisms

Pages 156–158

Standards: 7.G.B.6, 7.EE.B.4b

Goals:
Students will use the area of the base to develop a formula for the volume of polygonal prisms.
Students will solve real-world and mathematical problems involving the volume of polygonal prisms.

Prior Knowledge Required:
Is familiar with cubic units of measurement
Can find the area of a rectangle, triangle, parallelogram, and trapezoid
Knows that the volume of a rectangular prism is \( V = l \times w \times h \)
Can multiply and divide decimals and fractions
Can substitute numbers for variables in a formula
Can solve inequalities

Vocabulary: base, congruent, cubic units, face, inequality, polygon, prism, right prism, volume

Materials:
right prisms with a variety of bases or BLM Nets of 3-D Shapes (1) to (8) (pp. R-68–75)
21 connecting cubes for demonstration
a set square or a protractor for demonstration
BLM Volume of a Pyramid (pp. R-79–82, see Extension 2) photocopied on cardstock,
one set for every 2 or 3 students
tape for every 2 or 3 students (see Extension 2)
about 1 1/4 cups dry rice for every 2 or 3 students (see Extension 2)

(MP.7) Review bases and faces of prisms. SAY: Prisms are 3-D shapes that have two identical opposite faces called bases. Hold up a triangular prism (or a prism made from BLM Nets of 3-D Shapes (3) or (4)). ASK: What shapes make the faces of this prism? (two triangles, three rectangles) Which shape would be considered the base? (triangle) Why? (there are two congruent triangles opposite to each other) Rotate and flip the shape and point out that the base is the base no matter which way the shape is oriented. SAY: Prisms are named for the shape of their base, so this is a triangular prism. Give a student a prism with a different base, and ask the student to say what shape makes the base and to put the prism base-down on a desk in a central location. Repeat with different prisms. Students should see prisms with the following bases: parallelogram (BLM Nets of 3-D Shapes (5)), trapezoid (BLM Nets of 3-D Shapes (6)), pentagon (BLM Nets of 3-D Shapes (7)), hexagon (BLM Nets of 3-D Shapes (8)). ASK: For all of these prisms, what shape are the other faces? (rectangles) Hold up a rectangular prism (BLM Nets of 3-D Shapes (1)) and a cube (BLM Nets of 3-D Shapes (2)). ASK: Which face is the base? (it could be any face) SAY: For a rectangular prism or a cube, it doesn’t matter which face we call the base. Leave the prisms on a desk at the front of the classroom.
Review finding the volume of a rectangular prism with $V = l \times w \times h$. ASK: To find the volume of a rectangular prism, what information do you need? (length, width, height) What is the formula for the volume of a rectangular prism? (Volume = length × width × height) Hold up a triangular prism and ASK: Why can’t we use that formula to calculate the volume of this prism? (triangular prisms don’t have length or width) SAY: Let’s find a general formula for volume that can be used for all prisms.

Develop the formula $\text{Volume} = \text{area of base} \times \text{height}$. Build the shape shown below with seven connecting cubes:

![Diagram of a prism built with connecting cubes](image)

ASK: What is the volume of this shape? (7 cubes) Add two more layers. Point out that this 3-D shape is a prism because it has two congruent bases and sides that are rectangles. ASK: What is the volume now? (21 cubes) SAY: To get the volume, you multiply the number of cubes in the bottom layer by the number of layers. Write on the board:

$$\text{Volume} = \text{area of bottom layer} \times \text{number of layers} = \text{area of base} \times \text{height of prism}$$

SAY: This is the same formula we got for rectangular prisms too. Write on the board:

$$\text{Volume of rectangular prism} = \frac{\text{length} \times \text{width} \times \text{height}}{\text{area of base}}$$

SAY: The volume of a prism is the area of the base times the height of the prism, even when the base isn’t a rectangle. There is also a short form for this formula, where $B$ is the area of the base and $h$ is the height of the prism. Write on the board:

$$\text{Volume} = \text{area of base} \times \text{height}, \text{or } V = B \times h$$

SAY: length × width × height is specific to the volume of rectangular prisms, but $B \times h$ can be used to find the volume of any prism.

Using the formula $\text{Volume} = \text{area of base} \times \text{height}$. Draw on the board:
ASK: What is the area of the base? (3 cm²) The height of the prism? (6 cm) So what is the volume? (18 cm³) Show the steps on the board:

\[ V = B \times h \]
\[ = 3 \times 6 \]
\[ = 18 \text{ cm}^3 \]

Remind students that as long as the units are consistent, such as cm and cm², they can just plug in the numbers and multiply. If students are struggling, write the steps on the board, as shown below:

\[ V = 3 \text{ cm}^2 \times 6 \text{ cm} \]
Calculation: \( 3 \times 6 = 18 \)
So, \( V = 18 \text{ cm}^3 \)

**Exercises:** Calculate the volume of the prism.

a)  

b) A triangular prism has \( B = 12 \text{ in}^2, h = 5 \text{ in} \)

c) A trapezoidal prism has \( B = 10 \text{ m}^2, h = 7.2 \text{ m} \)

**Answers:** a) 28 cm³, b) 60 in³, c) 72 m³

**Review the area of polygons.** Draw on the board:

Ask volunteers to write the formula they would use to find the area of each shape. \((b \times h + 2, \ell \times w, (b_1 + b_2) \times h + 2, b \times h)\) Have volunteers draw and label the measurements used in the formulas for the shapes, as shown below:

**Exercises:** Calculate the area of the polygon.

a)  

b)  

c)  

d)  

**Answers:** a) 6 in², b) 18 in², c) 20 in², d) 48 in²
Finding the volume of polygonal prisms. Draw on the board:

SAY: Imagine that this trapezoid is the base of a prism that has height 6 cm. To find the volume of the prism, first find the area of the base, then multiply it by the height of the prism. Write on the board:

\[ B = \]
\[ h = 6 \text{ cm} \]
\[ V = \]

ASK: How do we calculate the area of the trapezoidal base? \(((b_1 + b_2) \times h \div 2)\) Write the calculation on the board and allow students to use a calculator to work it out. \(((7.5 + 9.5) \times 5 \div 2 = 42.5 \text{ cm}^2)\) Remind students that units must be written in the answer, but not necessarily in the calculation.

ASK: How do we calculate the volume? \((42.5 \text{ cm}^2 \times 6 \text{ cm})\) Write the calculation on the board, then find the answer. \((255 \text{ cm}^3)\)

Exercises: Use a calculator to find the volume of the prism with the base shown and height 5 cm.

a) [Diagram: Base 8 cm, Height 5 cm, 3 cm]

b) [Diagram: Base 9.3 cm, Height 12.1 cm, 2 cm]

Answers: a) 160 cm³, b) 562.65 cm³

(MP.6) Point out that it’s important to identify the height of the prism and to not get it confused with the height of the base. Look at the display of 3-D shapes on the desk. SAY: When a prism is standing on its base, the height is how tall it is. Point to the height of one of the prisms. Then, tip that prism over and SAY: Now, when the prism is on its side, its height is not how tall it is—that is the height of the base. Instead, its height is the distance between the two bases, and when a shape is on its side like this, that distance is horizontal. Height is always measured along an edge. Use a set square or a protractor to show students that the height meets the base at a 90° angle. SAY: For this reason, these prisms are called right prisms.

Exercises:
1. Bold an edge that shows the height of the prism.

a) [Diagram: Edge highlighted]

b) [Diagram: Edge highlighted]

c) [Diagram: Edge highlighted]

d) [Diagram: Edge highlighted]
**Selected answers:** a) ![Image](image1.png)  

b) ![Image](image2.png)  
c) ![Image](image3.png)  
d) ![Image](image4.png)

2. Circle the height of the prism. Then, calculate the volume of the prism.

a)

![Image](image5.png)

**Answers:** a) 15 ft, 900 ft³; b) 5 m, 100 m³

(MP.7) Finding the area of polygonal prisms by decomposing the base. Remind students that when they’re presented with a shape that doesn’t have a formula for area, they can divide the shape into shapes that do have formulas. Draw on the board:

![Image](image6.png)

ASK: How could you divide this shape? (make a rectangle and a triangle) Show the division, find the unknown measurements, and label the shape, as shown below:

![Image](image7.png)

Write on the board:

\[ A_1 = \]
\[ A_2 = \]
\[ \text{Total area} = \]

Ask students what calculations are needed to find the area of each shape and write them on the board:

\[ A_1 = 8 \text{ in} \times 4 \text{ in} = 32 \text{ in}^2 \]
\[ A_2 = 8 \text{ in} \times 3 \text{ in} + 2 = 12 \text{ in}^2 \]
\[ \text{Total area} = 32 \text{ in}^2 + 12 \text{ in}^2 = 44 \text{ in}^2 \]

ASK: If this were the base of a prism with height 10 in, what would be the volume? (440 in³)
**Exercises:** a) Decompose the shape. Find the area of each part to find the total area.

\[\text{i) } \begin{array}{c}
\text{15 cm} \\
\text{12 cm} \\
\text{4 cm}
\end{array} \quad \text{ii) } \begin{array}{c}
\text{3 in} \\
\text{7 in} \\
\text{10 in} \\
\text{6 in}
\end{array}\]

b) Calculate the volume of the prism with the base shown in part a) and the given height.

\[\text{i) } h = 10 \text{ cm} \quad \text{ii) } h = 12 \text{ in}\]

**Answers:**

\[\text{i) } A_1 = 55 \text{ cm}^2 \quad A_2 = 48 \text{ cm}^2 \quad \text{Total area} = 103 \text{ cm}^2 \]

\[\text{ii) } A_1 = 75 \text{ cm}^2 \quad A_2 = 28 \text{ cm}^2 \quad \text{Total area} = 103 \text{ cm}^2 \]

\[\text{Total area} = 103 \text{ cm}^2 \]

\[\text{Total area} = 71 \text{ in}^2 \]

**MP.1, MP.4) Solving inequalities to find a missing dimension.** Draw on the board:

\[V \geq 60 \text{ ft}^3\]

SAY: We’re missing some information about this prism, but we know that the volume must be at least 60 ft³. We can write and solve an inequality to find the unknown dimension. To find the volume of a prism, we need to find the area of the base, then multiply it by the height. Write on the board:

\[B = \text{area of base} \quad h = \text{height}\]

ASK: What is the area of the base of this prism? (6x + 2 = 3x) What is the height of the prism? (5)

Write these answers on the board. SAY: We know that the volume must be at least 60 ft³, and that volume equals \(B \times h\). Write on the board:

\[V \geq 60 \text{ ft}^3 \quad B \times h \geq 60 \text{ ft}^3\]
SAY: Let’s substitute and solve the inequality. Have students explain each step as you write on the board, as shown below:

\[
3x(5) \geq 60 \\
15x \geq 60 \\
x \geq \frac{60}{15} \\
x \geq 4 \text{ ft}
\]

SAY: So the height of the triangular base must at least 4 ft for the volume to be at least 60 ft\(^3\).

**Exercises:** Write and solve an inequality to find the unknown dimension.

a) \(V > 60 \text{ cm}^3\)  

b) \(V\) is no more than 75 m\(^3\)  

c) \(V\) is at least 6 in\(^3\)

**Answers:** a) \(x > 10 \text{ cm}\), b) \(x \leq 3 \text{ m}\), c) \(x \geq 0.5 \text{ in}\)

**Word problems practice.**

**(MP.4) Exercises:**
1. The base of a free-standing punching bag is an octagon. The area of the base is 3.5 ft\(^2\) and the height is 3 ft.
   a) What is the volume of the punching bag?
   b) A 30 kg bag of sand fills \(\frac{2}{3}\) ft\(^3\). How many bags of sand do you need to fill the punching bag?

**Answers:** a) 10.5 ft\(^3\), b) \(10.5 + \frac{2}{3} = 15 \frac{3}{4}\), so 16 bags of sand are needed

2. a) What is the unknown dimension of the fish tank in the picture if the volume of the tank is 84,000 cm\(^3\)?
   b) If 1,000 cm\(^3\) of water weigh about 2.2 lb, how heavy would the water in the tank be if the tank was 80% full?

**Answers:**
   a) height of triangular base is 40 cm
   b) 80% of 84,000 cm\(^3\) = 67,200 cm\(^3\). 67,200 cm\(^3\) + 1,000 cm\(^3\) = 67.2 units.
   \(67.2 \times 2.2 \text{ lb} = 147.84 \text{ lb}\). The water in the tank would weigh about 148 lb.
(MP.1, MP.2) Extensions
1. A cylinder is a prism with a circular base. Like all prisms, the formula for the volume of a cylinder is Volume = area of base × height. Since the area of a circle = \( \pi r^2 \), then the volume of a cylinder = \( \pi r^2 \times h \).

Find the volume of the cylinder. Use \( \pi = 3.14 \), and write your answers to two decimal places.

\begin{align*}
a) & \quad \text{volume} = 90 \text{ cm}^3 \\
b) & \quad \text{volume} = 323.5 \text{ cm}^3 \\
c) & \quad \text{volume} \approx 288.49 \text{ cm}^3 \\
d) & \quad \text{volume} \approx 2,127.66 \text{ cm}^3 \\
\end{align*}

Answers: a) \( V = 90 \text{ cm}^3 \), b) \( V = 323.5 \text{ cm}^3 \), c) \( V \approx 288.49 \text{ cm}^3 \), d) \( V = 2,127.66 \text{ cm}^3 \)

(MP.7) 2. Work with a partner or two. Make the nets from BLM Volume of a Pyramid. Tape the sides securely. Match a pyramid with the prism that has the same base and same height to make two pairs of shapes.

a) Take one pair of shapes. Fill the pyramid with rice. Pour the rice into the prism. How many times do you have to refill the pyramid to fill up the prism?
b) Repeat with the other pair of shapes. Did you get the same result?
c) Check with another group. Did they get the same result as your group?
d) Describe the relationship between the volume of a prism and the volume of a pyramid with the same base and height.
e) The formula to find the volume of a pyramid is Volume = area of base × height ÷ 3 or \( V = Bh ÷ 3 \). Use the formula to find the volume of the pyramid. All measurements are in inches. Write answers to one decimal place.

\begin{align*}
i) & \quad \text{volume} = 480 \text{ in}^3 \\
ii) & \quad \text{base is a 5 by 5 square, height of pyramid is 4} \\
iii) & \quad \text{base is a 6 by 4 rectangle, height of pyramid is 12} \\
iv) & \quad \text{base is a triangle with base 8 and height 5, height of pyramid is 12} \\
\end{align*}

Bonus: A cone is a pyramid with a circular base. Since the area of a circle is \( \pi r^2 \), then the volume of a cone is \( V = \pi r^2 \times h ÷ 3 \). Use \( \pi = 3.14 \), and write your answers to two decimal places.

\begin{align*}
i) & \quad \text{volume} \approx 167.47 \text{ in}^3 \\
ii) & \quad \text{volume} \approx 615.44 \text{ in}^3 \\
\end{align*}

Answers: a) 3 times; b) yes; c) groups should get the same results; d) the volume of the prism is three times as much as the volume of the pyramid; e) i) \( V = 480 \text{ in}^3 \), ii) \( V \approx 33.3 \text{ in}^3 \), iii) \( V = 96 \text{ in}^3 \), iv) \( V = 80 \text{ in}^3 \); Bonus: i) \( V \approx 167.47 \text{ in}^3 \), ii) \( V \approx 615.44 \text{ in}^3 \)
G7-26 Nets and Surface Area of Prisms

Pages 159–160

Standards: 7.G.B.6

Goals:
Students will use nets to find the surface area of rectangular prisms and triangular prisms.

Prior Knowledge Required:
Is familiar with square units of measurement
Is familiar with nets of prisms
Can find the area of rectangles and triangles

Vocabulary: back, base, bottom, face, front, left side, net, prism, right side, square units, surface area, top

Materials:
overhead projector
transparency of BLM Quarter-Inch Grid Paper (p. R-83)
grid paper or BLM Quarter-Inch Grid Paper (p. R-83)
rulers
medium-sized cereal box

NOTE: Before class, measure the length, width, and height of a medium-sized cereal box and write the measurements in permanent marker near the edges in large print. Label the faces with the terms “front,” “back,” “right side,” “left side,” “top,” and “bottom.” You will also need this box for the next lesson. In the sample answers for this lesson, we use a box with dimensions 9.5 by 3 by 13.5 (in inches).

Review volume. SAY: In the first two lessons of this unit, you were calculating the volume of prisms—this means that you were finding the amount of space inside them. Hold up the cereal box and have a volunteer use a calculator to find its volume. (9.5 in \times 3 \text{ in} \times 13.5 \text{ in} = 384.75 \text{ in}^3) Make sure the student uses cubic units in the answer.

Introduce surface area. SAY: There is another calculation you can do with this box: you can determine the area of its surface. This would tell us, for example, how much cardboard we would need to make the box, how much wrapping paper we would need to cover the box, or how much paint we would need to paint the box. Point out that the term area is used for 2-D shapes, so when we talk about 3-D shapes, we say surface area. Surface area is the total area of all the surfaces, or faces, of a 3-D shape. ASK: What units do you think we use to measure surface area? (square units) PROMPT: What units do we use to measure area?

(MP.6) Drawing the net of a rectangular prism. Hold up the cereal box and ASK: How many faces does this box have? (6) What shapes are the faces? (rectangles) How many different
sizes of faces are there? (3) SAY: We are going to draw the net of this box. Remind students that a net is a pattern that can be folded to make a 3-D shape. Show students the front of the box, then trace around the front near the top of the board. Write “front” in the middle of the rectangle on the board. Ask a volunteer to tell you the dimensions of the rectangle. The picture should look like this:

```
9.5
Front
13.5
```

Tip the box down so the bottom of the box is flat on the board. SAY: Now I will trace the bottom of the box. Draw the rectangle, write “bottom” in the middle, and ask students to tell you the dimensions. The picture should look like this:

```
9.5
Front
13.5
```

```
9.5
Bottom
3
```

Continue to trace all the faces of the box until you have the complete net drawn on the board, as shown below:

```
13.5
Left side
3
9.5
Front
13.5

9.5
Right side
3
```

```
9.5
Bottom
3
```

```
9.5
Back
13.5
```

```
9.5
Top
3
```

Leave this net displayed on the board.
**Exercises:** Copy the net of the rectangular prism on grid paper. On the net, label each face and write the length of each edge. All measurements are in inches.

![Net of a rectangular prism](image)

**Answers:**

![Labelled faces of the net](image)

**(MP.4) Using the net to find the surface area of a rectangular prism.** SAY: The net of the cereal box shows the shape of every face, or surface, that makes the cereal box. So we can use the net to find the surface area of the box. Next to the net of the cereal box, write on the board:

- **Front:**
- **Back:**
- **Right side:**
- **Left side:**
- **Top:**
- **Bottom:**

Tell students that they may use calculators. ASK: What is the area of the front of the box? (128.25 in²) Write the calculation on the board, as shown below:

Front: 13.5 in \( \times \) 9.5 in = 128.25 in²

**Exercises:** Write a multiplication statement for the area of each face of the cereal box.

**Answers:**

Front: 13.5 in \( \times \) 9.5 in = 128.25 in²  
Back: 13.5 in \( \times \) 9.5 in = 128.25 in²  
Right side: 3 in \( \times \) 13.5 in = 40.5 in²  
Left side: 3 in \( \times \) 13.5 in = 40.5 in²  
Top: 9.5 in \( \times \) 3 in = 28.5 in²  
Bottom: 9.5 in \( \times \) 3 in = 28.5 in²
ASK: If surface area is the total area of all faces, then what is the surface area of this prism? (394.5 in\(^2\)) Write on the board:

Surface area = 394.5 in\(^2\)

**Exercises:** Using the rectangular prism nets you drew in earlier exercises, write a multiplication statement for the area of each face of the prism. Then find the surface area.

**Answers:**

a) Front: 3 in \(\times\) 2 in = 6 in\(^2\)
   
   Back: 3 in \(\times\) 2 in = 6 in\(^2\)
   
   Right side: 1 in \(\times\) 2 in = 2 in\(^2\)
   
   Left side: 1 in \(\times\) 2 in = 2 in\(^2\)
   
   Top: 3 in \(\times\) 1 in = 3 in\(^2\)
   
   Bottom: 3 in \(\times\) 1 in = 3 in\(^2\)
   
   Surface area = 22 in\(^2\)

b) Front: 1 in \(\times\) 1 in = 1 in\(^2\)
   
   Back: 1 in \(\times\) 1 in = 1 in\(^2\)
   
   Right side: 1 in \(\times\) 1 in = 1 in\(^2\)
   
   Left side: 1 in \(\times\) 1 in = 1 in\(^2\)
   
   Top: 1 in \(\times\) 1 in = 1 in\(^2\)
   
   Bottom: 1 in \(\times\) 1 in = 1 in\(^2\)
   
   Surface area = 6 in\(^2\)

**MP.6) Drawing the net of a triangular prism.** Draw on the board:

![Image of a triangular prism net]

ASK: What type of prism is this? (triangular) How many faces does a triangular prism have? (5) What shapes are the faces? (2 triangles and 3 rectangles) How many different sizes of faces are there? (the 2 triangles are congruent and the 3 rectangles are all different) On the board, draw the parts of the net one by one. Ask students to tell you what dimensions to make each face. The picture should look like this:

![Image of a triangular prism net with dimensions]

SAY: The names for the faces of a triangular prism are less specific than for a rectangular prism: we’ll just call them “bases” and “faces.” ASK: Which shape are the bases? (triangles) The faces? (rectangles) Label the net on the board as shown below:

![Image of a triangular prism net with labels]
Leave this net displayed on the board.

**Exercises:** Copy the net of the triangular prism on 1/4 inch grid paper. On the net, label each face and write the length of each edge. All measurements are in inches. Use one grid square per inch.

![Net of the Triangular Prism]

**Answers:**

![Labelled Net with Measurements]

**(MP.1) Using the net to find the surface area of a triangular prism.** Write on the board:

Base 1: 
Base 2: 
Face 1: 
Face 2: 
Face 3: 

Refer back to the net on the board. SAY: The base is a triangle. ASK: How do we calculate the area of a triangle? (base $\times$ height $\div$ 2) What is the area of the base? (6 in$^2$) Write the calculation on the board:

Base 1: $3\text{ in} \times 4\text{ in} \div 2 = 6\text{ in}^2$

**Exercises:** Write a multiplication equation for the area of each face of the prism. 
**Answers:**
Base 1: $3\text{ in} \times 4\text{ in} \div 2 = 6\text{ in}^2$
Base 2: $3\text{ in} \times 4\text{ in} \div 2 = 6\text{ in}^2$
Face 1: $10\text{ in} \times 3\text{ in} = 30\text{ in}^2$
Face 2: $10\text{ in} \times 4\text{ in} = 40\text{ in}^2$
Face 3: $10\text{ in} \times 5\text{ in} = 50\text{ in}^2$
ASK: What is the surface area of this prism? (132 in²) Write on the board:

Surface area of prism = 132 in²

Exercises: Using the net you drew in the previous exercises, write a multiplication equation for the area of each face of the prism. Then find the surface area.

Answers:
Base 1: 8 in × 6 in ÷ 2 = 24 in²
Base 2: 8 in × 6 in ÷ 2 = 24 in²
Face 1: 8 in × 3 in = 24 in²
Face 2: 6 in × 3 in = 18 in²
Face 3: 10 in × 3 in = 30 in²
Surface area = 120 in²

(MP.3) Using nets to find the surface area of a prism. ASK: How does drawing the net help you find the surface area of a prism? (it shows you all the faces of a prism so you can find each area, then you add the areas to find the surface area)


![Net of a prism](image)

Answers:
Front: 12 in²
Back: 12 in²
Top: 6 in²
Bottom: 6 in²
Right side: 8 in²
Left side: 8 in²
Surface area = 52 in²

(MP.4) Extensions
1. a) Given the net of a trapezoidal prism, find its surface area.
Hint: area of a trapezoid = (base 1 + base 2) × height ÷ 2

![Net of a trapezoidal prism](image)
b) Given the net of a prism with a parallelogram base, find its surface area. 
Hint: area of a parallelogram = base × height

\[
\text{Area} = \text{base} \times \text{height}
\]

\[
\begin{array}{c}
\text{Top} \\
\text{Front} \\
\text{Right} \\
\text{Back} \\
\text{Left} \\
\text{Bottom}
\end{array}
\]

3 cm 

10 cm 

5 cm 

4 cm 

\text{Surface area} = 16 \text{ ft}^2

\[
\begin{array}{c}
\text{Top} \\
\text{Front} \\
\text{Right} \\
\text{Back} \\
\text{Left} \\
\text{Base}
\end{array}
\]

8 in 

12 in 

12 in 

\text{Surface area} = 170 \text{ cm}^2

\[
\begin{array}{c}
\text{Base} \\
\text{Front} \\
\text{Right} \\
\text{Left} \\
\text{Back}
\end{array}
\]

144 in²

\text{Surface area} = 336 \text{ in}^2

\text{(MP.7)} 

2. You don’t have to draw a net to find the surface area. You could just draw the faces separately.

a) The diagram shows all the faces of a prism. Name the prism.

i) 

\[
\begin{array}{c}
\text{Front} \\
\text{Right} \\
\text{Back} \\
\text{Left}
\end{array}
\]

ii) 

\[
\begin{array}{c}
\text{Front} \\
\text{Right} \\
\text{Left}
\end{array}
\]

iii) 

\[
\begin{array}{c}
\text{Front} \\
\text{Right} \\
\text{Left}
\end{array}
\]
b) How many of each face would you need to make the prism? Write the number on the face.

i) 
ii) 
iii)

(Images of rectangular and triangular prisms)

2 ft 3 ft
5 ft
3 ft 5 ft
2 ft 3 ft

2 m
2 m
2 m
2.8 m
2.8 m
3 m

10 cm
10 cm
5 cm
5 cm

3 m
3 m

2.8 m
2 m

2 m

iii) The area of the pentagon is 43 cm². SA = ___ × 43 + ___ × 10 × 5 = ____________

(c) Use the answers from part b) to fill in the blanks. Find the surface area of the prism.

i) SA = ___ × 5 × 3 + ___ × 2 × 3 + ___ × 2 × 5 = ____________

ii) SA = ___ × (2 × 2 + 2) + ___ × 2 × 3 + ___ × 2.8 × 3 = ____________

iii) The area of the pentagon is 43 cm². SA = ___ × 43 + ___ × 10 × 5 = ____________

(d) Find the surface area using as few calculations as you can. Hint: Draw the faces.

i) SA = 6 × 2 × 2 = 24 cm²

ii) SA = 2 × 2.3 × 2.3 + 4 × 2.3 × 4.6 = 52.9 cm²

iii) SA = 2 × 8 × 2.5 + 2 × 10.5 × 2.5 + 2 × 8 × 10.5 = 260.5 ft²

iv) SA = 2 × 3 × 3 + 2 + 2 × 3 × 7 + 4.2 × 7 = 80.4 m²

Answers:
a) i) rectangular prism, ii) triangular prism, iii) cube
b) i) two of each rectangle; ii) two triangles, two 2 by 3 rectangles, one 2.8 by 3 rectangle; iii) two pentagons, five rectangles
c) i) 2, 2, 62 ft², ii) 2, 2, 1, 24.4 m², iii) 2, 5, 336 cm²
d) i) SA = 6 × 2 × 2 = 24 cm², ii) SA = 2 × 2.3 × 2.3 + 4 × 2.3 × 4.6 = 52.9 cm², iii) SA = 2 × 8 × 2.5 + 2 × 10.5 × 2.5 + 2 × 8 × 10.5 = 260.5 ft², iv) SA = 2 × 3 × 3 + 2 + 2 × 3 × 7 + 4.2 × 7 = 80.4 m²

(MP.8) 3. Find 0.63 – 0.54. Explain how you got your answer.
Answer: I found 0.63 – 0.54 = 0.09, 0.6363 – 0.5454 = 0.0909, and 0.636363 – 0.545454 = 0.090909. I realized that the pattern will continue because I am doing the same calculations at each place value. So 0.63 – 0.54 = 0.09.
Goals:
Students will find the surface area of rectangular prisms without drawing nets.

Prior Knowledge Required:
Is familiar with square units of measurement
Can find the area of rectangles
Can multiply whole numbers, fractions, and decimal numbers
Can substitute numbers for variables in a formula
Can identify congruent polygons

Vocabulary: congruent, face, prism, square units, surface area

Materials:
medium-sized cereal box

NOTE: Before class, use a permanent marker to label the faces of a medium-sized cereal box with the terms front, back, right side, left side, top, and bottom. You can use the cereal box from Lesson G7-26.

Identifying congruent faces on rectangular prisms. SAY: In the last lesson, we used nets to find the surface area of prisms. Today we will find ways to calculate the surface area without drawing nets, because that can be time consuming. Remind students that to find surface area, add together the areas of each face. Hold up the cereal box and SAY: Rectangular prisms have three pairs of congruent faces. ASK: How could this fact make calculating surface area simpler? (you don’t have to work out the area of all six faces because some areas are the same) Which face has the same area as the front? (the back) Which face has the same area as the top? (the bottom) Which face has the same area as the right side? (the left side) Point out that opposite sides are congruent.

(MP.2) Finding the area of hidden faces of a rectangular prism. Draw on the board:

![Diagram of a rectangular prism with areas labeled]

SAY: We are given the area of the three visible faces of this prism. But three faces are hidden. ASK: What is the area of the bottom? (12 in²) The back? (20 in²) The left side? (15 in²)
**Exercises:** The area of each visible face is given. What is the area of each hidden face?

a) ![Rectangle Prism](image1.png)

b) ![Cube](image2.png)

**Bonus:** This is a cube.

Back:
Bottom:
Left side:
Right side:

**Answers:** a) 24 cm², 48 cm², 32 cm²; b) 27 m², 12 m², 36 m²; c) 49 in²

(MP.6) **Using congruent faces to find the surface area of a rectangular prism.** Draw on the board:

![Rectangular Prism](image3.png)

ASK: What is the area of the front? (32 cm²) The top? (40 cm²) The right side? (20 cm²) Write each area on the face on the board. SAY: If we double the areas of the visible faces, then we will find the areas of the hidden faces, too. Write on the board:

Front + Back = 32 cm² × 2 = 64 cm²
Top + Bottom = 40 cm² × 2 = 80 cm²
Right + Left sides = 20 cm² × 2 = 40 cm²

ASK: What is the surface area of this prism? (64 + 80 + 40 = 184 cm²)

**Exercises:**

a) Draw the prism. Write the area of each visible face directly on the face.

i) ![Rectangle Prism](image4.png)

ii) ![Rectangle Prism](image5.png)

b) Double each area from part a) to find the total area of each pair of opposite faces. Then find the surface area.
Answers:

a) i) Front: 21 in²
   Top: 14 in²
   Left side: 6 in²
   ii) Front: 80 cm²
   Top: 72 cm²
   Right side: 90 cm²

b) i) Front + Back = 21 in² × 2 = 42 in²
   Top + Bottom = 14 in² × 2 = 28 in²
   Left + Right sides = 6 in² × 2 = 12 in²
   Surface area = 82 in²
   ii) Front + Back = 80 cm² × 2 = 160 cm²
   Top + Bottom = 72 cm² × 2 = 144 cm²
   Right + Left sides = 90 cm² × 2 = 180 cm²
   Surface area = 484 cm²

(MP.4) Developing a formula for the surface area of a rectangular prism. Review how to find the surface area of a rectangular prism. Write on the board:

\[ \text{Surface area} = 2 \times (\text{area of front}) + 2 \times (\text{area of top}) + 2 \times (\text{area of right side}) \]

SAY: Let's write a formula using length, width, and height. Draw on the board:

\[ \text{length} \quad \text{width} \quad \text{height} \]

Point to the prism and ASK: How do we calculate the area of the front? (length × height) On the board, write “2 × (length × height)” under “(area of front).” Repeat with area of top and area of right side, as shown below:

\[ \text{Surface area} = 2 \times (\text{area of front}) + 2 \times (\text{area of top}) + 2 \times (\text{area of right side}) \]
\[ = 2 \times (\text{length} \times \text{height}) + 2 \times (\text{length} \times \text{width}) + 2 \times (\text{width} \times \text{height}) \]

SAY: To simplify the formula, let's write length as \( \ell \), width as \( w \), and height as \( h \). Remind students that you don’t have to write the multiplication signs or brackets between the numbers and variables. Continue writing on the board:

\[ \text{SA} = 2\ell h + 2\ell w + 2wh \]

SAY: We can use this formula to find the surface area of a rectangular prism.

Using the formula to find the surface area of a rectangular prism. SAY: We have a rectangular prism with these dimensions. Write on the board:

\[ \ell = 4 \text{ cm} \quad w = 3 \text{ cm} \quad h = 6 \text{ cm} \]

SAY: To use the formula, substitute the dimensions for the variables. Have students tell you what to replace each variable by, and write their answers on the board, as shown below:

\[ \text{SA} = 2(4)(6) + 2(4)(3) + 2(3)(6) \]
\[ = 48 + 24 + 36 \]
\[ = 108 \text{ cm}^2 \]
(MP.1) Point out that each dimension is paired up once with each of the other two dimensions in this calculation: 4 is paired with 6 once, 4 is paired with 3 once, and 3 is paired with 6 once.

**Exercises:** Use the formula to find the surface area of the rectangular prism.

a) [Diagram of a rectangular prism with dimensions 3 in, 10 in, and 5 in.]

b) \( \ell = 6 \text{ m}, w = 4 \text{ m}, h = 1.5 \text{ m} \)

**Bonus:** \( \ell = \frac{14}{2} \text{ ft}, w = \frac{3}{4} \text{ ft}, h = \frac{2}{3} \text{ ft} \)

**Answers:**

a) \( SA = 2(3)(5) + 2(3)(10) + 2(10)(5) = 30 + 60 + 100 = 190 \text{ in}^2 \)
b) \( SA = 2(6)(1.5) + 2(6)(4) + 2(4)(1.5) = 18 + 48 + 12 = 78 \text{ m}^2 \)

**Bonus:** \( SA = 2(9/2)(2/3) + 2(9/2)(3/4) + 2(3/4)(2/3) = 6 + 6 \frac{3}{4} + 1 = 13 \frac{3}{4} \text{ ft}^2 \)

SAY: The formula tells you to find the sum of double the area of each visible face. You could also double the sum of the areas of each face. Write on the board:

\[
SA = 2 \times (\text{area of front} + \text{area of top} + \text{area of right side}) = 2(\ell h + \ell w + wh)
\]

Tell students that to find surface area, they can use either formula, or they can use the method where they list each pair of faces. Whichever method they choose, they must check that they have found the sum of six surfaces, because a rectangular prism has six faces.

(MP.3) **Exercises:**

1. Which of the following will give the surface area of a prism? Explain.
   a) \( \text{(area of top} + \text{area of left side} + \text{area of back)} \times 2 \)
   b) \( \ell \times w \times h \)
   c) \( \text{(area of bottom} + \text{area of right side} + \text{area of left side)} \times 2 \)
   d) \( 2 \times \text{area of left side} + 2 \times \text{area of bottom} + 2 \times \text{area of back} \)
   e) \( \text{area of front} + \text{area of right side} + \text{area of top} \)
   f) \( \text{(area of bottom} + \text{area of right side} + \text{area of front)} \times 2 \)
   g) \( 2 \times \text{area of front} + 2 \times \text{area of top} + 2 \times \text{area of back} \)

**Answer:** a), d), and f) because we need to choose one area from each pair of sides—top and bottom, left and right, front and back—and double the sum of the 3 areas

2. The areas of the front, top, and right faces of a prism add to 450 cm\(^2\). How can you find the surface area of the prism?

**Answer:** Double 450 cm\(^2\) to get 900 cm\(^2\).
(MP.4) Word problems practice.

Exercises:
1. a) Bob wants to wrap a gift in a rectangular box that measures 15 inches by 9.5 inches by 2 inches. What surface area needs to be covered by wrapping paper?
   b) Bob estimates that it will take an extra 10% of wrapping paper to be able to overlap the edges. How much paper does he need?
   Answers: a) 383 in², d) 10% of 383 = 38.3, 383 + 38.3 = 421.3 in²

2. a) Anne is making a jewelry box. She wants to use as little wood as possible. Which box should she make: one that has length 11 inches, width 6 inches, and height 3 inches, or one that has length 10 inches, width 5 inches, and height 4 inches?
   b) Anne is going to put felt on the sides and bottom of the box that she chose from part a) to protect the jewelry. How much felt does she need?
   c) Felt is sold in rectangular sheets that measure 9 inches by 11 inches. How many sheets of felt does Anne need? Draw out the faces and show how they could fit on the sheets of felt.
   Answers: a) The first box has $SA = 234 \text{ in}^2$ and the second box has $SA = 220 \text{ in}^2$, so the second box requires 14 in² less wood to make; b) 170 in² of felt; c) she needs 2 sheets that could be divided as follows:

Extensions
1. Edges $a$, $b$, and $c$ have lengths that are whole numbers. The surface area of each face is written directly on the face. What are the lengths for edges $a$, $b$, and $c$?
   Answer: $a = 6 \text{ m}$, $b = 3 \text{ m}$, $c = 2 \text{ m}$

2. Each box in the chart has a volume of 100 m³. Which boxes can be made if you only have 200 m² of cardboard?

<table>
<thead>
<tr>
<th>length (m)</th>
<th>width (m)</th>
<th>height (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Box 1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Box 2</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Box 3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Box 4</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Box 5</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Box 6</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Box 7</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
Answers: Box 1 has SA = 304 m², Box 2 has SA = 258 m², Box 3 has SA = 250 m², Box 4 has SA = 240 m², Box 5 has SA = 208 m², Box 6 has SA = 160 m², Box 7 has SA = 130 m². Only Box 6 and Box 7 can be made with 200 m² of cardboard.

(MP.6) 3. Make a poster that explains the steps you follow to find the surface area of a rectangular prism. Give examples to demonstrate your step-by-step method.

(MP.1, MP.4, MP.6) 4. Jane’s doctor recommended a daily limit of 30 mL of added sugar. Jane bought 5/2 cups of sugar and used 3/4 of it to bake a cake. How many pieces should she cut the cake into to make sure that each piece contains at most 30 mL of sugar? Show your answer as an inequality and as a complete sentence, and explain the steps you used to solve the problem. Use 1 cup = 240 mL.

Sample solutions:
• To find the amount of sugar Jane used, I calculated 5/2 × 3/4 = 15/8. So Jane used 15/8 cups of sugar in the cake. Since 1 cup is 240 mL, I calculated 15/8 × 240 = 450, so she used 450 mL of sugar in the cake. The amount of sugar in each piece is 450 mL ÷ x, where x is the number of pieces that she cuts the cake into. Since each piece needs to contain at most 30 mL of sugar, 450 ÷ x ≤ 30. I tried different numbers for x in a table to see what x has to be to make the inequality true. As x got larger, 450 ÷ x got smaller, which makes sense because dividing by a larger number gets a smaller quotient. Since 450 ÷ x = 30 for x = 15, then 450 ÷ x ≤ 30 for x ≥ 15. So Jane has to cut the cake into at least 15 pieces.
• I multiplied both sides of the inequality 450 ÷ x ≤ 30 by x to get 450 ≤ 30x or 30x ≥ 450. I know x is positive because it’s the number of pieces, so I don’t have to change the direction of the inequality sign. Then x ≥ 450 ÷ 30, so x ≥ 15. So Jane has to cut the cake into at least 15 pieces.
• I changed the inequality 450 ÷ x ≤ 30 to x ÷ 450 ≥ 1 ÷ 30. I can do that because you can think of 450/x and 30/1 as fractions, and if you can compare two positive fractions then you can compare their inverses (or reciprocals) by reversing the inequality. For example, since 2 < 3, then 1/2 > 1/3. Once I got that x/450 ≥ 1/30, I multiplied both sides by 450 to get x ≥ 450/30 = 15, so x ≥ 15. So Jane has to cut the cake into at least 15 pieces.

Redirecting students: Encourage students to use precise language (MP.6) in their explanations. Some examples of imprecise or incorrect language that students might use include incorrectly referring to the inequality as an equation, imprecisely saying that the inequality will “work” instead of “will be true,” incorrectly saying “less than” when they mean “at most” or “less than or equal to,” or avoiding math terms such as “inverse” or “reciprocal.”
G7-28  Surface Area of Polygonal Prisms and Pyramids

Pages 163–165

Standards: 7.G.B.6

Goals:
Students will find the surface area of polygonal prisms and pyramids.

Prior Knowledge Required:
Is familiar with square units of measurement
Can find the area of rectangles, triangles, parallelograms, and trapezoids
Can identify congruent polygons
Is familiar with right pyramids

Vocabulary: base, composite, congruent, edge, face, prism, pyramid, square units, surface area, vertex

Materials:
a rectangular prism, a triangular prism, a trapezoidal prism, and a square-based pyramid or BLM Nets of 3-D Shapes (1), (3), (6), and (9) (pp. R-68, R-70, R-73, R-76)
5 different colors of chalk

Review surface area of rectangular prisms. Hold up a rectangular prism and ASK: How do you find the surface area of this prism? (find the area of each face and add them; add the area of the front, the top, and one side, and double the sum; use the formula) Write on the board:

rectangular prism: \( SA = 2lh + 2lw + 2wh \) or \( SA = 2(lh + lw + wh) \)

(MP.1) Finding the surface area of polygonal prisms. Hold up a triangular prism and a trapezoidal prism and ASK: Why can’t we use the formula for a rectangular prism to find the surface area of these prisms? (they don’t have three pairs of congruent sides; they don’t have length, width, and height) To find the surface area of a polygonal prism, you just have to find the sum of the areas of all the surfaces.

Show students the triangular prism and ASK: How many faces does this shape have? (5) Draw on the board:

```
4 cm
\( \Delta \)
5 cm
1 cm
6 cm
10 cm

Area of base 1 =
Area of base 2 =
Area of bottom =
Area of right side =
Area of left side =
```
With colored chalk, lightly shade one base and ASK: What is the area of this base? (12 cm²) PROMPT: How do we find the area of a triangle with base 6 cm and height 4 cm? 
(6 × 4 ÷ 2 = 12 cm²) Write the calculation on the board in the same color chalk next to “Area of base 1.” Use a different color of chalk and lightly shade the other base. ASK: What is the area of this base? (12 cm²) Write this on the board next to “Area of base 2.” With a third color of chalk, lightly shade the bottom and ASK: What shape is this face? (rectangle) What is the area? (10 × 6 = 60 cm²) Write this on the board next to “Area of bottom” and continue for the last two faces, using two more colors of chalk. The board will look like this:

- Area of base 1 = 6 × 4 ÷ 2 = 12 cm²
- Area of base 2 = 12 cm²
- Area of bottom = 10 × 6 = 60 cm²
- Area of right side = 10 × 5 = 50 cm²
- Area of left side = 10 × 5 = 50 cm²

ASK: What is the surface area of the prism? (12 + 12 + 60 + 50 + 50 = 184 cm²) SAY: To find the area of a polygonal prism, just find the area of each face and add the areas together. Point out that it doesn’t matter which face you start with, just make sure you don’t miss any.

If students struggle to match the dimensions to the faces, write the dimensions of the face as part of the list, as shown below:

- Area of base 1 \((b = 6 \text{ cm}, h = 4 \text{ cm}) = \)
- Area of bottom \((ℓ = 10, w = 6) = \)

You could also simplify the descriptions to “Area 1,” “Area 2,” etc., and tell students that it doesn’t matter which area they find first.

**Exercises:** Find the surface area of the prism. Hint: Sketch the prism and shade each face as you find its area.

**a)**

- 9 in
- 12 in
- 15 in

**Answers:**

- Area of base 1 = 12 × 9 ÷ 2 = 54 in²
- Area of base 2 = 54 in²
- Area of bottom = 12 × 20 = 240 in²
- Area of right side = 15 × 20 = 300 in²
- Area of left side = 9 × 20 = 180 in²
- Surface area = 828 in²

**b)**

- 6 m
- 13 m
- 10 m
- 20 m

**Answers:**

- Area of base 1 = (13 + 5) × 6 ÷ 2 = 54 m²
- Area of base 2 = 54 m²
- Area of bottom = 13 × 20 = 260 m²
- Area of right side = 10 × 20 = 200 m²
- Area of top = 5 × 20 = 100 m²
- Area of left side = 6 × 20 = 120 m²
- Surface area = 788 m²

(MP.7) Point out that although there is not a formula for the surface area of a polygonal prism, there are still ways we can simplify the calculations. ASK: How can we find the surface area in
the least number of steps? (multiply the area of one base by 2 because there are always two bases, look for other congruent faces) Referring to the example on the board, point out which faces are congruent and how you would plan the calculation. Write on the board:

\[
SA = 2 \times \text{area of base} + 2 \times \text{area of right side} + \text{area of bottom}
\]

\[
= 2 \times 12 + 2 \times 50 + 60
\]

\[
= 184 \text{ cm}^2
\]

**Exercises:** Find the surface area using as few calculations as possible. All measurements are in centimeters.

a) ![Diagram](image)

![Diagram](image)

**Answers:**

a) \(SA = 2 \times \text{area of base} + 2 \times \text{area of right side} + \text{area of bottom} = 2 \times 48 + 2 \times 40 + 64 = 240 \text{ cm}^2\),

b) \(SA = 2 \times \text{area of base} + 2 \times \text{area of right side} + \text{area of top} + \text{area of bottom} = 2 \times 28 + 2 \times 100 + 80 + 200 = 536 \text{ cm}^2\)

**MP.1) Finding the surface area of pyramids.** Remind students that a pyramid has one base (a square, rectangle, triangle, etc.) and faces that are triangles. Hold up a square-based pyramid and ASK: How many faces does the pyramid have? (5) What shapes are the faces? (1 square, 4 triangles) Which faces are congruent? (the 4 triangles) So to find the surface area of the pyramid, what should we do? (find the area of the base and add 4 times the area of one face) Write on the board:

\[
SA = \text{area of base} + 4 \times \text{area of face}
\]

Draw on the board:

![Diagram](image)

ASK: What is the area of the base? \((12 \times 12 = 144 \text{ in}^2)\) What is the area of one face? \((12 \times 10 \div 2 = 60 \text{ in}^2)\) So what is the area of the four faces? \((4 \times 60 = 240 \text{ in}^2)\) Write on the board:

\[
SA = 144 + 4 \times 60
\]

\[
= 144 + 240
\]

\[
= 384 \text{ in}^2
\]

SAY: There is no formula for surface area that covers all pyramids. So you should look for congruent faces and make a plan, like we did here. Make sure you find the area of each face.
Exercises: Find the surface area of the pyramid.

a) Square base
   (10 cm × 10 cm)

b) Rectangle base
   (18 in × 10 in)

Bonus: Equilateral triangle base
   (sides 6 m, height 5.2 m)

Solutions:

a) \[ SA = 10 \times 10 + 4 \times 10 \times 11 ÷ 2 = 320 \text{ cm}^2 \]
b) \[ SA = 18 \times 10 + 2 \times 18 \times 13 ÷ 2 + 2 \times 10 \times 15 ÷ 2 = 564 \text{ in}^2 \]
c) \[ SA = 6 \times 5.2 ÷ 2 + 3 \times 6 \times 10 ÷ 2 = 105.6 \text{ m}^2 \]

(MP.2) Converting units of measurement to find surface area. Draw on the board:

 área de bases = 
 área de lados = 
 área de base = 

SAY: Let’s find the surface area in as few steps as possible. ASK: How many bases are there? (2) Are any faces congruent? (yes) How many congruent faces are there? (2) SAY: Since there are two bases, we can find the area of one base and double it, and then do the same for the sides. Have students tell you how to find each of the areas. Write the answers on the board, with units, as shown below:

\[
\begin{align*}
\text{Area of bases} &= 2 \times 4 \text{ ft} \times 18 \text{ in} ÷ 2 \\
\text{Area of sides} &= 2 \times 5 \text{ ft} \times 2.5 \text{ ft} \\
\text{Area of bottom} &= 4 \text{ ft} \times 5 \text{ ft}
\end{align*}
\]

ASK: Are all the units the same? (no) SAY: We must make the units the same before calculating the areas. ASK: How many measurements do we have to convert if we use inches? (3) If we use feet? (1.5 ft) SAY: Now we can do the calculations without units. Add the headings “Surface area” and “Calculations (in feet)” on the board, as shown below:

<table>
<thead>
<tr>
<th>Surface area:</th>
<th>Calculations (in feet):</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area of bases = (2 \times 4 \text{ ft} \times 18 \text{ in} ÷ 2)</td>
<td></td>
</tr>
</tbody>
</table>
Have a volunteer do the calculations in feet on the board, as shown below:

**Surface area:**
- Area of bases = \(2 \times 4 \text{ ft} \times 18 \text{ in} \div 2\)
- Area of sides = \(2 \times 5 \text{ ft} \times 2.5 \text{ ft}\)
- Area of bottom = \(4 \text{ ft} \times 5 \text{ ft}\)

**Calculations (in feet):**
- \(2 \times 4 \times 1.5 \div 2 = 6\)
- \(2 \times 5 \times 2.5 = 25\)
- \(4 \times 5 = 20\)

ASK: So what is the surface area? (51 ft\(^2\))

**Exercises:** Find the surface area of the prism. Hint: 12 in = 1 ft, 100 cm = 1 m

- a)
- b)
- c)

**Answers:**
- a) 34,700 cm\(^2\) or 3.47 m\(^2\),
- b) 600 in\(^2\) or 4 \(\frac{1}{6}\) ft\(^2\),
- c) 1,152 in\(^2\) or 8 ft\(^2\)

**MP.7 Finding the surface area of composite shapes.**

SAY: Now that you’re experts at finding the surface area of a prism, let’s find the surface area of a composite 3-D shape. Draw on the board:

SAY: There is no formula that tells you how to calculate the surface area of this shape. You just need to make a plan. There are a number of different ways to approach this problem. I’ll show you one. First, let’s divide the 3-D shape into two rectangular prisms and find the missing measurements. Show the separation and edge lengths on the board, as shown below:
SAY: Let’s find the surface areas of two parts as though they were separate prisms. Have students guide you through the calculations as you write on the board:

\[
\begin{align*}
\text{SA of top prism: } & 2(5)(8) + 2(4)(8) + 2(5)(4) = 184 \text{ in}^2 \\
\text{SA of bottom prism: } & 2(12)(6) + 2(8)(6) + 2(8)(12) = 432 \text{ in}^2
\end{align*}
\]

SAY: We need to subtract the overlapping area. ASK: What is the area that overlaps? PROMPT: Shade the 5 by 8 rectangle. \((5 \times 8 = 40 \text{ in}^2)\) Point out that this area should not be counted for the top or for the bottom, so we need to subtract it twice from the sum of the surface areas of the top and bottom. Write on the board:

\[
\text{SA} = \text{SA of top prism} + \text{SA of bottom prism} - 2 \times \text{(overlapping area)} \\
= 184 + 432 - 2(40) \\
= 536 \text{ in}^2
\]

**Exercises:** Make a plan to find the surface area of the composite shape. Then find the surface area. All measurements are in inches.

a) Rectangular prism: \(\ell = 10, w = 5, h = 6\)

b) Rectangular prism has \(\text{SA} = 280 \text{ in}^2\), cube has \(\text{SA} = 24 \text{ in}^2\), overlapping area is \(2 \times 4 \text{ in}^2\), Total \(\text{SA} = 280 + 24 - 8 = 296 \text{ in}^2\)

c) Trapezoidal prism has \(\text{SA} = 672 \text{ in}^2\), cube has \(\text{SA} = 54 \text{ in}^2\), overlapping area is \(2 \times 9 \text{ in}^2\), Total \(\text{SA} = 672 + 54 - 18 = 708 \text{ in}^2\)

**(MP.2, MP.4) Word problems practice.**

**Exercises:** Two tents are on sale. They are both two-person tents.

a) How much floor space would each person have in each tent?

b) How much material is needed to make each tent? Remember: There is material on the bottom of each tent.

c) If the company charges $157 for the material to make Tent B, how much should the company charge for the material to make Tent A?
Solutions:
a) Tent A: \( 4 \text{ m} \times 3.5 \text{ m} = 14 \text{ m}^2 \) or \( 7 \text{ m}^2 \) per person, Tent B: \( 2 \text{ m} \times 4 \text{ m} = 8 \text{ m}^2 \) or \( 4 \text{ m}^2 \) per person
b) Tent A: \( SA = 2 \times 4 \times 1.5 + 2 \times 3.5 \times 2.5 + 4 \times 3.5 = 37.5 \text{ m}^2 \),
Tent B: \( SA = 2 \times (2 \times 0.5 + 2 \times 1.5 + 2) + 2 \times 4 \times 0.5 + 2 \times 4 \times 1.8 + 2 \times 4 = 31.4 \text{ m}^2 \)
c) \( $157 \div 31.4 \text{ m}^2 = $5/\text{m}^2 \), so \( $5/\text{m}^2 \times 37.5 \text{ m}^2 = $187.50 \)

Extensions
(MP.6, MP.8) 1. When you make a net of a prism, you can join the non-base faces to make a large rectangle.
a) For each shape, what sides are the bases and what are the dimensions of the rectangle? Show how you did each calculation.

b) Fill in the blanks:
   One dimension of the rectangle is always the ______ of the prism. The other dimension of the rectangle is always the ______ of the base.
   Explain to a partner how you know.
c) Create a formula for the surface area of a prism, using your answer to part b). Explain your formula to a partner using math words.
d) Use your formula from part c) to find the surface area of each shape from part a).

Answers:
a) A: I used the two sides as bases so the rectangle is \( 2 + 3 + 2 + 3 = 10 \text{ in} \) by \( 5 \text{ in} \); B: the two triangles are the bases and the rectangle is \( 15 + 12 + 9 = 36 \text{ m} \) by \( 20 \text{ m} \); C: the two trapezoids are the bases and the rectangle is \( 1.25 + 1.5 + 1.25 + 3 = 7 \text{ ft} \) by \( 1 \text{ ft} \)
b) height, perimeter; explanation: I’m always adding the side lengths of the base to get one of the dimensions and that is the perimeter of the base. The other dimension is always the height of the prism.
c) Let \( SA \) be the surface area, \( B \) be the area of the base, \( P \) the perimeter of the base, and \( h \) the height of the prism. The surface area of a prism with given base and height is \( 2B + Ph \).
   Explanation: I added the areas of the two bases to the area of the rectangle.
d) A: \( 2 \times 6 + 10 \times 5 = 62 \text{ in}^2 \), B: \( 2 \times 54 + 36 \times 20 = 828 \text{ m}^2 \), C: \( 2 \times 4.5 + 7 \times 1 = 16 \text{ ft}^2 \)

Redirecting students: If students struggle with part a), encourage them to label side lengths that they know are equal to other given side lengths. For example, for the prism in B, the 15 m side is equal to the unlabeled side of the front triangle. ASK: When you unfold the triangle to make the rectangle, which sides go together to make one side of the rectangle? (all three sides of the triangle). In part c), if students say, for example, that \( B \) is the area, ASK: \( B \) is the area of what? Encourage students to say more precisely how they are defining the variables.
2. A cylinder has a net like this:

![Diagram of a cylinder net]

The area of the base is \(\pi r^2\) and the perimeter of a circle is the circumference. So the formula to find the surface area of a cylinder is \(SA = 2B + Ch\) or \(SA = 2\pi r^2 + 2\pi rh\). Find the surface area of a cylinder with radius 4 cm and height 8 cm. Use \(\pi = 3.14\) and write your answers to two decimal places.

**Solution:** \(B \approx 50.24\) cm\(^2\), \(C \approx 25.12\) cm, \(h = 8\) cm, \(SA = 2 \times 50.24 + 25.12 \times 8 \approx 301.44\) cm\(^2\)

**(MP.2) 3. a)** Write an algebraic expression for the volume and surface area of the prism.

i) 

![Diagram of a rectangular prism]

**ii)** 

![Diagram of a triangular prism]

b) Use the algebraic expressions from part a) to find the volume and surface area for \(x = 5\) cm.

c) Use the algebraic expressions from part a) to find the volume and surface area for \(x = \frac{1}{2}\) in.

d) Solve for \(x\) from part a) if:

i) the volume of the triangular prism is 288 ft\(^3\).

ii) the surface area of the triangular prism is 352 m\(^2\).

**Answers:**

a) i) \(V = 12x^3\), \(SA = 38x^2\); ii) \(V = 96x\), \(SA = 64x + 96\)

b) i) \(V = 1,500\) cm\(^3\), \(SA = 950\) cm\(^2\); ii) \(V = 480\) cm\(^2\), \(SA = 416\) cm\(^2\)

c) i) \(V = 1.5\) in\(^3\), \(SA = 9.5\) in\(^2\); ii) \(V = 48\) in\(^2\), \(SA = 128\) in\(^2\)

d) i) \(x = 3\) ft, ii) \(x = 4\) m
**G7-29 Volume and Surface Area**

Pages 166–168

Standards: 7.G.B.6, 7.EE.B.4a

Goals:
Students will solve problems involving volume and surface area of polygonal prisms.

Prior Knowledge Required:
Can find the volume and surface area of polygonal prisms
Is familiar with square and cubic units of measurement

Vocabulary: congruent, cube, dimension, prism, surface area, T-table, volume

Review volume and surface area. ASK: What is volume? (the amount of space inside a 3-D object) How do we calculate volume? (area of base × height) When would we calculate volume in real life? (sample answer: filling a pool with water) What is surface area? (the sum of the areas of all the surfaces of a 3-D object) How do we calculate surface area? (add the areas of each face; use nets to keep track of faces) When would we calculate surface area in real life? (sample answer: painting a room)

(MP.2) Exercises: Would you calculate volume or surface area to determine the quantity?
- a) the amount of soil in a garden box
- b) the amount of paint needed to paint the sides of a garden box
- c) the amount of cardboard needed to make a cereal box
- d) the amount of cereal that can be put in a box
- e) the amount of drywall needed for the walls of a room
- f) the amount of air in a room

Answers: a) volume, b) surface area, c) surface area, d) volume, e) surface area, f) volume

Finding volume and surface area for rectangular prisms. Remind students that rectangular prisms have three pairs of congruent faces, which allows us to simplify the calculations for volume and surface area. ASK: How do we calculate the volume of a rectangular prism? (length × width × height) How do we calculate the surface area of a rectangular prism? (2 × area of top + 2 × area of front + 2 × area of right side, or $2lh + 2lw + 2wh$) Write the formulas on the board. Remind students that writing the units at every step of the calculation helps to avoid multiplying dimensions in different units.

(MP.2, MP.6) Exercises: Find the volume and surface area of the rectangular prism.
- a) length = 0.5 m, width = 30 cm, height = 1.2 m
- b) $l = \frac{3}{4}$ ft, $w = 18$ in, $h = 4$ in

Answers: a) $V = 180,000$ cm³ or $0.18$ m³, $SA = 22,200$ cm² or $2.22$ m²; b) $V = 1,512$ in³ or $7/8$ ft³, $SA = 1,068$ in² or $7 5/12$ ft²
Bonus:
c) Do part a) again by converting the units differently. To check your answer, use $1 \text{ m}^3 = 1,000,000 \text{ cm}^3$ and $1 \text{ m}^2 = 10,000 \text{ cm}^2$.
d) Do part b) again by converting the units differently. To check your answer, use $1 \text{ ft}^3 = 1,728 \text{ in}^3$ and $1 \text{ ft}^2 = 144 \text{ in}^2$.

(MP.7) Finding volume and surface area for cubes. Remind students that a cube is a rectangular prism with all edges of equal length. ASK: How does that make our calculation for volume easier? (calculate length × length × length) So if a cube has edge a, what is the volume? ($a \times a \times a = a^3$)

Exercises: Find the volume of the cube with edge length of…
a) 4 cm  b) 1.1 m  c) $1\frac{1}{3}$ ft
Answers: a) 64 cm$^3$, b) 1.331 m$^3$, c) $64/27$ ft$^3$ or $2 10/27$ ft$^3$

ASK: How can we find the surface area of a cube? (find the area of one face and multiply by 6) SAY: If the edge is a, then what is the area of one face? ($a \times a = a^2$) The area of six faces? ($6 \times a \times a = 6a^2$)

Exercises: Find the surface area of the cube with edge length of…
a) 2 ft  b) 3.5 in  c) $2\frac{1}{2}$ m
Answers: a) 24 ft$^2$, b) 73.5 in$^2$, c) 37 1/2 m$^2$

Finding volume and surface area for polygonal prisms. Draw on the board:

[Diagram of a triangular prism]

ASK: How do we find the volume of a polygonal prism? (area of base × height) What measurements do we need to know to be able to calculate the volume of the triangular prism on the board? (base and height of the triangle, and height of the prism) How do we find the surface area of a polygonal prism? (add the area of each face) What measurements do we need to know to calculate the surface area of the triangular prism on the board? (the three sides of the triangle, the height of the triangle, and the height of the prism)

Exercises: Find the volume and surface area of the prism on the board.
Answers: $V = 312 \text{ cm}^3$, $SA = 360 \text{ cm}^2$
(MP.7) Finding volume and surface area for composite shapes. Draw on the board:

SAY: This is a small dollhouse. I want to know how much space is inside it for the dolls, and how much wood I need to make it. First, let’s find the volume. We will use the formula \( V = B \times h \).

ASK: What do you notice about the units? (they’re in feet and inches) Which units should we use? (inches, so there won’t be fractions or decimals) What two shapes make the base? (a rectangle and a triangle) Have a volunteer convert the dimensions and divide the shape, as shown below:

\[
B = 18 \text{ in} \times 24 \text{ in} + 24 \text{ in} \times 5 \text{ in} \div 2 = 432 \text{ in}^2 + 60 \text{ in}^2 = 492 \text{ in}^2
\]

Have another volunteer tell you how to find the volume, and write it on the board:

\[
V = B \times h = 492 \text{ in}^2 \times 21 \text{ in} = 10,332 \text{ in}^3
\]

SAY: To find how much wood I need to make the dollhouse, we calculate surface area. ASK: What calculation from finding volume will be helpful? (the area of the base) How many other faces are there? (5) Write on the board:

\[
\begin{align*}
\text{Area of bases} & \quad \text{Area 1} = 492 \text{ in}^2 \\
& \quad \text{Area 2} = 492 \text{ in}^2
\end{align*}
\]

\[
\begin{align*}
\text{Area of faces} & \quad \text{Area 3 (right top)} = \\
& \quad \text{Area 4 (left top)} = \\
& \quad \text{Area 5 (right side)} = \\
& \quad \text{Area 6 (left side)} = \\
& \quad \text{Area 7 (bottom)} =
\end{align*}
\]

ASK: What is the area of the two top faces? (21 in \(\times\) 13 in = 273 in\(^2\)) Write that amount for areas 3 and 4 on the board. ASK: What is the area of the two side faces? (21 in \(\times\) 18 in = 378 in\(^2\))
Write that amount for areas 5 and 6 on the board. ASK: What is the area of the bottom? 
\((21 \text{ in} \times 24 \text{ in} = 504 \text{ in}^2)\) Write that amount in area 7 on the board. ASK: What is the surface area of the prism? \((2,790 \text{ in}^2)\) Point out that you could simplify the calculations by pairing up the congruent faces and doubling the area of one of the faces.

**Exercises:** Find the volume and surface area of the shape.

![Volume and Surface Area Diagram]

**Bonus:** Find the volume and surface area of a 4 m cube with a 2 m square hole removed from the middle.

![Bonus Problem Diagram]

**Answers:** \(V = 476 \text{ in}^3, \ SA = 360 \text{ in}^2;\) Bonus: \(V = 48 \text{ m}^3, \ SA = 120 \text{ m}^2\)

**MP.1** Using an organized search to find dimensions of prisms with a given volume.

Write on the board:

Find all possible whole-number dimensions of a rectangle with area 30 cm\(^2\).

SAY: This problem has a lot of possible solutions, so we need an organized way to search for solutions. ASK: What are some ways to organize your solutions? (sample answers: use a T-table; list all factor pairs of 30; draw rectangles) ASK: Which way do you think will be the least efficient? (drawing rectangles because that takes time) SAY: Let’s make a T-table. Draw on the board:

<table>
<thead>
<tr>
<th>length (cm)</th>
<th>width (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

SAY: A very organized way would be to start with length 1 cm and find the width that makes area 30 cm\(^2\). ASK: What is the width? (30 cm) Write those dimensions in the table and continue for length 2 cm (width 15 cm), length 3 cm (width 10 cm), and length 5 cm (width 6 cm). ASK: How do you know we can stop here? (the next length would be 6 cm, but that would give us width 5 cm and that is the same rectangle as in the row above) SAY: There
are four rectangles with whole-number dimensions that can have area 30 cm². Point out that if we accepted decimal numbers for the dimensions, that there would be too many possibilities to list!

Write on the board:

Find all possible whole-number dimensions of a rectangular prism with volume 30 cm³.

(MP.3) ASK: Compared to the previous problem, how is this problem the same and how is it different? (we are still looking for numbers that multiply to 30, but this time there are 3 numbers involved) Will drawing prisms be a convenient strategy? (no) Why not? (we do not draw prisms accurately the way we do rectangles, and it will take too much time to draw all possible prisms)

SAY: Let's make a T-table again. Draw on the board:

<table>
<thead>
<tr>
<th>length</th>
<th>width</th>
<th>height</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

SAY: When we have three values that can change, it makes sense to solve a simpler problem first. In this case, we can set the first value, look at what the other two values can be in an organized way, then modify the first value, and repeat. So we will start with length 1 cm for the prism, and see what width and height the prism can have. ASK: What do we know about the area of the base of a prism with length 1 cm and volume 30 cm³? (B = 30 cm²) Have students list all possible widths and heights for a prism with length 1 cm and area of base 30 cm². Point out that this was a problem that they solved before—finding all rectangles that have area 30 cm². Have students tell you how to fill out the T-table, as shown below:

<table>
<thead>
<tr>
<th>length</th>
<th>width</th>
<th>height</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 cm</td>
<td>1 cm</td>
<td>30 cm</td>
</tr>
<tr>
<td>1 cm</td>
<td>2 cm</td>
<td>15 cm</td>
</tr>
<tr>
<td>1 cm</td>
<td>3 cm</td>
<td>10 cm</td>
</tr>
<tr>
<td>1 cm</td>
<td>5 cm</td>
<td>6 cm</td>
</tr>
</tbody>
</table>

ASK: Do all prisms with volume 30 cm³ have length 1 cm? (no) Tell students that they will now look for all prisms of length 2 cm. ASK: What will the area of the base be? (15 cm²) ASK: Does it make sense to include prisms with width 1 cm? (no) Why not? (because any prism with width 1 cm was already listed as a prism with length 1 cm) Have students find the prism with length 2 cm and add it to the chart, as shown below:

<table>
<thead>
<tr>
<th>length</th>
<th>width</th>
<th>height</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 cm</td>
<td>1 cm</td>
<td>30 cm</td>
</tr>
<tr>
<td>1 cm</td>
<td>2 cm</td>
<td>15 cm</td>
</tr>
<tr>
<td>1 cm</td>
<td>3 cm</td>
<td>10 cm</td>
</tr>
<tr>
<td>1 cm</td>
<td>5 cm</td>
<td>6 cm</td>
</tr>
<tr>
<td>2 cm</td>
<td>3 cm</td>
<td>5 cm</td>
</tr>
</tbody>
</table>
ASK: What length should we use next? (3 cm) What will the area of the base be? (10 cm²) What dimensions make an area of 10 cm²? (1 cm by 10 cm, 2 cm by 5 cm) Which rows in the chart already list these dimensions? (the third and fifth rows) SAY: The next lengths to use, 5 cm, 6 cm, and so on, have also been listed already. We can stop the search here and see that there are five different rectangular prisms with whole-number dimensions that have volume 30 cm².

Exercises:
1. a) Add a column to the T-table on the board and find the surface area of each prism.
   b) Which prism with volume 30 cm³ has the least surface area?

   **Answers:**

<table>
<thead>
<tr>
<th>length</th>
<th>width</th>
<th>height</th>
<th>surface area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 cm</td>
<td>1 cm</td>
<td>30 cm</td>
<td>122 cm²</td>
</tr>
<tr>
<td>1 cm</td>
<td>2 cm</td>
<td>15 cm</td>
<td>94 cm²</td>
</tr>
<tr>
<td>1 cm</td>
<td>3 cm</td>
<td>10 cm</td>
<td>86 cm²</td>
</tr>
<tr>
<td>1 cm</td>
<td>5 cm</td>
<td>6 cm</td>
<td>82 cm²</td>
</tr>
<tr>
<td>2 cm</td>
<td>3 cm</td>
<td>5 cm</td>
<td>62 cm²</td>
</tr>
</tbody>
</table>

   b) The prism with dimensions 2 cm, 3 cm, and 5 cm has the least surface area.

2. a) Use a T-table to find all rectangular prisms with volume 40 cm³.
   b) Which prism with volume 40 cm³ has the least surface area?

   **Answers:**

<table>
<thead>
<tr>
<th>length</th>
<th>width</th>
<th>height</th>
<th>surface area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 cm</td>
<td>1 cm</td>
<td>40 cm</td>
<td>162 cm²</td>
</tr>
<tr>
<td>1 cm</td>
<td>2 cm</td>
<td>20 cm</td>
<td>124 cm²</td>
</tr>
<tr>
<td>1 cm</td>
<td>4 cm</td>
<td>10 cm</td>
<td>108 cm²</td>
</tr>
<tr>
<td>1 cm</td>
<td>5 cm</td>
<td>8 cm</td>
<td>106 cm²</td>
</tr>
<tr>
<td>2 cm</td>
<td>2 cm</td>
<td>10 cm</td>
<td>88 cm²</td>
</tr>
<tr>
<td>2 cm</td>
<td>4 cm</td>
<td>5 cm</td>
<td>76 cm²</td>
</tr>
</tbody>
</table>

   b) The prism with dimensions 2 cm by 4 cm by 5 cm has the least surface area.

**Bonus:** Find the rectangular prism with volume 64 cm³ that has the least surface area.

**Answers:**

<table>
<thead>
<tr>
<th>length</th>
<th>width</th>
<th>height</th>
<th>surface area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 cm</td>
<td>1 cm</td>
<td>64 cm</td>
<td>258 cm²</td>
</tr>
<tr>
<td>1 cm</td>
<td>2 cm</td>
<td>32 cm</td>
<td>196 cm²</td>
</tr>
<tr>
<td>1 cm</td>
<td>4 cm</td>
<td>16 cm</td>
<td>168 cm²</td>
</tr>
<tr>
<td>1 cm</td>
<td>8 cm</td>
<td>8 cm</td>
<td>160 cm²</td>
</tr>
<tr>
<td>2 cm</td>
<td>2 cm</td>
<td>16 cm</td>
<td>136 cm²</td>
</tr>
<tr>
<td>2 cm</td>
<td>4 cm</td>
<td>8 cm</td>
<td>112 cm²</td>
</tr>
<tr>
<td>4 cm</td>
<td>4 cm</td>
<td>4 cm</td>
<td>96 cm²</td>
</tr>
</tbody>
</table>

The 4 cm cube has the least surface area.

SAY: Look at your answers for the previous exercises. ASK: For a fixed volume, what is the shape of the prism that gives you the least surface area? (when the dimensions are similar, such as in a cube) What is the shape of the prism that gives you the greatest surface area? (when there are 2 small dimensions and one really big dimension, such as in long and skinny prisms)
Using algebra to solve volume and surface area problems. Draw on the board:

![Diagram of a prism with dimensions 5 in, 4 in, and x in]

SAY: The volume of this prism is 60 in³. We can use an equation to find the unknown dimension. ASK: What equation can you write? \((5 \times 4 \times x = 60)\) Have volunteers explain step-by-step how to solve the equation. Write each step on the board, as shown below:

\[
\begin{align*}
5(4)(x) & = 60 \\
20x & = 60 \\
x & = \frac{60}{20} \\
x & = 3 \text{ in}
\end{align*}
\]

SAY: So the height must be 3 inches for the volume to be 60 in³. Point out that solving an equation is especially helpful when the numbers are not easy to calculate mentally.

Draw on the board:

![Diagram of a prism with dimensions 6 cm, 3 cm, and x cm]

SAY: The surface area of this prism is 72 cm². ASK: What formula do we use to find the surface area of a prism? \((2\ell h + 2\ell w + 2wh)\) Write on the board:

\[
\begin{align*}
\text{SA} & = 72 \text{ cm}^2 \\
2\ell h + 2\ell w + 2wh & = 72
\end{align*}
\]

ASK: What values can we substitute in the formula? \((\ell = 6 \text{ and } h = 3)\) Have a volunteer substitute the values and solve the equation, as shown below:

\[
\begin{align*}
2(6)(3) + 2(6)(x) + 2(x)(3) & = 72 \\
36 + 12x + 6x & = 72 \\
18x & = 72 - 36 \\
18x & = 36 \\
x & = \frac{36}{18} \\
x & = 2 \text{ cm}
\end{align*}
\]

SAY: So the width is 2 cm.
Exercises: Write and solve an equation to find the unknown dimension.

a) \( V = 84 \text{ in}^3 \)

\[
x \text{ in}
\]
\[
7 \text{ in}
\]
\[
4 \text{ in}
\]

b) \( SA = 152 \text{ cm}^2 \)

\[
x \text{ cm}
\]
\[
4 \text{ cm}
\]
\[
6 \text{ cm}
\]

Bonus: A rectangular prism has volume \( 1 \text{ cm}^3 \), width \( \frac{1}{2} \text{ cm} \), and height \( \frac{3}{4} \text{ cm} \). What is its length?

Answers: a) \( x = 3 \text{ in} \), b) \( x = 8 \text{ cm} \), Bonus: \( l = 2\frac{2}{3} \text{ cm} \)

(MP.4) Word problems practice.

Exercises:

1. A rectangular box without a lid has volume 40,000 cm\(^3\). The box is 40 cm wide and 50 cm long.
   a) What is the height of the box?
   b) What is the surface area of the box? Remember, there is no lid.
   c) The material for the box costs $12.35 per m\(^2\). How much will the material for the box cost?

   Answers: a) 20 cm, b) \( 40 \text{ cm} \times 50 \text{ cm} + 2 \times (40 \text{ cm} \times 20 \text{ cm} + 50 \text{ cm} \times 20 \text{ cm}) = 5,600 \text{ cm}^2 \)
   = 0.56 m\(^2\), c) 0.56 \times $12.35 \approx $6.92

2. You are designing a cereal box for a cereal company. The box needs to have a volume of 2,000 cm\(^3\). There are many possible boxes you could make with this volume.
   a) Verify that the three sets of measurements (in centimeters) below produce boxes with volume 2,000 cm\(^3\).
   
   \( 1 \times 1 \times 2,000 \)
   \( 2 \times 25 \times 40 \)
   \( 5 \times 25 \times 16 \)
   
   b) Calculate the surface area of each box in part a).
   c) If the material to make the box costs 25¢ per cm\(^2\), which box would you recommend?
   d) Find three more boxes with the same volume and calculate the surface area of each. Now which box would you recommend?
   e) The cereal company wants the front of the box to be at least 20 cm wide and 20 cm high, and the depth of the box to be at least 4 cm. Find two boxes satisfying these conditions that each have a volume of 2,000 cm\(^3\). Which box would you recommend?

   Answers: a) yes, all three sets equal 2,000 cm\(^3\); b) 8,002 cm\(^2\), 2,260 cm\(^2\), 1,210 cm\(^2\);
   c) 5 \times 25 \times 16; d) answers will vary; e) A: 20 \times 20 \times 5 and B: 20 \times 25 \times 4,
   Surface area of A = 1,200 cm\(^2\), Surface area of B = 1,360 cm\(^2\), so box A is better because it uses less material and would be cheaper to make

Extensions

(MP.6) 1. a) Find the volume and surface area of the rectangular prism.
   
   i) length = 8 cm, width = 8 cm, height = 4 cm
   ii) \( \ell = 6 \text{ in}, w = 6 \text{ in}, h = 6 \text{ in} \)

   b) In part a), does the volume equal the surface area? Explain.

   Answers: a) i) \( V = 256 \text{ cm}^3, SA = 256 \text{ cm}^2 \); ii) \( V = 216 \text{ in}^3, SA = 216 \text{ in}^2 \);
   b) no, the units are different

(MP.1) 2. Sketch two prisms with different bases that have volume 300 in\(^3\).

Sample answers: a rectangular prism with dimensions 10 in \( \times \) 5 in \( \times \) 6 in, a triangular prism with height 6 inches and a triangular base with base 10 inches and height 10 inches
3. a) Two rectangular prisms each have a volume of 8 in³. Prism A has dimensions 1 inch by 1 inch by 8 inches. Prism B has dimensions 1 inch by 2 inches by 4 inches. Predict which prism will have a smaller surface area.
b) Calculate the surface area of the two prisms. Was your prediction correct?
c) Find the dimensions of the prism with a volume of 8 in³ that has the least surface area.
d) Why might it be useful to know which prism has the least surface area for a given volume?

**Answers:**
- a) Prism A is longer and skinnier than Prism B, so I think Prism B will have a lesser surface area;
- b) Prism A: \( SA = 34 \text{ in}^2 \), Prism B: \( SA = 28 \text{ in}^2 \), my prediction was correct;
- c) a prism with dimensions 2 inches by 2 inches by 2 inches will have the least surface area (24 in²);
- d) this is useful because sometimes you want a container with the maximum area that uses the minimum amount of material

(MP.3, MP.5) 4. a) Draw all non-congruent triangles that have a right angle and two 4 cm sides. Explain how you know you found all of them. Use any tools you think will help.
b) In pairs, explain your answers to part a). Do you agree with each other? Discuss why or why not.

**Selected sample answers:**
- I used a compass and a set square to draw the triangle. I drew the right angle between the two 4 cm sides. I tried drawing only one of the 4 cm sides adjacent to the right angle, but when I measured the side opposite the right angle it was always longer than 4 cm, no matter how short I made the third side. I used a compass to figure out why. The circle in the picture below is centered at \( A \) and all points on it are the same distance from \( A \) as \( B \). Since the line joining \( AC \) intersects the circle, \( AC \) is longer than \( AB \).

![Triangle Diagram]

- I drew the triangle on centimeter grid paper because grid paper already has right angles on it. I measured the side opposite the right angle and it was always more than 4 cm. To understand why, I thought of playing football. If I’m at the 10-yard line and two teammates are at the 30-yard line and one of them is straight ahead, then that last person is closer to me. I know because I don’t have to throw the ball as hard to the player who’s straight ahead of me:

My teammates at the 30 yard line

Me at the 10 yard line

(MP.2) Whole-class follow-up: Take up both solutions above. Point out that the tricky part in the second solution is recognizing that you can join all the points to make a triangle. Encourage students to think of other real-world situations where they can use a hidden right triangle to show that the side opposite of a right angle is the longest side of the triangle, and then to explain their idea to a partner.
G7-30 Cross Sections of 3-D Shapes (Introduction)

Pages 169–170

Standards: 7.G.A.3

Goals:
Students will understand that a cross section of a rectangular prism or pyramid can be different shapes depending on how the cut is made, and will describe the cross sections by the number of sides.
Students will explore cuts made parallel to the base of prisms and pyramids.

Prior Knowledge Required:
Can differentiate between prisms and pyramids
Can name prisms and pyramids using the shape of their base
Understands the concepts of congruence and similarity

Vocabulary: congruent, cross section, edge, face, intersect, parallel, plane, rectangular prism, rectangular pyramid, side, similar, vertex

Materials:
prisms with a variety of bases or BLM Nets of 3-D Shapes (pp. R-68–78)
a kitchen knife
a square sandwich
BLM Nets for Cuts of Cubes and Rectangular Pyramids (pp. R-84–113), photocopied on cardstock (one net for each student plus an extra copy of each page in case of error)
tape
scissors
paper for tracing
index cards
small sticky notes or page flags
sheet of bristol board or chart paper
clean milk carton with the top cut off
colored water

Introduce planes. Point to a flat surface, such as the top of your desk. SAY: A plane is a flat two-dimensional surface, like this, except it goes on forever in every direction. Have students brainstorm objects in the classroom that would be planes if you were to extend them forever in every direction. (a sheet of paper, walls, a shelf of a bookcase, a side of a bookcase, any side of a paperback book) Then name the following objects and have students signal thumbs up if the object is part of a plane, or thumbs down if it is not: a computer mouse (no), a cup (no), a playing card (yes), a cube (no), a face of a cube (yes), a smooth skating rink (yes), the top of a smooth ice cube (yes), the glass surface of a window (yes), a smooth road (yes), a road with a speed bump (no).
Introduce cross sections. Show students a kitchen knife. Point to the blade and ASK: Is this part of a plane? (yes) How do you know? (it is flat) Show students a sandwich and ASK: Is this a prism? (yes) What kind of prism? (approximately square) Tell students that you are going to cut through the sandwich “prism” with the knife “plane.” Cut through the sandwich to make two triangular “prisms.” Show students how the knife went through the sandwich. SAY: This area shows the intersection between a vertical plane and a prism. ASK: What shape is it? (a rectangle) Draw on the board:

SAY: On the two-dimensional board, the shape of the cut doesn’t look like a rectangle. But in three-dimensional space, you see that the top of the sandwich is horizontal and the side of the cut is vertical, so they are at right angles to each other. So, all four angles are right angles, even though it doesn’t look that way on the board. Tell students that a cross section of a 3-D shape is the intersection of the 3-D shape with a plane.

Uses of cross sections in real life. SAY: A cross section is a view of the inside of something. Cross sections of a rectangular prism or rectangular pyramid can be many different shapes. Discuss how cross sections are used in real life using the examples below:
- in geography to see the land beneath the surface
- by construction workers to see more detail of a building
- by chefs to make food attractive
- by scientists to see inside a brain
- by biologists to see the ring pattern in a tree

Activities 1–2
1. BLM Nets for Cuts of Cubes and Rectangular Pyramids has 30 nets that make 15 shapes: eight prisms and seven pyramids. Give each student one net from the BLM photocopied onto cardstock, scissors, and tape. Explain to students that their net is either part of a cube or part of a rectangular pyramid (a pyramid with a rectangular base) that has been sliced by a plane. Explain how to make the shapes using the following steps:

Step 1: The shape students will make is shown on the lower right corner of the BLM. Students cut out the shape and set it aside.

Step 2: Students trace the shaded face onto another sheet of paper, which will become the cross section of their 3-D shape. They label each side measurement and set the paper aside.

Step 3: Each student puts together a net for their part of a cube or pyramid. Students who finish early can help their peers.
Tips for making the net:
- fold along all the edges first
- use small pieces of tape
- line up the edges carefully before taping
- to tape pieces together when they are less flexible, place several pieces of tape onto the faces at a time, rather than taping one piece, then putting the next piece of tape on
- be sure the shaded face is on the outside

Step 4: Students move around the class and try to match the cross sections—the shaded faces—with other students’ cross sections, to find the part that completes their 3-D shape. (the pairs are: A4, B11, C10, D14, E2, F6, G1, H8, I5, J7, K15, L9, M12, N13, O3)

Tips for finding the matching cross section:
- Students who made pyramids go to one side of the room and students who made prisms go to the other side.
- Students ask each other mathematical questions such as: How many sides does your cross section have? How many pairs of parallel sides? How many right angles?
- Match the pictures from the lower-right corner of the BLM.
- Try to put the two shapes together to make a cube or pyramid.

When students have all found their partners, provide them with an index card and small sticky notes or page flags. Students can use the index card to see how the plane cuts the cube or rectangular pyramid so that they recognize that the shape they made is actually a cross section, and that it matches the shape on the lower-right corner of the BLM. Students can then remove the index card and use the sticky notes to hold their shape together temporarily.

Step 5: As a class, create a poster titled “Cross Sections of Cubes and Pyramids” on Bristol board or chart paper. Start the poster by dividing the paper into eight regions: “0 sides,” “1 side,” “2 sides,” “3 sides,” “4 sides,” “5 sides,” “6 sides,” “more than 6 sides.” Partners tape one picture of their shape cut from the lower-right corner of the BLM, and one drawing of the 2-D cross section side-by-side in the correct region of the poster, according to the number of sides that their cross section has. (0 sides: none; 1 side: none; 2 sides: none; 3 sides: H8, I5, J7; 4 sides: A4, B11, C10, F6, G1, K15, L9, M12, O3; 5 sides: E2, N13; 6 sides: D14; more than 6 sides: none)

2. Display a milk carton (about 1 L or smaller) with the top cut off and some water in it (for a 7 cm by 7 cm carton, a 1 cm depth should do). Use food coloring to add color to the water. Show small groups of students how the surface of the water in the square prism can make different shapes, as shown below:
Challenge students to make other more specific shapes, such as an equilateral triangle or a rectangle that is not a square.

(End of activities)

When students finish Activity 1, you will have a class set of cross sections of 3-D shapes. Choose three shapes, one with a cut that is vertical (such as I5), one with a cut that is horizontal (such as L9), and one with a cut on an angle (such as M12), and use an index card to show how the plane made by the index card cuts each shape. Point out that the plane can be horizontal, vertical, or on an angle.

NOTE: In this lesson, naming shapes according to the number of sides (triangle, quadrilateral, etc.) is precise enough. In the next lesson, students will describe the quadrilaterals more precisely as squares, rectangles, trapezoids, or parallelograms.

Give each student or group of students a prism and a pyramid (e.g., from the class set you just put together, or made from BLM Nets of 3-D Shapes) and an index card for the exercises below. Have students show their answers simultaneously, then take up the various possibilities as a class.

Exercises:
1. Demonstrate how the intersection of a prism and an index card can be:
   a) a vertex of the prism          b) an edge of the prism          c) a face of the prism

Sample answers:

a)  

b)  

 c)  

2. Demonstrate how the intersection of a pyramid and an index card can be:
   a) a vertex of the pyramid       b) an edge of the pyramid       c) a face of the pyramid
   Sample answers:
   a) ![Vertex of the Pyramid]
   b) ![Edge of the Pyramid]
   c) ![Face of the Pyramid]

   Point out that now students can add more examples to the poster they made earlier. Draw the two examples of “0 sides” (1.a) and 2.a)) and “1 side” (1 b) and 2 b)) on the poster.

A cross section cannot have more sides than the prism. ASK: How many sides have you seen a cross section of a cube have so far? (0, 1, 3, 4, 5, or 6) Tell students that it is impossible for a cross section of a cube to have two sides. Write the following descriptions on the different regions of the poster:

- 0 sides: point
- 1 side: line segment
- 2 sides: impossible
- 3 sides: triangle
- 4 sides:
- 5 sides:
- 6 sides:
- more than 6 sides:

Have volunteers add the names of the shapes in the next three regions based on the number of sides. (4 sides: quadrilateral, 5 sides: pentagon, 6 sides: hexagon)

ASK: Can the cross section have seven sides? (no) Show students two index cards and SAY: Let’s use these index cards to represent planes. Using the index cards to demonstrate, SAY: Two planes can be parallel or they can intersect ASK: When they intersect, what does the intersection look like? (a line) SAY: A plane can intersect a face of a cube in at most one line segment because the face of the cube is part of a plane. Since a cube has only six faces, the cross section cannot have more than six sides.

(MP.3) A cross section cannot have more sides than the pyramid. ASK: How many sides can a cross section of a rectangular pyramid have? (0, 1, 3, 4, or 5) SAY: Two sides is still impossible. ASK: Why can’t the cross section have six sides? (because the pyramid only has 5 faces and a plane can’t intersect a face in more than one line segment)
Refer back to the poster. SAY: Six sides is possible for a cube, but not for a pyramid. More than six sides is impossible for both. Write “impossible” in the “more than 6 sides” region of the poster.

**Comparing horizontal cuts of prisms and pyramids.** Show students prisms and pyramids, (e.g., made from BLM Nets of 3-D Shapes). Place them so that they are all standing on their bases. Hold up several shapes in turn, and have volunteers name the shape using the shape of the base and whether it is a pyramid or prism. (e.g., rectangular pyramid, pentagonal prism) For each shape, SAY: Imagine cutting through the shape horizontally, parallel to the base. ASK: What is the shape of a horizontal cross section? (the same shape as the base) Demonstrate that this is the case using G1 and L9, two horizontal cuts from the class set you just made.

SAY: A horizontal cross section always makes the same shape as the base, but sometimes the size changes. For which 3-D shapes does the size of the cross section stay the same, no matter how high up the shape you cut it? (prisms) Show students how this is the case for shape G1. ASK: How does the size change for pyramids if you cut it with a plane that is higher up? (the size gets smaller) Show students how this is the case for shape L9. SAY: For a pyramid, the cross section gets smaller and smaller as you go up the pyramid, until it becomes just a single point, which is the top vertex of the pyramid.

**(MP.7) Exercises:** The diagram shows three horizontal cross sections of a pyramid or prism. Name the shape.

- a) 
- b) 
- c) 
- d) 
- e) 
- f) 

**Bonus:**

- g) 
- h) 

**Answers:** a) square prism, b) square pyramid, c) pentagonal prism, d) triangular prism, e) pentagonal pyramid, f) rectangular pyramid, Bonus: g) cylinder, h) cone

Write on the board:

<table>
<thead>
<tr>
<th>Prism</th>
<th>Pyramid</th>
</tr>
</thead>
<tbody>
<tr>
<td>same size and shape</td>
<td>same shape, different size</td>
</tr>
</tbody>
</table>

ASK: What is the word for two geometric shapes that are the same size and shape? (congruent) What is the word for geometric shapes that are the same shape but different sizes? (similar)
Continue writing on the board:

Horizontal cross sections

<table>
<thead>
<tr>
<th>Prism</th>
<th>Pyramid</th>
</tr>
</thead>
<tbody>
<tr>
<td>same size and shape</td>
<td>same shape, different size</td>
</tr>
<tr>
<td>congruent</td>
<td>similar</td>
</tr>
</tbody>
</table>

SAY: You have seen how a plane can intersect cubes and pyramids at different angles and the different shapes of the cross sections. In the next lesson, we will study these shapes in more depth.

**Extensions**

**(MP.1, MP.7)** 1. Scramble the parts of cubes and pyramids made from BLM Nets for Cuts of Cubes and Rectangular Pyramids and have students put them together again as a puzzle. Students will need to pay close attention to the shape of the cross section as they do the puzzle. **Answer:** the shapes pairs are A4, B11, C10, D14, E2, F6, G1, H8, I5, J7, K15, L9, M12, N13, and O3

**(MP.7)** 2. Which shapes are possible cross sections of a cylinder?

a) [Circle]  b) [Triangle]

[Circle]  d) [Oval]

**Bonus:** For each possible cross section, describe what type of slice you would make to get the shape.

**Answers:** a), c), d); Bonus: to get a circle, slice vertically; to get a rectangle, slice across the circle; to get an oval, slice on an angle in a way that misses the circles.

3. What shapes can you make by slicing a sphere?

**Answer:** a circle or a point

**(MP.1)** 4. Describe how to cut a cake into eight congruent pieces by making three straight cuts.

**Answer:** Cut the cake vertically into four slices. Then cut horizontally through the cake, dividing each of the four slices into top and bottom pieces.

**(MP.7)** 5. Cut the figure horizontally at different levels. Show what all the possible horizontal slices look like.

a) [Figure]  b) [Figure]  c) [Figure]  

[Right side]  [Front]  [Right side]
**Bonus:** The Willis Tower (formerly known as the Sears Tower) in Chicago is essentially made up of nine rectangular prisms. Use the Internet to research the design of the Willis Tower. Draw the horizontal cross sections at floors 50, 66, 90, and 110.

**Answers:**

a) 

b) 

c) 

**Bonus:** Floor 50  
Floor 66  
Floor 90  
Floor 110

(MP.2) 6. The temperature decreases by 4°C each hour. At 2:00 p.m., the temperature was 4°C. At 3:00 p.m., the temperature was 0°C. At what time was the temperature \( \frac{2}{3} \)°C? Show your work. Explain what each step means in the situation. What fact did you not need to use?

**Sample answers:**

- I used \( h \) for the number of hours since 3 p.m. Temp. at 3 p.m. + temp. change each hour \( h \) = given temperature. Since the temperature at 3 p.m. is 0, I didn’t need that part of the equation, so I just wrote:

  \[
  \text{temperature change each hour} \times h = \text{given temperature}
  \]

\( h = \frac{8}{3} \div (-4) = -\frac{2}{3} \). This means that the number of hours since 3 p.m. is \(-\frac{2}{3}\), so the time is \(\frac{2}{3}\) of an hour before 3 p.m. \(\frac{2}{3}\) of an hour is 40 minutes, so the time is 2:20 p.m. I did not need to use that the temperature at 2 p.m. was 4°C.

- I used \( h \) for the number of hours since 2 p.m. My equation was:

\[
4 + (-4h) = \frac{2}{3}, \text{ so } -4h = \frac{8}{3} - 4 = \frac{8}{3} - \frac{12}{3} = \frac{4}{3}, \text{ so } h = \left( -\frac{4}{3} \right) \div (-4) = \frac{1}{3}
\]

The time is \(\frac{1}{3}\) hour, or 20 minutes, after 2 p.m., so the time was 2:20 p.m. I didn’t need to use the fact that the temperature at 3:00 p.m. was 0°C.

Whole-class follow-up: Have several volunteers present solutions. For each solution, ask the rest of the class to discuss in pairs the following questions: Is the argument correct? Is there a part of the argument they do not understand? Could the argument be improved to make it clearer? Compare the solutions. ASK: Which way seems easier to you? Why? Which way is faster? (the first solution is faster because there was only one step needed to solve the equation)
Standards: 7.G.A.3

Goals:
Students will recognize perpendicular lines on the surface of a prism.
Students will describe cross sections of rectangular prisms and pyramids as specifically as possible.

Prior Knowledge Required:
Understands that a cross section of a 3-D shape is its intersection with a plane
Can identify prisms and pyramids
Can name prisms and pyramids for the shape of their base

Vocabulary: congruent, cross section, equilateral, face, horizontal, isosceles, parallel, perpendicular, plane, rectangular prism, rectangular pyramid, scalene, vertex, vertical

Materials:
shapes made in Activity 1 of Lesson G7-30
BLM Vertical Cuts of Cubes (p. R-114)
BLM Cross Section Shapes (p. R-115)
BLM Edges of Cross Sections (p. R-116)
BLM Cross Sections (p. R-117)
poster made in Lesson G7-30
an open box
marker
a cube or cube made from BLM Nets of 3-D Shapes (2) (p. R-69)
strip of colored paper
tape
two index cards
BLM Fake Net (p. R-118–119, see Extension 2)

Vertical cross sections of rectangular prisms are quadrilaterals or line segments. Tell students that a prism can be cut in many ways, but for now you want to focus on vertical cuts. Draw on the board:
SAY: However the plane meets the top face when you cut straight down, it will meet the bottom face exactly the same way. Any cross section of a cube can miss a face completely, can be a face, or can meet the face in a line segment or in a point. If the cross section meets the face in a line segment, like in the first four drawings on the board, the cross section will have four sides: the two vertical cuts and the two line segments. If the cross section meets the top face at just a vertex, such as the front right top vertex, then the cross section will be a vertical line that meets the bottom edge in the corresponding vertex (the front right bottom vertex), like in the fifth picture on the board.

Exercises: Complete BLM Vertical Cuts of Cubes.
Selected answers:
f)  
g)  
k)  

Vertical cross sections of rectangular prisms are rectangles or line segments. Display an open box with the open part facing students. Use a marker to draw a ray on a vertical edge and several rays on a horizontal face that meet the vertical ray, as shown below:

Point out that any vertical line is perpendicular to any horizontal line that it touches. You can demonstrate this by fitting an index card in a corner of the box and rotating it so that the edges of the index card coincide with the rays. Draw the diagram above on the board to show what the angles look like in two dimensions.

SAY: Not all of the angles look like right angles in two dimensions, but that’s just because it’s hard to show something in two dimensions when it’s really in three dimensions. But any line on a horizontal face of a cube is perpendicular to a vertical line that it meets. So, in fact, all four angles of the vertical cross section are right angles. This is true not only for cubes, but for any rectangular prism. ASK: What do we call a quadrilateral with four right angles? (a rectangle) SAY: So all the vertical cross sections are rectangles, unless the cross section is just an edge. The two parts of the cube, after you cut it, are prisms, and their side faces are all rectangles.
(MP.7) A plane intersects parallel planes in parallel lines. Tell students that a plane intersects parallel planes in parallel lines. Place the box against the board on an angle. Trace the top edge and bottom edge on the board to show that the lines are parallel, as shown below:

Parallel lines on cross sections of rectangular prisms. SAY: If a plane meets two other parallel planes, it does so in parallel lines. Let’s investigate what this says about cross sections of rectangular prisms. Point out that the front and back of a prism are parts of parallel planes, so if a cross section has sides on both the front and back, the sides must be parallel. ASK: What other pairs of faces are parallel on a rectangular prism? (top and bottom, left and right) SAY: So if a cross section has sides on the left and right side of a rectangular prism, for example, those sides are parallel.

Give students BLM Cross Section Shapes. Tell students that the shapes shown are all the cross sections from the shapes they put together in the previous lesson.

Exercises: Complete BLM Cross Section Shapes.
Selected answers:

c)  

e)  

f)  

n)  

Show students shape C1 from the shapes made in Activity 1 of Lesson G7-30. Draw on the board:

SAY: The sides of the cross section in the top and bottom faces are parallel because the top face is parallel to the bottom face. Mark the parallel lines in the picture. ASK: Which faces are the other two sides in? (the left face and the back face) Are those faces parallel? (no) SAY: So this quadrilateral has only one pair of parallel sides. ASK: What is the name for a quadrilateral with exactly one pair of parallel sides? (trapezoid) SAY: When you see pictures of cross sections on paper, the hardest part can be figuring out which sides are in parallel faces, or even which face a side of the cross section is in.
**Exercises:** Complete BLM Edges of Cross Sections.
**Answers:** a) top, b) left, c) bottom, d) front, e) front, f) bottom, g) back, h) left, i) right, j) bottom, k) top, l) front, m) left, n) back, o) top, p) right, q) top, r) back, s) bottom, t) front

Tell students that being able to recognize which face an edge is in will help them identify pairs of parallel sides in cross sections because sides in parallel faces are automatically parallel. Refer students to part c) of BLM Edges of Cross Sections and point out that there are two vertical lines. ASK: Are they in parallel faces? (no) Which faces are they in? (front and left side) SAY: The vertical lines are not in parallel faces, but they are parallel because they are both vertical. Point out that this is shape A4 from the shapes they made in the last lesson and that this cross section is indeed a rectangle.

**(MP.7) Exercise:** On the first example of each different cross section on BLM Edges of Cross Sections (a), (c), (e), (g), (i), (k), (n), and (q)), mark the pairs of parallel sides with the same number of arrows.
**Answers:**

Describing four-sided cross sections of rectangular prisms as parallelograms or trapezoids. Tell students that knowing when sides are parallel helps to describe precisely the cross sections that are quadrilaterals. SAY: There are two ways to tell if sides of a cross section of a rectangular prism are parallel. When two sides are parallel to the same edge of the cube, then they are parallel to each other. When two sides are in opposite faces, they also must be parallel.

**(MP.6) Exercises:** Describe the quadrilateral as a parallelogram or a trapezoid.
(MP.1) **Bonus**: Draw the 2-D cross section.
**Answers**: a) parallelogram, b) trapezoid, c) parallelogram, d) parallelogram, e) trapezoid, f) parallelogram, Bonus: Selected answer: e)

---

(MP.6) **Identifying cross sections as rectangles**. SAY: When the shape is a parallelogram, we can sometimes describe it even more specifically as a rectangle. All rectangles are parallelograms, but not all parallelograms are rectangles, so identifying the shape as a rectangle is more specific than identifying it as a parallelogram. ASK: How can you tell if something is a rectangle and not just a parallelogram? (it has four right angles) Tape a colored strip of paper to the top and front faces of a cube (or a cube made from BLM Nets of 3-D Shapes (2)), to create the line segments shown below:

Remind students that any line segment in the top face is perpendicular to any vertical line it touches. Rotate your cube in various ways, and point out that the lines are still perpendicular, as shown in the examples below:

SAY: If you have two line segments that meet each other on adjacent faces, and one of them is parallel to an edge of the prism, then the line segments are perpendicular.

**Exercises**: Look at the cross sections from the previous exercises.
  a) Which cross sections are rectangles?
  b) Which cross sections are parallelograms that are not rectangles?
**Answers**: a) parts a) and c), b) parts d) and f)

**Identifying cross sections as squares**. SAY: When the shape is a rectangle, you can sometimes describe it even more specifically as a square. ASK: How can you tell if a rectangle is a square? (all its sides are equal) What is an easy way to make a cross section of a cube that is a square? (make the cross section parallel to the base) PROMPT: The base is a square, so how can I make a cross section congruent to the base? Draw on the board:
SAY: Any cross section made parallel to a face of a cube is a square, but there are other ways, too. Draw on the board:

![Cross Section Diagram]

SAY: This is definitely a rectangle because two sides are parallel to an edge. It’s a square as long as the sides are equal.

Give students BLM Cross Sections. Tell students that if a side of the cross section looks like it was drawn parallel to an edge, they can assume that it was.

(MP.6) Exercises: Complete BLM Cross Sections.
Answers: a) rectangle, b) trapezoid, c) rectangle, d) parallelogram, e) parallelogram, f) trapezoid, g) trapezoid, h) hexagon, i) square, j) rectangle, k) square, l) parallelogram, m) rectangle, n) pentagon, o) square, p) triangle

Students who finish early can choose a special quadrilateral from the poster and write its precise name on the poster made in the previous lesson.

(MP.6) The cross section has to be the whole intersection. Tell students that since a cross section is the intersection of a plane with a 3-D shape, it must be the whole intersection. Show students a cube and an index card. Place the cube on a table. Draw on the board:

![Cube and Index Card Diagram]

Have a volunteer use a vertical cut to show you how the vertical edge can be the cross section. ASK: Can the horizontal edge be the cross section when using a vertical cut? (accept all guesses) Demonstrate trying to do this. ASK: What happens if I cut vertically through the horizontal edge indicated? (it passes through the entire side face) SAY: So the edge shown is only part of a cross section. Draw on the board:

![Vertical Cut Diagram]

(MP.3) ASK: Can this vertical line be a cross section? (no) How do you know? (because the plane would intersect the entire back face)
Exercise: Which triangle can be a cross section of the prism?
A.  

B.  

Answer: B. A is only part of a cross section.

Review horizontal cross sections of rectangular pyramids. ASK: What can be the horizontal cross section of a rectangular pyramid? (a rectangle similar to the base, or the top vertex of the pyramid) Demonstrate this using shape L9 from the shapes created in Activity 1 of Lesson G7-30.

(MP.6) Describing vertical cross sections of pyramids as points, triangles, trapezoids, or general quadrilaterals. Pass around shapes I5, J7, and K15 and point out that these pyramids were chosen because they have vertical cuts.

Exercises: Think of a rectangular pyramid.
 a) What special quadrilateral can you make using vertical cuts?
 b) If you used a vertical cut to cut off a vertex, what kind of shape would that make?
 c) How can you make a triangle with a vertical cut if you don’t cut off a vertex?
 d) Can a vertical cross section be a point?
 e) Can a vertical cross section be a line segment?
 Bonus: How can you make a vertical cut that is a quadrilateral but not a trapezoid?
 Answers: a) trapezoid; b) triangle; c) pass through the top vertex; d) yes, it can be any base vertex; e) yes, it can be any base edge; Bonus: cut straight down without cutting parallel to an edge of the base

Introduce non-vertical cross sections of pyramids. SAY: Now that we understand horizontal and vertical cross sections, let’s look at other kinds of cross sections. Pass around the pyramids that have cuts that are neither vertical nor horizontal (M12, N13, O3) before assigning the exercises below.

Exercises: Think of the rectangular pyramids you made.
 a) How can you make a non-vertical cross section be a trapezoid?
 b) How can you make a non-vertical cross section be a triangle?
 c) What is the greatest number of sides a non-vertical cross section can have?
 d) Can a non-vertical cross section be a point?
 e) Can a non-vertical cross section be a line segment?
 Answers: a) cut parallel to one of the edges, going in a slightly upward or downward direction; b) pass through the top vertex or cut off any base vertex; c) 5; d) yes; e) yes

Conclusion. Tell students that they have learned more precisely how to describe shapes of cross sections, and in particular how to tell when line segments on 3D shapes are parallel or perpendicular.
Extensions
1. A plane cuts through a cube at AC and meets the vertex D.

![Diagram of a cube with a plane cutting through it] 

a) Are triangles ABC, ABD, and CBD congruent? How do you know?
b) What kind of triangle is the cross section ACD?

**Answers:** a) angle B is 90°, and AB = BC = BD since this is a cube, so the triangles are congruent; b) since the triangles are congruent, AC = AD = CD, so ACD is an equilateral triangle.

2. Give students BLM Fake Net, scissors, and tape. The “cross section” of the cube consists of five points that are actually not on a plane and so is not a true cross section. Have students make the net and look at the “cross section.” Part of it looks almost horizontal and another part looks almost vertical, so it is not on a plane. When students finish, emphasize that not just any five points on a cube make a cross section.

**(MP.7)** 3. A horizontal slice halfway up the 3-D shape makes cross sections as shown.

a) What shape is the cross section and what are its dimensions?

![Cross sections of 3-D shapes] 

b) Find the area of the cross section. Use 3.14 for π. Write your answer to two decimal places.

**Answers:** a) i) rectangle, 4 in by 3 in; ii) rectangle, 2.5 in by 1 in; iii) circle, diameter 6 in; iv) circle, radius 2.5 in; b) i) 12 in², ii) 2.5 in², iii) 28.26 in², iv) 19.63 in²

**(MP.7)** 4. This is a top view of a rectangular pyramid:

![Top view of a rectangular pyramid] 

A line segment can be drawn to represent a vertical cut, like this:

![Diagram of a line segment representing a vertical cut] 

This cut would represent a triangular cross section because the plane cuts through two faces and the base of the prism.
Match the top views of the pyramids (A–D) to the vertical cuts (1–4).

A. B. C. D.

1. 2. 3. 4.

Answers: A 2, B 3, C 1, D 4

(MP.2, MP.4, MP.6) 5. When Marla turned 5 years old, she was 105 cm tall. She grew 7 cm each year since then. Marla is at a fair and she just went on a ride that requires each rider to be at least 140 cm tall. What might her age be? Write your answer as a full sentence.

Answer: Let \( x \) be Marla’s age in years. \( 7(x - 5) \) is the amount she grew since turning 5 years old. She was 105 cm tall when she turned 5 years old, so her current height is \( 7(x - 5) + 105 \). The information we are given tells us that she is at least 140 cm tall now, so I wrote an inequality: \( 7(x - 5) + 105 \geq 140 \). I solved the inequality and got \( x \geq 10 \), so Marla is at least 10 years old.
Nets of 3-D Shapes (1)
Nets of 3-D Shapes (2)
Nets of 3-D Shapes (3)
Nets of 3-D Shapes (4)
Nets of 3-D Shapes (5)
Nets of 3-D Shapes (6)
Nets of 3-D Shapes (7)
Nets of 3-D Shapes (8)
Nets of 3-D Shapes (9)
Nets of 3-D Shapes (10)
Nets of 3-D Shapes (11)
Volume of a Pyramid (1)
Volume of a Pyramid (2)
Volume of a Pyramid (3)
Volume of a Pyramid (4)
Quarter-Inch Grid Paper
Nets for Cuts of Cubes and Rectangular Pyramids (1)
Nets for Cuts of Cubes and Rectangular Pyramids (2)
Nets for Cuts of Cubes and Rectangular Pyramids (3)
Nets for Cuts of Cubes and Rectangular Pyramids (5)
Nets for Cuts of Cubes and Rectangular Pyramids (6)
Nets for Cuts of Cubes and Rectangular Pyramids (7)
Nets for Cuts of Cubes and Rectangular Pyramids (8)
Nets for Cuts of Cubes and Rectangular Pyramids (9)
Nets for Cuts of Cubes and Rectangular Pyramids (10)
Nets for Cuts of Cubes and Rectangular Pyramids (11)
Nets for Cuts of Cubes and Rectangular Pyramids (12)
Nets for Cuts of Cubes and Rectangular Pyramids (13)
Nets for Cuts of Cubes and Rectangular Pyramids (14)
Nets for Cuts of Cubes and Rectangular Pyramids (15)
Nets for Cuts of Cubes and Rectangular Pyramids (16)
Nets for Cuts of Cubes and Rectangular Pyramids (17)
Nets for Cuts of Cubes and Rectangular Pyramids (18)
Nets for Cuts of Cubes and Rectangular Pyramids (19)
Nets for Cuts of Cubes and Rectangular Pyramids (20)
Nets for Cuts of Cubes and Rectangular Pyramids (21)
Nets for Cuts of Cubes and Rectangular Pyramids (22)
Nets for Cuts of Cubes and Rectangular Pyramids (23)
Nets for Cuts of Cubes and Rectangular Pyramids (24)
Nets for Cuts of Cubes and Rectangular Pyramids (25)
Nets for Cuts of Cubes and Rectangular Pyramids (26)
Nets for Cuts of Cubes and Rectangular Pyramids (27)
Nets for Cuts of Cubes and Rectangular Pyramids (28)
Nets for Cuts of Cubes and Rectangular Pyramids (29)
Nets for Cuts of Cubes and Rectangular Pyramids (30)
Vertical Cuts of Cubes

1. Draw vertical lines to complete the vertical cross section.

a)  

b)  

c)  

d)  

e)  

f)  

g)  

h)  

i)  

j)  

k)  

l)  

Bonus ➤ Finish drawing the vertical cross section. You will need to draw both vertical sides and the bottom side.

m)  

n)  

o)  

p)  

Bonus ➤ Finish drawing the cross section straight across the cube, from left to right or from right to left.

q)  

r)  

s)  

t)
Cross Section Shapes

1. Lines that appear parallel are in fact parallel. Mark parallel lines with the same number of arrows.

   a)  
   b)  
   c)  
   d)  
   e)  
   f)  
   g)  
   h)  
   i)  
   j)  
   k)  
   l)  
   m)  
   n)  
   o)
Edges of Cross Sections

1. Shade the face that the thick black side of the cross section lies in.

   a) ![Diagram](image1)
   b) ![Diagram](image2)
   c) ![Diagram](image3)
   d) ![Diagram](image4)

   e) ![Diagram](image5)
   f) ![Diagram](image6)
   g) ![Diagram](image7)
   h) ![Diagram](image8)

   i) ![Diagram](image9)
   j) ![Diagram](image10)
   k) ![Diagram](image11)
   l) ![Diagram](image12)

   m) ![Diagram](image13)
   n) ![Diagram](image14)
   o) ![Diagram](image15)
   p) ![Diagram](image16)

   q) ![Diagram](image17)
   r) ![Diagram](image18)
   s) ![Diagram](image19)
   t) ![Diagram](image20)

**Bonus** Shade a line parallel to the black line whenever possible.
Cross Sections

1. Identify the cross section of the cube as specifically as you can. If it is not a quadrilateral, name it by its number of sides.

   a)  
   b)  
   c)  
   d)  

   e)  
   f)  
   g)  
   h)  

   i)  
   j)  
   k)  
   l)  

   m)  
   n)  
   o)  
   p)  

   ____  
   ____  
   ____  
   ____  

   ____  
   ____  
   ____  
   ____  

   ____  
   ____  
   ____  
   ____  

   ____  
   ____  
   ____  
   ____  
Fake Net (1)
Fake Net (2)
Unit 7  Statistics and Probability: Statistics

This Unit in Context
In Grade 6, students learned to calculate the mean, median, and range of a collection of data. They extended their knowledge of how to summarize numerical data sets to include measures of variability, such as interquartile range and mean absolute deviation (6.SP.B.5c).

In this unit, students will continue their study of the measurements of central tendency. They will use measures of center and measures of variability of numerical data from random samples to draw informal comparative inferences about two populations (7.SP.B.4). They will also informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability (7.SP.B.3). In Grade 8, their study will expand to include investigating patterns of association in bivariate data.

Mathematical Practices in This Unit
In this unit, you will have the opportunity to assess MP.1 to MP.4, MP.6, and MP.8. The MP labels in this unit flag both opportunities to develop the mathematical practice standards and opportunities to assess them. Below is a list of where we recommend assessing each standard, as well as some examples of how students can show that they have met a standard.

**MP.1:** Extension in SP7-16
In the extension in SP7-16, students make sense of a non-routine problem when they make a plan of all the steps they need to do: find the ratio of recipes and what each ingredient would be without compensating for an egg, find the volume of egg that needs to be compensated for, find the volume of milk that needs to be added, and then find the total amount of milk needed for the recipe. They persevere in solving the problem when they carry out their plan.

**MP.2:** SP7-18 Extension 3, SP7-18 Question 6 on AP Book 7.2 p. 196, exercises on pp. S-62 and S-63 in SP7-19
In SP7-18 Question 6 on AP Book 7.2 p. 196, students reason abstractly and quantitatively when they estimate, from two dot plots, the mean of each population, which population has a greater MAD, and how large the overlap is between the two populations.

**MP.3:** SP7-14 Extension 4, activity in SP7-17, exercises on p. S-62 in SP7-19

**MP.4:** Extension in SP7-16, SP7-18 Extension 3

**MP.6:** SP7-14 Extension 4

**MP.8:** Question 5 in Performance Task: Percentage Discounts on p. S-105
In Question 5 in Performance Task: Percentage Discounts on p. S-105, students notice and express regularity when they determine the greatest percentage discount they can get on two items at a store that offers a sale of “buy one item and get an item of equal or lesser value for free.”
Unit 7  Statistics and Probability: Statistics

Introduction
In this unit, students will find the measures of central tendency (mean, median, and mode) and measures of variability (range, mean absolute deviation or MAD, and IQR). They will identify the five features of box plots, read and create box plots, and match box plots with dot plots. Students will investigate how adding or removing an outlier affects measures of central tendency (mean) and variability (MAD).

Students will understand the necessity of using samples in place of a whole population, and that the samples need to be random to be representative. They will also understand that the sample size needs to be large so reliable inferences can be drawn from the sample.

Students will assess the degree of visual overlap of two numerical data distributions with similar variability. They will measure the difference between the centers by expressing it as a multiple of a measure of variability. Students will draw informal comparative inferences about two populations. They will understand that the lower the degree of overlap in the samples, the more confident they can be that the two populations are different.

NOTE: The answers provided for calculating the MAD assume that students round the answers for the mean as asked. If your students have access to a MAD calculator (for example, at http://www.alcula.com/calculators/statistics/mean-absolute-deviation/), their answers will be more precise than the answers shown here.

Fraction notation. We show fractions in two ways in our lesson plans:

Stacked: \( \frac{1}{2} \)  Not stacked: 1/2

If you show your students the non-stacked form, remember to introduce it as new notation.
SP7-10  Mean, Median, and Frequency

Pages 174–175

Standards: 7.SP.A.1

Goals:
Students will find the mean, median, and mode.
Students will find the range of a data set and draw dot plots.

Prior Knowledge Required:
Can compare and order whole numbers
Can divide multi-digit whole numbers by one-digit numbers
Can compare and order whole numbers
Can solve a simple equation to find an unknown value

Vocabulary: average, center, data point, data set, data value, frequency, frequency chart, mean, median, mode, set of data, tally chart

Materials:
15 blocks

Calculating the mean. SAY: Five people want to evenly share the plums they picked. One way to share is for people with a lot of plums to give some to people with less, until everyone has the same number of plums. Mathematicians call this finding the mean or average number of plums. Draw on the board:

<table>
<thead>
<tr>
<th>Plums Picked</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mona</td>
</tr>
<tr>
<td>Ray</td>
</tr>
<tr>
<td>Jen</td>
</tr>
<tr>
<td>Ed</td>
</tr>
<tr>
<td>Alice</td>
</tr>
</tbody>
</table>

SAY: Each number is called a data value or data point. A group or collection of data values is called a data set or a set of data. The data values in a set all refer to the same thing. In this case, 4, 2, 6, 1, and 2 all refer to the number of plums each person picked.

Ask for five volunteer “plum pickers.” Give them blocks to represent the plums (4, 2, 6, 1, and 2). Write on the board:

\[\_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_\]

One by one, in any order, have students give you all their blocks. Write the numbers in the blanks. ASK: How many blocks do I have now? (15) Write on the board:

Sum = 15
SAY: Now I’d like to share the blocks evenly. ASK: How many people do I have to divide them between? (5) Write on the board:

Number of data values = 5

ASK: How do we use 15 and 5 to find the mean? (15 ÷ 5) Write on the board:

15 ÷ 5 = ___

ASK: What is the mean? (3) Write it in the blank.

Exercises: Calculate the mean. Signal your answers with your fingers.

a) 4, 7, 2, 1, 1    b) 6, 9, 4, 3, 3

c) 10, 12, 3, 7    Bonus: 0, 8, 15, −5, 14, 19, 12

Answers: a) 3, b) 5, b) 8, Bonus: 9

Refer students to parts a) and b) in the previous exercises and ASK: What is the relationship between the data values in parts a) and b)? (each data value in b) is 2 more than the corresponding data value in a)) How can you compare the mean of the two data sets? (the mean of the data set in b) is the mean of the data set in a) plus 2) SAY: Whenever you add the same number to or subtract the same number from all data values, the mean will add (or subtract) with the same number.

Exercises:

a) Find the mean of 0, 0, 1, 4, 6, 8, 8, 10, 11, 12.
b) Add 5 to each data point in part a). Recalculate the mean. How did it change?

Answers: a) 6; b) 11, the mean increased by 5

Finding the median of a data set. SAY: When we put sets of data in order, we call the one in the middle, or center, position the median. Write on the board:

6, 8, 11, 14, 19, 26, 27

ASK: Is the set in order? (yes) SAY: 14 is in the center, so it is the median. Underline 14.

Being organized to order data and find the median. Write on the board:

23, 11, 9, 16, 27, 4, 20, 12, 8

SAY: Let’s order these numbers from least to greatest in an organized way. ASK: What is the smallest number? (4) Write “4” under 23 and cross out 4 in the original set. SAY: We cross out the 4 to show we’ve used it. ASK: What number is next? (8) Write “8” beside the 4. Continue until the entire set is in order. (4, 8, 9, 11, 12, 16, 20, 23, 27) Leave this organized set of data on the board.
Have students complete the following exercises using the same method.

**Exercises:** Order the numbers.

a) 16, 12, 23, 42, 7, 33, 10, 41  
b) 203, 322, 230, 302, 233, 320  

**Answers:** a) 7, 10, 12, 16, 23, 33, 41, 42; b) 203, 230, 233, 302, 320, 322

SAY: Let’s find the median of the set we ordered earlier. To do this, we count inward one number at a time from each end. Cross out 4 and 27, then 8 and 23. Continue until only the 12 is left. SAY: So 12 is the median. Circle the 12.

Write on the board:

```
3 5 6 12 14
```

SAY: Sets with an odd number of data points always have one number in the center position. Let’s find the median of a set with an even number of numbers. Write on the board:

```
2 3 7 8
```

Cross out 2, then 8. ASK: Why can’t we cross out 3 and 7? (there will be no numbers left)  
SAY: Sets with an even number of data points have two center numbers. The median is halfway between the two center numbers. Circle the 3 and 7 together. SAY: An easy way to get the number halfway between two numbers is to add the numbers, then divide by 2. Tell students that they will learn why this works in the next lesson. SAY: The median number doesn’t need to be in the set. ASK: What number is halfway between 3 and 7? (5) Write “5” on the board below the circled 3 and 7, and SAY: The median is 5.

**Exercises:** Order the numbers, then find the median.

a) 4, 7, 2, 1, 5  
b) 9, 12, 3, 7  
c) 19, 1, 26, 4, 28, 17  
**Bonus:** 320, 602, 362, 630, 302, 360, 620, 203

**Answers:** a) 4, b) 8, c) 18, Bonus: 361

**Mean and median as centers of data sets.** SAY: The mean and median both tell us about the center of a set, but we find them in different ways. One uses calculations (the mean) and the other uses ordering (the median). Write on the board:

```
14, 2, 5
```

Have students find the mean and then the median. (7, 5)

**Exercises:** Find the mean and median.

a) 11, 1, 15  
b) 1, 17, 6, 8  
**Bonus:** 108, 106, 110, 108

**Answers:** a) mean = 9, median = 11; b) mean = 8, median = 7; Bonus: mean = 108, median = 108
Finding the frequency of a data value. SAY: Frequency is the number of times something happens. In a data set, the frequency is the number of times a value occurs. Write on the board:

7, 7, 4, 4, 7, 5, 6, 5

SAY: The set has eight data points, but only four data values: 4, 5, 6, and 7. There are three 7s, so the frequency of 7 is 3. ASK: What is the frequency of 4? (2) Of 5? (2) Of 6? (1) SAY: The value that appears the most often or with the greatest frequency is called the mode. ASK: What is the mode of the data set? (7) Leave the data set on the board.

Organizing data in a tally or frequency chart. Draw on the board:

<table>
<thead>
<tr>
<th>Data Value</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

SAY: This is a tally chart or frequency chart. We write all numbers in the range in the Data Value column. In other words, we start at 4 and write all the numbers up to and including 7. Fill in the Data Value column. SAY: Now let’s put the data into the chart, starting with the Tally column. ASK: What is the first data point in the set? (7) Draw one tally mark for 7. Cross out the first 7 in the set. SAY: I crossed out the 7 to remind myself that I’ve tallied it already. ASK: What number is next? (7) Draw a second tally beside the first one. Cross out the second 7 in the set. Have students help you fill in the rest of the Tally column. SAY: Now let’s complete the Frequency column. We write “2” for 4 because there are two 4s. ASK: What do we write for 5? (2) For 6? (1) For 7? (3) The completed chart should look like this:

<table>
<thead>
<tr>
<th>Data Value</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Write on the board:

4, 7, 4, 4, 7

Point at the two data sets on the board and ASK: What do you notice about the highest and lowest values of these two sets? (they are the same)

Explain that you want to make another chart for the new data. Draw on the board another tally chart with the same headings as the previous one. Have students help you fill in the Data Value column.
SAY: Now let's fill in the Tally and Frequency columns. ASK: What is the first data point in the set? (4) How do we show that in the tally column? (mark one tally beside 4) Draw a tally mark and cross out the first 4. ASK: What comes next? (7) Draw a tally mark in tally column beside 7. Have students help you fill in the remainder of the chart. (see completed chart below)

<table>
<thead>
<tr>
<th>Data Value</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Tell students that for the following exercises they will need to know how to show five tallies. SAY: To show five tallies, draw four regular tallies and one tally across. Draw on the board: 

Exercises: Transfer the data into a chart. Cross out each number as you mark it in the chart.

a) 3, 2, 5, 3, 5, 5  

b) 11, 10, 10, 12, 15, 12, 10, 10, 11, 10

Answers:

a)  

<table>
<thead>
<tr>
<th>Data Value</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b)  

<table>
<thead>
<tr>
<th>Data Value</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

(MP.8) Using frequency to calculate the mean. Write on the board:

3, 4, 9, 8, 6, 6, 9, 6, 3, 6

ASK: How did we calculate the mean of a set at the beginning of this lesson? (we added the values and divided by the number of values) SAY: Now we are going to calculate the mean using frequencies. Draw on the board a chart with Data Value, Tally, and Frequency headings. Have students help you fill in the chart for the data set on the board, one column at a time. The completed chart should look like this:

<table>
<thead>
<tr>
<th>Data Value</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
SAY: If we add all the numbers one by one, we will have a very long addition that looks like this.
Write on the board:

\[3 + 4 + 9 + 8 + 6 + 6 + 9 + 6 + 3 + 6\]

SAY: Instead, we can add all the 3s, then all the 4s, and so on. Write on the board:

\[(\_ \times 3) + (\_ \times 4) + (\_ \times 5) + (\_ \times 6) + (\_ \times 7) + (\_ \times 8) + (\_ \times 9)\]

\[= \_ + \_ + \_ + \_ + \_ + \_ + \_\]

Point at the first blank and ASK: What is the frequency of 3? In other words, how many 3s are there? (2) Write “2” in the first blank. Continue until you have filled in the blanks with the frequency of each value. ASK: What is \(2 \times 3\)? (6) Write “6” in the first blank. Continue until you have filled in all the blanks. (6 + 4 + 0 + 24 + 0 + 8 + 18)

SAY: Doing it this way means you get to do a much shorter addition. Have students do the addition. ASK: What is the total? (60) How many data points are in the set? (10) How do we find the mean? (60 ÷ 10) What is the mean? (6)

**Exercises:** Put the data into a chart and calculate the mean.

a) 4, 4, 6, 5, 5, 6

b) 6, 2, 2, 7, 6, 6, 2, 1, 4

**Answers:**

a)

<table>
<thead>
<tr>
<th>Data Value</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[(2 \times 4) + (2 \times 5) + (2 \times 6)\]

\[= 8 + 10 + 12\]

\[= 30\]

Mean = 30 ÷ 6 = 5

b)

<table>
<thead>
<tr>
<th>Data Value</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[(1 \times 1) + (3 \times 2) + (0 \times 3) + (1 \times 4) + (0 \times 5) + (3 \times 6) + (1 \times 7)\]

\[= 1 + 6 + 0 + 4 + 0 + 18 + 7\]

\[= 36\]

Mean = 36 ÷ 9 = 4

**Bonus:** Make a chart counting by 100s to calculate the mean of 600, 400, 600, 600, 400, 1,000. **Answer:** 600
SP7-11  Mean, Median, and Dot Plots

Pages 176–177

Standards: 7.SP.A.1

Goals:
Students will find the range of a data set and draw dot plots.

Prior Knowledge Required:
Can compare and order whole numbers
Can divide multi-digit whole numbers by one-digit numbers
Can solve a simple equation to find an unknown value

Vocabulary: average, center, data set, data value, dot plot, frequency, mean, measures of central tendency, median, mode

Review using frequency to calculate the mean. Remind students that they can find the sum of the data values in a set by multiplying each data value by its frequency, then adding the products. To find the mean, divide the sum of the data values by the number of data values.

Write on the board:

2, 5, 3, 3, 2, 6, 2, 6, 6, 5

Ask a volunteer to find the mean of the data set using frequency, as shown below:

\[3 \times 2 + 2 \times 3 + 2 \times 5 + 3 \times 6 = 6 + 6 + 10 + 18 = 40\]
\[\text{Mean} = 40 \div 10 = 4\]

Exercises:

a) Ten people work in an office. They are paid different salaries, depending on their job.
   - Salesperson: $25,000 per year
   - Clerk: $20,000 per year
   - Assistant: $17,500 per year
   If there are 5 salespeople, 3 clerks, and 2 assistants, find the mean salary.

b) Predict whether the mean will increase or decrease if …
   - i) a clerk retires
   - ii) an assistant retires
   - iii) 2 salespeople are hired

c) Eleven people work in an office. They are paid different salaries for different jobs.
   - Salesperson: $18 per hour
   - Clerk: $17 per hour
   - Bookkeeper: $16 per hour
   - Assistant: $12 per hour
   If there are 5 salespeople, 2 clerks, 1 bookkeeper, and 3 assistants, find the mean salary per hour.

d) Predict whether the mean will increase or decrease if …
   - i) a clerk retires
   - ii) a salesperson retires
   - iii) a bookkeeper is hired
   - iv) 2 new assistants are hired
   - v) an assistants retires
   - vi) a clerk is hired

Bonus: 2 clerks and an assistant are hired
Answers: a) 22,000; b) i) increase, ii) increase, iii) increase; c) 16; d) i) decrease, ii) decrease, iii) decrease, iv) decrease, v) increase, vi) increase, Bonus: decrease

The “center” can mean different things. Explain to students that the mean, median, and mode are all called measures of central tendency because they all give a value that is the middle, or center, of a set in some way. Explain that the word “center” might mean median, mean, or mode, depending on the situation, but it is most often used to indicate the mean. Explain that there are sets where there is no mode. For example, the set 4, 5, 8, 11 has no mode because all frequencies are the same.

ASK: Can there be a set with no mean or no median? If students do not do so on their own, prompt them to think of survey responses. For example, tell students that you checked the eye color of 15 people and found that 5 people have hazel eyes, 3 have green, 4 have brown, and 3 have blue. ASK: Can we find the mean of this set? (no) The median? (no) Why not? (the data values are not numbers, they are colors) Can we find a mode? (yes) What is the mode? (hazel is the data value that appears the most often; hazel is the most common eye color) SAY: Survey responses like these are examples of data sets with no mean or no median.

SAY: Here is a problem with numerical data values where only the mode is a valuable measure of central tendency. Write on the board:

A store owner is deciding what sizes of shoes to order. He looks at his sales data.

<table>
<thead>
<tr>
<th>Shoe Size</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number Sold</td>
<td>2</td>
<td>12</td>
<td>35</td>
<td>28</td>
<td>20</td>
<td>9</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

ASK: How many shoes did he sell altogether? (111) Which size did he sell the most of? (7) What is the mode of the set of shoe sizes sold? (7) What is the median of the set of shoe sizes sold? (there are 111 data values, so the median is the 56th value, which is 8) The mean size of the shoes sold is about 7.9. Can you buy a pair of shoes of this size? (no) To decide what size of shoes to order the most of, which is most relevant: the mean, median, or mode? (the mode) Why? (it’s the shoe size that is sold the most often)

Exercises:
a) Which measures of central tendency does the following statement refer to?
A karate school will allow you to graduate to the next belt if you score more than 70 on at least half of the tests and have an average of at least 75.
b) Is any other measure of central tendency useful?

Answers:
a) median (70), mean (75)
b) no, you can have grades such as 10, 11, 12, 80, 80, 21, and 17 (out of 100) and get a mode of 80, but this does not accurately reflect your overall performance

Creating sets with given measures of central tendency. Tell students that you want to create a set with four data values with a mean of 5. You already know three data values are 3, 3, and 5. SAY: We can use an algebraic solution to find the remaining data value, which we
can represent with a variable, such as \( x \). Write on the board an equation for the mean and solve it, as shown below:

\[
(3 + 3 + 5 + x) \div 4 = 5
\]
\[
3 + 3 + 5 + x = 20
\]
\[
11 + x = 20
\]
\[
x = 20 - 11
\]
\[
= 9
\]

SAY: So the four data values are 3, 3, 5, 9.

**MP.4** Exercises: Ben’s marks from his first four math tests are 78, 83, 74, and 79. He has one more test to write. He wants a mean of at least 80 for all five tests.

a) What total does Ben need for all five tests?
b) What is the lowest mark that Ben needs to get on the last test?

Answers: a) 400, b) 86

**MP.2** Review dot plots. Draw on the board:

Number of Stories Each Student Wrote

<table>
<thead>
<tr>
<th>Number of Stories</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of People</td>
<td>H</td>
<td>H</td>
<td></td>
<td>H</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

ASK: What is this dot plot about? (the number of stories each student wrote) SAY: The number line shows that students wrote anywhere from 1 to 6 stories. Circle the column of data above 6. SAY: This shows that 2 students wrote 6 stories. ASK: How many students wrote 1 story? (8) How many students wrote 2 stories? (6) How many students wrote stories altogether? (28) What was the largest number of stories any student wrote? (6) How many more students wrote 1 story than wrote 2 stories? (2) How many students wrote either 5 or 6 stories? (4)
Ask a volunteer to plot the data from the chart on the board using the following template:

**Number of Plums Each Person Picked**

When the volunteer is done, ASK: How many people picked 1 plum? (5) How many people picked 3 or more plums? (5) What was the largest number of plums any one person picked? (5) Did anyone pick exactly 4 plums? (no) How many more people picked 1 plum than picked 5 plums? (2) How many people picked plums altogether? (14)

**(MP.4, MP.5) Mean, range, and mode.** Tell your students that you are going on a trip. You find out that the daily temperature at your destination (in °C) is predicted to be 15, 14, 11, 12, 11, 10, and 18. Have students find the highest temperature, the lowest temperature, and the range of temperatures. (18°C, 10°C, 8°C) ASK: Do you need to know all the temperatures to decide if you will need snow pants, or is knowing the lowest temperature enough? (no) Do you need all the temperatures to decide if you will need a pair of shorts, or is knowing the highest temperature enough? (no)

SAY: Suppose you only know the range of temperatures at your destination (8°C). ASK: What does this mean? (the temperature doesn’t change very much) How can this help you decide what kinds of clothes to pack? (sample answer: you might need either snow pants or shorts, but not both) What if the range at your destination is 20°C? What does that tell you? What will you pack? (the temperature changes a lot; you will need clothes to suit very different temperatures)

ASK: What is the mean of the set of temperatures? (13°C) If you know the range and the mean, but not the exact temperatures, do you know what kinds of clothes to pack? (yes) Will you need snow pants? (no) Will you need a bathing suit? (no) How do you know? (the temperatures cannot be below 5° or above 21° because the mean is 13°, which is somewhere in the middle of the set)

Ask students if they can decide what to pack for the trip if they only know the range, median, and the mode of the set. (yes, we know enough about the temperatures to decide what clothes to pack because the range is small)

**Extensions**
1. Find the mean. Write your answer as a mixed number.
   a) 3, 4, 6, 8  b) 4, 7, 8  c) 0, 3, 2, 7, 6  d) 1, 2, 5, 2, 6, 3
   **Answers:** a) 5 1/4, b) 6 1/3, c) 3 3/5, d) 3 1/6
2. a) Calculate the mean.
   i) 7, 2, 15 ii) 7, 2, 15, 0
b) Explain why the mean of the set in part i) is greater than the mean of the set in part ii).
   **Answers:** a) i) 8, ii) 6; b) both totals are 24, but in i) 24 has to be divided into 3 groups and in ii) it has to be divided into 4 groups. The fewer groups there are, the more there will be in each group, so the mean will be greater

(MP.3) 3. Yu is sales manager in a department store. Each week, she must maintain average (mean) daily sales of at least $8,500. Sales for Monday, Tuesday, Wednesday, and Thursday were $7,530, $8,470, $6,550, and $7,155. The store is not open on Sunday. What are the total sales? What sales will Yu need to make on Friday and Saturday to maintain her average daily sales of at least $8,500?
   **Answer:** The total sales are $29,705. Yu needs $21,295 in sales over two days.

4. You have written six history tests so far, and you will write three more. Your marks are 75, 71, 81, 62, 65, and 75.
   a) If you want a mean of at least 70, what total marks will you need to get on the last three tests? What must your average be for the last three tests?
   b) If you want a mode of at least 75, what marks will you need to get on the last three tests? What conditions must your three test scores satisfy?
   **Answers:** a) you will need a total of 201 marks on the last three tests, or at least an average of 67; b) any three different grades other than 62, 65, 72 will work, but since it is difficult to make sure you don’t get a specific grade, aim for any three grades above 75

5. Roy’s first eight math test scores (out of 20) are 13, 14, 18, 16, 14, 13, 14, and 18.
   a) Calculate the mean, median, and mode.
   b) Roy’s father tells him that if he increases his average score to 16 out of 20 after the next two tests, he will give him $200. By “average,” do you think Roy’s father means the mode, the mean, or the median?
   c) How can Roy increase his average to 16?
   **Answers:** a) mean: 15, median: 14, mode: 14; b) mean; c) Roy needs two tests with marks of 20 out of 20

6. A family drives at 110 kph for 6 hours, then 60 kph for 4 hours.
   a) How far did the family drive in the first 6 hours? In the last 4 hours? Altogether?
   b) Find the mean speed in kph. Hint: mean speed = distance traveled ÷ time traveled.
   (MP.1, MP.3) c) Is the mean speed closer to 110 kph or 60 kph? Why does this make sense?
   **Answers:** a) 660 km in the first 6 hours, 240 km in the last 4 hours, 900 km altogether; b) 90 kph; c) closer to 110 kph, because they drove longer at the higher speed
SP7-12  Shape and Symmetry of Dot Plots

Pages 178–180

Standards: preparation for 7.SP.A.2, 7.SP.B.3, 7.SP.B.4

Goals:
Students will draw and interpret the shape of dot plots and use dot plots to find the mean.
Students find deviation from the mean in a data set.

Prior Knowledge Required:
Can calculate the mean of a set of data
Can identify line-symmetric figures

Vocabulary: center, data set, data value, deviation, dot plot, frequency, guess and check, line of symmetry, mean, median, symmetrical, tally chart

Materials:
BLM Calculating Means from Dot Plots (p. S-65)
calculators

Symmetry in dot plots. Remind students that a shape is symmetrical if, when folded along a line of symmetry, the two sides fit over each other exactly. Draw on the board:

Draw a dashed vertical line at 12, as shown below:

Have students imagine folding the dot plot along this line. ASK: Do the two sides fit over each other exactly? (yes) Explain to students that this is because the vertical line at 12 is the line of symmetry.
The mean and median of symmetrical data sets. Draw on the board:

<table>
<thead>
<tr>
<th>Data Value</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(MP.1) Have students use the frequencies to find the total, and then the mean. (total: 120, mean: 12) Then have them find the median—remind students to write the data values in order first. ASK: What is the median? (12) What do you notice about the median and the mean? (they are the same) SAY: In symmetrical sets, the median and mean are always equal. Point out how the tallies in the chart are symmetric around the 12, the same way the dot plot is.

Exercises: Make a tally chart for the data. Look at the tallies and decide if the mean equals the median.

a) 27, 26, 27, 26, 29, 28, 28
b) 9, 13, 15, 9, 12, 15, 12, 9, 11, 15
c) 2, 7, 9, 5, 7, 11, 7, 11, 7, 2

Bonus: 160, 160, 140, 100, 140, 160, 160 (Hint: Make a chart that goes up by tens, not ones.)
Answers: a) no, b) yes, c) no, Bonus: no

Calculating the mean from a dot plot. SAY: You can find the mean of a symmetrical dot plot by finding the center value. But for dot plots that aren’t symmetrical, you have to calculate the mean. Draw on the board:

Point at 1 on the number line. ASK: How many dots are above 1? (5) Write “× 5” under the 1, as shown. SAY: The 5 means there are 5 ones in the set. ASK: What is 5 × 1? (5) Write “5” under “× 5.” To calculate the mean, we start by adding 1 five times. Point at the 2 on the number line. ASK: How many dots are above the 2? (0) Write “× 0” under the 2. ASK: What is 0 × 2? (0) Write “0” under “× 0.” SAY: There are no 2s in the data set, so there shouldn’t be any 2s when we calculate the mean. The picture should look like this:
Continue until you have done this for all numbers on the number line, then have students calculate the mean. (2) **NOTE:** When calculating the mean, students do not need to include zeros for the numbers that do not appear in the set. Instead of \( (1 \times 5 + 2 \times 0 + 3 \times 1 + \ldots + 6 \times 1) \div 7 \), students can just write the non-zero numbers: \( (1 \times 5 + 3 \times 1 + 6 \times 1) \div 7 \).

Have students complete **BLM Calculating Means from Dot Plots.** (1. a) 8, b) 13, c) 40, d) 22)

Explain to students that they can find the mean of a data set two different ways. Write on the board:

\[
1, 8, 6
\]

SAY: You can find the mean using division. Write on the board:

\[
\text{Mean} = \frac{\text{sum of data values}}{\text{number of data values}}
\]

\[
= \frac{15}{3}
\]

SAY: You can also find the mean using a picture. Draw on the board:

![Dot plot with numbers 1, 8, 6]  

Demonstrate how to move one ball from a rod with too many balls (by crossing the ball out) to a rod with fewer balls (by drawing another ball). Then have a volunteer continue until the mean is found; that is, until the rods all have the same number of balls. The final picture should look like this:

![Final dot plot]  

Draw a horizontal line in the picture to show where the mean is. ASK: When you moved some balls to other rods, which rods did you take the balls from—the rods with more balls than the mean, or the rods with fewer balls than the mean? (the rods with more balls than the mean) Which rods did you move them to? (the rods with fewer balls than the mean) ASK: Why does that make sense? (because the mean is between the lowest and highest value) Explain to students that if all the rods are going to have the same number of balls, they need to add balls to the rods that have less than the mean and take balls away from the rods that have more than the mean. ASK: Is the number of balls you took away from some rods equal to the number of
Have students use division to calculate the mean for the data set 3, 0, 4, 6, 2. (3) Then draw on the board:

ASK: How many balls are above the horizontal line? (4) How many empty spaces are below the line? (4) If you moved the balls above the mean to the spaces below the mean, how many would you have in each pile? (3, which is the mean)

Draw on the board:

ASK: Is the horizontal line in the right place? Does the line show the mean? (no) How do you know? (there are more spaces below the line than there are balls above the line) How can we move the line so there are fewer spaces below the line and more balls above the line? PROMPTS: Should we move the line up or down? (down) How far? (one “row”) Have a volunteer show this on the board. ASK: Is the line in the right place now? (yes) How do you know? (there are 5 balls above the line and 5 spaces below the line, so moving the 5 balls to the 5 spaces will make all the rods have the same number of balls)

Repeat with several other pictures, but have a volunteer guess where the mean is by drawing a horizontal line. Then have students check the guess independently by counting balls above the line and the spaces below the line. The numbers should be the same. Adjust the line accordingly as necessary.

**Finding the mean by guess and check.** Write on the board:

80, 86, 82, 86, 86

Tell students that you want to find the mean without actually adding the data values. SAY: My guess is that the mean is 83. ASK: Do you think my guess is reasonable? (yes, because it's
between the lowest and highest values) Tell students that you want to check if your guess is too high or too low without calculating the mean. ASK: Which values are above my guess? (86, 86, 86) Which values are below my guess? (80, 82) Write on the board:

Above my guess: 86, 86, 86  Below my guess: 80, 82

Point to each data value and have volunteers say how much it is above or below your guess of 83. Write their answers below the data values, then add to find the total above and total below, as shown below:

Above my guess: 86, 86, 86  Below my guess: 80, 82

<table>
<thead>
<tr>
<th>3</th>
<th>3</th>
<th>3</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total above: 3 + 3 + 3 = 9</td>
<td>Total below: 3 + 1 = 4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(MP.1, MP.5) ASK: Are there more below or above my guessed mean of 83? (more above) Is my guess too high or too low? (too low) So should my next guess be higher or lower? Why? (higher, because we need the total amounts below and above the mean to be the same) SAY: Let's guess 84 as the mean instead.

Have students find how many below or above 84 each data value is. (4 below, 2 above, 2 below, 2 above, 2 above) Have students add to find the totals below and above 84 for all of the data values in the set. (4 + 2 = 6 below and 2 + 2 + 2 = 6 above) SAY: Since the total below is equal to the total above, 84 is the mean. Have students add the five data values and divide by 5 to check. Ask students to think about which is easier: adding 4 + 2 and 2 + 2 + 2 or adding the five data values and dividing by 5.

(MP.7) Find the mean of another set using a similar method. Write on the board:

Set: 173, 180, 172, 176, 174

Have students make an educated guess for the mean. For example, 175. Write on the board:

Mean (guess): 175

Write the difference between each data value and the mean that you guessed, either above or below each data value, depending on whether the data value is greater than or less than the mean. Then find the total differences above and below the mean that you guessed. The final picture should look like this.

Set: 173, 180, 172, 176, 174

<table>
<thead>
<tr>
<th>5</th>
<th>1</th>
<th>Total above = 5 + 1 = 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

SAY: The total differences above and below the mean are equal, so our guess is correct and the mean is 175.
Exercises: Guess and check to find the mean. Write an addition statement to verify your guess.

a) 56, 68, 69, 59, 67, 65
b) 45, 46, 47, 45, 45, 52, 46, 50
c) 128, 131, 127, 129, 136, 131, 128

Answers: a) mean = 64, because 8 + 5 = 4 + 5 + 3 + 1; b) mean = 47, because 2 + 1 + 2 + 2 + 1 = 5 + 3; c) mean = 130, because 2 + 3 + 1 + 2 = 1 + 6 + 1

Students can double-check their answers by using a calculator to add the data values, then divide the sum by the number of data values.

Deviation from the mean. SAY: To find the deviation of a data value from the mean, you can subtract the mean from the data value. Write on the board:

\[
\begin{array}{cccccc}
4 & 2 & 3 & 0 & 6 \\
\end{array}
\]

SAY: The mean of the data set is 3. Let’s find the deviation of each data value from the mean. Write the subtraction for each value underneath the data set, as shown below:

\[
\begin{array}{cccccc}
4 & 2 & 3 & 0 & 6 \\
4 - 3 & 2 - 3 & 3 - 3 & 0 - 3 & 6 - 3 \\
\end{array}
\]

Then calculate the deviations and write on the board:

\[
\begin{array}{cccccc}
1 & -1 & 0 & -3 & 3 \\
\end{array}
\]

Explain to students that the deviation is not a new concept for them because they’ve already been finding the difference above the mean and below the mean. But now, below the mean has negative sign. Remind students that the total difference above and below the mean was equal, so the sum of all deviations from the mean is zero because the positive and negative deviations cancel each other out. As an example, add all deviations from the mean for the example on the board, as shown below:

\[
1 + (-1) + 0 + (-3) + 3 = 0
\]

(MP.1, MP.7) Exercises: Guess the mean, then check your guess by adding the deviations from the mean.

a) 3, 3, 4, 4, 5, 5, 6, 7, 8
b) 2, 2, 4, 5, 6, 7, 7, 15

Selected solution: b) Guessing 5 gives deviations of −3, −3, −1, 0, 1, 2, 2, 10, which gives a total that is positive, so the mean needs to be higher. Guessing 6 gives deviations of −4, −4, −2, −1, 0, 1, 1, 9, which gives a total of 0, so 6 is the mean.

Answers: a) 5, b) 6

Write on the board:

\[
1 + 1 = 2
\]

SAY: This addition sentence shows the amounts below a mean (point at 1 + 1) and the amounts above a mean (point at 2) for a data set with 3 data values. We can create a diagram with
mean 5 for this addition sentence. We will put the spaces below the mean on the left-hand side, and the balls above the mean on the right-hand side. Draw on the board:

![Diagram showing mean and data set]

ASK: What data set did this create? (4, 4, 7) How did I know the first data value was 4? (the first number in the addition sentence is 1, so the data value is 1 less than the mean 5) How did I know the last data value was 7? (the last number in the addition sentence is 2, so the data value is 2 above the mean 5) Have students confirm that the mean is 5 by independently adding the data values and dividing by the number of data values.

**Extensions**

(MP.3) 1. Tell students that data values can be negative. For example, data values that are temperatures. Have students use the addition statement 4 + 7 = 3 + 3 + 3 + 2 to build a data set with mean 5. (1, −2, 8, 8, 8, 7) The left side is used for below the mean and “4” means four below the mean. The mean is 5. The data value is 4 less than that, so the data value is 1. The 7 means seven below the mean, which is −2. The right side is used for above the mean and “3” means three above the mean, which is 8. Challenge students to build their own data sets with mean 5 by coming up with their own addition statements.

**Bonus:** Specify the number of data values to make it more difficult.

2. There is another way to find a set of 3 numbers with a mean of 5. For example, start with the data set 5, 5, 5, then add and subtract the same number from the data values. Every time you add 1 to a data value, you must subtract 1 from a different data value. Example:

5, 5, 5 → 4, 5, 6 → 3, 6, 6 → 2, 7, 6 → 2, 8, 5 → 3, 8, 4

a) Find a set of 3 numbers with a mean of 8.
b) Find a set of 4 numbers with a mean of 5.

3. Can you build a data set that satisfies the conditions? Either give an example or explain why it isn’t possible.
a) The mean score on a test is greater than 50, but more than half the students have marks less than 50.
b) The median score is greater than 50, but more than half the students have marks less than 50.
c) The mean score is greater than 90 (out of 100), but more than half the class has scores less than 50.

**Sample answers:** a) yes, 48, 49, 49, 49, 60 has mean 51; b) no, by definition, the median is the middle number so it will be less than 50; c) no, because even if all the marks less than 50 are 49 and all the marks greater than 50 are 100, the average will be less than the average between 49 and 100, so less than 74.5.
SP7-13  IQR and Box Plots

Pages 181–183

Standards: preparation for 7.SP.A.2, 7.SP.B.3, 7.SP.B.4

Goals:
Students will find the mean, median, range, and the interquartile range of a data set.
Students will identify the five features of box plots, and match box plots to dot plots.

Prior Knowledge Required:
Can find the range and the median of a set of data

Vocabulary: box plot, center, data point, data set, data value, dot plot, first quartile (Q1), interquartile range (IQR), lower half, mean, median, third quartile (Q3), upper half, value, variability

Materials:
transparency of BLM Box Plots for Large Data Sets (p. S-66)
BLM Reading Box Plots (p. S-67, optional)
BLM Matching Dot Plots with Box Plots (p. S-68, optional)
BLM Comparing IQRs (p. S-69, optional)
BLM Grade 7 Math Test Dot Plots (p. S-70, optional)

Review median. Remind students that the median is the number in the center position of an ordered set. Write on the board:

Set 1: 1, 4, 4, 6, 9, 11, 12
Set 2: 1, 4, 4, 6, 9, 11

ASK: When there are an odd number of data points, like in Set 1, how do we find the median? (it’s the number in the center position) What is the median of Set 1? (6) How do we find the median in sets with an even number of data points, like Set 2? (add the two middle numbers and divide by two) What is the median of Set 2? (5)

Dividing data sets in half. SAY: We use the median to divide sets in half, creating a lower half and upper half. Write on the board:

1, 2, 4, 9

SAY: This set has an even number of data points. ASK: What is the median? (3) How many data points are in each half? (2) Show the breakdown into halves, as shown below:

1, 2, 4, 9
lower half
upper half
SAY: Now we’ll do this with a set that has an odd number of data points. Write on the board:

1, 2, 4, 9, 13

ASK: What is the median? (4) ASK: Can we divide 5 points into 2 equal halves? (no) Explain that when sets have an odd number of data points, we divide them into equal halves by not including the median in either half. Show this on the board, as shown below:

```
1, 2, 4, 9, 13

lower half
upper half
```

Point out that the median, 4, isn’t in either half. ASK: When we leave out the median, how many data points are in each half? (2)

**Exercises:** Write the set. State the median. State the lower and upper halves, and label them.

a) 1, 2, 3, 4, 5, 6, 7, 8, 9
b) 1, 2, 3, 4, 5, 6, 7, 8

c) 6, 32, 45, 49, 51, 66, 104
d) 13, 23, 33, 43, 53, 63

**Bonus:** 650, 1,272, 1,338, 2,601, 4,039

**Answers:**
a) median = 5, lower half: 1, 2, 3, 4, upper half: 6, 7, 8, 9
b) median = 4.5; lower half: 1, 2, 3, 4; upper half: 5, 6, 7, 8

c) median = 49; lower half: 6, 32, 45; upper half: 51, 66, 104
d) median = 38; lower half: 13, 23, 33; upper half: 43, 53, 63

**Bonus:** median = 1,338; lower half: 650, 1,272; upper half: 2,601, 4,039

**MP.6 Finding the first and third quartiles.** Tell students that the median of the lower half is called the *first quartile* or *Q1*. The median of the upper half is called the *third quartile* or *Q3*. Q1 and Q3 can each be a data value, or not.

**Exercises:** Underline the lower half and the upper half of each part of the previous exercises, then find Q1 and Q3.

**Answers:**
a) Q1= 2.5, Q3 = 7.5; b) Q1= 2.5, Q3 = 6.5; c) Q1= 32, Q3 = 66; d) Q1= 23, Q3 = 53;

**Bonus:** Q1= 961, Q3 = 3,320

Write the data set shown below on the board. Mark the median, but do not add the quartile labels and their arrows:

```
lower half
3 4 6 7

upper half
9 19 19 25

first quartile Q1 = 5

M

third quartile Q3 = 19
```
SAY: "M" stands for median. ASK: What is the median? (8) Write it in. ASK: Looking at the lower half, where is its median or center? (between 4 and 6) Draw the arrow for the first quartile.

ASK: What is the median of the lower half? (5) SAY: The median of the lower half, or the first quartile, is 5. Label it "Q1." ASK: Looking at the upper half, where is its median? (between 19 and 19) Draw the arrow for the first quartile. SAY: The median of the lower half is between 19 and 19, so it is 19. The median of the upper half, or the third quartile, is 19. Label it "Q3."

**NOTE:** Another notation for median (M) is Q2.

**Exercises:** Copy the data set into your notebook. Mark the position of the median (M), the first quartile (Q1), and the third quartile (Q3). State their values.

| a) 15, 25, 35, 45, 55, 65, 75, 85, 95, 105, 115, 125 |
| Q1 = 40 | M = 70 | Q3 = 100 |

| b) 1, 4, 6, 8, 8, 11, 11, 17, 19, 23, 30, 30, 34, 37, 41, 42 |
| Q1 = 8 | M = 18 | Q3 = 32 |

**Finding the first and third quartiles for sets with an odd number of data values.** Write the data set below on the board with no labels, arrows, or brackets:

| lower half | upper half |
| 1 2 4 6 6 9 12 14 17 29 29 |

ASK: What is the median of this set? (9) Label it. What numbers are in the lower half? (1, 2, 4, 6, 6) ASK: What is the first quartile, or the median of the lower half? (4) Label it. ASK: What numbers are in the upper half? (12, 14, 17, 29, 29) What is the third quartile, or the median of the upper half? (17) Label it.

**Exercises:** Write the set. Mark the positions of the median (M), the first quartile (Q1), and the third quartile (Q3). State their values.

| a) 2, 11, 13, 14, 21, 23, 24, 28, 32 |
| Q1 = 12 | M = 21 | Q3 = 26 |

| b) 0, 4, 5, 10, 16, 17, 18, 18, 21, 22, 22 |
| Q1 = 5 | M = 17 | Q3 = 21 |

**Introducing the interquartile range.** SAY: So far in this lesson we have found the median and the first and third quartiles of data sets. Write on the board:

| 1 3 5 10 12 17 19 25 |
Have students help you find and label the median, Q1, and Q3. (11, 4, 18) Draw a line from Q1 to Q3. The final picture should look like this:

M = 11

1 3 5 10 12 17 19 25

Q1 = 4 Q3 = 18

SAY: This distance between Q1 and Q3 is the interquartile range or IQR. Write on the board:

Interquartile range = Q3 − Q1

SAY: We find the value of the interquartile range by subtracting Q1 from Q3. Have students help you find the IQR, as shown below:

Interquartile range = Q3 − Q1
= 18 − 4
= 14

SAY: The IQR tells us how the data in the set is arranged around the median. The wider this distance is, the more widely spread out the data is around the median. Explain that it is like the range but instead of showing the distance between the highest and lowest values in the set, it shows the distance between Q1 and Q3. It’s the range of the middle half (50%) of the data.

**Exercises:** Find the median, Q1, Q3, and IQR.

a) 5, 5, 6, 7, 7, 7, 17, 18, 18
b) 3, 4, 8, 8, 8, 13, 22, 24, 25

**Bonus:** 167, 245, 267, 379, 381, 604, 899, 907, 987, 1,291, 1,314

**Answers:**
a) M = 7, Q1 = 6, Q3 = 17, IQR = 11; b) M = 8, Q1 = 6, Q3 = 23, IQR = 17;
Bonus: M = 604, Q1 = 267, Q3 = 987, IQR = 720

**Finding the unknown when you know any two of the values for IQR, Q1, and Q3.** Explain that if you know any two of the values for IQR, Q1, or Q3 you can find the third value. Write on the board:

IQR = 45, Q1 = 10, Q3 = ?
IQR = Q3 − Q1

Have students help you substitute in the values of IQR and Q1, then solve the equation for Q3, as shown below:

IQR = Q3 − Q1
45 = Q3 − 10
45 + 10 = Q3
Q3 = 55
Exercises: Find the unknown value.
a) $Q_3 = 8$, IQR = 5, $Q_1 = ?$
b) $Q_1 = 22$, IQR = 16, $Q_3 = ?$
**Bonus:** $Q_1 = 421$, IQR = 579, $Q_3 = ?$
**Answers:** a) $Q_1 = 3$; b) $Q_3 = 38$; Bonus: $Q_3 = 1,000$

Comparing using the interquartile range. Draw on the board:

<table>
<thead>
<tr>
<th>Group 1 Quiz Marks</th>
<th>1</th>
<th>11</th>
<th>13</th>
<th>14</th>
<th>16</th>
<th>17</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 2 Quiz Marks</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>14</td>
<td>16</td>
<td>17</td>
<td>19</td>
<td>20</td>
</tr>
</tbody>
</table>

Explain that these are the marks two groups of students got on a quiz that was out of 20. SAY: We can use the interquartile ranges to help us understand how the quiz marks for each group compare with each other. ASK: Did the two groups get the same highest and lowest marks? (yes) Ask a volunteer to find the interquartile range for both groups, as shown below:

Group 1: $Q_1 = 12$, $Q_3 = 18$
IQR = $Q_3 - Q_1$
$= 18 - 12$
$= 6$

Group 2: $Q_1 = 5$, $Q_3 = 18$
IQR = $Q_3 - Q_1$
$= 18 - 5$
$= 13$

(MP.5) Explain to students that the interquartile range is more useful here than the range because the IQR gives information about the middle (50%) of the data.

Using box plots to summarize data set information. SAY: Often we need to compare more than just the interquartile range. One way to do that is by using box plots.

SAY: A box plot is a diagram that summarizes five important features of a data set: the lowest value, highest value, median (M), Q1, and Q3. Box plots also tell us about how spread out a data set is. Write on the board:

| 0 | 1 | 3 | 4 | 6 | 7 | 19 | 20 |

Draw a number line from 0 to 20 on the board below the data set. Have volunteers plot the data on a horizontal line above the number line. SAY: Let’s divide up this set by finding Q1, the median, and Q3. Have volunteers mark and label Q1, the median (M), and Q3. Draw the box around the plotted data and vertical lines at Q1, M, and Q3, as shown below.
Tell students that box plots look a lot like these diagrams. Draw the box plot above the diagram, as shown below:

SAY: In box plots the box doesn’t go around the entire set. It only goes around the interquartile range. It extends from Q1 to Q3, giving us information about how the data is arranged around the center. SAY: Since the highest and lowest values are important to know, we show them with these two horizontal lines. Point to the horizontal lines that extend from either end of the box in turn, and SAY: One of the lines is from Q1 to the lowest value in the set, and the other line is from Q3 to the highest value. Point out that the ends of these lines are precisely where the highest and lowest data values are.

**Finding the median from a dot plot.** Remind students that they can find the median of a dot plot by writing the dots as a list of data values and counting to the middle. Draw on the board:

![Dot plot example](image)

ASK: What is the median? (2.5)

**(MP.5) Why use box plots when dot plots are easy to draw?** Explain that with very small sets such as previous example, it’s hard to understand why a box plot is any more convenient than just creating a dot plot.

Project a transparency of **BLM Box Plots for Large Data Sets** on the board. Tell students that you are going to need their help to create a box plot for this data set. Ask students to count the data points. (31)

If you think they need to see the data in an ordered list to answer the following questions, have them help you write it on the board. ASK: If there are 31 data points, where is the median? (at the 16th data point) Have students help you find it and tell you its value. (14) Mark the median with an arrow and label it below the number line. ASK: How many data points are in the lower half? (15) If there are 15 data points in the lower half, where is Q1? (at the 8th data point) Have students help you find it and tell you its value. (12) Mark Q1 with an arrow and label it. ASK: Where is Q3? (at the 24th data point) Have students help you find it and tell you its value. (15) Mark Q3 with an arrow and label it. Ask students for the lowest value and the highest value. (2, 17) Mark and label the lowest and highest values.
Draw vertical dashed lines from each labeled point from the number line to above the dot plot. Explain that the lines will help you draw the box plot in the right position. Draw the box plot above the dot plot, as shown below:

ASK: If you only look at the box plot, and not the dot plot, can you tell where all the individual data points are? (no) SAY: That’s not a problem because we use box plots to summarize some of the most important features of data sets, not to show all the data as a dot plot would.

For extra practice, give students BLM Reading Box Plots, BLM Matching Dot Plots with Box Plots, BLM Comparing IQRs, and BLM Grade 7 Math Test Dot Plots. (see selected answers below)

BLM Reading Box Plots:

<table>
<thead>
<tr>
<th>Box Plot A</th>
<th>Median</th>
<th>Q1</th>
<th>Q3</th>
<th>IQR</th>
<th>Highest Value</th>
<th>Lowest Value</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Box Plot B</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>9</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>Box Plot C</td>
<td>12</td>
<td>7</td>
<td>13</td>
<td>6</td>
<td>17</td>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>Box Plot D</td>
<td>13</td>
<td>11</td>
<td>16</td>
<td>5</td>
<td>20</td>
<td>6</td>
<td>14</td>
</tr>
<tr>
<td>Box Plot E</td>
<td>11</td>
<td>4</td>
<td>14</td>
<td>10</td>
<td>18</td>
<td>2</td>
<td>16</td>
</tr>
<tr>
<td>Box Plot F</td>
<td>7</td>
<td>5</td>
<td>10</td>
<td>5</td>
<td>12</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>Box Plot G</td>
<td>12</td>
<td>10</td>
<td>14</td>
<td>4</td>
<td>14</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

BLM Matching Dot Plots with Box Plots: 1. C, B, A

BLM Comparing IQRs:
2. Set 1: IQR = 13, Q3 = 19, Q1 = 6, median = 10
   Set 2: IQR = 11, Q3 = 17, Q1 = 6, median = 12
   Set 3: IQR = 6, Q3 = 18, Q1 = 12, median = 17
   Set 4: IQR = 15, Q3 = 18, Q1 = 3, median = 16
BLM Grade 7 Math Test Dot Plots:

1. Class A

<table>
<thead>
<tr>
<th>Q1</th>
<th>Q3</th>
<th>IQR</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>95</td>
<td>20</td>
</tr>
</tbody>
</table>

Class B

<table>
<thead>
<tr>
<th>Q1</th>
<th>Q3</th>
<th>IQR</th>
</tr>
</thead>
<tbody>
<tr>
<td>85</td>
<td>95</td>
<td>10</td>
</tr>
</tbody>
</table>

2. Number of Data Points | Median | Range | IQR
---|---|---|---
Dot Plot 1: Class A | 23 | 90 | 30 | 20
Dot Plot 2: Class B | 23 | 90 | 30 | 10

3. a) Class B, b) Class B, c) yes

**Extension**

(MP.1) Create two different sets of data, each with 7 data points, lowest value = 4, highest value = 13, median = 10, IQR = 3. Use the diagram below as a guide.

_____ _____ _____ _____ _____
Q1 M Q3

**Sample answers:** 4, 8, 9, 10, 10, 11, 13; and 4, 10, 10, 10, 10, 13, 13, 13
SP7-14  Mean Absolute Deviation (MAD)

Pages 184–187

Standards: 7.SP.B.4

Goals:
Students will calculate and interpret the mean absolute deviation of sets of data.
Students will investigate how adding or removing an outlier affects the measures of central tendency and the range of a set of data.

Prior Knowledge Required:
Can calculate the mean and range of a set of data
Can find the absolute value of a number
Can add and subtract decimals
Can multiply and divide decimals by whole numbers

Vocabulary: absolute deviation, absolute value, box plot, clustered, data points, data set, data value, deviation, dot plot, interquartile range (IQR), mean, mean absolute deviation (MAD), measures of central tendency, median, mode, outliers, range, symmetric, variability

Materials:
BLM Five-Part Spinner (p. S-71), one per pair of students
a paper clip for each pair of students
access to software with statistical functions (e.g., MS Excel) for each student

Review finding the mean. Remind students that to find the mean of a data set, they divide the sum of the data points by the number of data points. ASK: What is the mean of 1, 6, and 20? (9)

Review finding the range. Write on the board:

6, 7, 13, 15
Range = highest − lowest

Remind students that when you find the difference between the highest and lowest values in a set, you are finding the range. ASK: What is the range of the set? (9)

Review absolute value. Remind students that the absolute value of a number is its distance from zero without considering whether it is left or right of zero. Draw on the board a number line from -5 to +5. Draw a dot at -3 and an arrow from 0 to -3. ASK: How many units is -3 from 0? (3 units) Write on the board:

|−3| = 3
SAY: We use this symbol, a line on each side of the number, to mean we are finding the number’s absolute value or its distance from zero. Repeat for +4. (|+4| = 4) The final picture should look like this:

```
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>+1</td>
<td>+2</td>
<td>+3</td>
<td>+4</td>
<td>+5</td>
</tr>
</tbody>
</table>
```

**Exercises:** Find the absolute value.

a) |−24|  b) |+24|  c) |237|  d) |−34,999|

e) |−3²|  f) |(−3)²|  g) |1|  h) |−2| 3 |

**Answers:** a) 24; b) 24; c) 237; d) 34,999; e) 9; f) 9; g) 1/5; h) 2/3

**Activity 1**

**Collecting data.** Have students work in pairs. Give each pair **BLM Five-Part Spinner**, a paper clip, and a pencil. Have each pair of students collect two sets of data, each made up of 10 spins. For the first set, one student spins and the other records the results. For the second set, the students switch roles. If the spinner lands on a line between two regions, they repeat the spin.

(end of activity)

(MP.7) **Another tool for describing sets.** Explain that when we work with sets of data, as in the spinner activity, we need to be able to describe the data. SAY: We can do this in several ways: explaining how the data was collected, counting the number of data points, finding the center, and finding the range. We also need to describe how the results vary, in other words, their variability. Draw on the board:

```
Dot Plot 1
  1 2 3 4 5

Dot Plot 2
  1 2 3 4 5
```

SAY: Let’s pretend these two dot plots are from the spinner activity you just did. Dot Plot 1 shows that the spinner landed on 1 four times, 3 twice, and 5 four times. Draw on the board the table below (without any values), and have students fill in the cells, as shown below:

<table>
<thead>
<tr>
<th></th>
<th>Number of Dots</th>
<th>Mean</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dot Plot 1</td>
<td>10</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Dot Plot 2</td>
<td>10</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

(MP.5) **ASK:** Do the dot plots have the same number of points? (yes) Do they have the same mean? (yes) Do they have the same range? (yes) Cover up but don’t erase the dot plots.

**ASK:** If you had not seen the dot plots at all, can you tell just by looking at the table that the dot plots are different? (no) Explain that the dot plots are clearly different, and since we sometimes
only see the tables, we need to include something in the table that shows the difference.
SAY: Both the mean and median are 3. Dot Plots 1 and 2 are both symmetric, so the mean and
median are equal. Circle the 3 on both number lines of the dot plots. ASK: Would you say the
points in one of these dot plots are more spread out from the mean than in the other dot plot?
(yes, they’re more spread out in Dot Plot 1) Leave the dot plots and the chart on the board.

**Absolute deviation.** SAY: To find the absolute deviation of any data point from the mean, you
find the deviation from the mean then find the absolute value of that. ASK: What is the mean of
3, 4, and 8? (5) Have students find the deviation from the mean of each value, as shown below:

<table>
<thead>
<tr>
<th>Value</th>
<th>Deviation from Mean</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3 - 5 = -2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4 - 5 = -1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>8 - 5 = 3</td>
<td>3</td>
</tr>
</tbody>
</table>

Draw on the board a number line from 3 to 8. Plot the data points and circle the mean of 5.
Draw a line from 3 to 5 and ASK: What is the deviation of 3 from the mean? (−2) SAY: We say
that the absolute deviation of 3 is 2. Write this on the board above the arrow. Repeat for 4 and
8. (1, 3) The final picture should look like this:

```
absolute deviation = |4 − 5| = 1
absolute deviation = |8 − 5| = 3
absolute deviation = |3 − 5| = 2
```

SAY: We just found the distance from each data point in the set to the mean. Emphasize that
the distance between two data values can’t be negative, and that the distance is actually the
absolute value of the difference.

Write on the board:

1, 2, 7, 10

ASK: What is the mean of this set? (5) Draw on the board a number line from 1 to 10. Have a
volunteer mark the points and circle the mean. Have volunteers draw arrows and label them to
show the absolute deviations. The final picture should look like this:

```
absolute deviation = 5
absolute deviation = 2
absolute deviation = 3
absolute deviation = 4
```

1  2  3  4  5  6  7  8  9  10
**Exercise:** Find the mean of the set 3, 8, 10. Draw a number line, then mark the points, mean, absolute deviation lines, and absolute deviations for each point.

**Answer:**

```
<table>
<thead>
<tr>
<th>Data Point</th>
<th>Mean</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>17</td>
<td>10</td>
<td>7</td>
</tr>
</tbody>
</table>
```

**Introduce mean absolute deviation (MAD).** SAY: Knowing the absolute deviation for every point is the same as knowing how far every point is from the mean. When we find the average of all these distances, or absolute deviations, we are finding the *mean absolute deviation*, or MAD. The mean absolute deviation is the average distance of all the points from the mean. It helps us describe the variability in a set by showing how tightly clustered or widely spread out the data in a set are from the mean. For instance, if all the points are close to the mean, the average distance will be small. If all the points are far from the mean, the average distance will be large. And if some points are close and others are far, the average distance will be somewhere in between. Explain that points can be any distance from the mean, including on the mean. Write on the board:

```
6, 7, 17
```

Write on the board a table with the headings Data Point, Mean, and Absolute Deviation. Fill in the Data Point column for the given data points. ASK: What is the mean of the data set? (10) Fill in the Mean column. ASK: What is the absolute deviation of the first data point, 6? PROMPT: How far from 10 is 6 on the number line? (4) Write “4” in the Absolute Deviation column for the first data point. Draw a number line to show the distance, as shown below:

Distance = 4

Repeat for the remaining data points, then write the total absolute deviation below the table. The completed table should look like this:

```
<table>
<thead>
<tr>
<th>Data Point</th>
<th>Mean</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>17</td>
<td>10</td>
<td>7</td>
</tr>
<tr>
<td>Total absolute deviation = 14</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```
ASK: How do we find the average of the three absolute deviations? (add them, then divide by 3)

Write on the board:

\[
\frac{4 + 3 + 7}{3} = \frac{14}{3} = 4\frac{2}{3}
\]

SAY: The mean absolute deviation for the data set is 4 2/3.

Exercises: Complete the table. Find the mean absolute deviation.

<table>
<thead>
<tr>
<th>Data Point</th>
<th>Mean</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Total absolute deviation = 6

MAD = 2

<table>
<thead>
<tr>
<th>Data Point</th>
<th>Mean</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Total absolute deviation = 12

MAD = 4

What the mean absolute deviation actually is. Explain that now you are going to draw number lines for parts a) and b) in the previous exercises to show what the mean absolute deviation actually is. Draw on the board two number lines from 1 to 11. Have volunteers mark the data points using the sets in the previous exercises, then find and circle the mean. (the mean for both is 5) Write the mean absolute deviation beside each number line. The number lines should look like this:

![Number line for part a) with MAD = 2](image)

![Number line for part b) with MAD = 4](image)

(MP.4) ASK: Which set of data has the larger mean absolute deviation? (b) In which set are the points more widely spread out around the mean? (b) SAY: Sets of data that are more widely spread out around their mean have a larger MAD, and sets of data that are more tightly clustered around the mean have a smaller MAD.
Comparing the MADs in two dot plots. Refer students back to the two dot plots from the beginning of the lesson. Explain that now you want students to help you calculate the mean absolute deviation for these dot plots. For each dot plot, draw on the board a table with the headings Data Point, Mean, and Absolute Deviation. ASK: What goes in the Mean column for both dot plots? (3) Write it in.

Refer students to Dot Plot 1 and have them tell you how to fill in the Data Point column, then the Absolute Deviation column. When you get to the two data points that are at 3, point out that since they are right on the mean, the distance from each to the mean is zero. Have students find the total of the deviations. (16) Write it below the table. The final picture should look like this:

<table>
<thead>
<tr>
<th>Data Point</th>
<th>Mean</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Total absolute deviation = 16

ASK: What do we divide 16 by to get the mean absolute deviation? (the total number of data points, 10) Write on the board:

\[ \text{MAD} = \frac{16}{10} = 1.6 = 1 \frac{3}{5} \]

SAY: The MAD for Dot Plot 1 is 1 3/5.
Repeat with Dot Plot 2. The final picture should look like this:

**Dot Plot 2**

<table>
<thead>
<tr>
<th>Data Point</th>
<th>Mean</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Total absolute deviation = 10

\[
\text{MAD} = \frac{10}{10} = 1
\]

**ASK:** Which dot plot has the larger mean absolute deviation? (Dot Plot 1) So which set of data is more widely spread out around its mean? (Dot Plot 1) Refer students back to the table from the beginning of the lesson. Draw a fourth column with the heading Mean Absolute Deviation, and write the MADs that you just found. The completed table should look like this:

<table>
<thead>
<tr>
<th>Number of Dots</th>
<th>Mean</th>
<th>Range</th>
<th>Mean Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dot Plot 1</strong></td>
<td>10</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td><strong>Dot Plot 2</strong></td>
<td>10</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

**MP.5** **ASK:** If we can only see the table, but not the dot plots, what more would we understand about how the dot plots are different? (Dot Plot 2 has a smaller mean absolute deviation, so the data points are more tightly clustered around the mean)

**Exercises:**

1. The table shows the absolute deviations for three sets of data.

<table>
<thead>
<tr>
<th>Set 1</th>
<th>1</th>
<th>0</th>
<th>4</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set 2</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Set 3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

a) Find the mean absolute deviation of each set.
b) Which set is most spread out around the mean?

**Answers:** a) Set 1 MAD = 2, Set 2 MAD = 3, Set 3 MAD = 0; b) Set 2
(MP.3) 2. Draw the data sets on separate number lines: Set 1 (0, 1, 8) and Set 2 (26, 27, 34).
Explain how the data sets can have different data points, but the same mean absolute deviation.

Set 1:  

Set 2:  

Answers: The data points in Set 1 are exactly the same distances from the mean as the data points in Set 2. The mean absolute deviation depends on distance from the mean, no matter where the mean is.

How outliers affect measures. An outlier is a value that is very different from the other values in the set, so that removing it from the set changes the range significantly. Explain to students that sometimes you need to decide when an outlier should and shouldn’t be included.
Sometimes an outlier is a mistake. For example, we might record the ages of all the people over 60 at a tennis club as 63, 65, 6, 68, 72, 71, 64, 63, 62, 65, 61.

ASK: Which piece of data is the outlier? (6) Would you include that data value when calculating the mean? Have students calculate the mean, the mode, and the median of the data with and without the outlier.

ASK: Which was most affected by the outlier, the mean, the mode, or the median? (the modes are 63 and 65, both with and without the outlier; the median is 64 with the outlier and 64.5 without the outlier; the mean changes more significantly, from 60 with the outlier to 65.4 without it) When you calculate the mean age including the outlier, how can you know immediately that there was a mistake somewhere? (the mean including the outlier is 60; the mean age of several people over 60 cannot be 60!)

Activity 2
Have students use software with built-in statistical functions (e.g., MS Excel) to investigate the result of changes in values on sets of data. For example, have students create a table with the following values:

| 75 77 69 75 90 75 73 65 68 84 |
| 65 73 71 75 70 95 97 65 72 86 |

Ask students to find the mean (average), median, and mode (all found among statistical functions) of the set. Then have them change, add, or remove data values to see the effect on the mean, the median, and the mode.

(end of activity)

Using dot plots and box plots to find central tendency. Draw on the board:
SAY: The dot plot and box plot both show the data set 1, 1, 2, 2, 2, 2, 3, 3, 4, 4, 4, 4, 4, 4, 5.

ASK: What is the median? (3) Which type of graph makes it easier to find the median, the box plot or the dot plot? (box plot) 
ASK: Which type of graph makes it easier to find the mode? (dot plot) 
Have students find the mean. (3) 
ASK: Which graph can you use to find the mean? (dot plot) 
Can you use the box plot to find the mean? (no) 
Can you determine from the box plot if the data set is symmetric? (no) 
From the dot plot? (yes,) 
Explain to students that box plots emphasize the middle 50% and IQR of the data set, so it’s easier to see variability; dot plots show all data values, so it’s easier to find the mean and mode.

Exercises:

![Box plots](image)

a) Find the median of each box plot.
b) Find the range and IQR of each box plot.
c) Which box plot has the lower variability?

**Answers:**
a) A: median = 10, B: median = 11; 
b) A: range = 12, IQR = 5; B: range = 9, IQR = 4; 
c) B has lower variability

**Extensions**

**MP.1** 1. Create a data set with 3 data points, mean = 5, and the total of the 3 absolute deviations = 6. Use the number line as a guide.

![Number line](image)

**Answer:** Answers may vary. Sample answer: 3, 4, 8

**MP.1** 2. The median is usually less affected by adding an outlier than the mean. However, this is not always the case. Consider a set with 1,000 values of 1 and 999 values of 5. The mean is $5,995 \div 1,999 = \text{about 2.999}$. The median is 1. Now add an outlier of 25. The new set has 2,000 values with mean $6,020 \div 2,000 = 3.01$, so the change in the mean is about .01. The median is halfway between the two middle values, 5 and 1, which is 3, so the change in the median is 2. This happens because the outlier is not large enough to affect the mean—there are too many small data values.

**MP.3** How large does the outlier in the set described above need to be to make the change in the mean more significant than the change in the median?

**Solution:** The median will be 3 no matter how large the outlier is. So we need the mean to change by at least 2. When a larger number than the mean is added, the mean increases. So the new mean should be at least 4.999. For the sake of simplicity, let the new mean be 5. There are 2,000 data values in the new set, so the total of the data values should be $5 \times 2,000 = 10,000$. The total of the data values in the new set = the total of the data values in the old set + outlier = 5,995 + outlier = 10,000. So the outlier needs to be at least 4,005.
3. Discuss whether the outliers should be included when calculating the measures of central tendency in the given situation.

a) Ivan records the ages of campers in the group he is responsible for at a summer camp: 6, 7, 7, 6, 77, 6, 7. Should the outlier be included in when calculating the mean or median? Why or why not?

b) The panel of judges for a school talent show includes three teachers and one representative from each grade. Eva is performing in the competition, and her best friend is one of the judges. Eva’s marks out of 10 are: 3, 4, 3, 5, 3, 4, 3, 4, 10, 3, 3. Should the outlier be included when finding Eva’s average mark?

c) At a school fundraiser, the following amounts of money (in dollars) were raised per grade: 150, 130, 127, 854, 99, 112.50, 154, 167. To find the average amount of money raised per grade, should the outlier be included?

**Answers:**
a) not for mean, but the outlier doesn’t change the median significantly; 
b) no; c) yes

(MP.3, MP.6) 4. a) What shape is the cross section of the cube? Explain how you know using math words.

![Cross section of a cube](image)

b) In pairs, discuss your answers to part a). Do you agree with each other? Discuss why or why not.

**Sample answer:** The cross section is part of a plane. When a plane meets two parallel faces, the lines they make are parallel or the plane would have to turn or be curved and then it wouldn’t be a plane anymore. So the two sides of the cross section that are in opposite faces of the cube are parallel. The other two sides are not in opposite faces and are not parallel. So the shape is a quadrilateral with one pair of parallel sides. That makes it a trapezoid.

**NOTE:** In part b), encourage partners to ask questions to understand and challenge each other’s thinking (MP.3) and use of vocabulary (MP.6)—see p. A-49 for sample sentence and question stems.
SP7-15  Bias in a Sample or Census

Pages 188–189

Standards: 7.SP.A.1

Goals:
Students will decide when it is appropriate to use a survey of the whole population and when it is appropriate to only take a sample.

Prior Knowledge Required:
Knows what a survey is and understands what it can be used for
Can conduct a survey

Vocabulary: biased sample, census, over-represented, population, random, representative sample, sample, source of bias, survey, timing, under-represented, whole population

(MP.5) A census or a sample? Tell students that sometimes they will want to know things about very large groups of people, but they may not be able to gather data for everyone in the population. For example, if we want to know how many people in the United States have read “The Wizard of Oz,” it wouldn’t be practical to ask everyone if they’ve read it. SAY: Collecting information by surveying the entire population is called a census.

SAY: I want to find the average shoe size of everyone in New York State. ASK: Should I ask, or survey, everyone in the state, or only some people the state? (only some people) Why? (it would be too difficult to survey everyone in the state; a sample will do)

Repeat with a few more examples:
• If I want to know what books my classmates read last week, should I survey the whole class or only a sample? (the whole class)
• If I want to know the average shoe size of people on my block, should I survey everyone on the block or only a sample? (everyone on the block)
• If I want to know how many people in the United States watched a particular television show, should I ask everyone in the United States or just a sample? (a sample)

(MP.5) Exploring biased samples. Tell students that you want to know if Grade 7 students in New York State prefer action movies or comedies. ASK: Can I ask only students in New York City? Why or why not? (no, because the survey is about the whole state) Can I ask only boys? (no, that sample would be biased because it doesn’t represent the whole population)

SAY: A biased sample is not similar to the whole population because some part of the population is not represented. In the above example, Grade 7 students in New York State are the whole population. If only boys are surveyed, then girls are not represented. If only New York City students are surveyed, then people from other cities and towns in the state are not represented. If only public school students are surveyed, then private school students are not
represented. However, the question might focus only on a specific population. For example, if I want to know if boys prefer action movies or comedies, it wouldn’t make sense to ask girls.

SAY: A Grade 1 to 8 school is planning a games party and wants to decide what games to buy. ASK Would the sample be biased if the school surveyed all of the Grade 2 students? (yes) All of the boys? (yes) Every tenth student, when listed in alphabetical order? (no) 3 people chosen at random from each classroom? (no)

Tell students that a sample that is similar to the whole population is called a representative sample. Finding a representative sample is often the most difficult part of conducting a survey. For example, an apartment building manager would like to reserve the games room once a week for a dance. Discuss the bias if, to decide what kind of music should be played at the dance, the building manager:

a) Asks the bridge club. (people in the bridge club are likely to be older than the average building population)
b) Asks the soccer club. (people in the soccer club are likely to be younger)
c) Asks the book club. (more women tend to join book clubs than men)
d) Asks the teen movie club. (only teenagers will be represented)
e) Puts a survey under every tenth door by apartment number. (no bias)
f) Lists the names of people living in the building in alphabetical order and picks every tenth person to ask. (no bias)
g) Asks people at the playground. (teenagers and adults are less likely to be represented)

(MP.5) Timing can create bias. Explain that even surveying people at the same location, at different times, can produce different biases. ASK: How would your samples be different if you surveyed people at the mall on a weekday morning or on a weekend morning? Some points to discuss are provided below:

• On weekday mornings, most teenagers and many working people would be excluded.
• People who go to malls on weekday mornings could be unemployed, retired, have a part time job, or start working later in the day.
• University students might have class schedules that allow them to go to the mall on a weekday morning.
• Some people go to the mall on weekdays to avoid the crowds on the weekend.
• Any group that was over-represented (given too much representation) on weekday mornings may be under-represented (not given enough representation) on the weekend.

Repeat with the following question: If a mall wants to decide whether to rent space to a pet store or a video game store, how and when should they conduct a survey? Some points to discuss are provided below:

• The mall should probably survey customers at different times and weight the results differently based on how crowded the mall is. If more customers visit the mall during the weekend, results from the weekend should be given more weight.
• The mall is interested in biasing their sample to their customers. If it happens that more elderly people go to that mall, they will want the survey to target older people.
SAY: Suppose there are three apartment buildings next to the mall. The mall decides to survey every tenth apartment unit in these apartment buildings instead of asking mall customers at different times of day. ASK: Is this a representative sample? (no, because only people living in apartments are represented; this will bias the sample against people who live in nearby houses; not all the people who live in the apartments necessarily like visiting malls)

(MP.5) The wording of a question can affect the results of the survey. Discuss the following three cases, in which the sample is representative but the results are biased.

Case 1: A town council is thinking of selling a city park and allowing a department store to be built in its place. Two groups ask different questions:
A: Are you in favor of having a new store that will provide jobs for 50 people in our town?
B: Are you in favor of keeping our neighbourhood quiet and peaceful?

ASK: Which question do you think was proposed by someone in favor of selling the park? (A) Which was proposed by someone against selling the park? (B) Have students write a survey question that is more neutral and does not already suggest an answer. (Sample answers: How do you feel about building a new store on the park site? Would you agree or disagree with building a new store on the park site? Are you in favor of or against building a new store on the park site?)

Case 2: A student council chooses music and food for school events. Most people enjoy the music, but not the food. Two surveys are shown below:

Survey A
1. Do you enjoy the music at the events?
2. Is the student council doing a good job?
3. Do you enjoy the food at the events?

Survey B
1. Do you enjoy the food at the events?
2. Is the student council doing a good job?
3. Do you enjoy the music at the events?

ASK: What is the same about the two surveys? (they have the same three questions) What is different? (the questions are in a different order) Which survey is more likely to suggest that student council is doing a good job? Why? (Survey A, because it starts with a question about the music, which is what students are known to like best) How can the order of the questions affect the results of a survey? (if the question about whether student council is doing a good job is asked after something students are happy or unhappy with, the results will be biased about how the student council is doing; but if they ask the question about student council performance first, there would be no bias)

Case 3: Give half the students a list of questions as follows:
What color do you add to yellow to make orange?
What color do you add to blue to make purple?
What color is blood?
What color is a stop sign?
What color is a strawberry?
What color is a poppy?
What traffic light color do you go on?

...
Have students pair up and ask their partners the list of questions. Ask students how many had a partner who answered red to the last question (without saying who their partner was). Emphasize that this is because they were being primed to answer “red” because of all the previous questions. As soon as they heard the beginning of the question “what color of traffic light,” students likely pictured a red traffic light. SAY: Being primed for an answer can affect the results of a survey.

Tell students that you want to survey two different younger classes, (you may wish to have students conduct the surveys using two different questions), as shown below:

Question A: Is it okay to watch television while having a conversation with a friend? Question B: Is it okay to have a conversation with a friend while watching television?

ASK: How are the questions the same? How are they different? (they use the same words, but in a different order) Which question do you think will get more “Yes” answers? Why? (Even though the logical content of the questions is the same, the pictures you get in your mind when answering the two questions are different. When answering Question A, you are likely to picture someone not really paying attention to their friend because they are so interested in the television show; when answering Question B, you may picture two people who are talking with the television on in the background, or two people who are talking about the television show they are watching together. No one will offend the television by not paying attention to it.)

Extensions
1. **Another source of bias.** Tell students that you stood outside a hockey arena and counted the fans who were wearing jerseys of each team playing that evening. The results were 60% home team jerseys and 40% away team jerseys. Can you conclude that 40% of the fans at the game supported the away team? Explain. Hint: Is a fan who is wearing a jersey more likely to support the home team or the away team? Why?
   **Sample answer:** A fan of the away team is more likely to be coming from out of town, and therefore more likely to make the extra effort to wear their team’s jersey. The fans who wear jerseys are thus a biased sample, more likely to be cheering for the Away team.

2. **Sample size can also affect the outcome of a survey.** SAY: I want to know if a coin toss is fair. ASK: If I toss a coin once and it comes up heads, can I conclude that it is more likely to come up heads than tails? (no) What if I toss it 3 times and get 2 heads? (no) What if I toss it 300 times and get 280 heads? (yes, the coin toss is not fair)

SAY: A friend tells me that most students in the United States are against school uniforms. ASK: Can I make that conclusion after I ask one random student if he or she is against school uniforms? What if I ask 3 random students and 2 of them are against school uniforms? What if I ask 300 random students and 280 of them are against school uniforms?

ASK: When a random sample of the population is chosen, is it more likely to be representative of the whole population if it is a large sample or a small sample? Why?
SP7-16 Using Samples to Draw Inferences

Pages 190–191

Standards: 7.SP.A.2, 7.RP.A.2

Goals:
Students work with random samples and learn about the importance of representative samples when drawing inferences.

Prior Knowledge Required:
Can calculate the mean of a set of data
Can identify line-symmetric figures
Understands ratios, percentages, and fractions
Can multiply fractions

Vocabulary: box plot, cross multiply, decimal fractions, decimal numbers, equation, fractions, inferences, population, proportion, representative sample, sample, variable

Review writing equations using cross multiplying. Remind students how to cross multiply to write an equation when there is a variable in one of the fractions. Write on the board:

\[
\frac{8}{50} = \frac{x}{300} 
\]

SAY: I don’t know what number x is, but if I use cross multiplication, I know that 50 times x is equal to 8 times 300. Write on the board:

\[
8 \times 300 = 50 \times x, \text{ so } 2,400 = 50x 
\]

Exercises: Cross multiply to write an equation for x.

a) \( \frac{x}{9} = \frac{20}{45} \)  
b) \( \frac{15}{3} = \frac{x}{4} \)  
c) \( \frac{6}{x} = \frac{2}{5} \)  
d) \( \frac{3}{75} = \frac{x}{500} \)

Answers: a) 45x = 180; b) 60 = 3x; c) 30 = 2x; d) 1,500 = 75x

Using cross multiplication to solve equations. Review equations such as \( 2 \times b = 12 \), and remind students that to solve this type of equation, they have to divide both sides of the equation by the coefficient of the unknown. For example, in the equation \( 2b = 12 \), the answer is \( b = 12 \div 2 \), so \( b = 6 \).

Exercises: Solve the equations you wrote for the previous exercises.

Sample solution: a) 45x = 180, x = 180 ÷ 45 = 4

Answers: a) x = 4, b) x = 20, c) x = 15, d) x = 20
**Cross multiplying when the answers are decimal numbers.** Tell students that sometimes when we cross multiply to solve for \( x \), the answer will be a decimal number. Remind students that we can write ratios in fraction form even when both terms are not whole numbers. Review writing fractions as decimal fractions, then as decimals. Write on the board:

\[
\begin{align*}
\frac{1}{2} &= \frac{5}{10} = 0.5 \\
\frac{1}{4} &= \frac{25}{100} = 0.25 \\
\frac{2}{4} &= \frac{1}{2} = 0.5 \\
\frac{3}{4} &= \frac{75}{100} = 0.75 \\
\frac{1}{5} &= \frac{2}{10} = 0.2 \\
\frac{2}{5} &= \frac{4}{10} = 0.4 \\
\frac{3}{5} &= \frac{6}{10} = 0.6 \\
\frac{4}{5} &= \frac{8}{10} = 0.8
\end{align*}
\]

**Exercises:** Solve for \( x \).

a) \( \frac{10}{3} = \frac{4}{x} \)  

b) \( \frac{x}{5} = \frac{3}{6} \)  

c) \( \frac{6}{192} = \frac{x}{2000} \)  

d) \( \frac{15}{100} = \frac{x}{235} \)

**Sample solution:** c) \( 12,000 = 192x; \) \( x = 12,000 \div 192 = 62.5 \)

**Answers:** a) \( x = 1.2 \), b) \( x = 2.5 \), c) \( x = 62.5 \), d) \( x = 35.25 \)

**Use proportion to draw inferences.** SAY: If you use a representative sample of the population, you can guess that the proportion in the whole population will be close to the proportion in the sample. Reasoning from a sample to the population is called making an inference about a population.

Tell students that they can use a proportion to make inferences. SAY: For example, every day, 4 airplanes land or take off every 3 minutes at London Heathrow Airport. ASK: How many airplanes land or take off per hour every day? PROMPT: There are 60 minutes per hour, so to find the answer you can write a proportion. Write on the board:

\[
\frac{\text{minutes}}{\text{airplanes}} = \frac{3}{4} = \frac{60}{x}
\]

Ask a volunteer to solve the proportion and find \( x \). (80)

SAY: Now consider another question that is more related to statistics. Carlos wants to know how many airplanes land or take off from 1 p.m. to 2 p.m. at Heathrow Airport. He counts the number of airplanes in the first 10 minutes of the period, from 1:00 p.m. to 1:10 p.m., and he finds that 8 airplanes land or take off during the period. Ask students to write the proportion for the question. Give them time to write it in their notebooks, then ask a volunteer to write the proportion on the board and solve it, as shown below:

\[
\frac{\text{minutes}}{\text{airplanes}} = \frac{10}{8} = \frac{60}{x}
\]

\[
10x = 60 \times 8
\]

\[
10x = 480
\]

\[
x = 48
\]
SAY: Based on the sample, 48 airplanes land or take off from 1 p.m. to 2 p.m. Emphasize that if Carlos collects the data on another day and he finds that 9 airplanes land or take off between 1:00 p.m. to 1:10 p.m., then his prediction would be different.

Explain to students that, in this lesson, we assume that the sample has the same or close to the same proportion as the whole population, and therefore we can use the sample to make a good estimate for the whole population. In the next lesson, we’ll look at how to decide if we can be confident that a sample is at least close to the same proportion as the whole population, such as large randomly selected samples.

(MP.4) Exercise: Lily wants to know if people in a small town with a population of 5,000 are in favor of or against opening a new library. She asks 150 people at random, and finds that 96 people like the idea of a new library and 54 people do not. Estimate how many people like the idea of a new library.

Answers: about 3,200

Drawing box plots. Start with an example. Write on the board:

5, 3, 8, 2, 5, 4, 2, 6, 3, 3, 5, 5, 7, 7, 6

SAY: I want to draw a box plot for this data set. To draw a box plot, I follow these steps. Have students help you complete Steps 1 and 2.

Step 1: Order the data. (2, 2, 3, 3, 3, 4, 5, 5, 5, 5, 6, 6, 7, 7, 8)

Step 2: Determine the lowest value, the highest value, the median (M), Q1, and Q3. (lowest value = 2, highest value = 8, M = 5, Q1 = 3, Q3 = 6)

Step 3: Draw a number line from the lowest value to the highest value. Draw on the board:

\[
\begin{array}{cccccccc}
2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

Step 4: Mark the lowest value, Q1, M, Q3, and the highest value above the number line. Show the extreme values with a dot, and draw small vertical lines for the Q1, M, and Q3. The picture should look like this:

\[
\begin{array}{cccccccc}
& | & | & | & \cdot \\
2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

Step 5: Draw a line from the lowest value to Q1, draw a box from Q1 to Q3, and draw another line from Q3 to the highest value. The final picture should look like this:

\[
\begin{array}{cccccccc}
& | & | & | & \cdot \\
2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]
**Exercises:** Draw a box plot for the set of data.

a) 4, 1, 5, 8, 6, 2, 3, 4, 9, 6, 4, 3, 9, 7, 5
b) 75, 73, 81, 82, 80, 77, 74, 75, 77, 75

**Answers:**

a) ![Box plot for data a)](image)
b) ![Box plot for data b)](image)

**Extension**

(MP.1, MP.4) Zack is making a cake for his sister’s birthday. Here is the recipe:

- 1 cup white sugar
- \(\frac{1}{2}\) cup all-purpose flour
- \(\frac{1}{2}\) cup butter
- \(\frac{3}{4}\) tsp baking powder
- 2 eggs
- \(\frac{1}{2}\) cup milk
- 2 teaspoons vanilla extract

The recipe says to use one 10-inch diameter round pan. Zack only has a 9-inch diameter round pan. He wants to make the cake have about the same depth in the pan so he doesn’t have to change the cooking time or the oven temperature. He has to use a whole number of eggs, so he decides to use 1 egg, which he measured to have a volume of \(\frac{3}{8}\) cup. He needs to keep the ratio of liquid ingredients to solid ingredients the same, so he compensates by adding extra milk. What is the final recipe he should use?

**Solution:**

The ratio of the new recipe using the 9-inch round pan to the original recipe using the 10-inch round pan is new : original = \(\pi \times 4.52 : \pi \times 5^2 = 4.52 : 5^2 = 81 : 100\). So he should make about \(\frac{4}{5}\) of the original recipe. He should use \(\frac{4}{5}\) of \(\frac{1}{2}\), or \(\frac{2}{5}\), of a cup of milk, and he should use \(\frac{4}{5}\) of 2 eggs, or \(\frac{4}{5}\) of \(\frac{3}{4}\) cup of eggs = \(\frac{3}{5}\) cup of eggs. But he is planning to only use \(\frac{3}{8}\) cup of eggs, so he is using too little egg by \(\frac{3}{5} - \frac{3}{8} = (24 - 15)/40 = 9/40\) cup of eggs. He needs to add \(\frac{9}{40}\) cup milk to \(\frac{2}{5}\) cup milk, so I added \(\frac{9}{40} + \frac{2}{5} = \frac{9}{40} + 16/40 = 25/40 = 5/8\) cup milk. To find all the other ingredients other than egg and milk, I multiplied the original amount of each one by \(\frac{4}{5}\). So his final recipe should be:

- 4/5 cup white sugar
- 1 1/5 cup all-purpose flour
- 2/5 cup butter
- 1 2/5 tsp baking powder
- 1 egg
- 5/8 cup milk
- 1 3/5 teaspoons vanilla extract

Redirecting students: Encourage students to make a plan of the steps they need to do before carrying out the plan: find the ratio of the recipes and what each ingredient would be without compensating for the egg, find the volume of egg he needs to make up for, find the volume of milk he needs to add, and then find the total amount of milk needed for the recipe. You may need to suggest that students look for a ratio that is close to 81 to 100 but is easier to work with.
SP7-17 Sampling Variability

Pages 192–193

Standards: 7.SP.A.2

Goals:
Students will learn that different samples have different variability.
Students will understand the advantage of using a relatively large sample.
Students will learn that larger samples have smaller variability of the sample mean.

Prior Knowledge Required:
Can calculate the mean of a set of data
Can identify line-symmetric figures

Vocabulary: biased sample, estimate, first quartile (Q1), fraction, interquartile range (IQR),
mean, mean absolute deviation (MAD), median, proportion, random sample, sample, survey,
third quartile (Q3), unbiased sample, variability

Materials:
BLM 300 Random Numbers from 0 to 5 (p. S-72)
BLM Sample Size (pp. S-73–74)
large bowl or jar
1/2 pound each of two colors of dried beans (e.g., red beans and white beans)
a page of Spanish text
access to an online random number generator for each student

(MP.5) A large sample produces better results than a small sample. Bring in a large bowl
or a jar containing dried beans in two different colors, about 1/2 pound of each color (e.g., red
beans and white beans). Make 40% of the beans one color (e.g., red) and 60% the other color
(e.g., white), but don’t tell students. Mix the beans thoroughly. Explain how this makes any
sample they pick a random or unbiased sample. For example, you would have a very biased
sample if you poured in all the red beans first, then poured the white beans on top, and you
didn’t mix them. Tell students that you want to know which proportion of beans is red and which
is white, without counting every single bean. Invite students to describe how you might do this.

Place the bowl at the front of the class or have students pass it around. Have students take
turns closing their eyes, choosing 10 beans, recording the results, and then putting the beans
back. Invite volunteers to tell how many of each color bean they chose. ASK: Do you think we
can estimate the fraction of red beans and white beans in the whole bowl based on the fraction
in a sample of 10? (no) Why not? (everyone got different answers, so we wouldn’t know whose
answer to take)
Have students pool their results in pairs, then in groups of four (pair up the pairs). Repeat with groups of eight, then groups of 16, then with the whole class. Students can record their results in a table, as shown below:

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>10</th>
<th>20</th>
<th>40</th>
<th>80</th>
<th>160</th>
<th>Whole Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red beans</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>White beans</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Percentage of red beans</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Tell students the exact proportion of beans of each color in the bowl and ASK: Which sample produced the best and most reliable estimate? (the largest one) SAY: Some small samples might be closer to the actual proportion, maybe even exact, but we can’t rely on them because the small samples are so different from each other.

Give students BLM 300 Random Numbers from 0 to 5 to complete the following exercises.

**Exercises:**
a) Find the sum of the numbers in each row, or sample size 10.
b) Find the mean of the numbers in each row, then draw a dot plot of the means.
c) Find the median, Q1, and Q3 for the means that you found in part b).
d) Find the IQR, mean, and the MAD for the means that you found in part b).
e) Combine every three rows of random numbers to make groups of a larger sample of size 30. Find the mean of each sample of size 30, then draw a dot plot of the means. Round answers to the nearest tenth.
f) Find the median, Q1, and Q3 for the means that you found in part e).
g) Find the IQR, mean, and MAD for the means that you found in part e).
h) Compare the IQR and MAD of sample size 10 with the IQR and MAD of sample size 30. Which has the smaller variability?

**Answers:**
a) 19, 22, 33, 27, 23, 35, 14, 24, 37, 34, 23, 21, 28, 27, 19, 28, 21, 22, 21, 20, 22, 28, 23, 30, 19, 31, 17, 29, 17, 34;
b) 1.9, 2.2, 3.3, 2.7, 2.3, 3.5, 1.4, 2.4, 3.7, 3.4, 2.3, 2.1, 2.8, 2.7, 1.9, 2.8, 2.1, 2.2, 2.1, 2.0, 2.2, 2.8, 2.3, 3.0, 1.9, 3.1, 1.7, 2.9, 1.7, 3.4;
c) median = 2.3, Q1 = 2.1, Q3 = 2.9; d) IQR = 0.6, mean = 2.493, MAD = 0.51;
e) 2.5, 2.8, 2.6, 2.5, 2.4, 2.1, 2.7, 2.2, 2.7

[Dot plot images]

f) median = 2.5, Q1 = 2.4, Q3 = 2.7; g) IQR = 0.3, mean = 2.5, MAD = 0.16; h) the sample size of 30 has the smaller IQR and MAD, so it has the smaller variability
Activity
Distribute a page of Spanish text. Put students into the same number of groups as there are paragraphs in the text, and assign a paragraph to each group. Ask each student to identify the most common letter in one sentence in their paragraph (make sure all sentences are used; some sentences may be assigned to more than one student, or vice versa). ASK: Is the most common letter the same for all sentences? Which letters are most common? How many letters were identify as the most common?

Now have each group identify the most common letter in their paragraph. Finally, have students identify the most common letter on the page.

(MP.3) ASK: Do you think the most common letter on this page will be the same as the most common letter on another page? In a whole book? In the Spanish language overall? Why is it better to use a large sample size to find the most common letter in a language than a small sample, such as a sentence or paragraph? (sample answer: When you know more information, you can be more confident you are right. Knowing which letters occur more often in a whole page is a lot of information, but knowing which occurs most in a sentence is only a little bit of information; with a small sample, it could just be coincidence that you picked a sentence with a lot of one letter (for example, the word “banana” has “a” as half its letters but that doesn’t mean half the letters in English are “a”, it’s just because we used such a small sample).)

(end of activity)

Using a sample to accurately represent a whole population. SAY: You are conducting a survey and need to ask a yes-or-no question. ASK: How many people should you survey so that the fraction of people in your sample who answer “yes” is close to the fraction in the whole population who would answer “yes”? Give students BLM Sample Size—they will explore this question as they complete the BLM. (selected answers: 1. 2,000/10,000 or 1/5; 3. B; 8. Yes; 10. 100)
SP7-18  Degree of Overlap of Two Populations with the Same Variability

Pages 194–196

Standards: 7.SP.B.3

Goals:
Students will evaluate the visual degree of overlap between two data sets, as displayed by box plots or dot plots.

Prior Knowledge Required:
Can evaluate the mean and MAD of a data set
Can evaluate the median and IQR of a data set
Can draw dot plots and box plots
Can read and interpret dot plots and box plots
Understands the real-world interpretation of center and variability
Knows that larger samples are less variable

Vocabulary: average, box plot, center, data point, data set, degree of overlap, dot plots, first quartile (Q1), frequency, interquartile range (IQR), mean, mean absolute deviation (MAD), median, outlier, population, third quartile (Q3), samples, variability

Materials:
calculators

NOTE: Allow students to use calculators throughout this lesson.

Introduce the importance of comparing populations. Give examples of situations in which people might be interested in comparing differences between populations, and discuss reasons why the populations might be different. For example:
• Are basketball players generally taller than soccer players? (people who are taller might be more likely to prefer basketball because it is easier to score a basket if you’re taller)
• Are words in a Grade 7 textbook generally longer than words in a Grade 2 textbook? (Grade 7 students are able to read longer words)
• Do people in one country tend to live longer than people in another country? (some countries have better health care systems)
• Does one company serve customers faster than another company? (one company might have a more efficient system)

Allow students to suggest other possible comparisons.

Introduce the importance of overlap. For each example discussed above, emphasize the fact that the comparisons are general. For example, not all basketball players are taller than all soccer players. ASK: Will Grade 7 textbooks have some words that are shorter than some
words in a Grade 2 textbook? (yes: “a” and “an” will be in the Grade 7 textbook) SAY: There will usually be some degree of overlap between the two populations. For example, Grade 7 textbooks will have some very short words and Grade 2 textbooks might have some longer words, such as “centimeter.”

SAY: Sometimes two populations will have a lot of overlap and sometimes only a little. ASK: Which of the two comparisons is more likely to have more overlap: comparing heights of 9-year-old girls and 9-year-old boys, or comparing heights of 9-year-old girls and 39-year-old women? (comparing heights of girls and boys) SAY: Statisticians want to know if there are general differences between two populations, but they also want to know how much overlap there is between them.

**Determining the degree of overlap visually.** SAY: You can sometimes tell just by looking at two box plots or dot plots how much overlap there is between the two data sets. It’s easier to tell if you plot the two data sets one on top of the other. Draw on the board:

![A and B dot plots](image)

ASK: Would you say that the data sets in A look as though they overlap a lot or a little? (a lot) What about in B? (a little) Keep these dot plots on the board for use later in the lesson.

**Using box plots to compare overlap in the middle half of the data.** Tell students that when sets are put in box plots, we can compare how much overlap there is in the middle half of the data. Draw on the board:

![Box plots](image)

Remind students that the boxes show the middle half of the data because 25% of the data is before the first box, 25% is in the first box, 25% is in the second box, and 25% is after the second box. SAY: To decide how much overlap there is in the boxes, imagine sliding the top box straight down over the bottom box, and see which parts turn dark gray. ASK: Would most of the boxes turn dark gray, or only a little? (most of them) SAY: When most of the boxes turn dark gray, there is high overlap; when only a small part of the boxes turn dark gray, there is small overlap; when none of the boxes turn dark gray, there is no overlap.
Exercises: Copy the box plots and color the part that overlaps. Describe the degree of overlap between the two data sets as high overlap, small overlap, or no overlap.

a)  

b)  

c)  

d)  

e)  

f)  

Answers: a) high overlap, b) small overlap, c) small overlap, d) high overlap, e) no overlap, f) small overlap

The variabilities and the difference between the means both affect the degree of overlap.

Tell students that in the previous examples, we compared data sets with the same variability, and the farther apart their centers were, the less they overlapped. Tell students that there is another factor that makes some pairs of data sets look as though they have high overlap and others look as though they have small overlap. SAY: Suppose I have two sticks, and I place them horizontally along a number line. Draw on the board:

0 1 2 3 4 5 6 7 8 9 10

SAY: Suppose I know that one of the sticks is centered at 4, and the other stick is centered at 6. ASK: Can you tell me how much they overlap? (no) What else do you need to know about the sticks? (how long they are) Continue drawing on the board:

0 1 2 3 4 5 6 7 8 9 10

SAY: Both pairs of sticks are centered at 4 and 6, but the first pair is longer. The first pair of sticks overlap a lot, whereas the second pair of sticks don’t overlap at all. Keep these diagrams on the board for use later in the lesson.

Tell students that when discussing overlap of data sets, they can use measures of center and variability, with the variability representing the “length” of the stick. When using box plots, the center is the median and the variability is the IQR. Refer students to the dot plots on the board. ASK: When you compare overlap of dot plots, would you use the mean or median for the center? (mean) Would you use the MAD or IQR as the measure of variability? (MAD)
NOTE: For part b) of the exercises below, you might give each student and each dot plot a number and assign students to one of the four dot plots so that all four are represented.

Exercises: Look at the dot plots for Data Set 1 and Data Set 2 on the board.
a) Calculate the mean of all four dot plots.
b) Choose one of the four dot plots and calculate the MAD.

Answers: a) means for A: 3 and 4, B: 3 and 7; b) MADs are all 4/3

When students finish, take up the answers and point out that all four MADs are the same. ASK: How could you have predicted that before checking? (the shapes are exactly the same, so how far each data set is on average from the mean will be the same for all four dot plots) Did you have to do any calculations to find the means, or was there an easier way? (the distributions are all symmetric, so they are all centered at their mean, which you can read from the dot plot)

(MP.1, MP.7) ASK: When the dot plots have a high overlap, which is larger, the MADs or the difference between the means? (MADs) When the dot plots have a small overlap, which is larger, the MADs or the difference between the means? (difference between the means) Refer students again to the sticks on the board. Point out that when the sticks are longer than the difference between the centers, there is a high overlap. When the sticks are shorter than the difference between the centers, there is no overlap.

(MP.2) Exercises: The dot plots below show the resting heart rates of babies less than 1 year old and of teenagers.

Babies:

Teenagers:

a) Describe the degree of overlap between the two dot plots.
b) Calculate the means of both data sets.
c) What is the difference between the means?
d) Without doing calculations, do you think both data sets have the same MAD? How do you know?
e) Calculate the MAD of both data sets.
f) Which is larger, the difference between the means or the MAD? Does this fit with your answer to part a)? Explain.

Answers: a) small overlap; b) 80 and 120; c) 40; d) yes, because the shapes of the distributions are identical; e) 10; f) the difference between the means is larger than the variability, which means there is a small overlap.
(MP.2) Comparing populations with different variability and substantial overlap. Tell students that, so far in this lesson, we have only compared data with the same variability. But we can also compare data sets with different variability.

Draw on the board:

**Time Sara Takes to Get to School (in Minutes)**

![Dot plots](image)

SAY: Sometimes Sara walks to school and sometimes she takes the bus. One day when Sara walked to school, she ran into someone she knew and stopped to chat for 10 minutes. Have a volunteer point out which data point on the board represents this situation. (the outlier) Tell students that Sara needs to get to school on time for an important test and is deciding whether to take the bus or walk. ASK: Should we include the outlier when calculating the average time it takes to get to school by walking? (no) Why not? (she has control over whether she stops to chat) SAY: There might be other factors, such as having to wait at traffic lights, that she doesn’t have control over, but we don’t have to count the outlier when comparing the data sets for this purpose. Erase the outlier from the board before having students complete the exercises below.

**Exercises:** Look at the dot plots for Time Sara takes to get to school.

a) Find the mean of both data sets.

b) Which method gets Sara to school faster on average, taking the bus or walking?

c) Find the MAD of both data sets.

d) Which method is more reliable, the bus or walking? Use your answer to part c) to justify your answer.

e) If you were Sara and needed to get to school on time for an important test, would you walk or take the bus? Why?

f) How would you describe the overlap between the two sets by looking at the dot plots—no overlap, small overlap, or high overlap?

**Answers:** a) walking: 20, taking the bus: 19; b) taking the bus; c) walking: 0.6, taking the bus: 1.6; d) walking because the MAD is much lower; e) I would walk because I can control how long it takes to get to school; f) answers will vary: both “small overlap” and “high overlap” are acceptable answers at this stage for the following reasons: there is small overlap because there is a lot of data in the “taking the bus” dot plot that does not overlap with the range of the “walking” dot plot, there is high overlap because the “walking” dot plot is completely contained in the “taking the bus” dot plot.
Tell students that you want to find the median of both sets because you want to draw box plots. SAY: There are 10 points, and 10 is even, so the median is halfway between the middle two values. Have volunteers circle the middle two values on each data set, and have other volunteers tell you the median walking time and the median taking the bus time (20 minutes, 18.5 minutes). Draw a line between the two middle values for each data set.

The plots should look like this:

![Box plots showing median times for walking and taking the bus.](image)

Before they complete the exercises below, remind students that Q1 is the median of the lower half and Q3 is the median of the upper half of the data.

**Exercises:**

a) Find Q1 and Q3 for both data sets.
b) Draw box plots for both data sets.
c) How would you describe the degree of overlap from looking at the box plots?

**Answers:**

a) walking: Q1 = 19, Q3 = 21, taking the bus: Q1 = 18, Q3 = 21
b) 

![Walking box plot](image)

![Taking the bus box plot](image)

c) high overlap, because 2/3 of the boxes would turn dark gray when you merge them

**Comparing populations with different variability and small overlap.** Draw on the board:

![Height data for students and teachers](image)
Tell students that at a K–8 school, ten randomly chosen students and ten randomly chosen teachers recorded their heights. **ASK:** By looking at the dot plots, and without doing any calculations, which would you say is more variable, the heights of the students or the heights of the teachers? (the students) **Why does that make sense?** (students are still growing) Have volunteers estimate, without doing any calculations, the mean of each data set. (student height seems to be centered close to 50, and teacher height seems to be centered close to 70)

**ASK:** In these dot plots, is it rare for a student’s height to be 5 inches away from the mean? (no, it happens quite a lot) Is it rare for a teacher’s height to be 5 inches away from the mean? (yes, it only happens once) **SAY:** That is one way to describe the fact that teacher height is less variable than student height. The teachers are more likely to be close to their mean height than students are. **ASK:** Are there any students close to the teacher mean? (yes) Are there any teachers close to the student mean? (no) **SAY:** The student population varies a lot more in height than the teacher population does, so it’s not surprising to find some students near the mean for teachers. It would be more surprising to find teachers near the mean for students.

**Exercises:** Here are two sentences from two different math textbooks:

- Fill in the blank for the expanded form.

  In the product of a number and a variable, the multiplication sign is usually dropped.

  a) One sentence came from a Grade 2 textbook and the other from a Grade 7 textbook. Which is which? How do you know?
  b) Write a data set for each grade consisting of the lengths of the words from each sentence.
  c) Draw the dot plots.

**Answers:** a) the second one is from Grade 7, Grade 2 students don’t multiply numbers and variables; b) Grade 2 word lengths: 4, 2, 3, 5, 3, 3, 8, 4; Grade 7 word lengths: 2, 3, 7, 2, 1, 6, 3, 1, 8, 3, 14, 4, 2, 7, 7
c) 

**Grade 2:**

```
0     1     2     3     4     5     6     7     8     9    10   11    12   13  14    15   16
      ●     ●     ●     ●     ●     ●     ●     ●     ●     ●     ●     ●     ●      ●     ●     ●
```

**Grade 7:**

```
0     1     2     3     4     5     6     7     8     9    10   11    12   13  14    15   16
      ●     ●     ●     ●     ●     ●     ●     ●     ●     ●     ●     ●     ●      ●     ●     ●
```

Tell students that you want them to find the mean and the MAD of both sets of data from the previous exercises. Remind students that they can use frequency plots to help add the data values. Below each column of the Grade 2 dot plot, write the total value of the dots in that column. (see answers below)

```
2      9     8     5                 8
```

**SAY:** Instead of adding all eight values, now we can just add these five values. This doesn’t save much time for the Grade 2 data, but it will for the Grade 7 data.
Exercises: Complete the totals row for Grade 7. Find the mean of both data sets, to one decimal place.

Answers:
Grade 2 mean: \((2 + 9 + 8 + 5 + 8) ÷ 8 = 32 ÷ 8 = 4\)
Grade 7 mean: \((2 + 6 + 9 + 4 + 6 + 21 + 8 + 14) ÷ 15 = 70 ÷ 15 ≈ 4.7\)

Remind students that the MAD is the average distance from the mean. Use the Grade 2 data set from the previous exercises to write on the board:

Grade 2 data set: 2 3 3 3 4 4 5 8
Distance from the mean: 2 1 1 1 0 0 1 4

ASK: What is the sum of all the distances from the mean? (9) How many data values are there? (8) SAY: So the average of the distances or MAD is 10 ÷ 8 = 1.25. Write this on the board.

Exercises:
a) Find the MAD of the Grade 7 data set, to one decimal place.
b) Which grade has the higher mean? The higher MAD?

Answers: a) 2.8, b) Grade 7 has the higher mean and the higher MAD

Extensions
(MP.7) 1. Here are two dot plots:

a) Do the dot plots have any data points in common?
b) Draw the box plots.
c) Describe the degree of overlap in the box plots.

Answers:
a) no
b)  

When students finish, emphasize that the degree of overlap between two sets is not about how many data points they have in common, but an overall picture of how the values overlap.
(MP.1) 2. Teach students that they can use frequencies to find the MAD the same way they use frequencies to find the mean, because the MAD is the mean of the deviations. Have students find the mean and the MAD of the following set by inputting the frequencies into the table:

\[
\text{Data Value: } 2, 2, 3, 3, 3, 4, 5, 5, 6, 6, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 30
\]

<table>
<thead>
<tr>
<th>Data Value</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Totals</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Distance from Mean</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>24</td>
</tr>
<tr>
<td>Totals of Distances</td>
<td>12</td>
<td>12</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>5</td>
<td>24</td>
</tr>
</tbody>
</table>

Answers:

Mean: \(\frac{6 + 12 + 4 + 15 + 48 + 35 + 30}{25} = 6\)

MAD: \(\frac{12 + 12 + 2 + 3 + 0 + 5 + 24}{25} = 2.32\)

(MP.2, MP.4) 3. John skis down a hill and loses 1,000 feet of altitude each minute. He finishes the hill in 12 minutes and ends up at \(\frac{3}{5}\) mile below sea level. Is the top of the hill high enough to be considered high altitude, very high altitude, or extremely high altitude according to the table below?

<table>
<thead>
<tr>
<th>High Altitude</th>
<th>8,000 to 12,000 feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very High Altitude</td>
<td>12,000 to 18,000 feet</td>
</tr>
<tr>
<td>Extremely High Altitude</td>
<td>18,000+ feet</td>
</tr>
</tbody>
</table>

Use positive and negative numbers to show your work. Write what each step means in the situation.

**Answer:** 3/5 of a mile is 3/5 of 5,280 feet or 3,168 feet, so John ends up at an altitude of \(-3,168\) feet. Let \(x\) be the altitude at the top of the hill. The height of the hill is 12,000 feet because it takes him 12 minutes to ski down and he loses 1,000 feet of altitude each minute. This height is also the difference between the altitudes at the top and bottom, so \(x - (-3,168) = 12,000\), or \(x + 3,168 = 12,000\), so \(x = 12,000 - 3,168 = 8,832\) feet. According to the table, the top of the hill is high enough to be considered high altitude.
**SP7-19 Using Mean and MAD to Compare Populations**

Pages 197–199

**Standards:** 7.SP.B.3, 7.SP.B.4

**Goals:**
Students will learn in general terms how confident they can be that two populations are different based on looking at samples.
Students will express the difference between the means as a multiple of the MAD.

**Prior Knowledge Required:**
Can evaluate the mean and MAD of a data set
Can evaluate the median and IQR of a data set
Can draw dot plots and box plots
Can read and interpret dot plots and box plots
Understands the real-world interpretation of center and variability
Can determine the degree of visual overlap
Understands the relationship between visual overlap and comparing the difference between the center to the variability

**Vocabulary:** average, box plot, data set, degree of overlap, dot plot, first quartile (Q1), frequency, interquartile range (IQR), mean, mean absolute deviation (MAD), median, third quartile (Q3), variability

**Materials:**
calculators
BLM 300 Random Numbers from 0 to 9 (p. S-75)

**NOTE:** Allow students to use calculators for all exercises in this lesson.

**Introduce the difference between the means as a multiple of the MAD.** Draw on the board:

![Dot plots for math and science tests]

**SAY:** Rob wrote 10 math tests and 10 science tests. These are his marks. **ASK:** Is Rob better at math or science? (math) How confident are you about that? (very confident) Why are you so confident? (because the dot plots are very different) Do these sets overlap? (no) What is the difference between the means? (45.5) Which is larger: the largest MAD or the difference
between the means? (the difference between the means) SAY: Not only is the difference between the means larger than the MAD, but in fact it is almost six times as large! Write on the board:

\[
\frac{\text{difference between the means}}{\text{MAD}} = \frac{45.5}{8} \approx 5.7
\]

SAY: When the difference between the means is a large multiple of the MAD, the graphs will have very small overlap. In this lesson, we will investigate why.

**Review simulating real-world situations.** Ask students to imagine that there is a bag of 100 marbles and 30 marbles are red. This could represent many real-life situations, such as 30% of the population supporting a specific candidate for an election, or 30% of the population texting for longer than a given amount of time. ASK: How can we simulate picking a red marble using a random number generator? (choose a random number from 1 to 100 and find the probability that the number is between 1 and 30; or choose a random number from 1 to 10 and find the probability that the number is 1, 2, or 3) If the second option doesn’t come up, PROMPT: Are there smaller numbers I can use instead of 30 and 100 that will have the same probability? (3 and 10) Write on the board:

\[
\frac{30}{100} = \frac{3}{10}
\]

Tell students that instead of picking numbers from 1 to 10, you will pick numbers from 0 to 9. SAY: The only benefit is that it is easier to search in a computer file for 0 to 9 than 1 to 10, because the number of occurrences of 1 gets confused with the 10s. Write on the board:

- 0, 1, or 2 = Red marble
- 3, 4, 5, 6, 7, 8, or 9 = Not a red marble

SAY: 30 Grade 7 students take turns selecting a sample of 10 marbles, recording how many marbles were red, then putting their marbles back. ASK: How can we simulate picking 10 marbles from the bag? (pick 10 random numbers from 0 to 9) Give students **BLM 300 Random Numbers from 0 to 9**. Tell students that each row represents a sample of 10 picks from the bag. Have students record beside each row how many from each sample represents a red marble (i.e., is between 0 and 2). Have students check with a partner that they have the same data. (3, 3, 5, 3, 4, 2, 5, 0, 2, 5, 6, 2, 6, 4, 1, 4, 2, 1, 3, 2, 1, 4, 5, 3, 3, 4, 5, 1, 2, 1)

**Exercises:** Use the data from each row of BLM 300 Random Numbers from 0 to 9.

a) Copy and complete the frequency chart for the number of red marbles.

<table>
<thead>
<tr>
<th>Number of Red Marbles</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Add the frequencies from part a). Make sure they add to 30.

c) Make a dot plot showing how many times each number of red marbles occurred.

d) Find the mean to one decimal place, then calculate the MAD of the data.
Selected answers:

a)  

<table>
<thead>
<tr>
<th>Number of Red Marbles</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>1</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

b) 

c) 

d) mean = 3.1, MAD = 1.4

Comparing samples of size 10. Tell students that samples of size 10 vary a lot because they are so small. Then tell students that another 30 samples of size 10 were picked from another bag of marbles. We’ll call the original bag from the exercises “Bag 1” and we’ll call this new bag “Bag 2.” Draw on the board (the dot plot for Bag 1 is the answer to part c) of the exercises above):

Exercises: Look at the dot plots for Bag 1 and Bag 2.

a) Find the mean and the MAD from Bag 2 to one decimal place.

b) Is the variability for both sets about the same?

b) Yes, c) Yes

Comparing samples of different sizes. When students finish, tell them that instead of plotting the number of red marbles, they could instead plot the proportion of red marbles. SAY: So, if 4 out of the 10 marbles picked were red, then the proportion of red marbles in the sample is 0.4. This is useful if you want to compare samples of size 10 to samples of size 30, for example. The sample of size 30 will likely have more red marbles than the sample of size 10, but it is the proportion of red marbles that we want to compare. Students will need to understand this concept in order to complete Question 2 on AP Book 7.2 p. 198.

Using the means and the largest MAD to discuss overlap. Tell students that there is high overlap when the difference between the means is smaller than the largest MAD. Refer students to the dot plots for Bag 1 and Bag 2. ASK: What is the difference between the two means? (0.6)
What are the two MADs? (1.1 and 1.4) From our description of high overlap, do these dot plots have high overlap? (yes)

Tell students that even though the means are different, when there is high overlap, it might be just a coincidence that the means are different. You can’t be very confident that the two bags are different at all; the difference might just be due to the fact that all samples vary. Tell students that in fact, Bag 2 also has exactly 30 red marbles, so it’s just like taking another 30 samples of size 10 from the same bag.

Tell students you took another 30 samples of size 10 from yet another bag. Draw on the board (the dot plot for Bag 1 is the same as above):

(MP.2) ASK: Does the overlap look more or less than the other pair? (less) Do you think it’s more likely or less likely that the two bags have the same amount of red marbles? (less likely) Tell students that one way of describing how likely it is for two sets of samples to be from similar populations is by describing how much they overlap. SAY: We can describe the degree of overlap by describing how the difference between the means compares to the largest MAD. Have students calculate the mean and MAD from Bag 3, to one decimal place. (mean = 5.4, MAD = 1.3) Summarize the means and MADs for Bags 1 and 3 on the board, as shown below:

Bag 1: mean = 3.1, MAD = 1.4
Bag 3: mean = 5.4, MAD = 1.3

ASK: What is the difference between the means? (2.3) What is the largest MAD? (1.4) Is the difference between the means larger or smaller than the largest MAD? (larger) So is there high overlap? (no) Is that consistent with your prediction when you looked at the dot plots? (yes) SAY: Because the overlap isn’t high, we can be reasonably sure that the two bags have different amounts of red marbles. Tell students that, in fact, Bag 3 has 60 red marbles. So, although the sample we chose had smaller than average amounts of red marbles, it was still big enough to show the difference between the bags with 30 red marbles.

NOTE: To provide an easier version of the exercises below, you can provide students with the means and the MADs of both data sets.
(MP.2, MP.3) Exercises: Calculate the means and the MADs of both data sets to one decimal place. Is there enough evidence to believe that the populations are actually different? Explain how you know.

a) Did 12-year-old boys in 1995 watch more TV than 12-year-old boys in 2007?

b) Do boys play more video games than girls?

c) Do females live longer than males?

d) Do girls spend more time on homework than boys?
Answers:
a) 1995 mean = 2.5, 1995 MAD = 0.9; 2007 mean = 2, 2007 MAD = 1; the difference between
the means is 0.5, which is less the largest MAD, so there is not enough evidence
b) Boys mean = 52, Boys MAD = 24; Girls mean = 21.5, Girls MAD = 16.7; the difference
between the means is 30.5, which is greater than the largest MAD, so there is enough evidence
c) Female mean ≈ 79.1, Female MAD ≈ 2.1; Male mean ≈ 73.7, Male MAD ≈ 2.2; the difference
between the means is 5.4, which is greater than both MADs, so there is enough evidence
d) Girls mean = 2.5, Girls MAD = 0.7; Boys mean = 2, Boys MAD = 0.7; the difference between
the means is 0.5, which is less than both MADs, so there is not enough evidence

Writing the difference between the means as a multiple of the largest MAD. Tell students
that they can be more precise than just saying whether the difference between the means is
larger or smaller than the largest MAD. They can actually write the difference between the
means as a multiple of the largest MAD. SAY: It’s important to use the larger MAD because
the difference needs to be larger than both MADs to be confident that the difference between
the means is meaningful. Refer back to the example of the three bags of marbles. Write on
the board:

Bag 1: mean = 3.1, MAD = 1.4
Bag 2: mean = 2.5 and MAD = 1.1
Bag 3: mean = 5.4, MAD = 1.3

\[
\frac{\text{Mean of Bag 1} - \text{Mean of Bag 2}}{\text{largest MAD}} = \frac{0.6}{1.4} \approx 0.43
\]

\[
\frac{\text{Mean of Bag 3} - \text{Mean of Bag 1}}{\text{largest MAD}} = \frac{2.3}{1.4} \approx 1.64
\]

SAY: The difference between the means in Bags 1 and 2 is only half the largest MAD, so there
is a lot of overlap. The difference between the means in Bags 1 and 3 is more than the largest
MAD, so the difference in the means is meaningful. The larger that multiplier is, the more
meaningful the difference is.

Have students compare Bags 2 and 3 by finding the difference between the means as a multiple
of the largest MAD. (2.9 ÷ 1.3 ≈ 2.23) ASK: Is the difference between Bags 2 and 3 more or less
meaningful than the difference between Bags 1 and 3? (more meaningful) How do you know?
because 2.23 is larger than 1.64)

(MP.2) Exercises:
a) For each situation in the exercises above, calculate the difference between the means as a
multiple of the largest MAD, to one decimal place.
b) For which situation are you most certain that the means are really different?
**Answers:** a) part a) $0.5 \div 0.9 \approx 0.6$, part b) $30.5 \div 24 \approx 1.3$, part c) $5.4 \div 2.2 \approx 2.5$, part d) $0.5 \div 0.7 \approx 0.7$; b) part c): life expectancies of females is generally greater than life expectancies of males.

**Using box plots to decide whether the difference in medians is meaningful.** Remind students that they can also use box plots to compare degree of overlap. SAY: The more overlap there is between samples, the harder it is to say for sure that the populations are really different. The samples might be different just because random samples vary.

**Exercises:** In which case would you describe the degree of overlap as higher? In which case would you be more confident that the populations are actually different?

a) **A.**

b) **A.**

**Answers:** a) more overlap in A, more confident B that the populations are actually different in B; b) more overlap in B, more confident that the populations are actually different in A.

**Extension**

a) Visually, how would you describe the overlap?

b) What are the medians?

c) What are the IQRs?

d) Which is larger: the difference between the medians or the largest IQR?

**Answers:** a) small overlap, b) both 10, c) both 6, d) the largest IQR.

**(MP.4)** When students finish, point out that this shows a limitation in using the difference between the medians to describe the degree of overlap; there might be very small overlap even when the medians are exactly the same. Point out that if there is no difference between the medians, you don’t need to decide if the difference is meaningful.
Calculating Means from Dot Plots

1. Multiply each data value by its frequency. Add the results of the multiplications. Divide by the number of data points in the dot plot.

a) 5 6 7 8 9 10
   \[ \times 4 \times 0 \times \underline{\times} \times \underline{\times} \times = \underline{\text{total}} \]
   \[ \text{total} \div \underline{\text{number of data points}} = \underline{\text{mean}} \]

b) 11 12 13 14 15 16 17

c) 10 20 30 40 50

d) 5 10 15 20 25 30 35 40
Box Plots for Large Data Sets
1. Use the box plots to fill in the table.

<table>
<thead>
<tr>
<th>Median</th>
<th>Q1</th>
<th>Q3</th>
<th>IQR</th>
<th>Highest Value</th>
<th>Lowest Value</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Box Plot A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box Plot B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box Plot C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box Plot D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box Plot E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box Plot F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Box Plot G</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Matching Dot Plots with Box Plots

1. Match each dot plot with one box plot.

- [ ] Write the box plot letter here

A. 

B. 

C. 

- 

- 

- 

-
Comparing IQRs

1. First, plot all four sets of data below in dot plots and try to rank them from highest to lowest IQR just by looking at them. Hint: A low IQR means the data points are more tightly clustered around the median.
   (highest IQR) _______ _______ _______ _______ (lowest IQR)

2. Use the sets of data to find the median, Q1, and Q3. Calculate the IQR. Do your answers match the order in which you ranked the IQRs in part 1?

   Set 1: 2 4 8 12 18 18 20 20
   IQR = Q3 − Q1
   median

   Set 2: 2 2 10 11 13 16 18 20
   IQR = Q3 − Q1
   median

   Set 3: 2 8 16 17 17 17 19 20
   IQR = Q3 − Q1
   median

   Set 4: 2 3 3 15 17 18 18 20
   IQR = Q3 − Q1
   median
Grade 7 Math Test Dot Plots

1. Use the dot plots to fill in the tables.

**Dot Plot 1:**
Class A Marks on a Grade 7 Math Test

![Dot Plot 1](image1)

<table>
<thead>
<tr>
<th>Q1</th>
<th>Q3</th>
<th>IQR</th>
</tr>
</thead>
</table>

**Dot Plot 2:**
Class B Marks on a Grade 7 Math Test

![Dot Plot 2](image2)

<table>
<thead>
<tr>
<th>Q1</th>
<th>Q3</th>
<th>IQR</th>
</tr>
</thead>
</table>

2. Use the dot plots and your answers from Question 1 to fill in the tables.

<table>
<thead>
<tr>
<th>Number of Data Points</th>
<th>Median</th>
<th>Range</th>
<th>IQR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dot Plot 1: Class A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dot Plot 2: Class B</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. a) Which class has the smaller IQR? ________________

b) Which class has more tightly clustered data points around the median? ________________

c) Can you answer part a) just by looking at the dot plots? _______
Five-Part Spinner

1. Spin 10 times and record the results in the table called “Data Set 1.”

2. Spin 10 times and record the results in the table called “Data Set 2.”

3. Make a dot plot for Data Set 1.

4. Make a dot plot for Data Set 2.
# 300 Random Numbers from 0 to 5

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 1 | 1 | 2 | 2 | 0 | 0 | 3 | 4 | 2 | 4 |   |   |   |   |   |   |   |   |   |   |
| 5 | 1 | 1 | 4 | 1 | 0 | 5 | 4 | 1 | 0 |   |   |   |   |   |   |   |   |   |   |
| 5 | 1 | 2 | 3 | 5 | 2 | 4 | 1 | 5 | 5 |   |   |   |   |   |   |   |   |   |   |
| 0 | 0 | 5 | 0 | 4 | 2 | 5 | 3 | 5 | 3 |   |   |   |   |   |   |   |   |   |   |
| 0 | 5 | 2 | 4 | 2 | 4 | 1 | 2 | 1 | 2 |   |   |   |   |   |   |   |   |   |   |
| 2 | 5 | 4 | 4 | 5 | 4 | 5 | 5 | 0 | 1 |   |   |   |   |   |   |   |   |   |   |
| 0 | 0 | 2 | 5 | 0 | 1 | 2 | 2 | 2 | 0 |   |   |   |   |   |   |   |   |   |   |
| 0 | 0 | 1 | 3 | 4 | 3 | 4 | 4 | 4 | 1 |   |   |   |   |   |   |   |   |   |   |
| 5 | 4 | 4 | 3 | 4 | 5 | 3 | 0 | 4 | 5 |   |   |   |   |   |   |   |   |   |   |
| 5 | 5 | 4 | 0 | 2 | 4 | 3 | 5 | 4 | 2 |   |   |   |   |   |   |   |   |   |   |
| 1 | 0 | 1 | 4 | 2 | 2 | 2 | 5 | 5 | 1 |   |   |   |   |   |   |   |   |   |   |
| 1 | 2 | 3 | 2 | 2 | 0 | 4 | 2 | 1 | 4 |   |   |   |   |   |   |   |   |   |   |
| 5 | 2 | 5 | 2 | 3 | 5 | 2 | 0 | 1 | 3 |   |   |   |   |   |   |   |   |   |   |
| 5 | 4 | 1 | 4 | 4 | 2 | 1 | 5 | 1 | 0 |   |   |   |   |   |   |   |   |   |   |
| 2 | 1 | 1 | 0 | 2 | 5 | 1 | 2 | 1 | 4 |   |   |   |   |   |   |   |   |   |   |
| 1 | 2 | 4 | 5 | 1 | 1 | 4 | 5 | 3 | 2 |   |   |   |   |   |   |   |   |   |   |
| 3 | 5 | 3 | 1 | 1 | 2 | 0 | 0 | 1 | 5 |   |   |   |   |   |   |   |   |   |   |
| 5 | 0 | 4 | 5 | 1 | 0 | 2 | 2 | 3 | 0 |   |   |   |   |   |   |   |   |   |   |
| 1 | 4 | 5 | 1 | 3 | 1 | 2 | 0 | 4 | 0 |   |   |   |   |   |   |   |   |   |   |
| 2 | 1 | 5 | 0 | 1 | 4 | 5 | 0 | 2 | 0 |   |   |   |   |   |   |   |   |   |   |
| 1 | 1 | 4 | 2 | 5 | 1 | 2 | 3 | 1 | 2 |   |   |   |   |   |   |   |   |   |   |
| 3 | 1 | 5 | 2 | 2 | 2 | 4 | 2 | 3 | 4 |   |   |   |   |   |   |   |   |   |   |
| 2 | 1 | 4 | 0 | 2 | 2 | 4 | 1 | 2 | 5 |   |   |   |   |   |   |   |   |   |   |
| 5 | 5 | 5 | 5 | 2 | 1 | 0 | 1 | 2 | 4 |   |   |   |   |   |   |   |   |   |   |
| 0 | 3 | 4 | 0 | 2 | 5 | 0 | 2 | 3 | 0 |   |   |   |   |   |   |   |   |   |   |
| 3 | 3 | 3 | 4 | 5 | 4 | 3 | 0 | 1 | 5 |   |   |   |   |   |   |   |   |   |   |
| 0 | 1 | 2 | 2 | 4 | 5 | 1 | 1 | 1 | 0 |   |   |   |   |   |   |   |   |   |   |
| 0 | 5 | 5 | 4 | 5 | 0 | 2 | 4 | 1 | 3 |   |   |   |   |   |   |   |   |   |   |
| 0 | 0 | 2 | 4 | 1 | 2 | 2 | 2 | 0 | 4 |   |   |   |   |   |   |   |   |   |   |
| 2 | 3 | 3 | 5 | 1 | 4 | 5 | 5 | 5 | 1 |   |   |   |   |   |   |   |   |   |   |
Sample Size (1)

1. A city has a population of 10,000 people. Suppose that 2,000 people in the whole city would answer “yes” to a survey question.

What fraction of people in the whole city would answer “yes”? ______

2. You can assign each person in the city from Question 1 with a number from 1 to 10,000. Use an online random number generator to randomly choose 5 numbers between 1 and 10,000.

How is this similar to randomly choosing 5 people to survey? __________

3. Which fraction below is the same as your answer to Question 1? ______
   A. The fraction of odd numbers in the set 1, 2, 3, 4, … , 10,000.
   B. The fraction of multiples of 5 in the set 1, 2, 3, 4, … , 10,000.
   C. The fraction of perfect squares in the set 1, 2, 3, 4, … , 10,000.

4. The samples in the table were generated using a random number generator. Circle the multiples of 5 and complete the table. Remember: A number is divisible by 5 if its ones digit is 0 or 5.

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>Sample</th>
<th>Number of Multiples of 5 in the Random Sample</th>
<th>Fraction of Multiples of 5 in the Random Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6,347, 3,071</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>4,157, 6,191</td>
<td>1</td>
<td>1/3</td>
</tr>
<tr>
<td>5</td>
<td>386, 2,704, 9,605, 1,552, 4,097</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>7,455, 6,420, 1,742, 1,174, 1,176</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1,996, 3,356, 7,804, 924, 1,237, 4,206, 9,031, 7,483, 9,444, 6,906</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>4,086, 9,086, 2,394, 7,978, 4,313, 7,094, 8,527, 4,375, 3,893, 3,732</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Create a random sample of 20 numbers from 1 to 10,000 using a random number generator.
Sample Size (2)

6. How many numbers from your random sample from Question 5 are multiples of 5? 

What fraction of numbers from your random sample are multiples of 5? 

7. Combine your random sample with the random samples of other students in your class, as indicated in the table below. Fill in the table.

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>Number of Multiples of 5</th>
<th>Fraction of Multiples of 5 in Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 (my results)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40 (with 1 other person)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60 (with 2 other people)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80 (with 3 other people)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 (with 4 other people)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8. Look at the table in Question 7. As the sample size gets larger, does the fraction of multiples of 5 in the sample get closer to the actual fraction of multiples of 5 in the whole population?

9. Do you think the fraction of “yes” answers in a large enough random sample would be close to the actual fraction of “yes” answers in the whole population?

10. How large is “large enough”? Is a sample size of 40 large enough? 80? 100?

11. How would you change the sampling if \( \frac{4}{5} \) of people in the city answered “yes”?

Would it change your answer in Question 10? Explain.
300 Random Numbers from 0 to 9

<table>
<thead>
<tr>
<th>6 8 3 0 2 1 3 9 5 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 7 9 0 3 5 8 3 6 2</td>
</tr>
<tr>
<td>8 8 2 1 7 5 5 1 2 0</td>
</tr>
<tr>
<td>0 7 4 7 5 3 1 5 6 0</td>
</tr>
<tr>
<td>3 4 3 9 2 0 2 8 7 0</td>
</tr>
<tr>
<td>2 7 6 4 9 9 3 4 1 5</td>
</tr>
<tr>
<td>5 3 8 9 1 2 3 0 2 1</td>
</tr>
<tr>
<td>5 9 6 5 7 5 4 4 4 5</td>
</tr>
<tr>
<td>9 6 9 5 6 5 1 1 7 9</td>
</tr>
<tr>
<td>9 0 2 0 8 7 2 3 5 2</td>
</tr>
<tr>
<td>4 5 0 2 9 1 0 3 0 2</td>
</tr>
<tr>
<td>8 5 6 3 4 3 2 8 0 9</td>
</tr>
<tr>
<td>1 0 4 0 2 5 9 2 1 6</td>
</tr>
<tr>
<td>5 8 8 6 1 1 4 1 0 6</td>
</tr>
<tr>
<td>5 5 7 9 5 9 6 7 3 1</td>
</tr>
<tr>
<td>6 2 2 8 3 7 5 6 1 2</td>
</tr>
<tr>
<td>7 8 8 8 1 8 9 2 6 8</td>
</tr>
<tr>
<td>9 7 8 6 2 9 9 3 9 3</td>
</tr>
<tr>
<td>5 5 2 8 2 4 4 8 2 9</td>
</tr>
<tr>
<td>1 7 4 8 3 1 9 9 6 9</td>
</tr>
<tr>
<td>4 5 6 7 8 6 9 2 5 6</td>
</tr>
<tr>
<td>0 3 0 2 6 3 4 2 4 3</td>
</tr>
<tr>
<td>4 0 8 1 8 4 1 6 2 0</td>
</tr>
<tr>
<td>6 6 8 5 3 0 6 4 1 1</td>
</tr>
<tr>
<td>5 6 2 9 1 3 5 5 2 4</td>
</tr>
<tr>
<td>3 4 0 7 1 9 3 4 1 0</td>
</tr>
<tr>
<td>5 1 1 1 8 6 6 7 0 0</td>
</tr>
<tr>
<td>2 4 6 5 9 6 9 9 9 6</td>
</tr>
<tr>
<td>8 6 8 9 2 5 7 7 9 1</td>
</tr>
<tr>
<td>3 9 9 6 0 6 6 6 5 8</td>
</tr>
</tbody>
</table>
PS7-8  Breaking a Problem into Cases

Teach this lesson after: 7.2 Unit 7

Standards: 7.EE.A.1, 7.EE.B.4a, 7.NS.A.1c, 7.G.A.2

Goals:
Students will learn how to break a problem into cases to find all possible answers.

Prior Knowledge Required:
Knows that the sum of the angles in a triangle is 180°
Can evaluate the absolute value of a number
Can solve equations
Can locate integers on a number line
Understands that the absolute value of the difference between two numbers is the distance between them on a number line
Can determine factors of small numbers
Can draw triangles using a ruler and a protractor

Vocabulary: absolute value, negative, parallel, parallelogram, perfect square, positive, prime number, trapezoid

Review breaking a problem into cases. Write on the board:

A triangle has the following properties:
Two angles are equal.
There is a 30° angle.

What could be the angles in the triangle?

Challenge students to sketch the triangle and label any unknown angles as x, but not to solve the problem. Have a volunteer draw their sketch on the board. ASK: Did anyone draw a different triangle? If yes, ASK: How is it different? Have a volunteer draw their different sketch on the board. If nobody drew a different triangle, draw the other case (either a triangle with two 30° angles or a triangle with a 30° angle and two equal angles) on the board, as shown below:

![Diagram]

SAY: You have to be careful here because I didn’t tell you whether 30° was one of the equal angles or not. So you have to consider both cases.
Write on the board:

**Case 1:** Two angles are 30°.  **Case 2:** The two other angles are equal.

Point to each sketch and ASK: Is this Case 1 or Case 2? (students can signal their answer using one or two fingers) Label the sketches with the case to which they belong, as shown below:

Case 1: \[\begin{array}{c}
\text{30} \\
\text{x} \\
\text{30}
\end{array}\]

Case 2: \[\begin{array}{c}
\text{30} \\
\text{x} \\
\text{x}
\end{array}\]

SAY: Now you can make an equation for each case, based on what the angles have to add to. ASK: What do all the angles in a triangle have to add to? (180°) Have volunteers write an equation for each case, as shown below:

\[x + 60 = 180\quad 2x + 30 = 180\]

Have other volunteers solve the equations, as shown below:

\[x = 120\quad 2x = 150, \text{ so } x = 75\]

Write on the board:

\[30°, 30°, 120°\quad 30°, 75°, 75°\]

SAY: This question has two possible answers, and they are both correct.

**(MP.3) Exercises:** What could be the angles of the triangle?

a) Two angles are equal. One angle is 40°.

b) Two angles are equal. The smallest angle is 15° smaller than the largest angle.

c) Two angles are equal. The largest angle is twice the smallest angle.

d) Two angles are equal. The largest angle is four times the smallest angle.

e) One angle is 54°. The largest angle is twice the smallest angle. Hint: Be sure to consider all three cases.

**Selected solution:** e) 54° is either the smallest, the middle, or the largest angle. If it is the smallest angle, then the largest is 108°, and the other angle is 180° – 108° – 54° = 18°, which is smaller than 54°, so this is not possible. If 54° is the largest angle, then the smallest is 27°, and the other angle is 180° – 54° – 27° = 79°, which is larger than 54°, so this case is also not possible. So, 54° must be the middle angle, and the three angles, x, 54°, and 2x, add to 180°, so 3x + 54 = 180, 3x = 126, and x = 42. The three angles are 42°, 54°, and 84°.

**Answers:** a) 40°, 40°, 100° or 40°, 70°, 70°; b) 55°, 55°, 70° or 50°, 65°, 65°; c) 45°, 45°, 90° or 36°, 72°, 72°; d) 20°, 80°, 80°, or 30°, 30°, 120°
Review absolute value. Write on the board:

\[ |3| \quad |\text{-}4| \quad |5| \quad |\text{-}5| \quad |0| \quad |0.8| \quad |\text{-}0.8| \]

Have volunteers say what the absolute value of each number is. (3, 4, 5, 5, 0, 0.8, 0.8)

Solving absolute value equations. Write on the board:

\[ |x| = 5 \]

SAY: \( x \) is a number so that its absolute value is 5. ASK: What could \( x \) be? (5 or \( \text{-}5 \)) SAY: There are two solutions, because \( x \) could be positive or \( x \) could be negative.

Exercises: Solve for \( x \). Find both solutions.

a) \( |x| = 3 \) \hspace{1cm} b) \( |x| = 7 \)

Answers: a) \( x = 3 \) or \( \text{-}3 \), b) \( x = 7 \) or \( \text{-}7 \)

SAY: Now I’m going to make it a bit harder. Write on the board:

\[ |x - 1| = 5 \]

SAY: \( x - 1 \) is a number so that its absolute value is 5. ASK: What could \( x - 1 \) be? (5 or \( \text{-}5 \)) SAY: There are two cases: \( x - 1 \) could be positive or \( x - 1 \) could be negative. Write on the board:

Case 1: \hspace{1cm} Case 2:
\[ x - 1 = 5 \quad x - 1 = \text{-}5 \]

SAY: You can solve these two cases. Have volunteers solve both equations for \( x \), as shown below:

Case 1: \hspace{1cm} Case 2:
\[ x - 1 = 5 \quad x - 1 = \text{-}5 \]
\[ x = 5 + 1 \quad x = \text{-}5 + 1 \]
\[ x = 6 \quad x = \text{-}4 \]

SAY: Absolute value equations are different from linear equations because they have two solutions. But you can find them the same way—you just have to use cases.

(MP.3) Exercises: Solve for \( x \). Find both solutions.

a) \( |x + 2| = 1 \) \hspace{1cm} b) \( |x - 3| = 4 \) \hspace{1cm} c) \( |x - 5| = 5 \) \hspace{1cm} d) \( |x + 4| = 3 \)

Answers: a) \( -1 \) or \( \text{-}3 \), b) \( 7 \) or \( \text{-}1 \), c) \( 10 \) or \( 0 \), d) \( \text{-}1 \) or \( \text{-}7 \)
Once you find the solutions, you should check that they both solve the equation. Write on the board:

\[ |x - 1| = 5 \quad \text{Check } x = 6 \text{ and } x = -4 \]

\[ |6 - 1| = |5| = 5 \]

\[ |-4 - 1| = |-5| = 5 \]

**Exercises:**

**MP.1** 1. Check your answers to the previous exercises. Make sure both solutions work.

**Selected solution:** a) \(|-1 + 2| = |1| = 1\) and \(|-3 + 2| = |-1| = 1\)

**MP.3** 2. Why does \(|x - 5| = 0\) have only one solution? Make up another absolute value equation that has only one solution.

**Answer:** Because 0 is its own opposite, the two cases, \(x - 5 = 0\) and \(x - 5 = -0\), are really the same case. Sample equation: \(|x + 3| = 0\) also has only one solution.

**Interpreting the meaning of \(|x - a|\) as the distance from \(x\) to \(a\) on a number line.** Remind students that the absolute value of the difference between two numbers is the distance between them on a number line. Draw on the board:

![Number line diagram]

**Exercises:** Determine the absolute value of the difference between the two numbers. Make sure it is equal to the distance between the numbers on the number line on the board.

a) \(|4 - 1|\)  
b) \(|1 - 4|\)  
c) \(|-5 - (-3)|\)  
d) \(|-3 - (-5)|\)

e) \(|1 - (-3)|\)  
f) \(|-3 - 1|\)  
g) \(|5 - (-5)|\)  
h) \(|-5 - 5|\)

**Answers:** a) 3, b) 3, c) 2, d) 2, e) 4, f) 4, g) 10, h) 10

Write on the board:

\[ |x - 1| = 3 \text{ means the distance from } ____ \text{ to } ____ \text{ is } ____ . \]

Have a volunteer fill in the blanks. (x, 1, 3)
Draw on the board:

\[ \begin{array}{cccccccc}
-5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
\end{array} \]

ASK: Which two points are a distance of 3 away from 1? (4 and −2) Label these points with an X on the number line. SAY: So this is what you should get when you solve the equations. Write on the board:

\[ \begin{align*}
    x - 1 &= 3 \\
    x - 1 &= -3
\end{align*} \]

SAY: Either \( x - 1 \) is +3, or it’s −3, because its absolute value is 3. Have volunteers solve the equations, as shown below:

\[ \begin{align*}
    x &= 3 + 1 \\
    x &= -3 + 1 \\
    x &= 4 \\
    x &= -2
\end{align*} \]

ASK: Are these the same answers we got using the number line? (yes)

**Exercises:*** Solve the equation in two ways, by using algebra and by using a number line. Make sure you get the same answer both ways.

a) \( |x - 2| = 4 \)  

b) \( |x - (-2)| = 4 \)

**Answers:** a) 6 and −2, b) −6 and 2

**Problem Bank**

**MP.3** 1. Glen multiplied a number by 3 and added 2. His answer was a factor of 10. What numbers could he have started with?

**Answer:** There are four cases: \( 3x + 2 \) is one of 1, 2, 5, and 10. The possibilities are \( x = -1/3, x = 0, x = 1, \) and \( x = 8/3 \).

**MP.3** 2. Will multiplied a positive integer by 3 and added 2. He told his answer to Grace, who said: “That’s a factor of 24.” What number did Will start with?

**Solution:** \( 3x + 2 \) is one of 1, 2, 3, 4, 6, 8, 12, and 24. The only equation that gives a positive-integer answer is \( 3x + 2 = 8 \) with answer \( x = 2 \), so Will must have started with 2.

**MP.3** 3. Abdul multiplied a positive integer by 5 and subtracted 3. His answer is a factor of 21. What positive integer did he start with?

**Solution:** \( 5x - 3 \) is one of 1, 3, 7, and 21. The only equation that gives a positive-integer answer is \( 5x - 3 = 7 \), which gives the answer \( x = 2 \). So Abdul must have started with 2.

**MP.3** 4. Roy subtracted 4 from a positive integer, multiplied the result by 10, and then subtracted 1. His answer was a two-digit perfect square. What number did he start with?

**Solution:** \((x - 4) \times 10 - 1\) is one of 16, 25, 36, 49, 64, and 81. Only \((x - 4) \times 10 - 1 = 49\) gives a positive-integer answer, \( x = 9 \), so Roy must have started with 9.
(MP.3) 5. How many solutions does $|x| = -3$ have? Explain.

**Answer:** None, because the absolute value (the distance of a number from 0) is never negative.

(MP.1, MP.3) 6. Find all two-digit numbers that are a prime number times the sum of their digits.

**Solution:** There are several cases to consider: $10x + y = 2x + 2y$, $10x + y = 3x + 3y$, $10x + y = 5x + 5y$, and $10x + y = 7x + 7y$. $10x + y = 11x + 11y$ would give $10y = -x$, which is not possible, so we can stop at 7—anything higher doesn’t work. Try each case in turn:

Case 1: $10x + y = 2x + 2y$. Then $8x = y$, so $x = 1$ and $y = 8$, so 18.

Case 2: $10x + y = 3x + 3y$. Then $7x = 2y$, so $x = 2$ and $y = 7$, so 27.

Case 3: $10x + y = 5x + 5y$. Then $5x = 4y$, so $x = 4$ and $y = 5$, so 45.

Case 4: $10x + y = 7x + 7y$. Then $3x = 6y$, or $x = 2y$, so 21, 42, 63, and 84.

So, all the two-digit numbers that are a prime number times the sum of their digits are 18, 21, 27, 42, 45, 63, and 84.

(MP.3) 7. A trapezoid has exactly three equal sides.

a) Are the two parallel sides equal? Explain how you know.

b) Sketch a trapezoid where the shorter parallel side is equal to the other two.

c) Sketch a trapezoid where the longer parallel side is equal to the other two.

d) Try to draw, using a compass and protractor, a trapezoid where the longer parallel side is equal to the other two and where the angle between the longer parallel side and the other two sides is …

i) $45^\circ$  

ii) $60^\circ$  

iii) $70^\circ$

e) Fill in the blanks: To be able to sketch a trapezoid as in part d), the angle must be between ____ and ____.

**Answers:**

a) No, because that would be a parallelogram.

b) 

c) 

d) i) $45^\circ$: the sides cross each other:  

ii) $60^\circ$: the sides form a triangle, not a trapezoid  

iii) $70^\circ$: this forms a trapezoid

e) $60^\circ$ and $90^\circ$. Anything larger than $90^\circ$ would result in the larger side being the shorter side.  

$90^\circ$ would result in the trapezoid being a rectangle. $60^\circ$ would result in the trapezoid being a triangle. Less than $60^\circ$ would result in the trapezoid’s sides crossing each other.
(MP.3) 8. a) In a triangle, two sides are equal. One side is three times another. Are the two longer sides equal or the two shorter sides? How do you know?
b) In a triangle, two angles are equal. One angle is three times another. All angles are whole numbers. What are the three angles in the triangle?

Answers:
a) The two longer sides are equal. If the shorter sides were equal, they wouldn’t add to more than the longer side, so it wouldn’t be a triangle.
b) Either the two smaller angles are equal or the two larger angles are equal. If the two smaller angles are equal, then \(x + x + 3x = 180\), so \(5x = 180\), \(x = 36\), and the angles are \(36^\circ\), \(36^\circ\), and \(108^\circ\). If the two larger angles are equal, then \(x + 3x + 3x = 180\), \(7x = 180\), so \(x\) is not a whole number.

9. Find all ten-digit numbers that use each digit from 0–9 once, such that the first \(n\) digits form a number divisible by \(n\).

Sample solution: Write the number as \(A,BCD,EFG,HIJ\).

- J is 0 and E is 5.
- B, D, F, and H are even, so A, C, E, G, and I are all odd.
- Since C is odd, D must be 2 or 6 to make CD a multiple of 4.
- Since G is odd, H must be 2 or 6 to make GH a multiple of 4.
- So, D and H are 2 and 6 in some order and B and F are 4 and 8 in some order.
- Since ABC and ABC,DEF are multiples of 3, so is \(ABC,DEF - 1,000 \times ABC = DEF\)
- The four possibilities for DEF are 254, 258, 654, and 658. Of these, only 258 and 654 give multiples of 3.

Case 1: \(DEF = 258\). Then \(B = 4\) and \(H = 6\). Now FGH is a multiple of 8 and F is even, so F00 is a multiple of 8. Then FGH – F00 = GH is also a multiple of 8. Since H is 6, G is 1, 5, or 9. G cannot be 5 because E is 5, so G is 1 or 9.

Now, ABC is a multiple of 3, so A4C is a multiple of 3 and A and C must be two of 1, 3, 7, and 9. The only combinations where \(A + 4 + C\) is a multiple of 3 are ABC = 147 and ABC = 741. That forces G to be 9, because 1 is taken. But neither 1,472,589 nor 7,412,589 are divisible by 7.

Case 2: \(DEF = 654\). Then \(B = 8\) and \(H = 2\). Since F is even, again GH is a multiple of 8. So, GH is 32 or 72. So G = 3 or 7.

ABC is a multiple of 3, so the possibilities for ABC are 183, 189, 381, 387, 389, 783, 789, 981, and 987. 387 and 783 use both 3 and 7, so we can eliminate those possibilities.

So, the possibilities for multiples of 7 are:

- 1,836,547
- 1,896,543
- 1,896,547
- 3,816,547
- 3,896,547
- 7,896,543
- 9,816,543
- 9,816,547
- 9,876,543

Trying each in turn, 3,816,547 is the only multiple of 7 that works, so 3,816,547,290 is the only number that works.
PS7-9  Choosing Strategies

Teach this lesson after: 7.2 Unit 7


Goals:
Students will choose between different problem-solving strategies.

Prior Knowledge Required:
Can create a factor tree
Can find a prime factorization
Can apply the divisibility rules for 2 through 11
Can break a problem into cases
Can create a tape diagram to solve a problem
Can solve equations
Can search systematically for an answer
Can determine what percentage increase a given increase represents

Vocabulary: composite, divisible, factor, factor tree, prime, surface area

Materials:
BLM Amusement Park (pp. S-98–101, see Performance Task)
BLM Percentage Discounts (pp. S-104–107, see Performance Task)

NOTE: The following Problem Bank questions reflect all the problem-solving strategies used in the problem-solving lessons for Grade 7. Choose among the following questions based on which problem-solving lessons you have taught.

Problem Bank
(MP.1, MP.7) 1. Solve the problems in two ways. Make sure you get the same answer using both ways.
   A. The value of my dimes is twice the value of my quarters. How many times as many dimes as quarters do I have?
   B. The value of my dimes is \(\frac{4}{5}\) the value of my quarters. How many times as many dimes as quarters do I have?
   a) Use searching systematically.
   b) Use algebra.
Solutions:

a) A. 

<table>
<thead>
<tr>
<th>Quarters</th>
<th>Value of Quarters (¢)</th>
<th>Value of Dimes (¢)</th>
<th>Number of Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td>50</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>100</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>75</td>
<td>150</td>
<td>15</td>
</tr>
</tbody>
</table>

So, I have 5 times as many dimes as quarters.

B. 

<table>
<thead>
<tr>
<th>Quarters</th>
<th>Value of Quarters (¢)</th>
<th>Value of Dimes (¢)</th>
<th>Number of Dimes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>40</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>75</td>
<td>60</td>
<td>6</td>
</tr>
</tbody>
</table>

So, I have twice as many dimes as quarters.

b) A. If there are \( x \) quarters, they are worth 25\( x \) cents. The dimes are worth twice as much, so they are worth 50\( x \) cents. So there are 5\( x \) dimes, and the number of dimes is 5 times the number of quarters.

B. If there are \( x \) quarters, they are worth 25\( x \) cents. The dimes are worth 4/5 of 25\( x \) or 20\( x \) cents. So there are 2\( x \) dimes, and the number of dimes is twice the number of quarters.

2. A Canadian dollar is worth 80% of a US dollar.

a) What percent of a Canadian dollar is a US dollar?

b) An American tourist visits Canada and makes a purchase costing $30 Canadian. He pays with an American $50 bill. How much Canadian change should he expect back?

**Answers:**

a) 125%; b) $50 US = $50 \times 1.25 \text{ Cdn} = $62.50 \text{ Cdn}, so he should expect $32.50 back.

3. A square is cut in half parallel to an edge. By what percentage did the total perimeter increase?

**Answer:** Let the side length be \( x \). The original perimeter is 4\( x \) and the new perimeter of the two half-squares is 6\( x \), so the perimeter increased by 2\( x \), which is half the original perimeter. That means it increased by 50%.

4. A cube is cut in half vertically. By what percentage did the surface area increase?

**Answer:** Let the side length be \( x \). The original surface area is 6\( x^2 \). The new surface area is 8\( x^2 \), an increase of 2\( x^2 \), which is 1/3 or 33.\( \frac{1}{3} \) % of the original surface area. The surface area increased by 33%.
(MP.1, MP.3, MP.4, MP.5, MP.7) 5. Decide which problem-solving strategy (or strategies) to use from the list below. Then solve the problem.

- Searching systematically
- Guessing, checking, and revising
- Looking for a pattern
- Using structure
- Using algebra
- Using logical reasoning
- Breaking a problem into cases

a) A power of 10 has exactly 100 factors. Which power of 10 is it?
b) Find all pairs of digits A and B so that the three-digit number A3B is divisible by 36.
(MP.7) c) A scalene triangle has perimeter 11 and whole-number side lengths. What are the side lengths of the triangle?
(MP.3) d) The four-digit number ABCD is a multiple of 11. If A = D, what can you say about B and C? Explain.
e) How many ways can you rearrange the digits 2, 3, 7, and 9 to make a four-digit number that is …
   i) a multiple of 11? ii) a multiple of 3?
f) What is the sum of integers 1 – 2 + 3 – 4 + 5 – 6 + … + 49 – 50?
g) The sum of the numbers from -40 to N is 170. What is N?
h) The sum of the numbers from -40 to N is 1,705. What is N?
(MP.1, MP.4) i) At the end of a tennis lesson, each student picks up 5 balls, leaving 4 for the instructor to pick up. The next day, the instructor brings the same number of tennis balls, but 2 fewer players show up. At the end of that lesson, each student picks up 7 balls, leaving only 2 for the instructor to pick up. How many tennis balls did the instructor bring each day?

Solutions:
a) Look for a pattern.
10 has 4 factors: 1, 2, 5, 10
100 has 9 factors: 1, 2, 4, 5, 10, 20, 25, 50, 100
1,000 has 16 factors: 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 125, 200, 250, 500, 1,000
Continuing this pattern, the number of factors of 10^n is the square of n + 1, so 10^9 has 10^2 = 100 factors.
b) Divide the problem into cases. B is 2 or 6 because 3B is a multiple of 4.
Case 1: B is 2. Then A + 3 + 2 = A + 5 is a multiple of 9, so A is 4.
Case 2: B is 6. Then A + 3 + 6 is a multiple of 9, so A is 9.
c) Use systematic search and then use structure. List all the ways three different numbers can add to 11: 1, 2, 8; 1, 3, 7; 1, 4, 6; 2, 3, 6; 2, 4, 5
The only combination that could form the sides of a scalene triangle is 2, 4, 5.
d) Use reasoning. A – B + C – D is a multiple of 11, but A = D, so C – B is a multiple of 11. But B and C are both single digits, so they can’t differ by more than 9, so C – B must be 0, meaning that B = C.
e) i) Use structure and then use systematic search. There are 8 ways, because you need to find two pairs of digits that differ by a multiple of 11, and the only one that works is \((7 + 9) - (2 + 3) = 11\), so the eight ways are 2,739, 2,937, 3,729, 3,927, 7,293, 7,392, 9,273, 9,372; ii) Use structure and then use systematic search. The sum of the digits is \(2 + 3 + 7 + 9 = 21\), a multiple of 3, so the question is really asking how many ways you can rearrange the four digits, since all of them work. There are 24 ways, six that start with each digit. The six that start with 2, for example, are 2,379, 2,397, 2,739, 2,793, 2,937, 2,973.

f) Use structure. The sum is the sum of twenty-five \(-1\)s: \((-1) + (3 - 4) + (5 - 6) + \ldots + (49 - 50)\), so the answer is \(-25\).

g) Use structure and systematic search. To get rid of all the negative numbers, the sum must go up to at least \(N = 40\). This gives \(41 + 42 + \ldots = 170\). Trying each possible ending number in turn quickly arrives at \(N = 44\), since \(41 + 42 + 43 + 44 = 170\).

h) Use structure and guessing, checking, and revising. As in part g), the sum simplifies to \(41 + 42 + \ldots = 1,705\). Try last number 50: the sum is \(41 + 42 + \ldots + 50 = 40 \times 9 + 50 + 1 + 2 + \ldots + 9 = 360 + 50 + 45 = 455\), too low. Try last number 60: \(41 + 42 + \ldots + 49 + 50 + \ldots + 59 + 60 = 40 \times 9 + 50 \times 10 + 60 + 2 \times (1 + 2 + \ldots + 9) = 360 + 500 + 60 + 90 = 1,010\), too low. Try last number 70. This works, so \(N = 70\).

i) Use algebra. Let \(x\) be the number of students in the lesson on the first day. Then the number of balls on the first day is \(5x + 4\) and the number of balls on the second day is \(7(x - 2) + 2\). These are equal, because the instructor brings the same number of balls both days, so \(5x + 4 = 7(x - 2) + 2\), \(5x + 4 = 7x - 14 + 2\), \(4 + 14 - 2 = 7x - 5x\), \(16 = 2x\), so \(x = 8\). There are 8 students in the class on the first day, so the number of balls is 44 = 5 \(\times\) 8 + 4. As confirmation, this is also equal to 7 \(\times\) 6 + 2.

6. Here are some branching patterns for factor trees.

a) How many different branching patterns can you find for a factor tree of 24?
b) Pick a branching pattern from part a). How many different factor trees of 24 have that pattern?

Solutions:
a) There are five branching patterns.

b) You can split 24 into two non-primes:

```
2 / 2 \
 3
```

```tex
\begin{array}{c}
2 \\
4 \\
\end{array} \\
\begin{array}{c}
6 \\
\end{array} \\
\begin{array}{c}
2 \\
2 \\
3 \\
3 \\
\end{array}
```
You can split 24 into a prime number on the left and a composite number on the right (two branching patterns):

```
   24
  /   \
 /     \ 
3     8
   /   \
  2     4
     /   \
    2    2
```

Using 2 on the left and 12 on the right gives the same two patterns.

You can split 24 into a composite number on the left and a prime number on the right (two branching patterns):

```
   24
  /   \
 /     \ 
8     3
   /   \
  2     4
     /   \
    2    2
```

Using 12 on the left and 2 on the right gives the same two patterns.

b) Pick branching pattern 2. There are four of this type:

```
   24
  /   \
 /     \ 
3     8
   /   \
  2     12
     /   \
    2    6
```

7. Each side length of a cube increases by 120%.
   a) By what percentage does the volume increase?
   b) By what percentage does the surface area increase?

**Answers:**

a) If the original side length is \( x \), the new side length is \( x + 1.2x = 2.2x \). So the volume is \((2.2x)^3 = 10.648x^3 = x^3 + 9.648x^3\), so the volume increased by 964.8%.

b) The original surface area is \( 6x^2 \) and the new surface area is \( 6(2.2x)^2 = 6x^2(4.84) = (original
type) + 3.84 \times (original
type), so the surface area increased by 384%.

8. One ice cube in an ice cube tray has a rectangular 3 cm by 5 cm base and is 2.2 cm tall. Water expands 10% when it freezes. If the ice is exactly level with the top of the tray when it’s frozen, how deep was the tray filled with water?

**Answer:** 2 cm
9. An opaque cube is made of 27 smaller cubes.
   a) What is the maximum number of cubes an observer could see at least one face of at any given time?
   b) The cube is painted red. How many smaller cubes have …
      i) exactly one face painted red?
      ii) exactly two faces painted red?
      iii) exactly three faces painted red?
      iv) no faces painted red?
      v) more than three faces painted red?
   Answers: a) 19; b) i) 6, ii) 12, iii) 8, iv) 1, v) 0

10. Three cubes are stacked as shown. The side lengths are 3 cm, 2 cm, and 1 cm.
    What is the total surface area of the new shape?

   Sample solutions:
   Solution 1: The surface area of the three cubes if they weren’t stacked is $6(1^2) + 6(2^2) + 6(3^2) = 84$. Subtract $2(1^2) + 2(2^2) = 10$ for the four hidden faces. The total surface area is 74 cm$^2$.
   Solution 2: $1^2 + 2^2 + 3^2 = 14$, so front area + back area + right side area + left side area = $4 \times 14$ cm$^2 = 56$ cm$^2$. The top side and the bottom side are both 9 cm$^2$. $56 + 9 + 9 = 74$, so the total surface area is 74 cm$^2$.

11. a) Is $\pi^2$ more or less than 9? How do you know?
    b) Compare the fractions.
       i) $\frac{3}{4}$ and $\frac{2}{3}$
       ii) $\frac{6}{7}$ and $\frac{7}{9}$
       iii) $\frac{3}{5}$ and $\frac{2}{3}$
       iv) $\frac{\pi}{4}$ and $\frac{2}{\pi}$
    c) Did doing parts i–iii) help you solve iv)? Explain.
    d) Compare the fractions mentally.
       i) $\frac{\pi}{4}$ and $\frac{3}{4}$
       ii) $\frac{2}{\pi}$ and $\frac{2}{3}$
       iii) $\frac{2}{3}$ and $\frac{3}{4}$
    e) Write the fractions in order from least to greatest: $\frac{2}{\pi}$, $\frac{\pi}{4}$, $\frac{2}{3}$, $\frac{3}{4}$

   Answers: a) more, $\pi > 3$, so $\pi^2 = \pi \times \pi > 3 \times 3 = 9$; b) i) $9/12 > 8/12$, ii) $54/63 > 49/63$,
            iii) $9/15 < 10/15$, iv) $\pi^2(4\pi) > 8/(4\pi)$; c) sample explanation: yes, it helped me know to multiply
            the denominators to get the common denominator; d) i) $\pi/4 > 3/4$, because $\pi > 3$, ii) $2/ \pi < 2/3$,
            because $\pi > 3$, iii) $2/3 < 3/4$, because $2/3$ is 1/3 less than 1, and $3/4$ is less than 1 by only 1/4,
            so 3/4 is the greater fraction; e) $2/\pi < 2/3 < 3/4 < \pi/4$

12. Find the dimensions of all rectangular prisms that have whole-number side lengths and a surface area of 22 cm$^2$.
   Answers: 1 cm × 1 cm × 5 cm and 1 cm × 2 cm × 3 cm.
13. Jay is thinking of a number. He did the following operation: (original number) – (sum of digits). He said his answer was either 9,999 or 10,000. Which one must have been his answer? Hint: Try doing the operation for very small numbers. What property does your answer always have? Can you explain why the answer will always have that property? Which number, 9,999 or 10,000, has that property?

**Bonus:** Find a number Jay could have been thinking of.

**Solutions:** Try for several two-digit numbers: 18 – 9 = 9, 23 – 5 = 18, 71 – 8 = 63, 56 – 11 = 45, 13 – 4 = 9; all the answers are multiples of 9. To see why this is true, write a two-digit number as 10a + b, then subtract 10a + b – (a + b) = 9a; this also works for three-digit numbers: 100a + 10b + c – (a + b + c) = 99a + 9b, also a multiple of 9. In fact, this reasoning works for any number of digits. So, John’s answer must have been 9,999, as that is divisible by 9.

**Bonus:** Sample answer: 10,001

14. Your friend picks a number from 1 to 1,000. You have to guess the number by asking yes or no questions about the number. Determine the least number of questions that need to be asked to guarantee that you will find the secret number.

**Solution:** Use the strategy “looking for a pattern.” Start with the number of questions needed for picking a number from 1 to 2: You just need to ask one question: Is it 1? If yes, done; if no, done, because the answer is 2. The number of questions needed for picking a number from 1 to 3 or 4 is 2: Is it greater than 2? If yes, Is it 3? If no, is it 1? The number of questions needed for picking a number from 1 to 5, 6, 7, or 8 is 3: Is it greater than 4? If yes, use two questions to determine which of 5, 6, 7, or 8 the number is and if no, use two questions to determine which of 1, 2, 3, or 4 the number is. The number of questions needed for picking a number from 1 to 16 (or from 1 to 9, 10, …, 16) is 4: First narrow it down to choosing among 8 numbers by asking if it is greater than 8. Then you can solve the problem by using three more questions, as above. Indeed, any time you multiply the number of numbers by 2, you need one more question. 1,000 is greater than 2^9, so you need 10 questions.

15. a) Find the next term in the pattern: 2, 3, 5, 8, 13, 21, 34.

b) Evaluate the first three terms in the pattern:

\[
\frac{1}{1+\frac{1}{2}} = \frac{1}{1+\frac{1}{\frac{1}{2}}}, \quad \frac{1}{1+\frac{1}{\frac{1}{2}}} = \frac{1}{1+\frac{1}{2}}
\]

**Bonus:** If the \(n\)th term in the sequence from part a) is \(A_n\), what is the \(n\)th term in the sequence from part b)?

**Answers:** a) 55; b) 2/3, 3/5, 5/8; c) 8/13; Bonus: \(A_n/A_{n+1}\)

16. a) Find a four-digit number so that the number above each digit tells how many times the digit occurs in the four-digit number.
b) Repeat part a) for …
i) a five-digit number.

| 0 | 1 | 2 | 3 | 4 |

ii) a seven-digit number.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 |

iii) an eight-digit number.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

iv) a nine-digit number.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

v) a 10-digit number.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

c) Calculate the sum of the digits for each part of a) and b). Why could you have predicted this?

**Answers:**
a) 1,210; b) i) 21,200, ii) 3,211,000, iii) 42,101,000, iv) 521,001,000, v) 6,210,001,000; c) the sum of the digits is a) 4; b) i) 5; ii) 7; iii) 8; iv) 9; v) 10; I could have predicted this because each digit tells how many times the number above it occurs, so the sum of the digits has to tell how many times all the digits occur together, and that must be the number of digits in the number.

17. a) Follow these steps for various three-digit numbers:

Step 1: Write the digits in increasing order, including any leading zeros (e.g., 700 becomes 007).
Step 2: Write the digits in decreasing order.
Step 3: Subtract the smaller number from the larger number.
Step 4: Repeat these steps with the new number (if the new number is a two-digit number, put a zero in front). Continue until the answer repeats.

Try this for …

i) 327   ii) 654   iii) 217   iv) 403   v) 600
vi) 112   vii) 111   viii) 444   ix) 555

Predict: When does the answer reach 495? When does it reach something else?

b) i) If you write the three-digit number with increasing digits as $100x + 10y + z$, what is the three-digit number with decreasing digits?

ii) Show that the answer when subtracting the two numbers is always a multiple of 99.

iii) How many multiples of 99 are there up to 1,000, starting at $99 \times 0 = 0$?

iv) Starting with each multiple of 99, what number gets reached?
Answers:

a) i) 327 becomes 732 − 237 = 495, which becomes 954 − 459 = 495, and the answer repeats;
   ii) 654 goes to 198, which goes to 792, which goes to 693, which goes to 594, which goes to 495;
   iii) 495;
   iv) 495;
   v) 495; vi) 99, which goes to 495; vii) 0; viii) 0; ix) 0; Predict: when not all the digits are the same, it reaches 495, when all the digits are the same, it reaches 0.

b) i) \(x + 10y + 100z\); ii) \(100x + 10y + z - (x + 10y + 100z) = 99x - 99z = 99(x - z)\), so this is always a multiple of 99; iii) 11, starting at 0 × 99 = 0 and going up to 10 × 99 = 990; iv) 0 goes to 0, but any other multiple of 99 goes to 495.

18. If \(4x : 5y = 3 : 2\), what is \(x : y\) ?

Answer: \(x : 5y = 3/4 : 2 = 3 : 8\), so \(x : y = 3 : 8/5 = 15 : 8\).

19. The product of the numbers from \(-10\) to \(N\) is positive. What can you say about \(N\)?

Answer: \(N\) is negative and odd.

20. How many integers have a reciprocal between …

a) 0.4 and 0.7?

b) \(\frac{13}{95}\) and \(\frac{53}{95}\)?

Solutions:

a) The reciprocal of 0.4 = 2/5 is 5/2 and the reciprocal of 0.7 = 7/10 is 10/7. The only integer between 10/7 and 5/2 is 2, so the answer is 1.

b) The reciprocal of 13/95 is 95/13 = 7 4/13 and the reciprocal of 53/95 is 95/53 < 2, so the integers from 2 to 7 all have reciprocals between 13/95 and 53/95.

21. A triangle has perimeter 333 cm. Which of these can be the length of its longest side?

A. 110 cm  B. 166 cm  C. 167 cm  D. 331 cm

Solution: Not A, because if that was the longest, the perimeter would be at most 330 cm. Not C or D, because the sum of the shortest two sides have to be longer than the longest side; but if the longest side is at least 167, then the shortest two sides have to add to more than 167, which would make the perimeter more than 334 cm. So the answer is 166 cm.

22. a) The average of two numbers is 30, and the average of three other numbers is 40. What is the average of all five numbers?

b) The average of two numbers is \(x\) and the average of three other numbers is \(x + 10\). What is the average of all five numbers?

c) How can you get your answer in part a) from your answer in part b)?

Solutions: a) the sum of all five numbers is \(60 + 120 = 180\), so their average is 36; b) the sum of the two numbers is \(2x\) and the sum of the three other numbers is \(3x + 30\), so the sum of all five numbers is \(5x + 30\) and their average is \(x + 6\); c) substituting \(x = 30\) gets the average as \(30 + 6 = 36\).
23. You have a 3 L pail, a 5 L pail, an 11 L pail, a 20 L pail, and a river to dump water into or to get water from.
   a) How can you measure 29 L?
   b) How can you measure 7 L?

   **Answers:**
   a) Fill the 20 L pail, the 11 L pail, and the 3 L pail, to make a total of 34 L. Pour water from, say, the 20 L pail to fill the 5 L pail. Then the total water in the 20 L pail, the 11 L pail, and the 3 L pail is 29 L.
   b) Fill the 20 L and the 3 L pails. Pour from the 20 L pail to fill the 5 L and 11 L pails. There remains 4 L in the 20 L pail. That together with the 3 L pail makes 7 L.

24. You have a 2 L pail, a 5 L pail, and a river.
   a) How can you make the 5 L pail have exactly 4 L in it?
   b) When the 5 L pail has exactly 4 L in it, how can you make the 2 L pail have 1 L in it?

   **Answers:**
   a) Fill the 2 L pail and pour into the 5 L pail. Do this again.
   b) Fill the 2 L pail. Pour the contents into the 5 L pail until the 5 L pail is full. There will now be 1 L in the 2 L pail.

25. You have a 3 L pail, a 7 L pail, and a river.
   a) Suppose you want to put 5 L in the 7 L pail. How would it help to have 1 L already in the 3 L pail?
   b) How could you put 1 L in the 3 L pail so that you can then do the steps from part a) to put 5 L in the 7 L pail?

   **Answers:**
   a) Fill the 7 L pail completely, and then pour as much as you can into the 3 L pail to fill it up. Since there was already 1 L in the 3 L pail, you must have poured 2 L from the 7 L pail, leaving 5 L in the 7 L pail.
   b) Fill the 7 L pail completely, and then pour 3 L into the 3 L pail. Empty the 3 L pail into the river and again pour 3 L from the 7 L pail into the 3 L pail. There is now 1 L in the 7 L pail. Empty the 3 L pail, and pour the 1 L from the 7 L pail into the 3 L pail.

26. How can you measure 16 L by using a 7 L pail, a 22 L pail, and a river?

   **Solution:** In the pictures below, the bigger pail represents the 22 L pail and the smaller pail represents the 7 L pail. The numbers represent the amount of water in the pail (in L).

   ![Diagram](image-url)

   **Step 1:** Fill the 22 L pail from the river.
   **Step 2:** Fill the 7 L pail from the 22 L pail.
   **Step 3:** Empty the 7 L pail into the river.
   **Step 4:** Fill the 7 L pail from the 22 L pail.

   **MP.1, MP.3**

   27. How can you measure 1 L by using a 7 L pail, a 22 L pail, and a river?
Step 5: Empty the 7 L pail into the river.

\[
\begin{array}{c|c}
8 & 0 \\
\end{array}
\]

Step 6: Fill the 7 L pail from the 22 L pail.

\[
\begin{array}{c|c}
1 & 7 \\
\end{array}
\]

Step 7: Empty the 7 L pail into the river.

\[
\begin{array}{c|c}
1 & 0 \\
\end{array}
\]

Step 8: Pour the 1 L from the 22 L pail into the 7 L pail.

\[
\begin{array}{c|c}
0 & 1 \\
\end{array}
\]

Step 9: Fill the 22 L pail from the river.

\[
\begin{array}{c|c}
22 & 1 \\
\end{array}
\]

Step 10: Fill the 7 L pail from the 22 L pail.

\[
\begin{array}{c|c}
16 & 7 \\
\end{array}
\]

Now you have measured 16 L.

(MP.1) 27. The Tower of Hanoi problem. There are three rods and some disks. All the disks are different sizes and they are stacked on one rod from smallest to largest. The goal is to move the stack to another rod by moving only one disk at a time from the top of one stack to the top of another stack. No disk can be placed on top of a smaller disk. Solve the problem for the given number of disks, using the smallest number of moves you can, and complete the chart.

<table>
<thead>
<tr>
<th>Number of Disks</th>
<th>Number of Moves Required to Solve the Puzzle</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Predict how many moves would be required if you started with 8 disks.

NOTE: The problem is easier to do using concrete materials. If you don’t have disks of different sizes, use cubes and tape numbers to them in order from 1 to the number of “disks.” The rule then becomes: “No cube can be placed on top of a smaller numbered cube.” You will have to imagine the rods.

Answer: \(2^8 - 1\), or 255, moves would be required.
Performance Task: Amusement Park

Materials:
BLM Amusement Park (pp. S-98–101)

Performance Task: Amusement Park. Provide students with BLM Amusement Park. Tell students that in this performance task they will work with equations, inequalities, bias in surveys, and circumference of circles.

Answers:
1. 800
2. a) 24, b) 23, c) 18,400
3. a) 180 + 15x, b) $630, c) $180, d) at least 8 hours
4. a) both estimates will affect the amount of money the owner will make: more people means more money, and a greater ratio of adults to children also means more money; b) i) the estimate will be biased because it's likely more people will come overall, but also more children will come because they are free every day instead of just on weekends; ii) the ratio of adults to children should not be biased, but the total number of people will be more than average; c) $26,692
5. 301.44 yards
Bonus: a) 3, 5, 7, 9, 10, 12, 14, 17, 19, 24; b) 2, 5, 10; sample explanation: The lowest-scoring region, obtained 3 times, gives a total of 6, so the lowest-scoring region must be 2. The highest-scoring region, obtained 3 times, gives a total of 30, so the highest-scoring region must be 10. To get the second lowest scoring total, use 9 = 2 + 2 + (second lowest scoring region), so the second lowest scoring region is 5.
Amusement Park (1)

You work at an amusement park. The Ferris wheel ...

- allows one complete rotation per ride.
- has 20 cabins that each carry 40 people.
- takes 20 minutes to make one complete rotation and stops for 5 minutes between rides.

1. How many people can go on each ride?

2. The Ferris wheel runs from 10 a.m. to 8 p.m.
   a) At most, how many Ferris wheel rides can the park offer each day?

   b) The Ferris wheel sometimes stops for longer periods when maintenance is needed or if someone needs assistance getting on or off. Subtract one ride from your answer in part a) to create a more realistic maximum number of rides.

   c) On one summer day, the Ferris wheel was constantly full. Using your answer from part b), how many people rode the Ferris wheel that day?
Amusement Park (2)

3. You work at the amusement park and are on call for six 5-hour shifts a week. You only need to go in if they need you, but they pay you $30 for each shift that you are on call, plus $15 for each hour that you work.

   a) How much money would you make in a week if you worked $x$ hours?

   b) What is the most money you could make in a week?

   c) What is the least money you could make in a week?

   d) How many hours would you need to work to make at least $300 in one week?
Amusement Park (3)

4. An adult ticket costs $20 and a child ticket costs $12. The owner of the Ferris wheel wants you to estimate ...

• the ratio of adults to children, on average.
• the number of people that ride the Ferris wheel, on average.

a) Why might the owner be interested in each estimate?

b) You want to pick 30 days in the year and check how many adult tickets and how many child tickets were sold. Explain how bias might be introduced in each sample below. Would the estimate be biased?

i) All 30 days are during summer vacation.

ii) All 30 days are bright sunny days throughout different times of the year.

c) You picked 30 days randomly. The average number of adult tickets sold was 956 and the average number of child tickets sold was 631. How much money can you expect the Ferris wheel to bring in each day?

5. The bottom of the Ferris wheel is 4 yards above the ground. The top of the Ferris wheel is 100 yards above the ground. What is the circumference of the Ferris wheel? Use 3.14 for \( \pi \).
Amusement Park (4)

**Bonus** A game at the amusement park allows a player to toss three beanbags aimed at a target, with points as shown below:

![Diagram of a target with points 1, 3, and 8]

a) If all three beanbags hit the target, what are all the possible scores a player can get?

b) On a similar target, the three scoring regions are not labeled, but the possible scores, assuming all beanbags hit the target, are 6, 9, 12, 14, 15, 17, 20, 22, 25, and 30. What points are assigned to each scoring region? Explain how you know.
Performance Task: Percentage Discounts

Materials:
BLM Percentage Discounts (pp. S-104–107)

Performance Task: Percentage Discounts. Provide students with BLM Percentage Discounts. Tell students that in this performance task, they will work with percentages of numbers in a real-world context by comparing two types of discounts, a direct percentage discount, and one based on requiring the customer to buy one item at full price before getting a discount.

Answers:
1. a) $12; b) $18; c) 60%, sample explanations: 100% – 40% = 60%, or solve the problem “18 is what percent of 30?” to get 60
2. a) $55; b) $12 and $21; c) $33; d) 40%, sample explanations: by the distributive property, 40% of $20 + 40% of $35 is 40% of $55, or solve the problem “22 (the amount off) is what percent of 55?” to get 40
3. a) 100, 60, 90, A; b) 100, 60, 75, A; c) 100, 60, 60, same; d) 100, 60, 50, B
4. a) 100, 90, 10; b) 100, 75, 25; c) 100, 60, 40; d) 100, 50, 50
5. A 50% discount is the greatest percentage discount you can get. This happens when both items are the same price.
6. a) yes, sample explanation: the discount price is always 0.6 times the original price, so the constant of proportionality is 0.6; b) no, sample explanation: the discount price can be different for the same total price if the cheaper item is a different price
7. a) 20% off, b) 10% off, c) 25% off
Bonus: a) 30% off, b) 15% off, c) x/2% off
Bonus: less; it will cost 98% of the current price in the new year because 0.7 × 1.4 = 0.98.
Percentage Discounts (1)

Store A has a sale: Get 40% off all items.

1. Amit buys a sweater with original price $30.
   a) What is the amount of the discount?

   b) What is the sale price, after the discount?

   c) What percentage of the original price is the sale price? Explain how you know.

2. Bev buys two items with original prices $20 and $35.
   a) What is the total price before the sale?

   b) What is the sale price of each item?

   c) What is the total price after the sale?

   d) What percentage off the total price did she get? Explain how you know.
Percentage Discounts (2)

Store A has a sale: Buy any number of items for 40% off each item.

Store B has a sale: Buy one item and get an item of equal or lesser value for free.

3. Which store gives a lower price if you buy two items at the given price?

<table>
<thead>
<tr>
<th>Item 1</th>
<th>Item 2</th>
<th>Total Price before Discount ($)</th>
<th>Total Price after Discount: Store A ($)</th>
<th>Total Price after Discount: Store B ($)</th>
<th>Which Store has the Lower Price?</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>$10</td>
<td>$90</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b)</td>
<td>$25</td>
<td>$75</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c)</td>
<td>$60</td>
<td>$40</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d)</td>
<td>$50</td>
<td>$50</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. For each part in Question 3, calculate the percentage discount you would get overall at Store B.

<table>
<thead>
<tr>
<th>Original Total Price ($)</th>
<th>Final Price ($)</th>
<th>Percent Discount</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>d)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. At Store B, what is the greatest percentage discount you could get on two items? When does this happen?
Percentage Discounts (3)

6. a) Is the total price proportional to the discount price at Store A? Explain how you know.

b) Is the total price proportional to the discount price at Store B? Explain how you know.

7. Store C has a sale: Buy one item and get an item of equal or lesser value for half price. What is the percentage discount off the original total price if you buy two items for ...

a) $30 and $20?

b) $8 and $32?

c) $15 and $15?
Percentage Discounts (4)

Bonus ►

What is the greatest percentage discount you can get at a store if the store announces the following sale?
Buy one item and get an item of equal or lesser value for …

a) 60% off.

b) 30% off.

c) $x\%$ off.

Bonus ►

Rick wants to buy a tennis racket. In the new year, the store selling the racket will increase the price by 40% and then have a 30% discount sale. Will the tennis racket cost more or less in the new year?
1 cm Grid Paper
Geometry: Angles and Areas – AP Book 7.2: Unit 4

AP Book G7-11

1. a) Supplementary

2. a) $\angle B$
   b) $\angle D$
   c) $\angle A$
   d) $\angle C$

3. a) Supplementary
   b) Complementary
   c) Complementary
   d) Supplementary

4. a) $x + 60 = 90$
   $x = 90 - 60$
   $x = 30$
   So $x = 30^\circ$
   b) $x + 75 = 90$
   $x = 90 - 75$
   $x = 15$
   So $x = 15^\circ$
   c) $x + 54 = 180$
   $x = 180 - 54$
   $x = 126$
   So $x = 126^\circ$
   d) $5x + x = 180$
   $6x = 180$
   $x = 180 / 6$
   $x = 30$
   So $x = 30^\circ$
   e) $3x + 5x + 2x = 180$
   $10x = 180$
   $x = 180 / 10$
   $x = 18$
   So $x = 18^\circ$
   f) $x + 57 = 90$
   $x = 90 - 57$
   $x = 33$
   So $x = 33^\circ$

5. Teacher to check.

6. a) $2x + 10 + 120 = 180$
   $2x = 180 - 130$
   $2x = 50$
   $x = 50 / 2$
   $x = 25$
   So $x = 25^\circ$
   b) $2x + 50 = 90$
   $2x = 90 - 50$
   $2x = 40$
   $x = 40 / 2$
   $x = 20$
   So $x = 20^\circ$
   c) $x + x + 40 = 180$
   $2x = 180 - 40$
   $x = 140 / 2$
   $x = 70$
   So $x = 70^\circ$
   d) $3x + 30 = 180$
   $3x = 180 - 30$
   $x = 150 / 3$
   $x = 50$
   So $x = 50^\circ$
   e) $3x + x = 90$
   $4x = 90$
   $x = 90 / 4$
   $x = 22.5$
   So $x = 22.5^\circ$

7. a) $80^\circ$
   b) $100^\circ$
   c) $10^\circ$
   d) No. Since $\angle A$ is greater than $90^\circ$, $\angle E + \angle A$ cannot equal $90^\circ$.

AP Book G7-12

1. a) Adjacent
   b) Vertical
   c) Adjacent
   d) Vertical

2. a) $x = 60^\circ$, $y = 60^\circ$
   b) $x = 30^\circ$, $y = 30^\circ$
   c) $x = 90^\circ$, $y = 90^\circ$

3. The measures of two vertical angles are equal.

4. a) $x = 50^\circ$, $y = 105^\circ$
   b) $x = 70^\circ$, $y = 110^\circ$
   c) $x = 48^\circ$, $y = 129^\circ$

5. a) $2x = 30$
   $x = 30 / 2$
   $x = 15$
   So $x = 15^\circ$
   b) $x - 15 = 60$
   $x = 60 + 15$
   $x = 75$
   So $x = 75^\circ$
   c) $3x = 75$
   $x = 75 / 3$
   $x = 25$
   So $x = 25^\circ$
   d) $2x + 10 = 90$
   $2x = 90 - 10$
   $x = 80 / 2$
   $x = 40$
   So $x = 40^\circ$
   e) $90 - x = 62$
   $90 - 62 = x$
   $28 = x$
   So $x = 28^\circ$
   f) $5(x - 6) = 120$
   $x = 6 = 120 + 5$
   $x = 24 + 6$
   $x = 30$
   So $x = 30^\circ$
Geometry: Angles and Areas – AP Book 7.2: Unit 4

AP Book G7-13
page 109

1. a) complementary
   b) supplementary
   c) vertical
   d) complementary

2. a) \( x = 70^\circ \)
   \( y = 80^\circ \)
   \( z = 30^\circ \)
   b) \( x = 55^\circ \)
   \( y = 120^\circ \)
   \( z = 85^\circ \)
   c) \( x = 80^\circ \)
   \( y = 30^\circ \)
   \( z = 110^\circ \)

3. a) \( x + 40 = 360 \)
   \( x = 360 - 40 \)
   \( x = 320 \)
   So \( x = 320^\circ \)
   b) \( x + x + x = 180 \)
   \( 3x = 180 \)
   \( x = 180 / 3 \)
   \( x = 60 \)
   So \( x = 60^\circ \)
   c) \( 2x + 3x + 4x = 180 \)
   \( 9x = 180 \)
   \( x = 180 / 9 \)
   \( x = 20 \)
   So \( x = 20^\circ \)

4. a) \( x + 100 + 35 = 180 \)
   \( x = 180 - 135 \)
   \( x = 45 \)
   So \( x = 45^\circ \)
   b) \( x + 5x + 3x = 180 \)
   \( 9x = 180 \)
   \( x = 180 / 9 \)
   \( x = 20 \)
   So \( x = 20^\circ \)

5. a) \( x = 60^\circ \), \( y = 120^\circ \)
   b) \( x = 92^\circ \), \( y = 48^\circ \)
   \( z = 132^\circ \)
   c) \( x = 65^\circ \), \( y = 65^\circ \)
   d) \( x = 55^\circ \), \( y = 35^\circ \)
   \( z = 55^\circ \)
   e) \( x = 60^\circ \), \( y = 30^\circ \)
   \( z = 60^\circ \)
   f) \( x = 40^\circ \), \( y = 50^\circ \)
   \( z = 40^\circ \)

AP Book G7-14
page 112

1. a) 6
   b) 4
   c) 10

2. a) 24
   b) 20
   c) 10

3. a) \( (4 \times 4) + 2 + (4 \times 7) + 2 \)
   \( = (4 \times 11) + 2 \)
   \( = 44 + 2 \)
   \( = 22 \)
   So the area is 22 cm²
   b) \( (5 \times 4) + 2 + (5 \times 7) + 2 \)
   \( = (5 \times 11) + 2 \)
   \( = 55 + 2 \)
   \( = 27.5 \)
   So the area is 27.5 cm²
   c) \( (h \times 4) + 2 + (h \times 7) + 2 \)
   \( = (h \times 11) + 2 \)
   So the area is \( (h \times 11) + 2 \) cm²

4. \( (h \times 10) + 2 - (h \times 3) + 2 \)
   \( = (h \times 7) + 2 \)
   So the area is \( (h \times 7) + 2 \) cm²

5. a) \( (24 \times 5) + 2 \)
   \( = (20 \times x) + 2 \)
   \( 120 = 20x \)
   \( 120 + 20 = x \)
   \( 6 = x \)
   b) \( (8.9 \times 4.5) + 2 \)
   \( = (4 \times (x + 2)) + 2 \)
   \( 40.05 = 4 \times (x + 2) \)
   \( 40.05 + 4 = x + 2 \)
   \( 10.01 - 2 = x \)
   \( 8.01 = x \)
   So \( x = 80 \) mm
   c) \( 130° \)
   d) \( 55° \)

6. a) \( 65° \)
   b) \( 80° \)
   c) \( 130° \)

AP Book G7-15
page 115

1. a) parallel
   b) not parallel
   c) parallel

2. a) P
   b) P
   c) T

3. a) \( 15 \) cm²
   b) \( 20 \) in²

4. a) \( A = 14 \) cm²
   b) \( A = 10 \) in²
   c) \( A = 21 \) ft²

5. a) \( A = 2 \times 2 = 4 \) cm²
   \( A = 4 \times 1 = 4 \) cm²
   b) \( A = 1 \times 4 = 4 \) in²
   \( A = 5 \times 0.8 = 4 \) in²

| Copyright © 2015 Jump Math, Not to Be Copied, CC Edition | Answer Keys for AP Book 7.2 | U-30 |
6. b) 5(21) = 35(x)  
105 = 35x  
105 ÷ 35 = x  
3 = x  
So x = 3 in  
c) 5(4) = 5x  
20 + 5 = x  
4 = x  
So x = 4 in  
d) 2(6) = 1.5x  
12 + 1.5 = x  
8 = x  
So x = 8 in

AP Book G7-16

1. a) \( h = 4 \) units  
A of P = 28 units²  
A of T = 14 units²

b) \( b = 9 \) units  
\( h = 4 \) units  
A of R = 36 units²  
A of T = 18 units²

2. b) \( (6 + 2) \times 5 + 2 \)  
= 8 \times 5 + 2  
= 40 + 2  
= 20

So area = 20 in²

c) \( (5 + 7) \times 3 + 2 \)  
= 12 \times 3 + 2  
= 36 + 2  
= 38

So area = 18 in²

3. a) circle 3, 4, 8  
\( A = (4 + 8) \times 3 + 2 \)  
\( A = 12 \times 3 + 2 \)  
\( A = 36 + 2 \)  
\( A = 18 \) cm²

b) circle 6, 12, 16  
\( A = (6 + 16) \times 12 + 2 \)  
\( A = 22 \times 6 \)  
\( A = 132 \) cm²

c) circle 4, 7, 13  
\( A = (13 + 7) \times 4 + 2 \)  
\( A = 40 \) cm²

4. a) \( 10 + 4 = 14 \) units²  
\( 4 + 4 + 6 = 14 \) units²

b) \( 4 + 4 \times 6 = 14 \) units²

8. a) \( 1 \times 1.4 = 1.4 \) m²  
\( 1.4 \times 80 = 112 \) m²  
\( 112 \times 26 = 2,912 \)

It will cost $2,912 to replace the glass in all 80 windows.

AP Book G7-17

page 119

1. a) i) \( 5.8 \) cm²  
ii) \( 580 \) mm²

b) \( 500 \) mm²

2. a) \( A = 62 \) ft²  
\( B = 38 \) ft²  
\( C = 24 \) ft²

b) Yes.  
\( 38 \) ft²  
\( 580 \) mm²

3. \( (9 \times 8) – (2 \times 5) = 62 \) ft²

4. \( 1 \) cm = 10 mm,  
so 1 cm² = 100 mm².

So \( 4 \) in  
\( 5 \) in  
\( 6 \) in  
\( 7 \) in

5. a) i) \( 5.8 \) cm²  
ii) \( 580 \) mm²

b) Yes.  
\( 5.8 \) cm² = 580 mm²

and 5.8 cm² × 100

\( 580 \) mm²

6. a) \( A = 2[(3 \times 4) + 2] \)  
\( = 2(12 + 2) \)  
\( = 2(6) \)  
\( = 12 \) ft²

b) \( A = [(3 \times 3) \times 2] \)  
\( + (3 \times 6) + (3 \times 3) \)  
\( = (9 + 18 + 9) \)  
\( = 45 \)  
\( = 50 \) ft²

7. a) \( x = 10 \) ft  
\( x = 10 \) ft

b) \( 6(3 + x) + 2 \)  
\( \geq 150 \)  
\( 6(2x + 20) \geq 150 \times 2 \)  
\( 2x + 20 \geq 300 + 6 \)  
\( 2x \geq 50 - 20 \)  
\( x \geq 30 + 2 \)  
So \( x \geq 15 \) ft

c) \( 20(10 + 10 - x) + 2 \geq 150 \)  
\( 20(20 + x) \geq 150 \times 2 \)  
\( 20 - x \geq 300 + 20 \)  
\( 20 \geq 15 + x \)  
\( 20 - 15 \geq x \)  
\( 5 \geq x \)  
So \( x \leq 5 \) ft
4. \(72\, \text{cm}^2\)
5. a) 9
   b) Room A: 240 ft\(^2\)
       Room B: 189 ft\(^2\)
       Room A is bigger than Room B.
6. a) 106,400 mi\(^2\)
   b) 106,400 mi\(^2\)
7. | \(\ell\) | \(w\) | \(A\) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>24</td>
<td>864</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>54</td>
</tr>
</tbody>
</table>
8. b) 864 cm\(^2\)
   c) 24 cm\(^2\)
   d) 6 cm\(^2\)
   e) 54 cm\(^2\)
9. 45 m\(^2\)
10. 9.6 cm\(^2\)
11. 946 yd\(^2\)
12. 50 m\(^2\)

AP Book G7-20

1. Teacher to check.
2. Teacher to check.
3. a) \(r = 12\, \text{mm}\)
   \(d = 24\, \text{mm}\)
   b) \(r = 9\, \text{mm}\)
   \(d = 18\, \text{mm}\)
   c) \(r = 6\, \text{mm}\)
   \(d = 12\, \text{mm}\)
4. Teacher to check that the line is drawn through the center of the circle.
   a) \(r = 10\, \text{mm}\)
   \(d = 20\, \text{mm}\)
   b) \(r = 13\, \text{mm}\)
   \(d = 26\, \text{mm}\)
   c) \(r = 12\, \text{mm}\)
   \(d = 24\, \text{mm}\)
6. a) Teacher to check 2\(^{nd}\) drawn diameter.
   b) The center of the circle is where both diameters intersect.
7. Teacher to check drawn circle with diameter 7 cm.
8. Teacher to check.
9. You make a straight line.

AP Book G7-21

1. a) \(A = (3.14)(3)^2\)
   \(A = 28.26\, \text{yd}^2\)
   b) \(A = (3.14)(8)^2\)
   \(A = 200.96\, \text{ft}^2\)
   c) \(A = (3.14)(11)^2\)
   \(A = 379.94\, \text{mm}^2\)
   d) \(A = (3.14)(1.75)^2\)
   \(A = 9.62\, \text{in}^2\)
2. \(r = 5\, \text{ft}\)
   \(A = (3.14)(5)^2\)
   \(A = 78.5\, \text{ft}^2\)
   c) \(r = 4.2\, \text{cm}\)
   \(A = (3.14)(4.2)^2\)
   \(A = 55.39\, \text{cm}^2\)
   d) \(r = 16\, \text{in}\)
   \(A = (3.14)(16)^2\)
   \(A = 803.84\, \text{in}^2\)
3. \(b) A = 3 \times 36\)
   \(A = 27\, \text{cm}^2\)
   c) \(A = 1 \times 36\)
   \(A = 12\, \text{cm}^2\)
   d) \(A = 2 \times 36\)
   \(A = 24\, \text{cm}^2\)
### Geometry: Angles and Areas – AP Book 7.2: Unit 4

(continued)

<table>
<thead>
<tr>
<th>Question</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>4. a)</td>
<td>3.79 m²</td>
</tr>
<tr>
<td>4. b)</td>
<td>7.85 m²</td>
</tr>
<tr>
<td>4. c)</td>
<td>48.66 yd²</td>
</tr>
<tr>
<td>5. a)</td>
<td>Teacher to check.</td>
</tr>
<tr>
<td>5. b)</td>
<td>Teacher to check.</td>
</tr>
<tr>
<td>5. c)</td>
<td>Teacher to check.</td>
</tr>
<tr>
<td>5. d)</td>
<td>Teacher to check.</td>
</tr>
<tr>
<td>5. e)</td>
<td>Teacher to check.</td>
</tr>
</tbody>
</table>
| 5. f)   | i) radius  
          | ii) circumference |
| 6. a)   | 40° and 40° |
| 6. b)   | 80° and 80° or 140° and 20° |
| 6. c)   | In part a), the angle given was greater than 90°, so the two remaining angles must be acute.  
          | In part b), the angle given was less than 90° so the remaining angles could be either both acute, or one acute and one obtuse. |

### AP Book G7-22

**Page 132**

1. a) circle D  
   b) circle B and E

2. a) Teacher to check.  
    Base of parallelogram is 6 cm.  
    b) Teacher to check.  
    Base of right-angled trapezoid is 6 in.

3. a) i) A  
      ii) C  
      b) i) B  
      ii) D

4. a) B. 9x = 180°  
     So x = 20°  
     C. 4x = 180°  
     So x = 45°  
     D. 5x = 180°  
     So x = 36°  
     b) A  
     c) B and D  
     d) C

5. a) The acute angles measure 18° and 72°.  
    b) The acute angles measure 35° and 55°.

6. a) i) π × r × r  
      ii) π ÷ 2 = 2πr ÷ 2  
      Half of the circumference is half of 2πr, which is π × r.

### AP Book G7-23

**Page 134**

1. a) Circumference of a circle  
    b) Area of a triangle  
    c) Area of a trapezoid  
    d) Area of a circle  
    e) Area of a parallelogram or rectangle

2. Teacher to check diagrams.  
   a) A = b × h  
   b) A = (b₁ + b₂) × h + 2  
   c) A = b × h + 2

3. Given: base = 5 cm,  
   Find: h  
   Formula: A = b × h + 2

4. Given: r = 5 cm  
   Find: A  
   Formula: A = πr²

5. a) Area of circle A = 7.07 cm²  
    Area of circle B = 28.6 cm²

6. a) i) Area of unshaded part = (6)(6) ÷ 2  
         Area of shaded part = (3.14)(6)² ÷ 2  
         = 10.26 cm²
    ii) Area of unshaded part = (3.14)(1)²  
        Area of shaded part = (2)(2)  
        = 0.86 cm²
    iii) Area of unshaded part = (3)(4) ÷ 2  
         Area of shaded part = (3.14)(2.5)² ÷ 2  
         = 3.81 cm²

7. Total area:  
   \((l \times w) + (πr² + 2)\)  
   = (1.4)(0.9) + (π(0.45)² + 2)  
   = 1.26 + 0.32  
   = 1.58 m²
   = 1.58 × 25 = 39.5
   The glass will cost $39.50.

8. a) Area of the path:  
    \(\frac{1}{4}π(6)² - \frac{1}{4}π(5)²\)  
    = 28.26 - 19.63  
    = 8.63 m²
    b) Total area of flowerbeds:  
    (6)(8) - 8.63  
    = 39.37 m²
    c) Cost of tile:  
    (3)(8.63) = $25.89
    Cost of flowers:  
    (5)(39.37) = $196.85
    Total cost of garden:  
    25.89 + 196.85  
    = $222.74

9. a) 423.9 m  
    b) 1,413 seconds  
    or 23.55 minutes  
    (23 minutes 33 seconds)
    c) i) 800 people  
        ii) 13.25 m apart
    d) i) 1,120 people  
        ii) \(C = πd\)  
        (17.8)(28) = πd  
        498.4 + π = d  
        158.73 m = d  
        158.73 m > 135 m  
        so the High Roller is taller.

b) The area of the shaded part is less than the area of the unshaded part.
The Number System: Repeating Decimals and Terminating Decimals – AP Book 7.2: Unit 5

AP Book NS7-46

1. b) 30, 6
c) 60, 5
d) 70, 9
2. b) 115
c) 171
d) 322
3. b) 22
c) 26
d) 29
4. b) 7 R 22
c) 9 R 26
d) 7 R 29
5. b) 40
g) 52
h) 629
7. b) 1
c) 7
d) 3
8. a) 47 \\[ \begin{array}{c} 322 \\ 282 \\ 40 \\ 6 R 40 \\ 24 R 23 \\ 31 R 4 \\ 31 R 4 \\ 5 R 8 \\ 8 R 30 \\ 4 R 31 \\
7 \end{array} \]
9. a) 33 \\[ \begin{array}{c} 69.3 \\ 66 \\ 33 \\ 33 \\ 0 \\ 2.1 \\ 42 \\ 21 R 8.82 \\ 4 \\ 42 \\ 0 \\ 0.42 \\
11. 276 ÷ 12 = 23 tickets each
12. −1,080 + (−90) = 12 hours
13. a) 1
b) 2

AP Book NS7-47

1. a) negative number
b) 31 > 29
low
c) 50 > 36
low
d) negative number
high
2. b) 6
c) negative number
too high
8
d) 46 > 25
too low
6
e) 76 > 88
too low
5
f) negative
too high
5
3. a) \( \frac{6}{76} \div 468 \)
   
   456
   12
   6 R 12

   b) \( \frac{3}{43} \div 168 \)
   
   129
   39
   3 R 39

   c) \( \frac{3}{312} \div 186 \)
   
   166
   26
   3 R 26

   d) \( \frac{9}{65} \div 127 \)
   
   585
   27
   9 R 27

4. b) 6, 144
   
   negative number
   too high

   c) 2, 74
   
   18 < 37
   good

   d) 3, 111
   
   40 > 37
   too low

   e) 2, 66
   
   negative number
   too high

   f) 2, 98
   
   51 > 49
   too low

5. a) c, d, f
   
   b) e

   c) Answers will vary. Teacher to check.

   Sample answer:
   No. You can’t get an estimate that is too high because you’re dividing by a number that is greater than the actual divisor, which makes your estimated quotient less than what it actually is.

6. a) \( \frac{21}{43} \div 937 \)
   
   86
   77
   43
   34
   21 R 34

   b) \( \frac{38}{84} \div 3268 \)
   
   252
   748
   672
   76
   38 R 76

   c) \( \frac{64}{37} \div 2378 \)
   
   222
   158
   148
   10

   64 R 10

7. a) \( \frac{8}{28} \div 7.8 \)
   
   78
   76
   0

   b) \( \frac{12}{54} \div 648 \)
   
   54
   108
   108
   0

   c) \( \frac{1720}{25} \div 43000 \)
   
   25
   180
   175
   50
   50
   0

8. a) \( \frac{75}{21} \)
   
   b) \( -\frac{38}{20} \)
   
   c) \( -\frac{3.6}{20} \)
   
   d) \( \frac{0.065}{20} \)
   
   e) \( \frac{0.34}{20} \)
   
   f) \( -\frac{23.1}{20} \)

9. \( 210 + 28 = 7.50 \) each

10. \( 15.75 + 0.45 = 35 \) apples

11. \( 5.4 + 0.012 = 450 \) sheets

12. \( -15,480 + (-180) = 86 \) years

13. b) \( 1,974.65 + 71.35 = 2,046 \) ft per hour

   c) \( -28.76 - (-17.6) = -11.16\)F

   d) \( -11.16 + 3.6 = 3.1\)F per hour

3. a) i) \( \frac{555}{9} \)
   
   b) \( \frac{5000}{45} \)
   
   c) \( \frac{50}{45} \)
   
   d) \( \frac{5}{5} \)

   ii) \( 0.333 \)

   iii) \( \frac{0.444}{9} \)

14. a) 0.4

   b) 0.68

   c) 0.375

   d) 0.88…

   e) -0.77…

   f) \( \frac{26.00}{40} \)

   g) \( \frac{504}{560} \)

   h) \( \frac{504}{56} \)

   i) \( \frac{200}{200} \)

   j) \( \frac{18}{18} \)

   k) \( \frac{20}{20} \)

   l) \( \frac{20}{20} \)

   m) \( \frac{20}{20} \)

   n) \( \frac{20}{20} \)

   o) \( \frac{20}{20} \)

   p) \( \frac{20}{20} \)

   q) \( \frac{20}{20} \)

   r) \( \frac{20}{20} \)

   s) \( \frac{20}{20} \)

   t) \( \frac{20}{20} \)

   u) \( \frac{20}{20} \)

   v) \( \frac{20}{20} \)

   w) \( \frac{20}{20} \)

   x) \( \frac{20}{20} \)

   y) \( \frac{20}{20} \)

   z) \( \frac{20}{20} \)
The Number System: Repeating Decimals and Terminating Decimals – AP Book 7.2: Unit 5

(continued)

4. a) 0.5
b) 0.83

c) 0.13
d) 0.27
e) −0.32
f) −0.27
g) 0.72
h) 0.590

5. a) i) \(\frac{1}{11} = 0.09\)
ii) \(\frac{2}{11} = 0.18\)
iii) \(\frac{3}{11} = 0.27\)
iv) \(\frac{4}{11} = 0.36\)
b) i) \(\frac{5}{11} = 0.45\)
ii) \(\frac{6}{11} = 0.54\)
iii) \(\frac{7}{11} = 0.63\)
iv) \(\frac{8}{11} = 0.72\)
v) \(\frac{9}{11} = 0.81\)
vi) \(\frac{10}{11} = 0.90\)

AP Book NS7-50

page 146

1. a) i) 0.31, 0.34, 0.35, 0.40
ii) 0.740, 0.741, 0.747, 0.749, 0.750

2. b) circle 0.5 to 0.6
circle 0.58 to 0.59
circle 0.583

3. a) yes
b) yes

c) 0.31

AP Book NS7-51

page 148

1. a) i) 0.3

2. a) i) 0.625
ii) 0.583
iii) 0.61
iv) 0.52
v) 0.0065
vi) 0.28

b) \(\frac{5}{8}, \frac{13}{20}, \frac{13}{200}\)
c) \(\frac{625}{1000}, \frac{52}{100}, \frac{65}{10000}\)

a) 5 × 5
b) 2 × 2 × 2
c) \(2 \times 2 \times 5 \times 5 \times 5\)

4. a) i) 0.8
ii) 0.54
iii) 0.46
iv) 0.54
v) 0.954
vi) 0.55

b) \(\frac{4}{5}, \frac{27}{50}, \frac{11}{20}\)
c) \(\frac{4}{5}, \frac{27}{50}, \frac{11}{20}\)

5. a) i) \(\frac{3}{10}\)
ii) 7
iii) 11

b) i) yes
ii) no
iii) yes

c) i) yes
ii) no
iii) yes

6. a) 0.875
b) 0.75
c) \(\frac{56}{80} \div \frac{7}{10} = 0.7\)

AP Book NS7-49

page 145

1. circle 0.2222..., 0.512512512..., 0.0252525...

2. a) 0.33333333
b) 0.03333333
c) 0.00333333
d) 0.52525252
e) 0.81781781
f) 0.81717171
g) 0.92626262
h) 0.25373737
i) 7.23333333
j) 8.25935395

3. a) 0.8
The Number System: Repeating Decimals and Terminating Decimals – AP Book 7.2: Unit 5 (continued)

d) \(\frac{63}{77} = \frac{9}{11} = 0.81\) repeats

e) \(\frac{25}{55} = \frac{5}{11} = 0.45\) repeats

f) 0.2875 terminates

7. a) 3, 9, 27, 81, 243
    b) Yes. The denominators are all powers of 3.

8. a) \(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{5}{6}\)
    b) \(\frac{3}{6} = \frac{1}{2}\) or \(\frac{1}{2}\) is the only fraction whose denominator can be written as a product of 2s and 5s when in lowest terms.

c) \(\frac{1}{6} = 0.16\)
    \(\frac{1}{3} = 0.3\)
    \(\frac{1}{2} = 0.5\)
    \(\frac{2}{3} = 0.6\)
    \(\frac{5}{6} = 0.83\)

d) Only \(\frac{1}{2}\) terminates. My prediction was correct.

9. The numerators also have a factor of 3. So in lowest terms, the denominators are products of only 2s and 5s.

Bonus
Sample answers:

b) 3

c) 9

d) 100
e) 10
f) 7

AP Book NS7-52

page 150

1. a) \(\equiv 0.556\)

b) \(\equiv 0.4545\)
    \(\equiv 0.5\)
    \(\equiv 0.45\)
    \(\equiv 0.455\)
    \(\equiv 0.2555555\)

e) \(0.30\)

e) \(0.30\)

e) \(0.30303030\)
f) \(0.2\)

0.95

0.34343434
+ 0.61616161
0.95959595

0.598
0.34343434
+ 0.25555555
0.59888889

0.5
0.34343434
+ 0.22222222
0.36566666

0.22222222

0.355446
0.1111111111111111
0.1212121212121212
+ 0.1231231231231231
0.355463554643

0.6
0.34343434
+ 0.25555555
0.59888889

0.22222222

0.3554463554463

0.355446

0.1111111111111111
0.1212121212121212
+ 0.1231231231231231
0.355463554643

0.1111111111111111
0.1212121212121212
+ 0.1231231231231231
0.355463554643

0.6
0.34343434
+ 0.25555555
0.59888889

0.5
0.34343434
+ 0.22222222
0.36566666

0.5
0.34343434
+ 0.22222222
0.56566666

0.5
0.34343434
+ 0.22222222
0.56566666

0.1111111111111111
0.1212121212121212
+ 0.1231231231231231
0.355463554643

0.5
0.34343434
+ 0.22222222
0.36566666

0.5
0.34343434
+ 0.22222222
0.56566666

0.5
0.34343434
+ 0.22222222
0.56566666

0.5
0.34343434
+ 0.22222222
0.56566666

0.5
0.34343434
+ 0.22222222
0.56566666

0.5
0.34343434
+ 0.22222222
0.56566666
Geometry: Volume, Surface Area, and Cross Sections – AP Book 7.2: Unit 6

AP Book G7-24

page 152
1. b) 8 in³
c) 10 ft³
2. a) 4 × 5 = 20, 4 × 4 = 16
   b) 20, 16
c) 4, 3
d) 20 × 4 = 80,
   16 × 3 = 48
3. a) ℓ × w
c) ℓ × w × h
4. Teacher to check labels.
w = 6 cm, h = 2 cm
V = 4 cm × 6 cm × 2 cm
= 48 cm³
5. a) V = 3 in × 2 in × 4 in
   = 24 in³
b) V = 5 m × 2 m × 3 m
   = 30 m³
c) V = 55 ft × 20 ft
   × 40 ft
   = 44,000 ft³
6. a) V = 4 in × 5 in
   × 10 in
   = 200 in³
   It holds 200 in³.
   b) 1,200 + 200 = 6
   One bag will fill the bird feeder 6 times.
7. a) i) V = 5 × 4 × 3
     = 60 cm³
    ii) V = 4 × 3 × 5
     = 60 cm³
   b) The prisms have the same 3 numerical values for their dimensions, so the volumes are the same.
8. a) V = 1 \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}
   = \frac{1}{8} in³
   b) V = 2 \times \frac{13}{2} \times \frac{11}{3}
   = 47 \frac{2}{3} ft³
   c) V = 6.5 \times 3.5 \times 2
   = 45.5 cm³
9. V = \frac{63}{8} \times \frac{10}{3} \times \frac{59}{5}
   = 309 \frac{3}{4} in³
10. V = 4 \times 8 \times \frac{3}{4}
    = 24 ft³
   \times 6 = $144
11. a) 4 ft
    b) 12.5 in
12. a) i) V₁ = 4 in \times 2 in
      \times 2 in
      = 16 in³
      Total V
      = 30 in³ + 16 in³
      = 46 in³
         ii) V₁ = 3 in \times 2 in
      \times 3 in
      = 18 in³
      V₂ = 7 in \times 2 in
      \times 2 in
      = 28 in³
      Total V
      = 18 in³ + 28 in³
      = 46 in³
   b) Yes, because the whole shapes in parts i) and ii) are the same, they are just divided in different ways.
13. a) V₁ = 7 \times 4 \times 6
      = 168 in³
      V₂ = 5 \times 4 \times 11
      = 220 in³
      Total V = 388 in³
or
V₁ = 12 \times 4 \times 6
   = 288 in³
V₂ = 5 \times 5 \times 4
   = 100 in³
   Total V = 388 in³
   b) V₁ = 4 \times 11 \times 2
   = 88 in³
   V₂ = 10 \times 11 \times 5
   = 550 in³
   V₃ = 15 \times 11 \times 2
   = 330 in³
   Total V
   = 88 + 550 + 330
   = 968 in³
   or
   V₁ = 2 \times 5 \times 11
   = 110 in³
   V₂ = 7 \times 6 \times 11
   = 462 in³
   V₃ = 9 \times 4 \times 11
   = 396 in³
   Total V
   = 110 + 462 + 396
   = 968 in³
14. Teacher to check.
15. Circle the following:
   2 in by 3 in by 4 in,
   3 in by 4 in by 1 in,
   2 in by 5 in by 1.2 in
16. a) V ≤ 48
    b) width of prism
    V ≤ 48
    ℓ × w × h ≤ 48
   c) length of prism
    V ≤ 48
   b) V ≤ 14 in
17. ℓ × w × h ≤ 1,440
   15 × w × 12 ≤ 1,440
   180w ≤ 1,440
   w ≤ 1,440 + 180
   w ≤ 8 in
   Bonus
   V ≥ \frac{3}{4}
   ℓ × w × h ≥ \frac{3}{4}
   \ell × 1 \times \frac{1}{2} ≥ \frac{3}{4}
   \ell ≥ \frac{3}{4} + \frac{1}{2}
   \ell ≥ \frac{3}{4} \times \frac{2}{1}
   \ell ≥ \frac{6}{4}
   \ell ≥ 1 \frac{1}{2} ft

AP Book G7-25

page 156
1. b) hexagon
c) triangle
d) pentagon
e) triangle
f) pentagon
2. a) i) V = 32 in³
    ii) V = 14 in³
    iii) V = 14 in³
b) i) \frac{8}{in²} \times \frac{4}{in} \times \frac{32}{in³}
   ii) \frac{7}{in²} \times \frac{2}{in} \times \frac{14}{in³}
   iii) \frac{7}{in²} \times \frac{2}{in} \times \frac{14}{in³}
   c) V = \text{area of base} \times \text{height}
3. Circle the following:
   b) 11
c) 8
d) 2 (right vertical edge)
4. a) B = 3 in²
   h = 8 in
   V = 24 in³
   b) B = 6 cm²
   h = 5 cm
   V = 30 cm³
5. a) \( B = 7 \text{ in} \times 4 \text{ in} \)
   \[ V = 28 \text{ in}^2 \times 10 \text{ in} \]
   \[ = 280 \text{ in}^3 \]
   b) \( B = 3 \text{ cm} \times 4 \frac{1}{2} \text{ cm} \)
   \[ = 13 \frac{1}{2} \text{ cm}^2 \]
   \[ V = 11 \frac{2}{3} \times 13 \frac{1}{2} \text{ cm}^3 \]
   \[ = 157 \frac{1}{2} \text{ cm}^3 \]

6. a) \( h = 9 \text{ ft} \)
   \[ V = 24 \text{ ft}^2 \times 9 \text{ ft} \]
   \[ = 216 \text{ ft}^3 \]
   b) \( B = (5 + 3) \times 4.2 \div 2 \)
   \[ = 16.8 \text{ m}^2 \]
   \[ h = 4 \text{ m} \]
   \[ V = 16.8 \times 4 \]
   \[ = 67.2 \text{ m}^3 \]

7. a) i) \( A_2 = 5 \times 4 \)
    \[ = 20 \text{ ft}^2 \]
   \[ B = 18 + 20 \]
   \[ = 38 \text{ ft}^2 \]
    ii) \( A_1 = (9 + 6) \times 2 + 2 \)
     \[ = 15 \text{ in}^2 \]
   \[ A_2 = 9 \times 2 \]
   \[ = 18 \text{ in}^2 \]
   \[ B = 15 + 18 \]
   \[ = 33 \text{ in}^2 \]
   or
   \[ A_1 = 6 \times 4 \]
   \[ = 24 \text{ in}^2 \]
   \[ A_2 = (2 + 4) \times 3 + 2 \]
   \[ = 9 \text{ in}^2 \]
   \[ B = 24 + 9 \]
   \[ = 33 \text{ in}^2 \]
   b) \[ V = 38 \times 7 \]
   \[ = 266 \text{ ft}^3 \]
    ii) \[ V = 33 \times 5 \]
     \[ = 165 \text{ in}^3 \]

8. a) \( V \geq 45 \)
   \[ (6 \times 3) + 2 \times x \geq 45 \]
   \[ 9x \geq 45 \]
   \[ x \geq 45 \div 9 \]
   \[ x \geq 5 \text{ in} \]
   b) \( V \leq 56 \)
   \[ (3 + 5)x + 2 \times 3.5 \leq 56 \]
   \[ 8x + 2 \leq 56 + 3.5 \]
   \[ 4x \leq 16 \]
   \[ x \leq 16 + 4 \]
   \[ x \leq 4 \text{ ft} \]

5. Answers may vary.
   Teacher to check.
   Sample answer:
   The net of a prism shows each face of the prism,
   which makes it easier to calculate the area of each face.

6. a) \[ \text{SA} = 22 \text{ cm}^2 \]
   b) \[ \text{SA} = 24 \text{ cm}^2 \]
1. a) Area 1 = \(1 \times 2 \div 2\) = \(1\) in\(^2\)
Area 2 = \(1\) in\(^2\)
Area 3 = \(3 \times 2.2\) = \(6.6\) in\(^2\)
Area 4 = \(2 \times 3\) = \(6\) in\(^2\)
Area 5 = \(1 \times 3\) = \(3\) in\(^2\)

b) \(17.6\) in\(^2\)

2. a) Order may vary.
Sample answer:
top, bottom, left, right, front, back
Area 1 = \((3 + 1.5) \times 1 \div 2\) = \(2.25\) ft\(^2\)
Area 2 = \(2.25\) ft\(^2\)
Area 3 = \(1.25 \times 2\) = \(2.5\) ft\(^2\)
Area 4 = \(2.5\) ft\(^2\)
Area 5 = \(1.5 \times 2\) = \(3\) ft\(^2\)
Area 6 = \(3 \times 2\) = \(6\) ft\(^2\)

b) \(18.5\) ft\(^2\)

3. a) The formula is \(V = \ell \times w \times h\), but \(\ell = a\), \(w = a\), and \(h = a\), so the formula becomes \(V = a \times a \times a = a^3\).

4. a) \(18\) in = \(\frac{3}{2}\) ft
SA = \(2 \left(\frac{7}{2}\right) (2) + 2 \left(\frac{7}{2}\right) (3) + 2 \left(\frac{3}{2}\right) = 30\) \(\frac{1}{2}\) ft\(^2\)

b) \(7\) in = \(\frac{7}{12}\) ft
SA = \(2 \left(\frac{1}{1} + \frac{2}{3} \div 2\right) + \frac{3}{12} \div 2\) = \(\frac{2}{3}\) ft\(^2\)

5. Box A:
SA = \(4 \left(\frac{7}{2}\right) (4) + 2(4)(4)\) = \(152\) in\(^2\)

Box B:
SA = \(4 \left(\frac{6}{10.4} + 2\right) + \frac{6}{10.4} \times 2\) = \(160.8\) in\(^2\)

6. SA = \(3 \times 2 + 2 \div 3 \left[\left(\frac{1}{6}\right) + 2\left(2.5\right)\right] = 12\) \(\frac{1}{3}\) ft\(^2\)

Cost = \(12\) \(\frac{1}{3}\) \(\times 3 + 25\) = \(62\)

7. a) Top:
SA = \(2\left(\frac{3}{1.5} + 2\right) + 2\left(2.1\right)\) = \(26.1\) in\(^2\)

Bottom:
SA = \(4(8)\) + \(2(3)\) = \(114\) in\(^2\)

Total:
SA = \(26.1 + 114\) - \(2(3)\) = \(122.1\) in\(^2\)

b) Top:
SA = \((90)(90) + 4(90 \times 53 + 2)\) = \(17,640\) ft\(^2\)

Middle:
SA = \(6(90)(90)\) = \(48,600\) ft\(^2\)

Bottom:
SA = \(6(125)(125)\) = \(93,750\) ft\(^2\)

Total:
SA = \(17,640 + 48,600 + 93,750\) = \(159,040\) ft\(^2\)

b) \(V = 2 \times 4 \times 1.5\) = \(12\) ft\(^3\)
SA = \(2\left(\frac{2}{1.5}\right) + \left(\frac{2}{1.5}\right)\) = \(34\) ft\(^2\)

8. a) \(V = 100 \times 122.1\) \(\times 0.1\) = \(1,221\) cents = \(\$12.21\)

b) \(V = 25,518 + 127,590\) \(= 153,108\) ft\(^3\)
Cost = \(\frac{153,108}{127,590}\) \(\times 0.1\) = \(\$1.2\)

9. a) Top:
SA = \(90(90)\) = \(8100\) ft\(^2\)

Middle:
SA = \(6(90)(90)\) = \(5400\) ft\(^2\)

Bottom:
SA = \(6\left(\frac{125}{125}\right)\) = \(750\) ft\(^2\)

Total:
SA = \(8100 + 5400 + 750\) = \(14,250\) ft\(^2\)

b) \(V = 6 \times 18 \times 9\) = \(972\) in\(^3\)

SA = \(\frac{9}{2}\) \(\times \frac{3}{2} \times \frac{3}{4}\) = \(9\) \(\frac{9}{16}\) in\(^2\)

C) Cost = \(35,000 \times 0.1 \div 1,000\) = \(\$3.50\)

Bonus
Cost = \(56 \times 0.9\) = \(\$50.40\)
b) \[ SA = 2\ell w + 2\ell h + 2wh \]
\[ = 2(a)(a) + 2(a)(a) + 2(a)(a) = 2a^2 + 2a^2 + 2a^2 = 6a^2 \]

c) i) \[ V = 125 \text{ in}^3 \]
\[ SA = 150 \text{ in}^2 \]

ii) \[ V = 1,728 \text{ cm}^3 \]
\[ SA = 864 \text{ cm}^2 \]

iii) \[ V = \frac{1}{6} \text{ ft}^3 \]
\[ SA = \frac{1}{2} \text{ ft}^2 \]

4. Because the bases can be different shapes, and different shapes have different formulas for area.

5. a) \[ V = B \times h \]
\[ = 30 \times 5 \]
\[ = 150 \text{ in}^3 \]
\[ SA = 162 \text{ in}^2 \]

b) \[ V = B \times h \]
\[ = \frac{4}{3}(a) + 2 \times 4 \]
\[ = 24 \text{ m}^3 \]
\[ SA = 60 \text{ m}^2 \]

c) \[ V = \frac{1}{2}(1 + 0.7) + 2 \times 1.5 \]
\[ = 0.51 \text{ m}^3 \]
\[ SA = \frac{0.4(1.7 + 2) + (0.4)(1.5) + 0.4(1.7 + 2) + (0.5)(1.5) + (1.5)(0.7) + (1)(1.5)}{2} \]
\[ = 4.58 \text{ m}^2 \]

6. a) \[ \ell = 6 \text{ in} \]
\[ w = 3 \text{ in} \]
\[ h = 8 \text{ in} \]

b) The bigger juice box has 8 times the volume, so the price should be \[ 8 \times $0.50 = $4.00 \].

7. a) Box A: \[ SA = 24 \text{ in}^2 \]
Box B: \[ SA = 34 \text{ in}^2 \]
Box C: \[ SA = 28 \text{ in}^2 \]
Box A has the least surface area.

b) Answers will vary. Sample answer:
2 in by 3 in by 4 in,
6 in by 4 in by 1 in,
8 in by 3 in by 1 in.
The one with the least surface area is 2 in by 3 in by 4 in.

c) A cube will have the least surface area.

8. a) \[ V = 14 \times 24 + 2 \times 15 \]
\[ = 2,520 \text{ ft}^3 \]
\[ 2,520 \div 500 = 5.04 \]
5 people can stay in the cottage.

b) \[ B \times h \geq 6,000 \]
\[ 14 \times 24 + 2 \times h \geq 6,000 \]
\[ 168h \geq 6,000 \]
\[ h \geq 6,000 \div 168 \]
\[ h \geq 35.7 \text{ ft} \]
\[ 35.7 - 15 = 20.7 \text{ ft} \]
The cottage needs to be about 20.7 ft longer.

c) Area to cover
\[ = 24 \times 15 \times 2 \]
\[ = 720 \text{ ft}^2 \]
Area of 1 shingle
\[ = 1 \times 3 \]
\[ = 3 \text{ ft}^2 \]
\[ 720 + 3 \]
\[ = 240 \text{ shingles} \]

9. a) \[ V = (3 + 7) \times 25 + 2 \times 18 \]
\[ = 2,250 \text{ ft}^3 \]

b) \[ 2,250 = (30)w(5) \]
\[ 2,250 = 150w \]
\[ 2,250 + 150 = w \]
\[ w = 15 \text{ ft} \]

Answers will vary. Sample answer:
a rectangular prism with dimensions
10 ft by 15 ft by 15 ft

10. a) \[ V = 240 \text{ m}^3 \]
\[ SA = 280 \text{ m}^2 \]

b) \[ V = 448 \text{ m}^3 \]
\[ SA = 512 \text{ m}^2 \]

c) \[ V = 1,103.5 \text{ cm}^3 \]
\[ SA = 620.7 \text{ cm}^2 \]

5. Teacher to check.

In a, b, c, e, f, g, and i, all pairs of opposite sides of the cross section are parallel.

In d, only the left and right sides of the cross section are parallel.

In h, the cross section has no parallel sides.

6. a) e
b) g

c) d

d) h

e) a, b, c

f) f, i

7. a) rectangle
b) trapezoid

c) triangle
d) quadrilateral
e) triangle

f) line
g) point

h) triangle
1. a) 6
   b) 4
   c) $20 + \frac{8}{9}$
2. a) 5
   b) 9
   It is 4 more than the mean of the original data set.
   c) Answers may vary. Teacher to check.
      Sample answer: 4, 5, 8, 11
   d) $40 \div 8 = 5$
      Each data value is repeated.
      The means are the same.
   e) Prediction: The mean should still be 5.
      Check: $60 \div 12 = 5$
3. a) ii) Mean:
      $(38 + 39 + 41 + 46) \div 4$
      $= 164 \div 4$
      $= 41$
      Median: 40
      Range: 8
   b) ii) 11
   c) ii) 10
   d) Teacher to check.
   e) ii) 4
      iii) 2
4. No, because the middle value is the same regardless of whether the order is from smallest to largest or from largest to smallest.
5. US: consumption per person = 4,401,698 GWh/yr + 307 million = 0.014 GWh/yr/person Canada: consumption per person = 620,684 GWh/yr + 33 million = 0.019 GWh/yr/person
   Anwar did better in math.
   b) $\$32,375$
   c) The median, because most of the salaries are equal.
6. a) i)
<table>
<thead>
<tr>
<th>Value</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>
   b) ii) 11
   c) ii) 10
   d) Teacher to check.
   e) ii) 4
      iii) 2
7. a) $(3 + 5 + 5 + 6 + 6) \div 6$
      $= 30$
   b) $(1 \times 3) + (3 \times 5) + (2 \times 6)$
      $= 3 + 15 + 12$
      $= 30$
   c) Teacher to check.
   d) $30 + 6 = 5$
8. a) $40 + 10 = 4$
    b) $40 + 10 = 4$
    Bonus
    $5,000 + 10 = 500$

AP Book SP7-11

1. Science mean = $400 \div 5$
   $= 80$
   Math mean = $643 \div 8$
   $= 80.375$
   or $\frac{803}{10}$
   Anwar did better in math.
2. a) $\$32,375$
   b) $\$17,500$
   c) The median, because most of the salaries are equal.
3. The mean, because he wants the total amount of sales to be high.
4. a) $(2 + 19 + 7 + 4 + 15 + x) \div 6 = 10$
   $(47 + x) = 10 \times 6$
   $x = 60 - 47$
   $x = 13$
   b) $(80 + 93 + 91 + x) \div 4 = 90$
   $264 + x = 90 \times 4$
   $x = 360 - 264$
   $x = 96$
5. a) The mean, median, and mode for each data set is 10.
   b) no
   c) A: 0
      B: 18
      C: 18
   d) yes
   e) no
6. a) 4
   b) 3
   c) 4
   d) 25
   e) 7
7. a–b) $5,000 + 10 = 500$
   c) 2
   d) 11
8.  10 11 12 13 14
   |             |
   |             |
   |             |
   |             |
   |             |

AP Book SP7-12

1. 12 13 14 15 16 17 18 19 20
2. a) $32,375$
   b) $17,500$
   c) The median, because most of the salaries are equal.
   d) 14, 14
   e) They are all the same.
3. a) A  B  C  D
    | 2 3.5 3 4
    | Y N Y N
    | Y N Y N
   b) They are the same.
4. b) $1 + 3$
   c) $3 + 1 = 2 + 2$
   d) mean = 5
      $3 = 1 + 2$
   e) mean = 6
      $3 + 2 = 2 + 0 + 3$
   f) mean = 4
      $2 + 0 = 1 + 1$
5. b) 2, 2, 3, 6, 7
   c) 3, 3, 3, 3, 8
6. a) i) $-3, 1, 2$
    ii) $-3, -2, 2, 0, 3$
    iii) $-2, 0, 1, 1$
    b) i) $-3 + 1 = 2$

(continued)

ii) \((-3) + (-2) + 2 + 0 + 3 = 0\)

iii) \((-2) + 0 + 1 + 1 = 0\)

7. a) 5  
b) 5  
c) 11  
d) 20

8. a) 9 cm  
b) 3 eggs  
c) Albatross, Emperor Penguin, Flamingo, Common Loon  
d) no  
e) Generally, yes. The exception is the ostrich.

9. a) Ray should give Jack 3 apples and give Lily 1 apple.  
b) mean = 6  
That is how many apples they each have after sharing equally.  
c) No, because they all have 6 apples.

AP Book SP7-13
page 181

1. Teacher to check arrows.  
b) Q1 = 6.5  
M = 16  
Q3 = 23  
c) Q1 = 21  
M = 32  
Q3 = 41  
d) Q1 = 3  
M = 5.5  
Q3 = 7  

2. a) M = 7  
Q1 = 3.5  
Q3 = 10.5  
b) M = 50  
Q1 = 30  
Q3 = 70  
c) M = 16.5  
Q1 = 6.5  
Q3 = 23.5  

AP Book SP7-14
page 184

1. a) 5  
b) 7  
c) 0  

4. a) 

<table>
<thead>
<tr>
<th>R</th>
<th>IQR</th>
</tr>
</thead>
<tbody>
<tr>
<td>88</td>
<td>16</td>
</tr>
<tr>
<td>88</td>
<td>10</td>
</tr>
<tr>
<td>88</td>
<td>32</td>
</tr>
</tbody>
</table>

b) no  
They are all the same.  
c) Set 2  
The IQR is smallest for Set 2.

5. Set 1: IQR = 13  
Set 2: IQR = 27  
Set 3: Q3 = 1,000  
Set 4: Q3 = 57  
Set 5: Q1 = 26  

6. b) Median = 12  
IQR = 13.5 – 11.25 = 2  
Range = 14 – 10.25 = 3.75  

c) Median = \(\frac{12}{3}\)  
IQR = 2 – 1 = 1  
Range = \(\frac{2}{3} - \frac{1}{3}\) = 2

7. A and E  
B and F  
C and D  

8. a) 9 cm  
b) 3 eggs  
c) Albatross, Emperor Penguin, Flamingo, Common Loon  
d) no  
e) Generally, yes. The exception is the ostrich.

9. a) i) Michigan  
ii) Michigan  
iii) Michigan

6. a) 200  
b) 2, 4, 5, 7, 202  
200  
c) 2, 3, 7, 202  
200  
2, 3, 5, 7  
5  

7. a) circle 74  
b) mean: 15  
median: 3.5  
IQR: 3  
MAD: \(19\frac{2}{3}\) or 19.5  
c) 2, 3, 4, 5  
mean: 3.2  
median: 3  
IQR: 2.5  
MAD: 1.04  
d) mean  
e) MAD  
f) Answers may vary. Teacher to check.  
Sample answer:  
2, 3, 4, 5, 74 has mean: 88 + 5 = 17.6  
median: 4  
IQR: 37  
MAD: 22.56  
g) The removal of the outlier.

8. a) 5  
b) Sample answers:  
No, because she was probably just not feeling well that day.  
or  
Yes, because there is a unit that she didn’t learn well.

9. a) i) Michigan  
ii) Michigan  
iii) Michigan
b) Sample answer: Predict: Michigan has the greater mean and MAD. Check: Michigan has mean = 12.1 and MAD = 2.9. New York has mean = 8.1 and MAD = 2.5. My prediction was correct.

5. a) Not a good site because people at the beach tend to like swimming.
   b) Not a good site because people at a bookstore tend to like reading.
   c) Not a good site because people at a professional hockey game tend to like hockey.
   d) The best site, because you can find all types of people in a shopping mall.

6. a) Eddy’s, because all the students in his class know him.
   b) Kim, because more than half of the representative sample will vote for her.

7. Yes, because people will want to say they play more than they actually do.

8. a) B
   b) Sample answer: Should students be required to wear school uniforms?

AP Book SP7-17
page 192
1. a) 1/5 or 20%
   b) 400
   c) Range = 5
   d) Median = 1
   e) 1.5
   f) $1.50
   
2. a) 6
   b) 1/5
   c) Yes
   d) Median = 3
   e) The IQR of sample size 30 is smaller than the IQR of sample size 10.

AP Book SP7-18
page 194
1. a) close together
   b) IQR = 4
   c) No
   d) High
   e) No
   f) Small

2. a) Small
   b) High
   c) No
   d) High
   e) No
   f) Small

3. a) 2, 8, 6
   b) Yes, because all the distributions have the same shape, and hence the same MAD.

U-44
c) top data set:
   mean = 2
   MAD = \( \frac{2 + 1 + 1 + 0 + 0 + 1 + 1 + 2}{8} = 1 \)
   bottom data set:
   mean = 8
   MAD = \( \frac{2 + 1 + 1 + 0 + 1 + 1 + 2}{8} = 1 \)

d) In A: no, In B: yes
e) In A: the difference between the means
      In B: they are equal

4.  a)  A
    b)  A
    c)  B
    d)  small overlap

5.  a)  part i)
    b)  i)  less than
        ii)  greater than

6.  a)  Answers may vary.
      Teacher to check.
      Sample answer:
      Africa: 53
      Europe: 72
    b)  Africa
      The dots are more spread out.
    c)  no
    d)  yes
    e)  yes
    f)  no
    g)  small

7.  a)  Soccer:
      range = 10
      median = 157
      Q1 = 155
      Q3 = 161
      Basketball:
      range = 18
      median = 165
      Q1 = 159
      Q3 = 167
    b)  Teacher to check.
    c)  small overlap

### AP Book SP7-19

#### page 197

1.  a)  Teacher to check.
    b)  boys
    c)  | M | Q1 | Q3 | R |
       |---|---|---|---|
       | 37| 30| 43| 26 |
       | 23| 21| 35| 30 |

d)  Teacher to check.

e)  small

2.  a)  i)  0.45, 0.50, 0.55,
       0.60, 0.65, 0.70,
       0.75
       ii)  Teacher to check.
       iii)  mean = 0.56
            MAD = 0.09
    b)  ii)  Teacher to check.
        iii)  mean = 0.55
              MAD = 0.035
    c)  0.01 + 0.09 = 0.1
       0.1 < 1, so there is high overlap

3.  It is more likely to be a coincidence that the ages are different because there is more overlap in the ages than in the heights.

4.  a)  Teacher to check.
    b)  mean = 3.65
       MAD = 1.75
    c)  Teacher to check dot plot.
       mean = 9.9
       MAD = 1.12

5.  a)  20, 24
    b)  8, 14
    c)  Teacher to check box plots.
      small overlap
    d)  7.36, 13.92
    e)  13.92 - 7.36 = 6.56
       6.56 + 4.25 = 1.54

6.  a)  20, 24
    b)  8, 14
    c)  Teacher to check box plots.
      small overlap
    d)  7.36, 13.92
    e)  13.92 - 7.36 = 6.56
       6.56 + 4.25 = 1.54

7.  a)  The mean and MAD of the proportions are:
      Sample size 20:
      mean = 0.1825
      MAD = 0.0875
      Sample size 50:
      mean = 0.198
      MAD = 0.0224
      The means are close, but the sample of size 50 is much less variable.
      The true proportion is likely to be closer to the mean for 50 samples than for 20 samples, so estimate the true proportion to be 19.8% or about 20%.

8.  a)  20, 24
    b)  8, 14
    c)  Teacher to check dot plots.
      small overlap
    d)  7.36, 13.92
    e)  13.92 - 7.36 = 6.56
       6.56 + 4.25 = 1.54

9.  a)  Teacher to check.
    b)  mean = 3.65
       MAD = 1.75
    c)  Teacher to check dot plot.
       mean = 9.9
       MAD = 1.12

10. a)  Teacher to check.
       b)  mean = 3.65
           MAD = 1.75
       c)  Teacher to check dot plot.
           mean = 9.9
           MAD = 1.12
Unit 4: Geometry

Quiz (Lessons 11 to 13)

Name: ______________________

Date: ______________

1. Use an equation to find \( x \). Include the units in your answer.

   a) 
   \[ 51° \] 
   \[ x \]

   b) 
   \[ 120° \]
   \[ 2x \]

   c) 
   \[ 2x - 14° \]
   \[ 36° \]

   d)

   e) 

2. Evaluate the variables. Include the units.

   a) 
   \[ 110° \]
   \[ 50° \]
   \[ y \]
   \[ x \]
   \( x = \) _____, \( y = \) _____

   b) 
   \[ 55° \]
   \[ x \]
   \[ y \]
   \[ z \]
   \( x = \) _____, \( y = \) _____,
   \( z = \) _____

   c) 
   \[ 50° \]
   \[ y \]
   \[ x \]
   \[ z \]
   \( x = \) _____, \( y = \) _____,
   \( z = \) _____
Unit 4: Geometry

Quiz (Lessons 11 to 13)

1. a) \[ x = 90 - 51 \]
   \[ x = 39^\circ \]

   b) \[ 2x + 120 + x = 180 \]
   \[ 3x = 180 - 120 \]
   \[ 3x = 60 \]
   \[ x = 60 ÷ 3 \]
   \[ x = 20^\circ \]

   c) \[ 2x - 14 = 36 \]
   \[ 2x = 36 + 14 \]
   \[ 2x = 50 \]
   \[ x = 25^\circ \]

   d) \[ x + 90 = 110 \]
   \[ x = 110 - 90 \]
   \[ x = 20^\circ \]

   e) \[ x + 90 + 60 = 180 \]
   \[ x = 180 - 90 - 60 \]
   \[ x = 30^\circ \]

**Bonus**

\[ x + 4x + 4x = 180 \]
\[ 9x = 180 \]
\[ x = 180 ÷ 9 \]
\[ x = 20^\circ \]

2. a) \[ x = 70^\circ \]
   \[ y = 60^\circ \]

   b) \[ x = 35^\circ \]
   \[ y = 35^\circ \]
   \[ z = 55^\circ \]

   c) \[ x = 40^\circ \]
   \[ y = 50^\circ \]
   \[ z = 40^\circ \]
Unit 4: Geometry
Quiz (Lessons 14 to 18)

1. Find the area $A$ of the shape. Include the units.
   
   a) 
   
   b) 
   
   $A = \underline{\phantom{0}}$ \hspace{2cm} $A = \underline{\phantom{0}}$

2. The area is 24 cm$^2$. All measurements are in cm. Find $x$.
   
   a) 
   
   b) 
   
   $x = \underline{\phantom{0}}$ cm \hspace{2cm} $x = \underline{\phantom{0}}$ cm

3. Use the floor plans to find the area of each room in square feet.
   Which room is bigger, Room A or Room B?

   Room A: 
   
   Room B: 
   
   1 cm = 4 ft
   1 in = 1 yd

Bonus ► Find the area of the shaded shape.
All measurements are in inches.
1. a) $A = b \times h$
   $A = 3 \times 7$
   $A = 21 \text{ cm}^2$

   b) $A = \frac{(b_1 + b_2) \times h}{2}$
   $A = \frac{(7 + 11) \times 3}{2}$
   $A = \frac{18 \times 3}{2}$
   $A = \frac{54}{2}$
   $A = 27 \text{ in}^2$

2. a) $A = \frac{b \times h}{2}$
   $24 = \frac{(x + 5)(3)}{2}$
   $24 \times 2 = (x + 5)(3)$
   $48 = (x + 5)(3)$
   $\frac{48}{3} = x + 5$
   $16 = x + 5$
   $16 - 5 = x$
   $x = 11 \text{ cm}$

   b) $A = \left( \ell \times w \right) + \frac{(b_1 + b_2) \times h}{2}$
   $24 = \frac{(x + 5)}{2}$
   $24 \times 2 = (2 + 5) \times (2x)$
   $24 = 5x + \frac{7 \times 2x}{2}$
   $24 = 5x + \frac{14x}{2}$
   $24 = 5x + 7x$
   $24 = 12x$
   $\frac{24}{12} = x$
   $x = 2 \text{ cm}$

3. Room A: $1 \text{ cm} = 4 \text{ ft}$
   so, $2 \text{ cm} = 8 \text{ ft}$
   and $5 \text{ cm} = 20 \text{ ft}$
   Area $= \ell \times w$
   $= 8 \times 20$
   Area $A = 160 \text{ ft}^2$

   Room B: $1 \text{ in} = 1 \text{ yd}$
   so, $3 \text{ in} = 3 \text{ yd}$
   and $5 \text{ in} = 5 \text{ yd}$
   Area $= \ell \times w$
   $= 3 \times 5$
   Area $B = 15 \text{ yd}^2$

   3 ft = 1 yd, so $9 \text{ ft}^2 = 1 \text{ yd}^2$
   $(15)(9 \text{ ft}^2) = (15)(1 \text{ yd}^2)$
   $135 \text{ ft}^2 = 15 \text{ yd}^2$
   so Area B = $135 \text{ ft}^2$

   $135 \text{ ft}^2 < 160 \text{ ft}^2$
   so Room A is bigger.

**Bonus**

Area of trapezoid:
$A_1 = \frac{(b_1 + b_2) \times h}{2}$
$= \frac{(2 + 1 + 1)(4 + 2)}{2}$
$= \frac{(3 + 1) \times 6}{2}$
$= \frac{4 \times 6}{2}$
$= 12 \text{ in}^2$

Area of triangle:
$A_2 = \frac{b \times h}{2}$
$= \frac{(2 + 1)(3 + 2)}{2}$
$= \frac{3 \times 5}{2}$
$= \frac{15}{2}$
$= 7 \frac{1}{2} \text{ in}^2$

Area of overlap:
$A_3 = \ell \times w$
$= 1 \times 2$
$= 2 \text{ in}^2$

Total area:
$A_{\text{total}} = A_1 + A_2 - A_3$
$= 12 + 7 \frac{1}{2} - 2$
$= 17 \frac{1}{2} \text{ in}^2$
In all questions below, you may use a calculator. Use 3.14 for \( \pi \) and round your answer to two decimal places. Include the units in all your answers.

1. Find the circumference \( C \) and area \( A \) of the circle.

   a) \[ C = \quad \quad \quad \quad \quad A = \quad \quad \quad \quad \quad \]

2. A triangle has two equal angles. The other angle is 120°.

   What are the two equal angles in the triangle?

3. A circular window has diameter 1.2 m. If glass costs $25 per 1 m\(^2\), how much does the glass cost to make the window?

**Bonus** Find the area of the shaded part of the diagram.
Unit 4: Geometry

Quiz (Lessons 19 to 23)

1. \( C = \pi d \)
   \( = (3.14)(18) \)
   \( \approx 56.52 \text{ ft} \)
   \( A = \pi r^2 \)
   \( = (3.14)(9)^2 \)
   \( \approx 254.34 \text{ ft}^2 \)

2. \( 120 + x + x = 180 \)
   \( 2x = 180 - 120 \)
   \( 2x = 60 \)
   \( x = 60 / 2 \)
   \( x = 30^\circ \)
   The two equal angles measure \( 30^\circ \).

3. \( A = \pi r^2 \)
   \( = (3.14)(0.6)^2 \)
   \( \approx 1.1304 \text{ m}^2 \)
   \( 1.1304 \times 25 = 28.26 \)
   The glass costs $28.26.

Bonus
\[
A = \frac{1}{2} \pi (2)^2 - 2 \left( \frac{1}{2} \pi \left(1\right)^2 \right)
= \frac{1}{2} \pi (4) - 2 \left( \frac{1}{2} \pi \right)
= 2 \pi - \pi
= \pi
= 3.14 \text{ yd}^2
\]
Unit 4: Geometry
Test (Lessons 11 to 23)

In all questions below, you may use a calculator.

1. Write “complementary,” “supplementary,” “vertical,” or “adjacent.”
   a) ___________________________ angles add to 180°.
   b) ___________________________ angles add to 90°.
   c) ___________________________ angles are always equal.

2. Use an equation to find $x$.
   a) ![Diagram](image-a)
   b) ![Diagram](image-b)
   c) ![Diagram](image-c)

3. The area of the triangle is 30 in$^2$. Find $x$.
   ![Diagram](image-d)

   $x = \underline{\hspace{2cm}}$
Unit 4: Geometry
Test (Lessons 11 to 23)

4. The diagram shows a floor plan of a room. What is the area of the room?

5. In a right triangle, one acute angle is 5 times as large as the other acute angle. Draw a sketch to show the information and find all the angles in the triangle.

6. The diameter of a Ferris wheel is 80 m.
   a) What is the circumference of the wheel?

   b) The wheel turns about 0.3 m every second. How long does it take for the wheel to rotate once? Round your answer to the nearest minute.
7. a) Find the area of the shaded and the unshaded part of the diagram. Use 3.14 for $\pi$.

Unshaded part: Shaded part:

b) Does your answer make sense? Use the picture to explain.

Bonus► All three circles have the same radius. How do the areas of the shaded parts compare? Explain how you know.
1. a) supplementary  
b) complementary  
c) vertical  
2. a) \[ x = 180 - 90 - 41 \]  
   \[ x = 49^\circ \]  
b) \[ x + 30 = 135 \]  
   \[ x = 135 - 30 \]  
   \[ x = 105^\circ \]  
c) \[ x + x + x + 30 = 180 \]  
   \[ 3x = 180 - 30 \]  
   \[ 3x = 150 \]  
   \[ x = 150 + 3 \]  
   \[ x = 50^\circ \]  
3. \[ A = \frac{bh}{2} \]  
   30 = \( \frac{(3x+5)\times3}{2} \)  
   30(2) = (3x + 5)(3)  
   60 = (3x + 5)(3)  
   60 + 3 = 3x + 5  
   20 = 3x + 5  
   20 - 5 = 3x  
   15 = 3x  
   15 + 3 = x  
   \[ x = 5 \text{ in} \]  
4. \[ A = \frac{(b_1+b_2)\times h}{2} \]  
   \[ A = \frac{(7.5 + 17.5)\times10}{2} \]  
   \[ A = \frac{250}{2} \]  
   \[ A = 125 \text{ ft}^2 \]  
5. \[ x + 5x + 90 = 180 \]  
   \[ 6x = 180 - 90 \]  
   \[ 6x = 90 \]  
   \[ x = 90 + 6 \]  
   \[ x = 15^\circ \]  
   Sample diagram:  
   \[ \angle C = 90^\circ \]  
   \[ \angle B = 15^\circ \]  
   \[ \angle A = 75^\circ \]  
6. a) \[ C = \pi d \]  
   \[ C = (3.14)(80) \]  
   \[ C = 251.2 \text{ m} \]  
b) \[ \frac{251.2}{0.3} \]  
   \[ = 837.3 \text{ seconds} \]  
   \[ \frac{837.3}{60} = 13.96 \]  
   It takes about 14 minutes for the wheel to rotate once.  
7. a) Unshaded part:  
   \[ A = \pi r^2 \]  
   \[ A = (3.14)(3)^2 \]  
   \[ A = 28.26 \text{ m}^2 \]  
   Shaded part:  
   \[ A = (l \times w) - 28.26 \]  
   \[ A = (6 \times 6) - 28.26 \]  
   \[ A = 36 - 28.26 \]  
   \[ A = 7.74 \text{ m}^2 \]  
b) Yes, the answer makes sense.  
   Explanations may vary. Teacher to check.  
**Bonus**  
The shaded parts are equal.  
Explanations may vary. Teacher to check.
1. Divide.
   a) $265 \div 58 = \text{___________}$
   b) $7.82 \div (-2.3) = \text{___________}$

2. Use long division to write the fraction as a decimal. Stop adding zeroes to the dividend when the remainder is 0 or when you see a pattern.
   a) $\frac{14}{18} = \text{___________}$
   b) $\frac{-13}{40} = \text{___________}$

**Bonus**

Use long division to write $\frac{156}{108}$ as a decimal.

$$\frac{156}{108} = \text{___________}$$
Unit 5: The Number System

Quiz (Lessons 46 to 48)

1. a) \(4 \text{ R } 33\)
   
   \[
   \begin{array}{c|cc}
   \hline
   & 4 & \hline
   58 & 265 & 232 & 33 \\
   \hline
   \end{array}
   \]

   b) \(-3.4\)
   
   \[
   \begin{array}{c|cc}
   \hline
   & 3.4 & \hline
   23 & 78.2 & 69 & 92 & 92 & 0 \\
   \hline
   \end{array}
   \]

2. a) \(0.777\ldots\)
   
   \[
   \begin{array}{c|cc}
   \hline
   & 0.777 & \hline
   18 & 14.000 & 126 & 140 & 126 & 140 & 126 & 14 \\
   \hline
   \end{array}
   \]

   b) \(-0.325\)
   
   \[
   \begin{array}{c|cc}
   \hline
   & 0.325 & \hline
   40 & 13.000 & 120 & 100 & 80 & 200 & 200 & 0 \\
   \hline
   \end{array}
   \]

Bonus

\[
\frac{156}{108} = 1.444\ldots
\]

\[
\begin{array}{c|cc}
\hline
& 1.44 & \hline
108 & 156.00 & 480 & 432 & 480 & 432 & 48 \\
\hline
\end{array}
\]

COPYRIGHT © 2015 JUMP MATH: TO BE COPIED. CC
1. Write the decimal to six decimal places.
   a) \(0.\overline{7} \approx 0.\) _______ _______ _______ _______ _______ _______
   b) \(0.45 \approx 0.\) _______ _______ _______ _______ _______ _______

2. Use bar notation to write the repeating decimal.
   a) \(0.03333333\ldots = \) __________
   b) \(0.17417417\ldots = \) __________

3. Use long division to write \(\frac{5}{11}\) as a decimal. Use bar notation to write the decimal.

4. a) i) Counting by ones, \(8.54\) is between _______ and _______.
    ii) Counting by tenths, \(8.54\) is between _______ and _______

   b) i) Counting by tenths, \(0.\overline{7}\) is between _______ and _______.
    ii) Counting by hundredths, \(0.\overline{7}\) is between _______ and _______.

5. a) Write the fraction in lowest terms.
    i) \(\frac{18}{30} = \) __________
    ii) \(\frac{30}{45} = \) __________

   b) Can the fraction be written as a terminating decimal?
    i) __________
    ii) __________

Bonus: Can \(\frac{140}{175}\) be written as a terminating decimal? __________
Unit 5: The Number System

Quiz (Lessons 49 to 52)

1. a) 0.777777
   b) 0.454545

2. a) 0.03
   b) 0.174

3. \[ \begin{array}{c}
   11 \left) 5.000 \\
   \underline{4} \\
   60 \\
   \underline{55} \\
   50 \\
   \underline{44} \\
   \hline
   0 \end{array} \]

4. a) i) 8, 9
   ii) 8.5, 8.6
   b) i) 0.7, 0.8
   ii) 0.77, 0.78

5. a) i) \[ \frac{3}{5} \]
   ii) \[ \frac{2}{3} \]
   b) i) yes
   ii) no

Bonus

\[ \begin{array}{c}
   140 \qquad 4 \\
   175 \qquad 5 \\
   \hline
   \text{yes} \end{array} \]
1. Write the decimal to six decimal places.
   a) $0.\overline{5} \approx 0.\underline{5} \underline{5} \underline{5} \underline{5} \underline{5} \underline{5}$
   b) $0.2\overline{4} \approx 0.\underline{2} \underline{4} \underline{4} \underline{4} \underline{4} \underline{4}$
   c) $0.3\overline{6} \approx 0.\underline{3} \underline{6} \underline{6} \underline{6} \underline{6} \underline{6}$
   d) $0.50\overline{7} \approx 0.\underline{5} \underline{0} \underline{7} \underline{7} \underline{7} \underline{7}$

2. Use bar notation to write the repeating decimal.
   a) $0.12312312\ldots = \underline{0.1} \underline{2} \underline{3} \underline{1} \underline{2} \underline{3} \underline{1} \underline{2}$
   b) $0.83333333\ldots = \underline{0.8} \underline{3}$
   c) $1.75151515\ldots = \underline{1.7} \underline{5} \underline{1} \underline{5} \underline{1} \underline{5}$
   d) $0.90909090\ldots = \underline{0.9} \underline{0} \underline{9}$

3. a) Use long division to write $\frac{1}{6}$ to 3 decimal places.

   ![Long Division Diagram]

   $\frac{1}{6} \approx \underline{0.1} \underline{6}$

   b) If you were to continue dividing, what remainder would you get? Explain how you know.

   c) Use bar notation to write your answer.

   $\frac{1}{6} = \underline{0.1} \underline{6}$
Unit 5: The Number System
Test (Lessons 46 to 52)

4. a) Use long division to write \( \frac{7}{8} \) as a decimal.

\[
\begin{array}{c}
7 \\
8 \\
\hline
(\quad) \\
\hline
\end{array}
\]

\[
\frac{7}{8} = \underline{\underline{\quad}}
\]

b) How do you know that the division is finished at this point?

5. Use long division to write the fraction as a decimal. Use bar notation for repeating decimals.

a) \( \frac{-30}{-66} \) = \underline{\underline{\quad}}

b) \( \frac{-21}{56} \) = \underline{\underline{\quad}}

6. Write the thousandths marked with an X.

0.42

\[
\begin{array}{cccc}
\text{X} & \text{X} & \text{X} \\
\text{X} \\
\end{array}
\]

0.43

\[
\begin{array}{cccc}
\text{X} & \text{X} & \text{X} & \text{X} \\
\end{array}
\]
Unit 5: The Number System

Test (Lessons 46 to 52)

7. a) Circle the position of the number on the number lines.
   
i) 0.826

![Number line for 0.826]

ii) 0.5

![Number line for 0.5]

b) Does the number in part a) have an exact position on the third number line?

   i) ___________       ii) ___________

8. Can \( \frac{24}{30} \) be written as a terminating decimal? Explain.

9. Can \( \frac{40}{60} \) be written as a terminating decimal? Explain.

Bonus► Why doesn’t a repeating decimal like 0.4 have an exact position on a number line that measures tenths, hundredths, thousandths, etc.?
Unit 5: The Number System

Test (Lessons 46 to 52)

ADVANCED

10. Compare the decimals. Hint: Use the grids to line up the decimals.
   a) 0.249 ____________

   b) 0.725 ____________

11. Use bar notation to write your answer. Hint: Use the grids to line up the decimals.
   a) \( \overline{0.24} + \overline{0.35} = \) ____________

   b) \( \overline{0.759} - \overline{0.355} = \) ____________

Bonus ► \( 0.\overline{9} - 0.\overline{89} + 0.\overline{67} = \) ____________
1. a) 0.555555  
   b) 0.244444  
   c) 0.383636  
   d) 0.507070  
2. a) 0.723  
   b) 0.83  
   c) 1.751  
   d) 0.90  
3. a) \[
\begin{array}{c}
0.166 \\
6)
\hline
1.000 \\
6 \\
40 \\
36 \\
40 \\
36 \\
4 \\
0.166
\end{array}
\]
   b) 4, because I am always dividing 40 ÷ 6, which is 6 R 4.  
   c) 0.16  
4. a) \[
\begin{array}{c}
0.875 \\
8)
\hline
7.000 \\
64 \\
60 \\
56 \\
40 \\
40 \\
0 \\
0.875
\end{array}
\]
   b) The remainder is 0.  
5. a) \[
\begin{array}{c}
0.45 \\
66)
\hline
30.000 \\
264 \\
360 \\
330 \\
300 \\
264 \\
36 \\
0.454
\end{array}
\]
   b) -0.375  
6. 0.420, 0.421, 0.424, 0.428, 0.430  
7. a) Teacher to check.  
   b) i) yes  
   ii) no  
8. \[
\begin{array}{c}
24 \\
30 \div 5
\end{array}
\]
   Yes, because in reduced form, the denominator is 5 (has only 2s and 5s as a factor)  
   or \[
\begin{array}{c}
4 \\
8 \\
5 \\
10
\end{array} = 0.8 \text{ terminating}
\]
9. \[
\begin{array}{c}
40 \\
60 \div 3
\end{array}
\]
   No, because in reduced form, the denominator is 3 (doesn’t have only 2s and 5s as factors).  
   Bonus  
   Since there is no final digit, there is no exact position.  
ADVANCED  
10. a) >  
    b) <  
11. a) 0.59  
    b) 0.404  
   Bonus  
   0.7
Unit 6: Geometry
Quiz (Lessons 24 to 27)

You may use a calculator for all questions.

1. Find the volume of the prism.
   
   a) 
   \[ 6 \text{ m} \]
   \[ 8.5 \text{ m} \]
   \[ 4 \text{ m} \]

   b) 
   \[ 2 \text{ ft} \]
   \[ 5 \text{ ft} \]
   \[ 7 \text{ ft} \]

2. Find the surface area of the prism.

   \[
   \text{Surface Area} = 2(3 \times 4) + 2(3 \times 2) + 2(4 \times 2) = 24 + 12 + 16 = 52 \text{ cm}^2
   \]

Bonus ▶ A rectangular prism has height 2 cm and volume 40 cm$^3$. What could the dimensions of the base be? Give at least 2 possibilities.

- 3 cm and 4 cm
- 5 cm and 2 cm
1. a) \[ B = 6 \times 4 + 2 \]
   \[ = 12 \]
   \[ h = 8.5 \]
   \[ V = B \times h \]
   \[ = 12 \times 8.5 \]
   \[ = 102 \text{ m}^3 \]

   b) \[ V_1 = \ell \times w \times h \]
   \[ = 2(2)(5) \]
   \[ = 20 \text{ ft}^3 \]
   \[ V_2 = \ell \times w \times h \]
   \[ = 5(2)(4) \]
   \[ = 40 \text{ ft}^3 \]
   \[ V_{total} = V_1 + V_2 \]
   \[ = 20 + 40 \]
   \[ = 60 \text{ ft}^3 \]
   OR
   \[ V_1 = \ell \times w \times h \]
   \[ = 2(2)(1) \]
   \[ = 4 \text{ ft}^3 \]
   \[ V_2 = \ell \times w \times h \]
   \[ = 7(2)(4) \]
   \[ = 56 \text{ ft}^3 \]
   \[ V_{total} = V_1 + V_2 \]
   \[ = 4 + 56 \]
   \[ = 60 \text{ ft}^3 \]

2. \[ SA = 2(\ell h + \ell w + wh) \]
   \[ = 2(4(3) + 4(2) + 2(3)) \]
   \[ = 2(12 + 8 + 6) \]
   \[ = 2(26) \]
   \[ = 52 \text{ cm}^2 \]

Bonus
\[ V = \ell \times w \times h \]
\[ 40 = \ell \times w \times 2 \]
\[ 40 + 2 = \ell \times w \]
\[ 20 = \ell \times w \]
So \( \ell = 5, w = 4 \)
or \( \ell = 10, w = 2 \)
Unit 6: Geometry
Quiz (Lessons 28 to 31)

You may use a calculator for all questions.

1. Find the surface area.

```
12 m

4 m

8 m

5 m

6 m
```

2. What shape is the cross section?

a) The plane is parallel to the base of the rectangular prism.

b) The plane slices the rectangular pyramid vertically through the top vertex.

---

**Bonus** Explain why the cross section of a rectangular pyramid cannot be an octagon.
Unit 6: Geometry

Quiz (Lessons 28 to 31)

1. \( A_{\text{top}} = (b_1 + b_2) \times h \div 2 \)
   \[ (6 + 12) \times 4 \div 2 \]
   \[ = 36 \text{ m}^2 \]

\( A_{\text{bottom}} = A_{\text{top}} \)
\[ = 36 \text{ m}^2 \]

\( A_{\text{back}} = 12 \times 8 \)
\[ = 96 \text{ m}^2 \]

\( A_{\text{front}} = 6 \times 8 \)
\[ = 48 \text{ m}^2 \]

\( A_{\text{left side}} = 5 \times 8 \)
\[ = 40 \text{ m}^2 \]

\( A_{\text{right side}} = A_{\text{left side}} \)
\[ = 40 \text{ m}^2 \]

\( SA = 36 + 36 + 96 + 48 + 40 + 40 \)
\[ = 296 \text{ m}^2 \]

2. a) rectangle
   b) triangle

**Bonus**

The pyramid has 5 faces, so the cross section cannot have more than 5 sides.
Unit 6: Geometry
Test (Lessons 24 to 31)

You may use a calculator for all questions.

1. Find the volume.

\[
\text{Volume} = \frac{1}{2} \times 15 \text{ mm} \times \frac{1}{2} \text{ cm}\]

2. Find the surface area.

\[
\text{Surface Area} = 2 \times (12 \text{ in} \times 10 \text{ in}) + 2 \times (12 \text{ in} \times 8 \text{ in}) + 2 \times (12 \text{ in} \times 12 \text{ in})
\]

3. Find the unknown dimension so that the volume is at least 60 m\(^3\). Write the answer as an inequality.

\[
\text{Volume} = \frac{1}{2} \times 3 \text{ m} \times 10 \text{ m} \times x \text{ m}
\]

4. Rob is painting a birdhouse. He needs to paint all sides of the birdhouse except for the bottom side and the entrance on the right side. The entrance is shaded in the picture. If a small tube of paint covers about 55 in\(^2\), how many tubes would Rob need?
5. The plane is parallel to the base. Is the cross section similar or congruent to the base?

   a) ________________  b) ________________

6. A plane cuts the rectangular prism vertically through the bold line.

   a) Finish drawing the cross section.

   b) Describe the shape of the cross section. Explain how you know.

7. Mark the parallel edges of the cross section with the same number of arrows.

   a) ________________  b) ________________

**Bonus** Find 2 possible dimensions for a rectangular prism with volume 20 in³. Which has the smaller surface area?
1. 15 mm = 1.5 cm
   \[ V = \ell \times w \times h \]
   \[ = 1.5 \times 1.5 \times 1.5 \]
   \[ = 3.375 \]
   OR
   \[ V = \frac{2}{3} \times \frac{3}{2} \times \frac{3}{2} \]
   \[ = \frac{27}{8} \]
   \[ = \frac{3}{8} \]

2. \[ A_{\text{front}} = (b \times h) + 2 \]
   \[ = (12 \times 8) + 2 \]
   \[ = 96 + 2 \]
   \[ = 98 \text{ in}^2 \]
   \[ A_{\text{back}} = A_{\text{front}} \]
   \[ = 98 \text{ in}^2 \]
   \[ A_{\text{bottom}} = \ell \times w \]
   \[ = 12 \times 12 \]
   \[ = 144 \text{ in}^2 \]
   \[ A_{\text{left side}} = \ell \times w \]
   \[ = 12 \times 10 \]
   \[ = 120 \text{ in}^2 \]
   \[ A_{\text{right side}} = A_{\text{left side}} \]
   \[ = 120 \text{ in}^2 \]
   \[ S_A = 48 + 98 + 144 + 120 \]
   \[ + 120 \]
   \[ = 480 \text{ in}^2 \]

3. \[ V \geq 60 \text{ m}^3 \]
   \[ B \times h \geq 60 \]
   \[ (3 \times x + 2) \times 10 \geq 60 \]
   \[ 3x + 2 \geq 6 \]
   \[ 3x + 2 \geq 6 \]
   \[ 3x \geq 4 \]
   \[ x \geq 4 \]

4. \[ A_{\text{front}} = (8(12) + ((6 + 12) \times 4 + 2)) \]
   \[ = 96 + (18 \times 4 + 2) \]
   \[ = 96 + 36 \]
   \[ = 132 \text{ in}^2 \]
   \[ A_{\text{back}} = A_{\text{front}} \]
   \[ = 132 \text{ in}^2 \]
   \[ A_{\text{top}} = 6 \times 6 \]
   \[ = 36 \text{ in}^2 \]
   \[ A_{\text{left side}} = 6 \times 8 \]
   \[ = 48 \text{ in}^2 \]
   \[ A_{\text{left roof}} = 6 \times 5 \]
   \[ = 30 \text{ in}^2 \]
   \[ A_{\text{right roof}} = A_{\text{left roof}} \]
   \[ = 30 \text{ in}^2 \]
   \[ S_{A_{\text{paint}}} = 132 + 132 + 36 + 48 + 30 + 30 \]
   \[ = 408 \text{ in}^2 \]
   \[ 408 + 55 = 7.42 \]
   Rob would need about 8 tubes of paint.

5. a) congruent
   b) similar

6. a) Teacher to check.
   b) The cross section is a rectangle because the vertical lines are perpendicular to the horizontal lines of the top and bottom faces.

7. a)

   b)

---

**Bonus**

\[ V_1 = 20 \text{ in}^3 \]
\[ \ell \times w \times h = 20 \]
\[ 10 \times 2 \times 1 = 20 \]
\[ S_A_1 = 2(\ell h + \ell w + wh) \]
\[ = 2(1(10) + 1(2) + 2(10)) \]
\[ = 2(10 + 2 + 20) \]
\[ = 2(32) \]
\[ = 64 \text{ in}^2 \]
\[ V_2 = 20 \text{ in}^3 \]
\[ \ell \times w \times h = 20 \]
\[ 5 \times 4 \times 1 = 20 \]
\[ S_A_2 = 2(\ell h + \ell w + wh) \]
\[ = 2(4(5) + 4(1) + 1(5)) \]
\[ = 2(20 + 4 + 5) \]
\[ = 2(29) \]
\[ = 58 \text{ in}^2 \]

The prism with dimensions 1 in by 4 in by 5 in has a smaller surface area than the prism with dimensions 1 in by 2 in by 10 in.
Unit 7: Statistics and Probability

You may use a calculator for all questions.

1. Andy got 76, 74, and 84 on his first three science tests. What does he need to get on the next one to average 80 on all four tests?

2. a) Find the interquartile range (IQR) of the data sets.

   Set 1: 2, 6, 8, 10, 13, 17, 20, 24, 28, 34

   IQR = ________

   b) In which set are the data points more tightly clustered around the median? __________

3. Find the mean absolute deviation (MAD) of the data set: 10, 7, 4, 11

4. A teacher wants to know if school should start half an hour earlier.
   Is the sample representative or biased?
   a) Ten students chosen randomly from the school list. ______________________
   b) The ten students who arrive first in the morning. ______________________
   c) The ten students who arrive last in the morning. ______________________

Bonus ► Use the addition statement 5 + 2 = 3 + 4 to create a data set with mean 10.
Unit 7: Statistics and Probability

Quiz (Lessons 10 to 15)

1. \[
\frac{76 + 74 + 84 + x}{4} = 80
\]
   \[
   234 + x = 80(4)
   \]
   \[
   x = 320 - 234
   \]
   \[
   x = 86
   \]
   Andy needs 86 on his next test.

2. a) 16, 2
   b) Set 2

3. Mean = \[
\frac{10 + 7 + 4 + 11}{4} = 8
\]
   MAD = \[
\frac{2 + 1 + 4 + 3}{4} = 2.5
\]

4. a) representative
   b) biased
   c) biased

Bonus
5. 8, 13, 14
Unit 7: Statistics and Probability

Name: ______________________

Date: ________________

Quiz (Lessons 16 to 19)

You may use a calculator for all questions.

1. A quality control inspector examined 100 headphones and found 3 that were defective. How many headphones would he estimate to be defective in the shipment of 3,000 headphones?

2. Box plots A and B represent sets that have 500 data values.
   a) Which set has the greater median? __________
   b) Which set has the greater IQR? __________
   c) In which set are the data points more tightly clustered around the median? __________
   d) Do the two sets have no overlap, small overlap, or high overlap? ____________________

3. A bag has 300 blue marbles and 700 marbles that are other colors, but Ray doesn’t know this. He does two different experiments to determine the proportion of blue marbles that are in the bag. Match the experiments with their most likely results. Explain your answer.

   Experiments:                                     Results:
   10 samples of 5 marbles each                     mean = 0.31, MAD = 0.07
   10 samples of 20 marbles each                    mean = 0.26, MAD = 0.19

**Bonus**: One study found that girls spend a mean 80 minutes on homework each week with MAD 7 minutes, and boys spend a mean 60 minutes on homework each week with MAD 10 minutes. Divide the difference in means by the largest MAD. Is there high overlap between the data sets or not?
1. \[
\frac{3}{100} = \frac{90}{3,000}
\]
He would estimate that 90 would be defective.

2. a) A
b) B
c) A
d) high overlap

3. First experiment matches with second result. Second experiment matches with first result.
The larger sample size is more reliable because it will vary less (has a smaller MAD) and be closer to the mean.

Bonus
\[
\frac{20}{10} = 2
\]
There is not high overlap.
You may use a calculator for all questions.

1. Here are Tina’s most recent test results for science, math, and English.

   a) Complete the table.

<table>
<thead>
<tr>
<th>Science: 72, 79, 80, 83, 86</th>
<th>Math: 50, 80, 80, 95, 95,</th>
<th>English: 69, 70, 90, 91</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td></td>
<td></td>
</tr>
<tr>
<td>median</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MAD</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IQR</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   b) Can you distinguish between the subjects using the mean and the median? Explain.

   c) In which class does Tina have the most consistent test results? Explain how you know.

2. Twenty randomly selected students in Grade 7 were asked how much time they spent looking at a screen (computer, TV, etc.) last night:

   0 0 0 0 1 1 1 1 1 1 2 2 2 2 3 3 4 4 4

   a) What is the mean and median amount of time spent looking at a screen for this data set?

   Mean = _______  Median = _______

   b) There are 120 students in Grade 7. Based on the survey, how many Grade 7 students do you think spent 2 hours looking at a screen last night?

3. Circle the terms that measure the variability in a set of data.

   IQR  mean  median  mode  range  MAD
4. Sharon compares the delay (in minutes) for the last 20 flights at two different airports.

Airport A: 0, 0, 0, 0, 5, 5, 5, 5, 10, 10, 10, 15, 15, 20, 20, 25, 30, 35, 40, 50

Airport B: 0, 5, 10, 10, 15, 15, 25, 25, 25, 25, 30, 30, 35, 35, 40, 40, 40, 40, 45, 50

a) For each set of data, find the median, Q1, and Q3 and then draw the box plot.

\[
\begin{align*}
\text{M} &= _____ \\
\text{Q}_1 &= _____ \\
\text{Q}_3 &= _____ \\
\text{M} &= _____ \\
\text{Q}_1 &= _____ \\
\text{Q}_3 &= _____ \\
\end{align*}
\]

b) Is the overlap high or small between these two data sets? _________________________

c) Which airport would you prefer to fly from? Why?

5. Raj completes five high jumps at these heights (in cm): 140, 145, 140, 150, 125

a) What is his mean high jump height? __________

b) What is the mean absolute deviation (MAD)?

c) Mindy’s jumps have a mean of 137 cm and a MAD of 3 cm. What is the difference in the means of Raj’s jumps and Mindy’s jumps?

d) Divide your answer to part c) by the greater of the two MADs: ______ ÷ ______ = ______

**Bonus**► Pam got 75, 81, and 80 on her first three math tests. What does she need to average on her next two tests to get an average of 80 on all five tests?
Unit 7: Statistics and Probability

Test (Lessons 10 to 19)

1. a) Science

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>80</td>
</tr>
<tr>
<td>median</td>
<td>80</td>
</tr>
<tr>
<td>MAD</td>
<td>3</td>
</tr>
<tr>
<td>IQR</td>
<td>4</td>
</tr>
</tbody>
</table>

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>80</td>
</tr>
<tr>
<td>median</td>
<td>80</td>
</tr>
<tr>
<td>MAD</td>
<td>10</td>
</tr>
<tr>
<td>IQR</td>
<td>15</td>
</tr>
</tbody>
</table>

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>80</td>
</tr>
<tr>
<td>median</td>
<td>80</td>
</tr>
<tr>
<td>MAD</td>
<td>8</td>
</tr>
<tr>
<td>IQR</td>
<td>21</td>
</tr>
</tbody>
</table>

b) No. The means are all 80 and the medians are all 80.

c) Science, because both the MAD and the IQR are the smallest.

2. a) 1.8, 1.5

b) \[
\frac{24}{120} = 0.2
\]

24 students spent 2 hours looking at a screen last night.

3. Circle IQR, range, and MAD

4. a) 10, 5, 22.5

Teacher to check box plot.

b) small

c) Airport A, because there are shorter delays.

5. a) 140

b) \[
\frac{0 + 5 + 0 + 10 + 15}{5} = \frac{30}{5} = 6
\]

c) 3 cm

d) 3, 6, 0.5

Bonus

\[
\frac{75 + 81 + 80 + x}{5} = 80
\]

\[
236 + x = 80(5)
\]

\[
x = 400 - 236
\]

\[
x = 164
\]

Pam needs a total of 164 or an average of 82 on her next two tests.
### Scoring Guides for Sample Unit Quizzes and Tests

#### Unit 4: Geometry

**Quiz (Lessons 11 to 13), p. V-30**

**Common Core State Standards Emphasized:** 7.G.B.5, 7.EE.B.4a

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
</table>
| 1        | a) Gives correct equation  
Correctly solves equation  
Gives correct answer with units | $x + 51 = 90$  
$x = 90 - 51$  
x = $39^\circ$ | 1  
1  
0.5 |  |
|         | b) Gives correct equation  
Correctly simplifies equation  
Correctly solves equation  
Makes one computational error  
Gives correct answer with units | $2x + x + 120 = 180$  
$3x + 120 = 180$  
$3x = 180 - 120$  
$3x = 60$  
x = $60 \div 3$  
x = $20^\circ$ | 1  
0.5  
2  
(1)  
0.5 |  |
|         | c) Gives correct equation  
Correctly solves equation  
Makes one computational error  
Gives correct answer with units | $2x - 14 = 36$  
$2x = 36 + 14$  
x = $50 \div 2$  
x = $25^\circ$ | 1  
2  
(1)  
0.5 |  |
|         | d) Gives correct equation  
Correctly solves equation  
Gives correct answer with units | $x + 90 = 110$  
x = $110 - 90$  
x = $20^\circ$ | 1  
1  
0.5 |  |
|         | e) Gives correct equation  
Correctly solves equation  
Gives correct answer with units | $x + 90 + 60 = 180$  
x = $180 - 90 - 60$  
x = $30^\circ$ | 1  
1  
0.5 |  |
|         | Bonus: Gives correct answer | $x + 4x + 4x = 180$  
$9x = 180$  
x = $180 \div 9$  
x = $20^\circ$ | yes / no | /15 |
| 2        | a) Gives correct answer for $x$ with units  
Gives correct answer for $y$ with units | $x = 70^\circ$  
y = $60^\circ$ | 1  
1 |  |
|         | b) Gives correct answer for $x$ with units  
Gives correct answer for $y$ with units  
Gives correct answer for $z$ with units | $x = 35^\circ$  
y = $35^\circ$  
z = $55^\circ$ | 1  
1  
1 |  |
|         | c) Gives correct answer for $x$ with units  
Gives correct answer for $y$ with units  
Gives correct answer for $z$ with units | $x = 40^\circ$  
y = $50^\circ$  
z = $40^\circ$ | 1  
1  
1 |  |
|         | Total Points | /23 |
### Scoring Guides and Rubrics to accompany Sample Unit Quizzes and Tests for AP Book 7.2

#### Unit 4: Geometry

**Quiz (Lessons 14 to 18), p. V-32**

**Common Core State Standards Emphasized:** 7.G.A.1, 7.G.B.6, 7.EE.B.4a, 7.EE.B.4b, 7.RP.A.2

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a) Gives correct expression</td>
<td>$A = 7 \times 3$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 21 \text{ cm}^2$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Gives correct expression</td>
<td>$A = (7 + 11) \times 3 \div 2$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 27 \text{ in}^2$</td>
<td>1</td>
<td>/4</td>
</tr>
<tr>
<td>2</td>
<td>a) Gives correct equation</td>
<td>(\left(\frac{x + 5}{2}\right) \times 3 = 24)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{3x + 15}{2} = 24$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3x + 15 = 48$</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3x = 48 - 15$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x = 33$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x = 33 + 3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x = 11 \text{ cm}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Gives correct equation</td>
<td>$5x + \frac{(2+5)(2x)}{2} = 24$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$5x + 7x = 24$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$12x = 24$</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x = 2 \text{ cm}$</td>
<td>1</td>
<td>/8</td>
</tr>
<tr>
<td>3</td>
<td>Uses scale correctly for room A</td>
<td>$A = 8 \times 20$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct area of room A</td>
<td>$= 160 \text{ ft}^2$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Uses scale correctly for room B</td>
<td>$A = 3 \times 5$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct area of room B</td>
<td>$= 15 \text{ ft}^2$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer</td>
<td>Room A</td>
<td>1</td>
<td>/5</td>
</tr>
<tr>
<td>Bonus</td>
<td>Gives correct answer</td>
<td>Yes / No</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Total Points: /17**
### Scoring Guides for Sample Unit Quizzes and Tests

#### Unit 4: Geometry

(continued)

<table>
<thead>
<tr>
<th>Quiz (Lessons 19 to 23), p. V-34</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a) Gives correct expression for C</td>
<td>(3.14 \times 18)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer for C with units</td>
<td>(\approx 56.52 \text{ ft})</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct expression for A</td>
<td>(3.14 \times 9 \times 9)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer for A with units</td>
<td>(\approx 254.34 \text{ ft}^2)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Gives correct equation</td>
<td>(x + x + 120 = 180)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correctly simplifies equation</td>
<td>(2x + 120 = 180)</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correctly solves equation</td>
<td>(2x = 180 - 120)</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>(x = 60)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correctly solves equation</td>
<td>(x = 60 \div 2)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td>(x = 30^\circ)</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Gives correct radius</td>
<td>0.6 m</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct expression for area</td>
<td>(A \approx 3.14 \times 0.6 \times 0.6)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correctly solves equation</td>
<td>(\approx 1.1304 \text{ m}^2)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer</td>
<td>(1.1304 \times 25)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td>(= 28.26)</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
| Bonus | Gives correct answer | \[
\left(\frac{1}{2} \pi (2)^2\right) - 2 \left(\frac{1}{2} \pi (1)^2\right)
\]
| | | \[
= \frac{1}{2} \pi (4) - 2 \left(\frac{1}{2} \pi (1)\right)
\]
| | | \[
= 2\pi - \pi
\]
| | | \[
= \pi
\]
| | | \(\approx 3.14 \text{ yd}^2\) | yes / no | |

| Total Points | 13 |
### Scoring Guides for Sample Unit Quizzes and Tests
#### Unit 4: Geometry


<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a) Gives correct answer</td>
<td>supplementary</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>complementary</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>c) Gives correct answer</td>
<td>vertical</td>
<td>1</td>
<td>/3</td>
</tr>
<tr>
<td>2</td>
<td>a) Gives correct equation</td>
<td>$x + 41 + 90 = 180$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correctly solves equation</td>
<td>$x = 131 = 180$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>$x = 180 - 131$</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td>$x = 49°$</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Gives correct equation</td>
<td>$x + 30 = 135$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correctly solves equation</td>
<td>$x = 135 - 30$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td>$x = 105°$</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>c) Gives correct equation</td>
<td>$x + x + x + 30 = 180$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correctly simplifies equation</td>
<td>$3x + 30 = 180$</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correctly solves equation</td>
<td>$3x = 180 - 30$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>$3x = 150$</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td>$x = 150 + 3$</td>
<td>0.5</td>
<td>/10</td>
</tr>
<tr>
<td>3</td>
<td>Gives correct equation</td>
<td>$(3x + 5) \times 3 + 2 = 30$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correctly simplifies equation</td>
<td>$9x + 15 = 30 \times 2$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correctly solves equation</td>
<td>$9x + 15 = 60$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>$9x = 60 - 15$</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td>$9x = 45$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x = 45 + 9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x = 5 \text{ in}$</td>
<td>1</td>
<td>/5</td>
</tr>
<tr>
<td>4</td>
<td>Gives correct expression</td>
<td>$A = 10 \times (17.5 + 7.5) \div 2$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td>$A = 125 \text{ ft}^2$</td>
<td>1</td>
<td>/2</td>
</tr>
<tr>
<td>Question</td>
<td>How to Score</td>
<td>Answer</td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
<td>--------</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| 5        | Draws a right triangle | ![Diagram](image)
|          | Labels angles as $x$ and $5x$ |
|          | Gives correct equation | $5x + x + 90 = 180$ |
|          | Correctly solves equation | $6x + 90 = 180$ |
|          | Makes one computational error | $6x = 90$ |
|          | Gives correct angle measures | $x = 15^\circ$, $\angle B = 15^\circ$, $\angle A = 75^\circ$, $\angle C = 90^\circ$ |
| 6        | a) Gives correct expression | $C \approx (3.14)(80)$ |
|          | Gives correct answer with units | $C \approx 251.2$ m |
|          | b) Correctly uses a ratio to find the distance turned in one minute | $0.3 : 1 = 18 : 60$ |
|          | Correctly solves expression | $251.2 \div 18 \approx 13.96$ min |
|          | Gives rounded answer with units | 14 minutes |
| 7        | a) Gives correct expression for unshaded part | $A \approx (3.14)(3)(3)$ |
|          | Correctly solves expression for unshaded part | $A \approx 28.26$ m$^2$ |
|          | Gives correct expression for shaded part | $A \approx (6)(6)$ |
|          | Correctly solves expression for shaded part | $A \approx 7.74$ m$^2$ |
|          | b) Gives correct answer | Yes. |
|          | Gives correct explanation | The area of the circle should be less than the area of the square. |
| Bonus    | Gives correct answer and explanation | Shaded parts are equal in area. Teacher to check explanation. |
|          | | yes / no |

**Total Points**: /37
### Rubric for Unit 4: Geometry

Test (Lessons 11 to 23), p. V-36

<table>
<thead>
<tr>
<th>Common Core State Standard</th>
<th>Assessed by Question(s) …</th>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
<th>Level 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.G.A.1 Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.</td>
<td>4</td>
<td>Can answer few, if any, questions accurately and independently.</td>
<td>Can answer some questions accurately and independently.</td>
<td>Can answer most questions accurately and independently.</td>
<td>Can answer all or almost all questions, including bonuses, accurately and independently.</td>
</tr>
<tr>
<td>7.G.A.2 Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.G.B.4 Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.</td>
<td>6, 7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.G.B.5 Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.</td>
<td>2, 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.G.B.6 Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Rubric for Unit 4: Geometry

Test (Lessons 11 to 23), p. V-36 (continued)

<table>
<thead>
<tr>
<th>Common Core State Standard</th>
<th>Assessed by Question(s) …</th>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
<th>Level 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.EE.B.4a</td>
<td></td>
<td>Can answer few, if any, questions accurately and independently.</td>
<td>Can answer some questions accurately and independently.</td>
<td>Can answer most questions accurately and independently.</td>
<td>Can answer all or almost all questions, including bonuses, accurately and independently.</td>
</tr>
<tr>
<td>Solve word problems leading to equations of the form ( px + q = r ) and ( p(x + q) = r ), where ( p, q, ) and ( r ) are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?</td>
<td>2, 3, 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.RP.A.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Recognize and represent proportional relationships between quantities.</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Comments**
### Scoring Guides for Sample Unit Quizzes and Tests
### Unit 5: The Number System

#### Quiz (Lessons 46 to 48), p. V-40

**Common Core State Standards Emphasized:** 7.NS.A.2c, 7.NS.A.2d

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a) Divides correctly</td>
<td>4 [ \frac{567}{265} ]</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>[ \frac{232}{33} ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer</td>
<td>4 R 33</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Divides correctly</td>
<td>3.4 [ \frac{237}{82} ]</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>[ \frac{69}{92} ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer</td>
<td>-3.4</td>
<td>1</td>
<td>/6</td>
</tr>
<tr>
<td>2</td>
<td>a) Divides correctly</td>
<td>0.77 [ \frac{1874.00}{126} ]</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>[ \frac{140}{126} ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer</td>
<td>0.777…</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Divides correctly</td>
<td>0.325 [ \frac{4073.000}{12.0} ]</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>[ \frac{100}{80} ]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer</td>
<td>-0.325</td>
<td>1</td>
<td>/6</td>
</tr>
<tr>
<td>Bonus</td>
<td>Gives correct answer</td>
<td>1.44 [ \frac{10856.00}{108} ]</td>
<td>yes / no</td>
<td></td>
</tr>
</tbody>
</table>

| Total Points | /12 |

---

Scoring Guides and Rubrics to accompany Sample Unit Quizzes and Tests for AP Book 7.2
## Scoring Guides for Sample Unit Quizzes and Tests
### Unit 5: The Number System

**Quiz (Lessons 49 to 52), p. V-42**

**Common Core State Standards Emphasized:** 7.NS.A.1d, 7.NS.A.2c, 7.NS.A.2d

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a) Gives correct answer</td>
<td>0.777777</td>
<td>1</td>
<td>/2</td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>0.454545</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>a) Gives correct answer</td>
<td>0.03</td>
<td>1</td>
<td>/2</td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>0.174</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Divides correctly</td>
<td>0.45</td>
<td>2</td>
<td>/3</td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>(1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer using bar notation</td>
<td>(\frac{3}{5})</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>a) i) Gives correct answer</td>
<td>8, 9</td>
<td>1</td>
<td>/4</td>
</tr>
<tr>
<td></td>
<td>a) ii) Gives correct answer</td>
<td>8.5, 8.6</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) i) Gives correct answer</td>
<td>0.7, 0.8</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) ii) Gives correct answer</td>
<td>0.77, 0.78</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>a) i) Gives correct answer</td>
<td>(\frac{2}{3})</td>
<td>1</td>
<td>/4</td>
</tr>
<tr>
<td></td>
<td>a) ii) Gives correct answer</td>
<td>yes</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) i) Gives correct answer</td>
<td>no</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Bonus</td>
<td>Gives correct answer</td>
<td>yes</td>
<td>yes / no</td>
<td></td>
</tr>
</tbody>
</table>

**Total Points /15**
### Test (Lessons 46 to 52), p. V-44

**Common Core State Standards Emphasized:** 7.NS.A.1d, 7.NS.A.2c, 7.NS.A.2d

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a) Gives correct answer</td>
<td>0.555555</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>0.244444</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>c) Gives correct answer</td>
<td>0.363636</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>d) Gives correct answer</td>
<td>0.507070</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>a) Gives correct answer</td>
<td>0.123</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>0.83</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>c) Gives correct answer</td>
<td>1.751</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>d) Gives correct answer</td>
<td>0.90</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>a) Divides correctly</td>
<td>0.166</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Gives correct answer</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>The remainder will always be 4.</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Gives correct explanation</td>
<td>Each time 6 divides into 40, we get 6 R 4.</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>c) Gives correct answer</td>
<td>0.16</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>a) Divides correctly</td>
<td>0.875</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Gives correct answer</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>The remainder is 0.</td>
<td>1</td>
</tr>
</tbody>
</table>
### Scoring Guides for Sample Unit Quizzes and Tests

#### Unit 5: The Number System

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>a) Divides correctly Make one computational error</td>
<td>0.4545</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Gives correct answer using bar notation</td>
<td>0.45</td>
<td>1</td>
<td>/</td>
</tr>
<tr>
<td></td>
<td>b) Divides correctly Make one computational error</td>
<td>0.375</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Gives correct answer</td>
<td>−0.375</td>
<td>1</td>
<td>/</td>
</tr>
<tr>
<td>6</td>
<td>Gives five correct answers</td>
<td>0.420, 0.421, 0.424, 0.428, 0.430</td>
<td>3</td>
<td>/</td>
</tr>
<tr>
<td></td>
<td>Gives three or four correct answers</td>
<td>2</td>
<td>(2)</td>
<td>/</td>
</tr>
<tr>
<td></td>
<td>Gives one or two correct answers</td>
<td>Teacher to check.</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td>7</td>
<td>a) i) Circles correct position on all three number lines Circles correct position on one or two number lines</td>
<td>Teacher to check.</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>a) ii) Circles correct position on all three number lines Circles correct position on one or two number lines</td>
<td>Teacher to check.</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>b) i) Gives correct answer</td>
<td>yes</td>
<td>1</td>
<td>/</td>
</tr>
<tr>
<td></td>
<td>b) ii) Gives correct answer</td>
<td>no</td>
<td>1</td>
<td>/</td>
</tr>
<tr>
<td>8</td>
<td>Gives correct answer</td>
<td>Yes.</td>
<td>1</td>
<td>/</td>
</tr>
<tr>
<td></td>
<td>Gives correct explanation</td>
<td>( \frac{24}{30} = \frac{8}{10} = 0.8 )</td>
<td>1</td>
<td>/</td>
</tr>
<tr>
<td>9</td>
<td>Gives correct answer</td>
<td>No.</td>
<td>1</td>
<td>/</td>
</tr>
<tr>
<td></td>
<td>Gives correct explanation</td>
<td>( \frac{40}{60} = \frac{2}{3} = 0.6 )</td>
<td>1</td>
<td>/</td>
</tr>
<tr>
<td>Bonus</td>
<td>Gives correct answer</td>
<td>Since there is no final digit, there is no exact position.</td>
<td>yes / no</td>
<td>/</td>
</tr>
<tr>
<td>Advanced</td>
<td>a) Gives correct answer</td>
<td>&gt;</td>
<td>1</td>
<td>/</td>
</tr>
<tr>
<td>10</td>
<td>b) Gives correct answer</td>
<td>&lt;</td>
<td>1</td>
<td>/</td>
</tr>
</tbody>
</table>
### Scoring Guides for Sample Unit Quizzes and Tests

**Unit 5: The Number System**

(continued)

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Advanced 11</td>
<td>a) Gives correct answer</td>
<td>0.59</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>0.404</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bonus: Gives correct answer</td>
<td>0.7</td>
<td>yes / no</td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>Total Points</strong></td>
<td></td>
<td><strong>/37</strong></td>
<td></td>
</tr>
</tbody>
</table>
### Rubric for Unit 5: The Number System
Test (Lessons 46 to 52), p. V-44

<table>
<thead>
<tr>
<th>Common Core State Standard</th>
<th>Assessed by Question(s) …</th>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
<th>Level 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.NS.A.1d</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Apply properties of operations as strategies to add and subtract rational numbers.</td>
<td></td>
<td>10, 11</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.NS.A.2c</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Apply properties of operations as strategies to multiply and divide rational numbers.</td>
<td></td>
<td>3, 4, 5, 6, 7, 8, 9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.NS.A.2d</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.</td>
<td></td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Comments**
### Scoring Guides for Sample Unit Quizzes and Tests
Unit 6: Geometry

#### Quiz (Lessons 24 to 27), p. V-49

**Common Core State Standards Emphasized:** 7.G.B.6, 7.EE.B.4b

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a) Gives correct area of base</td>
<td>$6 \times 4 + 2 = 12$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Gives correct expression for volume</td>
<td>$V = 12 \times 8.5$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td>$= 102$ m$^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Gives correct expression for volume</td>
<td>$V = 7 \times 2 \times 4$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>$+ 2 \times 2 \times 1$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 56 + 4$</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 60$ ft$^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Gives correct expression for surface area</td>
<td>$SA = 2 \times (4 \times 2)$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ 2 \times (2 \times 3)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ 2 \times (4 \times 3)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>$= 16 + 12 + 24$</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 52$ cm$^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bonus</td>
<td>Gives correct answer</td>
<td></td>
<td>yes / no</td>
<td></td>
</tr>
</tbody>
</table>

**Total Points** 9

#### Quiz (Lessons 28 to 31), p. V-51

**Common Core State Standards Emphasized:** 7.G.A.3, 7.G.B.6, 7.EE.B.4a

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Gives correct expression for surface area</td>
<td>$SA = 2 \times (5 \times 8)$</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Makes one error in expression</td>
<td>$+ 6 \times 8 + 12 \times 8$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ 2 \times (4 \times (12 + 6) \div 2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 80 + 48 + 96$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ 72$</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= 296$ m$^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>a) Gives correct answer</td>
<td>rectangle</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>triangle</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Bonus</td>
<td>Gives correct answer</td>
<td></td>
<td>yes / no</td>
<td></td>
</tr>
</tbody>
</table>

**Total Points** 6
## Scoring Guides for Sample Unit Quizzes and Tests
### Unit 6: Geometry

#### Test (Lessons 24 to 31), p. V-53

**Common Core State Standards Emphasized:** 7.G.A.3, 7.G.B.6, 7.EE.B.4a, 7.EE.B.4b

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Correctly converts units</td>
<td>15 mm = 1.5 cm</td>
<td>1</td>
<td>/3</td>
</tr>
<tr>
<td></td>
<td>Gives correct equation</td>
<td>$V = 1.5 \times 1.5 \times 1.5$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td>$V = 3.375 \text{ cm}^3$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Gives correct area of bases</td>
<td>$2 \times ((12 \times 8) \div 2) = 48$</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>$2(10 \times 12) + 12 \times 12 = 384$</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Gives correct area of rectangular faces</td>
<td>$SA = 480 \text{ in}^2$</td>
<td>1</td>
<td>/5</td>
</tr>
<tr>
<td>3</td>
<td>Gives correct area of base</td>
<td>$3x + 2$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct expression for volume</td>
<td>$V = 3x + 2 \times 10 = 15x$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct inequality</td>
<td>$15x \geq 60$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correctly solves inequality</td>
<td>$x \geq 60 \div 15$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td>$x \geq 4 \text{ m}$</td>
<td>1</td>
<td>/5</td>
</tr>
<tr>
<td>4</td>
<td>Gives correct expression for area to be painted</td>
<td>$SA = 6 \times 6 + 2 \times (5 \times 6) + 6 \times 8 + 2 \times (12 \times 8) + 2 \times (4 \times (12 + 6) + 2) = 36 + 60 + 48 + 192 + 72$</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Makes one error in expression</td>
<td>$408 \div 55 \approx 7.4$</td>
<td>2</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>Gives correct answer with units</td>
<td>Rob needs 8 tubes of paint.</td>
<td>1</td>
<td>/6</td>
</tr>
<tr>
<td>5</td>
<td>a) Gives correct answer</td>
<td>congruent</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>similar</td>
<td>1</td>
<td>/2</td>
</tr>
<tr>
<td>6</td>
<td>a) Gives correct answer</td>
<td>Teacher to check.</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>rectangle</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct explanation</td>
<td>Opposite sides are parallel and angles are 90°.</td>
<td>1</td>
<td>/3</td>
</tr>
</tbody>
</table>
### Question 7

<table>
<thead>
<tr>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Marks one pair of parallel edges</td>
<td><img src="image1.png" alt="Diagram" /></td>
<td>1</td>
<td>/3</td>
</tr>
<tr>
<td>b) Marks two pairs of parallel edges Marks one pair of parallel edges</td>
<td><img src="image2.png" alt="Diagram" /></td>
<td>2 (1)</td>
<td></td>
</tr>
</tbody>
</table>

**Bonus**  
Gives correct answer  
Sample answer:  
Prism 1:  
$l = 5\text{ in}$,  
$w = 2\text{ in}$,  
$h = 2\text{ in}$  
$SA = 48\text{ in}^2$  
Prism 2:  
$l = 10\text{ in}$,  
$w = 1\text{ in}$,  
$h = 2\text{ in}$  
$SA = 64\text{ in}^2$  
Prism 1 has smaller surface area.  

**Total Points**  
/27
### Rubric for Unit 6: Geometry
Test (Lessons 24 to 31), p. V-53

<table>
<thead>
<tr>
<th>Common Core State Standard</th>
<th>Assessed by Question(s) …</th>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
<th>Level 4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>7.G.A.3</strong> Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.</td>
<td>5, 6, 7</td>
<td>Can answer few, if any, questions accurately and independently.</td>
<td>Can answer some questions accurately and independently.</td>
<td>Can answer most questions accurately and independently.</td>
<td>Can answer all or almost all questions, including bonuses, accurately and independently.</td>
</tr>
<tr>
<td><strong>7.G.B.6</strong> Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.</td>
<td>1, 2, 3, 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>7.EE.B.4a</strong> Solve word problems leading to equations of the form ( px + q = r ) and ( p(x + q) = r ), where ( p ), ( q ), and ( r ) are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>7.EE.B.4b</strong> Solve word problems leading to inequalities of the form ( px + q &gt; r ) or ( px + q &lt; r ), where ( p ), ( q ), and ( r ) are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. For example: As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Comments**
## Scoring Guides for Sample Unit Quizzes and Tests

### Unit 7: Statistics and Probability

**Quiz (Lessons 10 to 15), p. V-56**

**Common Core State Standards Emphasized:** 7.SP.A.1, 7.SP.B.4

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Gives correct equation</td>
<td>((x + 76 + 74 + 84) ÷ 4 = 80)</td>
<td>1</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td></td>
<td>Correctly solves equation</td>
<td>(x + 234 = 320)</td>
<td>2</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>(x = 320 - 234)</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(x = 86)</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td>2</td>
<td>a) Gives correct answer for Set 1</td>
<td>16</td>
<td>1</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td></td>
<td>Gives correct answer for Set 2</td>
<td>2</td>
<td>1</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>Set 2</td>
<td>1</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td>3</td>
<td>Gives correct mean</td>
<td>8</td>
<td>1</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td></td>
<td>Correctly finds the absolute deviation for all data points</td>
<td>2, 1, 4, 3</td>
<td>2</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td></td>
<td>Makes one computational error</td>
<td>((2 + 1 + 4 + 3) ÷ 4 = 2.5)</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td></td>
<td>Gives correct answer for MAD</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td>4</td>
<td>a) Gives correct answer</td>
<td>representative</td>
<td>1</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>biased</td>
<td>1</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td></td>
<td>c) Gives correct answer</td>
<td>biased</td>
<td>1</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td>Bonus</td>
<td>Gives correct answer</td>
<td>5, 8, 10, 13, 14 or 5, 8, 13, 14</td>
<td>yes / no</td>
<td>(\frac{1}{3})</td>
</tr>
</tbody>
</table>

### Total Points: \(\frac{1}{3}\)
Quiz (Lessons 16 to 19), p. V-58

**Common Core State Standards Emphasized:** 7.SP.A.2, 7.SP.B.3, 7.SP.B.4, 7.RP.A.2

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Gives correct ratio</td>
<td>( \frac{3}{100} = \frac{x}{3,000} )</td>
<td>1</td>
<td>/2</td>
</tr>
<tr>
<td></td>
<td>Gives correct answer</td>
<td>( x = 90 )</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>a) Gives correct answer</td>
<td>A</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) Gives correct answer</td>
<td>B</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>c) Gives correct answer</td>
<td>A</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>d) Gives correct answer</td>
<td>high overlap</td>
<td>1</td>
<td>/4</td>
</tr>
<tr>
<td>3</td>
<td>Gives correct answer for first experiment</td>
<td>second results</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct answer for second experiment</td>
<td>first results</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct explanation using MAD</td>
<td>The larger sample size has a smaller MAD and a mean closer to 0.300.</td>
<td>1</td>
<td>/4</td>
</tr>
<tr>
<td></td>
<td>Gives correct explanation using mean</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bonus</td>
<td>Gives correct answer</td>
<td>( \frac{20}{10} = 2 )</td>
<td>yes / no</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>There is not high overlap.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Total Points** /10
## Scoring Guides for Sample Unit Quizzes and Tests
### Unit 7: Statistics and Probability

(continued)

**Test (Lessons 10 to 19), p. V-60**

<table>
<thead>
<tr>
<th>Question</th>
<th>How to Score</th>
<th>Answer</th>
<th>Number of Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a) Gives three correct answers for mean</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Makes one error for mean</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives three correct answers for median</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Makes one error for median</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives three correct answers for MAD</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Makes one error for MAD</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives three correct answers for IQR</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Makes one error for IQR</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>80 80 80</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>80 80 80</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3 10 8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4 15 21</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) Gives correct answer</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct explanation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>No.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>The means are all 80 and the medians are all 80.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c) Gives correct answer</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct explanation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Science</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>The IQR and the MAD are the smallest.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>a) Gives correct mean</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct median</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) Gives correct ratio</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \frac{4}{20} = \frac{x}{120} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( x = 24 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>/4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Circles three correct answers</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Circles one or two correct answers</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>IQR, range, MAD</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>/2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>a) Gives correct median for Airport A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct Q1 for Airport A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct Q3 for Airport A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct median for Airport B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct Q1 for Airport B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gives correct Q3 for Airport B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Draws correct box plot for Airport A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Draws correct box plot for Airport B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Teacher to check.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>22.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>27.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>15</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>40</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) Gives correct answer</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>small overlap</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Question</td>
<td>How to Score</td>
<td>Answer</td>
<td>Number of Points</td>
<td>Total Points</td>
</tr>
<tr>
<td>----------</td>
<td>--------------</td>
<td>--------</td>
<td>------------------</td>
<td>--------------</td>
</tr>
</tbody>
</table>
| 4        | c) Gives correct answer  
Gives correct explanation | Airport A.  
It has a smaller mean and IQR, so it has shorter delays. | 1 | 1 |
| 5        | a) Gives correct answer | 140 | 1 | /11 |
|          | b) Correctly finds the absolute deviation for all data points  
Makes one computational error  
Gives correct answer for MAD  
Makes one computational error | 0, 5, 0, 10, 15  
(5 + 10 + 15) ÷ 5  
= 30 ÷ 5 = 6 | 2 | (1) 2 |
|          | c) Gives correct answer | 3 | 1 | /7 |
|          | d) Gives correct answer | 3 ÷ 6 = 0.5 | 1 | /7 |
| Bonus    | Gives correct answer | \[
\frac{75 + 81 + 80 + x}{5} = 80 \\
x + 236 = 400 \\
x = 400 - 236 \\
x = 164 \\
Pam must average 82 on her next two tests. | yes / no | /36 |
<table>
<thead>
<tr>
<th>Common Core State Standard</th>
<th>Assessed by Question(s) ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.SP.A.1 Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.</td>
<td>2</td>
</tr>
<tr>
<td>7.SP.A.2 Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.</td>
<td>2</td>
</tr>
<tr>
<td>7.SP.B.3 Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.</td>
<td>4</td>
</tr>
<tr>
<td>Common Core State Standard</td>
<td>Assessed by Question(s) …</td>
</tr>
<tr>
<td>----------------------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td>7.SP.B.4 Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.</td>
<td>1, 3, 4, 5</td>
</tr>
</tbody>
</table>

Comments
Grade 7 Common Core State Standards Curriculum Correlations

NOTE: The italicized gray JUMP Math lessons contain prerequisite material for the Common Core standards.

Domain

- RP  Ratios and Proportional Relationships
- NS  The Number System
- EE  Expressions and Equations
- G  Geometry
- SP  Statistics and Probability

Cluster

<table>
<thead>
<tr>
<th>7.RP</th>
<th>Ratios and Proportional Relationships</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.RPA</td>
<td>Analyze proportional relationships and use them to solve real-world and mathematical problems.</td>
</tr>
</tbody>
</table>

<p>| 7.RPA.1 | Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks 1/2 mile in each 1/4 hour, compute the unit rate as the complex fraction 1/2 / 1/4 miles per hour, equivalently 2 miles per hour. |</p>
<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>G7-9, 10</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>NS7-44</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>RP7-27 to 30</td>
</tr>
</tbody>
</table>

<p>| 7.RPA.2 | Recognize and represent proportional relationships between quantities. |</p>
<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>RP7-1 to 5, RP7-6 to 9</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>G7-8 to 10</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>RP7-36</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>G7-18, 23</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>SP7-16</td>
</tr>
</tbody>
</table>

<p>| 7.RPA.2a | Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin. |</p>
<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>EE7-20</td>
</tr>
</tbody>
</table>

<p>| 7.RPA.2b | Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships. |</p>
<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>RP7-28 to 30</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>EE7-19, 21, 22</td>
</tr>
</tbody>
</table>

<p>| 7.RPA.2c | Represent proportional relationships by equations. For example, if total cost ( t ) is proportional to the number ( n ) of items purchased at a constant price ( p ), the relationship between the total cost and the number of items can be expressed as ( t = pn ). |</p>
<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>EE7-18, 19, 21</td>
</tr>
<tr>
<td>7.RPA.2d</td>
<td>Explain what a point ((x, y)) on the graph of a proportional relationship means in terms of the situation, with special attention to the points ((0, 0)) and ((1, r)) where (r) is the unit rate.</td>
<td>Part</td>
</tr>
<tr>
<td>---------</td>
<td>------------------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7.RPA.3</th>
<th>Use proportional relationships to solve multistep ratio and percent problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.</th>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
<td>RP7-10, 11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>5</td>
<td>RP7-22 to 24</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>2</td>
<td>RP7-31, 32, 34, 35, 37</td>
</tr>
</tbody>
</table>
### 7.NS.1 The Number System

#### 7.NS.A Apply and extend previous understandings of operations with fractions.

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td><strong>7.NS-1, 10, 12</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>NS7-17</strong></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td><strong>NS7-18 to 21</strong></td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td><strong>SP7-12</strong></td>
</tr>
</tbody>
</table>

**7.NS.A.1** Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

**7.NS.A.1a** Describe situations in which opposite quantities combine to make 0. *For example, a hydrogen atom has 0 charge because its two constituents are oppositely charged.*

**7.NS.A.1b** Understand \( p + q \) as the number located a distance \(|q|\) from \(p\), in the positive or negative direction depending on whether \(q\) is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.

**7.NS.A.1c** Understand subtraction of rational numbers as adding the additive inverse, \( p - q = p + (-q) \). Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.

**7.NS.A.1d** Apply properties of operations as strategies to add and subtract rational numbers.

**7.NS.A.2** Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

**7.NS.A.2a** Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as \((-1)(-1) = 1\) and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.
<table>
<thead>
<tr>
<th>7.NS.A.2b</th>
<th>Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If ( p ) and ( q ) are integers, then ( -(p/q) = (-p)/q = p/(-q) ). Interpret quotients of rational numbers by describing real-world contexts.</th>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>7</td>
<td>NS7-29</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7.NS.A.2c</th>
<th>Apply properties of operations as strategies to multiply and divide rational numbers.</th>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>5</td>
<td>RP7-14, 15, RP7-16</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>7</td>
<td>NS7-25 to 27, 29 to 31</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>1</td>
<td>NS7-32 to 38, 40, 41</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>5</td>
<td>NS7-46, 47, 52</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7.NS.A.2d</th>
<th>Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.</th>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>1</td>
<td>NS7-40</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>5</td>
<td>NS7-48 to 52</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7.NS.A.3</th>
<th>Solve real-world and mathematical problems involving the four operations with rational numbers.</th>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>7</td>
<td>NS7-28 to 31</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>1</td>
<td>NS7-39, 44</td>
</tr>
</tbody>
</table>
### 7.EE  Expressions and Equations

#### 7.EE.A  Use properties of operations to generate equivalent expressions.

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>EE7-1, 3 to 5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>EE7-2, 6 to 12</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>NS7-43</td>
</tr>
</tbody>
</table>

#### 7.EE.A.1  Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.

#### 7.EE.A.2  Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. For example, \(a + 0.05a = 1.05a\) means that "increase by 5%" is the same as "multiply by 1.05."

### 7.EE  Expressions and Equations

#### 7.EE.B  Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>RP7-12, 17, 18</td>
</tr>
<tr>
<td></td>
<td></td>
<td>RP7-21, 22</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>EE7-19</td>
</tr>
</tbody>
</table>

#### 7.EE.B.3  Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. For example: If a woman making \$25 an hour gets a 10% raise, she will make an additional \(\frac{1}{10}\) of her salary an hour, or \$2.50, for a new salary of \$27.50. If you want to place a towel bar 9 3/4 inches long in the center of a door that is 27 1/2 inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.

#### 7.EE.B.4  Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>RP7-33</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>EE7-13, EE7-23 to 25</td>
</tr>
</tbody>
</table>

#### 7.EE.B.4a  Solve word problems leading to equations of the form \(px + q = r\) and \(p(x + q) = r\), where \(p\), \(q\), and \(r\) are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>EE7-14 to 17</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>G7-11 to 16, 22, 23</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>G7-29</td>
</tr>
</tbody>
</table>
7.EE.B.4b Solve word problems leading to inequalities of the form \( px + q > r \) or \( px + q < r \), where \( p, q, \) and \( r \) are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. For example: As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make, and describe the solutions.

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>EE7-26 to 28</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>G7-16</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>G7-24, 25</td>
</tr>
</tbody>
</table>
## 7.G  Geometry

### 7.G.A  Draw construct, and describe geometrical figures and describe the relationships between them.

<table>
<thead>
<tr>
<th>7.G.A.1</th>
<th>Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.</th>
<th>JUMP Math Grade 7 Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Part</td>
<td>Unit</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7.G.A.2</th>
<th>Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.</th>
<th>JUMP Math Grade 7 Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Part</td>
<td>Unit</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7.G.A.3</th>
<th>Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.</th>
<th>JUMP Math Grade 7 Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Part</td>
<td>Unit</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

### 7.G.B  Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

<table>
<thead>
<tr>
<th>7.G.B.4</th>
<th>Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.</th>
<th>JUMP Math Grade 7 Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Part</td>
<td>Unit</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7.G.B.5</th>
<th>Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.</th>
<th>JUMP Math Grade 7 Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Part</td>
<td>Unit</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7.G.B.6</th>
<th>Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.</th>
<th>JUMP Math Grade 7 Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Part</td>
<td>Unit</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>7.SP</td>
<td>Statistics and Probability</td>
<td></td>
</tr>
<tr>
<td>------</td>
<td>----------------------------</td>
<td></td>
</tr>
<tr>
<td>7.SPA</td>
<td><strong>Use random sampling to draw inferences about a population.</strong></td>
<td></td>
</tr>
<tr>
<td><strong>7.SPA.1</strong></td>
<td>Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.</td>
<td></td>
</tr>
<tr>
<td><strong>7.SPA.2</strong></td>
<td>Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. <em>For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.</em></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>JUMP Math Grade 7 Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7.SP</th>
<th>Statistics and Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>7.SRB</strong></td>
<td><strong>Draw informal comparative inferences about two populations.</strong></td>
</tr>
<tr>
<td><strong>7.SRB.3</strong></td>
<td>Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. <em>For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.</em></td>
</tr>
<tr>
<td><strong>7.SRB.4</strong></td>
<td>Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. <em>For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.</em></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>JUMP Math Grade 7 Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>
### 7.SP  Statistics and Probability

<table>
<thead>
<tr>
<th>7.SPC</th>
<th>Investigate chance processes and develop, use, and evaluate probability models.</th>
<th>JUMP Math Grade 7 Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.SPC.5</td>
<td>Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.</td>
<td>Part</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>7.SPC.6</td>
<td>Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.</td>
<td>Part</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>7.SPC.7</td>
<td>Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.</td>
<td>Part</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>7.SPC.7a</td>
<td>Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.</td>
<td>Part</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>7.SPC.7b</td>
<td>Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?</td>
<td>Part</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>7.SPC.8</td>
<td>Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.</td>
<td>Part</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>7.SPC.8a</td>
<td>Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.</td>
<td>Part</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>7.SPC.8b</td>
<td>Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., “rolling double sixes”), identify the outcomes in the sample space which compose the event.</td>
<td>Part</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>8</td>
</tr>
</tbody>
</table>
### 7.SP.C.8c

Design and use a simulation to generate frequencies for compound events. For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?

<table>
<thead>
<tr>
<th>Part</th>
<th>Unit</th>
<th>Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>SP7-8, 9</td>
</tr>
</tbody>
</table>